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EXISTENCE RESULTS FOR NONLINEAR ELLIPTIC PROBLEMS ON FRACTAL DOMAINS

M. FERRARA, G. MOLICA BISCI, AND D. REPOVŠ

ABSTRACT. Some existence results for a parametric Dirichlet problem defined on the Sierpiński fractal are proved. More precisely, a critical point result for differentiable functionals is exploited in order to prove the existence of a well determined open interval of positive eigenvalues for which the problem admits at least one non-trivial weak solution.

1. INTRODUCTION

The purpose of the present paper is to establish some existence results for the following Dirichlet problem

$$\begin{cases} \Delta u(x) + a(x)u(x) = \lambda g(x)f(u(x)), & x \in V \setminus V_0, \\ u|_{V_0} = 0, \end{cases} \qquad (S^{f,g}_{a,\lambda})$$

where V stands for the Sierpiński gasket in $(\mathbb{R}^{N-1}, |\cdot|)$, $N \geq 2$, V_0 is its intrinsic boundary (consisting of its N corners), Δ denotes the weak Laplacian on V and λ is a positive real parameter. We assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and that the variable potentials $a, g : V \to \mathbb{R}$ satisfy the following conditions:

(h_1) $a \in L^1(V, \mu)$ and $a \leq 0$ almost everywhere in V;
(h_2) $g \in C(V)$ with $g \leq 0$ and such that the restriction of g to every open subset of V is not identically zero.

Many physical problems on fractal domains lead to nonlinear models (for example, reaction-diffusion equations, problems on elastic fractal media or fluid flow through fractal regions), so it is appropriate to study nonlinear partial differential equations on fractals.

In recent years there has been an increasing interest in studying such equations, also motivated and stimulated by the considerable amount of literature devoted to the definition of a Laplacian operator for functions on fractal domains. Among the contributions to the theory of nonlinear elliptic equations on fractals we mention [9, 11, 13, 14, 18].

For instance, Falconer and Hu, in [11], considered Dirichlet problems defined on the Sierpiński fractal. More precisely, under certain hypotheses on the nonlinear term, the existence of at least one non-trivial solution was proved (see Theorems 3.5 and 3.18 of [11] and Remark 3.4 below).

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Further, in [13], Hu analyzed the following problem

$$\begin{cases} \Delta u(x) + a(x)u(x) = \lambda f(x, u(x)), & x \in K \setminus K_0, \\ u|_{K_0} = 0, \end{cases} \tag{$S^f_{a,\lambda}$}$$

where K is the Sierpiński gasket (of intrinsic boundary K_0) in $\mathbb{R}^2$ and $f : K \times \mathbb{R} \to \mathbb{R}$ is a continuous symmetric function satisfying some monotonicity properties. More precisely, in Theorem 2.2 of the above cited work, the existence of p–pairs of non-trivial solutions of $(S^f_{a,\lambda})$ was achieved in relation with the value of the p–th eigenvalue, say λ_p, of the problem

$$\begin{cases} \Delta u(x) + \lambda a(x)u(x) = 0, & x \in K \setminus K_0, \\ u|_{K_0} = 0. \end{cases}$$

Very recently, Breckner, Repovš and Varga [8] studied the existence of multiple solutions for the problem $(S^{f,g}_{a,\lambda})$ through variational methods. Their approach ensures the existence of at least three weak solutions under some hypotheses on the behaviour of the nonlinearity f.

Successively, Breckner, Rădulescu, and Varga [7] proved the existence of infinitely many solutions of problem $(S^{f,g}_{a,1})$ under the key assumption that the nonlinearity f is non-positive in a sequence of positive intervals (see Remark 3.2). See also the papers [2, 3, 4, 5, 6, 16] and references therein for related topics.

In this paper, requiring an asymptotic behaviour of the nonlinearity f at zero, we are able to determine a precise open interval of positive parameters λ, for which problem $(S^{f,g}_{a,\lambda})$ admits at least one non-trivial weak solution in the Sobolev space $H^1_0(V)$.

The proofs of our main results are based on a critical point theorem due to Ricceri [17] in the form given in [1].

Theorem 1.1. *Let X be a reflexive real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two sequentially weakly lower semicontinuous functionals. Assume also that Φ is strongly continuous and coercive. For every $r > \inf_X \Phi$, put*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{\left(\sup_{v \in \Phi^{-1}(]-\infty, r[)} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)}.$$

Then, for every $r > \inf_X \Phi$ and every $\lambda \in]0, 1/\varphi(r)[$, the restriction of the functional $I_\lambda := \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

Clearly, the abstract framework introduced in the above mentioned paper is adaptable to our context by using the geometric and analytic properties of the Sierpiński fractal as, for instance, the careful analysis of the Sobolev-type inequality (see, for instance, [11, Lemma 2.4] and Section 2)

$$\sup_{x,y \in V_*} \frac{|u(x) - u(y)|}{|x - y|^\sigma} \leq (2N + 3)\sqrt{W(u)}, \tag{1}$$

where

$$\sigma := \frac{\log((N + 2)/N)}{2 \log 2},$$

and V_* and W will be defined in the sequel.

A special case of our results reads as follows.

Theorem 1.2. *Let $f:\mathbb{R}\to\mathbb{R}$ be a non-negative continuous function. Assume that*

$$\lim_{\xi\to 0^+}\frac{\displaystyle\int_0^\xi f(t)dt}{\xi^2}=+\infty. \tag{h_0'}$$

Then the positive number λ^*, given by

$$\lambda^* := -\frac{1}{2(2N+3)^2\left(\displaystyle\int_V g(x)d\mu\right)}\sup_{\gamma>0}\frac{\gamma^2}{\displaystyle\int_0^\gamma f(t)dt},$$

is such that, for every $\lambda\in]0,\lambda^*[$, the elliptic Dirichlet problem $(S_{a,\lambda}^{f,g})$ admits at least one non-trivial weak solution $u_\lambda\in(H_0^1(V),\|\cdot\|)$. Furthermore, $\lim_{\lambda\to 0^+}\|u_\lambda\|=0$.

This paper is organized as follows. In Section 2 we recall the geometrical construction of the Sierpiński gasket and our variational framework. Successively, Section 3 is devoted to the proof of the main theorem. Finally, in the last section, we give an application of the obtained results.

We cite the very recent monograph by Kristály, Rădulescu and Varga [15] as a general reference for the basic notions used here.

2. Abstract Framework

Let $N\geq 2$ be a natural number and let $p_1,\dots,p_N\in\mathbb{R}^{N-1}$ be so that $|p_i-p_j|=1$ for $i\neq j$. Define, for every $i\in\{1,\dots,N\}$, the map $S_i\colon\mathbb{R}^{N-1}\to\mathbb{R}^{N-1}$ by

$$S_i(x)=\frac{1}{2}\,x+\frac{1}{2}\,p_i\,.$$

Let $\mathcal{S}:=\{S_1,\dots,S_N\}$ and denote by $F:\mathbb{P}(\mathbb{R}^{N-1})\to\mathbb{P}(\mathbb{R}^{N-1})$ the map assigning to a subset A of $\mathbb{R}^{N-1}$ the set

$$F(A)=\bigcup_{i=1}^N S_i(A).$$

It is known that there is a unique non-empty compact subset V of $\mathbb{R}^{N-1}$, called the *attractor* of the family $\mathcal{S}$, such that $F(V)=V$; see, Theorem 9.1 in Falconer [10].

The set V is called the *Sierpiński gasket* in $\mathbb{R}^{N-1}$ of *intrinsic boundary* $V_0:=\{p_1,\dots,p_N\}$. Let μ be the normalized restriction of the d-dimensional Hausdorff measure $\mathcal{H}^d$ on $\mathbb{R}^{N-1}$ to the subsets of V, so $\mu(V)=1$.

Further, the following property of μ will be useful in the sequel:

$$\mu(B)>0,\ \text{for every non-empty open subset } B \text{ of } V. \tag{2}$$

In other words, the support of μ coincides with V (see, for instance, Breckner, Rădulescu and Varga [7] for more details).

Denote by $C(V)$ the space of real-valued continuous functions on V and by

$$C_0(V):=\{u\in C(V)\mid u|_{V_0}=0\}.$$

The spaces $C(V)$ and $C_0(V)$ are endowed with the usual supremum norm $\|\cdot\|_\infty$. For a function $u\colon V\to \mathbb{R}$ and for $m\in\mathbb{N}$ let

$$W_m(u)=\left(\frac{N+2}{N}\right)^m \sum_{\substack{x,y\in V_m\\ |x-y|=2^{-m}}} (u(x)-u(y))^2, \tag{3}$$

where $V_m := F(V_{m-1})$, for $m\geq 1$. Put $V_* := \bigcup_{m\geq 0} V_m$ and note that $V=\overline{V_*}$.

We have $W_m(u)\leq W_{m+1}(u)$ for very natural m, so we can put

$$W(u)=\lim_{m\to\infty} W_m(u). \tag{4}$$

Define now

$$H_0^1(V):=\{u\in C_0(V)\mid W(u)<\infty\}.$$

It turns out that $H_0^1(V)$ is a dense linear subset of $L^2(V,\mu)$ equipped with the $\|\cdot\|_2$ norm. We now endow $H_0^1(V)$ with the norm

$$\|u\|:=\sqrt{W(u)}.$$

In fact, there is an inner product defining this norm: for $u,v\in H_0^1(V)$ and $m\in\mathbb{N}$ let

$$\mathcal{W}_m(u,v)=\left(\frac{N+2}{N}\right)^m \sum_{\substack{x,y\in V_m\\ |x-y|=2^{-m}}} (u(x)-u(y))(v(x)-v(y)).$$

Put

$$\mathcal{W}(u,v)=\lim_{m\to\infty} \mathcal{W}_m(u,v).$$

Then $\mathcal{W}(u,v)\in\mathbb{R}$ and the space $H_0^1(V)$, equipped with the inner product $\mathcal{W}$, which induces the norm $\|\cdot\|$, becomes a real Hilbert space.

Moreover,

$$\|u\|_\infty\leq (2N+3)\|u\|,\ \text{for every } u\in H_0^1(V), \tag{5}$$

and the embedding

$$(H_0^1(V),\|\cdot\|)\hookrightarrow (C_0(V),\|\cdot\|_\infty) \tag{6}$$

is compact.

For more details concerning the definitions and notions which lead in a natural way to the Sobolev space $H_0^1(V)$ we refer to Fukushima and Shima [12]. See also Sections 1.3 and 1.4 of [19] (this reference applies for $N=3$, but the cases $N\geq 4$ are straightforward generalizations of this one).

Remark 2.1. As pointed out by Falconer and Hu [11], we just observe that if $a\in L^1(V)$ and $a\leq 0$ in V then, by (5), the norm $\|\cdot\|_*$, defined by

$$\|u\|_* := \left(\mathcal{W}(u,u)-\int_V a(x)u(x)^2 d\mu\right)^{1/2}, \quad \forall u\in H_0^1(V),$$

is equivalent to $\|\cdot\|$.

We now state a useful property of the space $H_0^1(V)$ which shows, together with the facts that $(H_0^1(V),\|\cdot\|)$ is a Hilbert space and $H_0^1(V)$ is dense in $L^2(V,\mu)$, that $\mathcal{W}$ is a Dirichlet form on $L^2(V,\mu)$.

Lemma 2.1. *Let $h\colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz mapping with Lipschitz constant $L \geq 0$ and such that $h(0) = 0$. Then, for every $u \in H_0^1(V)$, we have $h \circ u \in H_0^1(V)$ and $\|h \circ u\| \leq L\|u\|$.*

Proof. It is clear that $h \circ u \in C_0(V)$. For every $m \in \mathbb{N}$ we have, by (3) and the Lipschitz property of h, that

$$W_m(h \circ u) \leq L^2 W_m(u).$$

Hence $W(h \circ u) \leq L^2 W(u)$, according to (4). Thus $h \circ u \in H_0^1(V)$ and $\|h \circ u\| \leq L\|u\|$. □

Following Falconer and Hu [11], we can define in a standard way a linear self-adjoint operator $\Delta\colon Z \to L^2(V,\mu)$, where Z is a linear subset of $H_0^1(V)$ which is dense in $L^2(V,\mu)$ (and dense also in $(H_0^1(V), \|\cdot\|)$), such that

$$-\mathcal{W}(u,v) = \int_V \Delta u \cdot v d\mu, \text{ for every } (u,v) \in Z \times H_0^1(V).$$

The operator Δ is called the (*weak*) *Laplacian* on V.

More precisely, let $H^{-1}(V)$ be the closure of $L^2(V,\mu)$ with respect to the pre-norm

$$\|u\|_{-1} = \sup_{\substack{h \in H_0^1(V)\\ \|h\|=1}} |< u, h >|,$$

where

$$< v, h >= \int_V v(x)h(x)d\mu,$$

$v \in L^2(V,\mu)$ and $h \in H_0^1(V)$. Then $H^{-1}(V)$ is a Hilbert space. Then the relation

$$-\mathcal{W}(u,v) =< \Delta u, v >, \quad \forall v \in H_0^1(V),$$

uniquely defines a function $\Delta u \in H^{-1}(V)$ for every $u \in H_0^1(V)$.

Finally, fix $\lambda > 0$. Let $a\colon V \to \mathbb{R}$, $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon V \to \mathbb{R}$ be as in the Introduction. We say that a function $u \in H_0^1(V)$ is a *weak solution* of $(S^{f,g}_{a,\lambda})$ if

$$\mathcal{W}(u,v) - \int_V a(x)u(x)v(x)d\mu + \lambda \int_V g(x)f(u(x))v(x)d\mu = 0,$$

for every $v \in H_0^1(V)$.

While we mainly work with the weak Laplacian, there is also a directly defined version. We say that $\Delta_s u$ is the *standard Laplacian* of u if $\Delta_s u : V \to \mathbb{R}$ is continuous and

$$\lim_{m\to\infty} \sup_{x \in V\setminus V_0} |(N+2)^m (H_m u)(x) - \Delta_s u(x)| = 0,$$

where

$$(H_m u)(x) := \sum_{\substack{y \in V_m\\ |x-y|=2^{-m}}} (u(y) - u(x)),$$

for $x \in V_m$. We say that $u \in C_0(V)$ is a *strong solution* of $(S^{f,g}_{a,\lambda})$ if $\Delta_s u$ exists and is continuous for all $x \in V \setminus V_0$, and

$$\Delta_s u(x) + a(x)u(x) = \lambda g(x)f(u(x)), \quad \forall\, x \in V \setminus V_0.$$

The existence of the standard Laplacian of a function $u \in H_0^1(V)$ implies the existence of the weak Laplacian Δu (see, for completeness, Falconer and Hu [11]).

Remark 2.2. If $a \in C(V)$, $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and $g \in C(V)$, then, using the regularity result Lemma 2.16 of Falconer and Hu [11], it follows that every weak solution of the problem $(S^{f,g}_{a,\lambda})$ is also a strong solution.

3. Main results

Define $F\colon \mathbb{R} \to \mathbb{R}$ by $F(\xi) = \displaystyle\int_0^\xi f(t)dt$ and fix $\lambda > 0$. The functional $I_\lambda\colon H^1_0(V) \to \mathbb{R}$ given by

$$I_\lambda(u) := \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_V a(x)u(x)^2 d\mu + \lambda \int_V g(x)F(u(x))d\mu, \tag{7}$$

for every $u \in H^1_0(V)$, will turn out to be the energy functional attached to problem $(S^{f,g}_{a,\lambda})$.

We have the following result contained in [11, Proposition 2.19] that we recall here in a convenient form.

Lemma 3.1. *The energy functional $I_\lambda\colon H^1_0(V) \to \mathbb{R}$ defined by relation (7) is a $C^1(H^1_0(V), \mathbb{R})$ functional. Moreover, for each point $u \in H^1_0(V)$,*

$$I'_\lambda(u)(v) = \mathcal{W}(u,v) - \int_V a(x)u(x)v(x)d\mu + \lambda \int_V g(x)f(u(x))v(x)d\mu, \ \ \forall\, v \in H^1_0(V).$$

In particular, $u \in H^1_0(V)$ is a weak solution of problem $(S^{f,g}_{a,\lambda})$ if and only if u is a critical point of I_λ.

The aim of the paper is to prove the following result concerning the existence of at least one non-trivial solutions of the problem $(S^{f,g}_{a,\lambda})$.

Theorem 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with $f(0) = 0$. Assume that*

$$-\infty < \liminf_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} \quad \textit{and} \quad \limsup_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} = +\infty. \tag{h$_0$}$$

Then the positive number λ^*, given by

$$\lambda^* := -\frac{1}{2(2N+3)^2\left(\displaystyle\int_V g(x)d\mu\right)} \sup_{\gamma>0} \frac{\gamma^2}{\max\limits_{|\xi|\leq\gamma} F(\xi)},$$

is such that, for every $\lambda \in]0, \lambda^*[$, the problem $(S^{f,g}_{a,\lambda})$ admits at least one non-trivial weak solution $u_\lambda \in H^1_0(V)$. Moreover,

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = 0,$$

and, for every $\bar\gamma > 0$, the function $\lambda \to I_\lambda(u_\lambda)$ is negative and strictly decreasing in

$$\left]0, -\frac{1}{2(2N+3)^2\left(\displaystyle\int_V g(x)d\mu\right)} \frac{\bar\gamma^2}{\max\limits_{|\xi|\leq\bar\gamma} F(\xi)}\right[.$$

Proof. Let us define the functionals $\Phi, \Psi : X \to \mathbb{R}$ by

$$\Phi(u) := \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_V a(x)u(x)^2 d\mu \quad \text{and} \quad \Psi(u) := -\int_V g(x)F(u(x))d\mu,$$

where X denotes the reflexive Banach space $H_0^1(V)$. Now, in order to achieve our goal, fix λ as in the conclusion.

With the above notations we have that $I_\lambda = \Phi - \lambda\Psi$. We seek for weak solutions of problem $(S_{a,\lambda}^{f,g})$ by applying Theorem 1.1. First of all we observe that, by Lemma 3.1, the functional $I_\lambda \in C^1(X, \mathbb{R})$.

Moreover, Φ is obviously coercive and, by using Lemma 5.6 in Breckner, Rădulescu and Varga [7], the functionals Φ and Ψ are weakly sequentially lower semicontinuous on X.

Since $0 < \lambda < \lambda^*$, there exists $\bar\gamma > 0$ such that

$$\lambda < \lambda^*(\bar\gamma) := -\frac{\bar\gamma^2}{2(2N+3)^2\left(\int_V g(x)d\mu\right)\max_{|\xi|\leq\bar\gamma} F(\xi)}. \tag{8}$$

Set $r := \dfrac{\bar\gamma^2}{2(2N+3)^2}$. Due to the compact embedding into $C_0(V)$, by (5), we have

$$\{v \in X : \Phi(v) < r\} \subseteq \{v \in X : \|v\|_\infty \leq \bar\gamma\}.$$

Therefore

$$\begin{aligned}
\varphi(r) &= \inf_{\Phi(u)<r} \frac{\sup_{\Phi(v)<r}\int_V (-g(x))F(v(x))d\mu + \int_V g(x)F(u(x))d\mu}{r - \Phi(u)} \\
&\leq \frac{\sup_{\Phi(v)<r}\int_V (-g(x))F(v(x))d\mu}{r} \\
&\leq -\left(\int_V g(x)d\mu\right)\frac{\max_{|\xi|\leq\bar\gamma} F(\xi)}{r} \\
&= -2(2N+3)^2\left(\int_V g(x)d\mu\right)\frac{\max_{|\xi|\leq\bar\gamma} F(\xi)}{\bar\gamma^2} = \frac{1}{\lambda^*(\bar\gamma)}.
\end{aligned}$$

Thanks to Theorem 1.1, there exists a function $u_\lambda \in \Phi^{-1}(]-\infty, r[)$ such that

$$I'_\lambda(u_\lambda) = \Phi'(u_\lambda) - \lambda\Psi'(u_\lambda) = 0,$$

and, in particular, u_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}(]-\infty, r[)$.

Now, we claim that the function u_λ cannot be trivial, i.e. $u_\lambda \neq 0$. Indeed, fix a non-negative function $u \in X$ such that there is an element $x_0 \in V$ with $u(x_0) > 1$. It follows that

$$D := \{x \in V \mid u(x) > 1\}$$

is a non-empty open subset of V (due to the continuity of u).

Define $h\colon \mathbb{R} \to \mathbb{R}$ as follows

$$h(t) := |\min\{t, 1\}|, \qquad \text{for all } t \in \mathbb{R}.$$

Then $h(0) = 0$ and h is a Lipschitz function whose Lipschitz constant L is equal to 1. Hence, by using Lemma 2.1, it follows that $v := h \circ u \in X$.

Moreover, $v(x) = 1$ for every $x \in D$, and $0 \leq v(x) \leq 1$ for every $x \in V$.
On the other hand, condition

$$-\infty < \liminf_{\xi \to 0^+} \frac{F(\xi)}{\xi^2}$$

implies the existence of real numbers $\rho > 0$ and ϱ such that

$$F(\xi) \geq \varrho \xi^2, \text{ for every } \xi \in [0, \rho[. \tag{9}$$

Further, condition

$$\limsup_{\xi \to 0^+} \frac{F(\xi)}{\xi^2} = +\infty$$

yields the existence of a sequence $\{\xi_n\}$ in $]0, \rho[$ such that $\lim\limits_{n \to \infty} \xi_n = 0$ and

$$\lim_{n \to \infty} \frac{F(\xi_n)}{\xi_n^2} = +\infty. \tag{10}$$

Now, we have that

$$\begin{aligned} I_\lambda(\xi_n v) = \frac{\xi_n^2}{2}\|v\|^2 - \frac{\xi_n^2}{2}\int_V a(x)v(x)^2 d\mu \quad &+ \quad \lambda F(\xi_n)\int_D g(x)d\mu \\ &+ \quad \lambda \int_{V \setminus D} g(x)F(\xi_n v(x))d\mu, \end{aligned}$$

for every $n \in \mathbb{N}$.

Using (9) and the fact that $g \leq 0$ in V, we get

$$\begin{aligned} I_\lambda(\xi_n v) \leq \frac{\xi_n^2}{2}\|v\|^2 - \frac{\xi_n^2}{2}\int_V a(x)v(x)^2 d\mu \quad &+ \quad \lambda F(\xi_n)\int_D g(x)d\mu \\ &+ \quad \lambda \varrho \xi_n^2 \int_{V \setminus D} g(x)v(x)^2 d\mu, \end{aligned}$$

for every $n \in \mathbb{N}$. Thus

$$\frac{I_\lambda(\xi_n v)}{\xi_n^2} \leq \frac{1}{2}\|v\|^2 - \frac{1}{2}\int_V a(x)v(x)^2 d\mu + \lambda \frac{F(\xi_n)}{\xi_n^2}\int_D g(x)d\mu + \lambda\varrho \int_{V \setminus D} g(x)v(x)^2 d\mu.$$

Condition (h_2) and (2) imply that

$$\int_D g(x)d\mu < 0,$$

so we get from (10) and the above inequality that

$$\lim_{n \to \infty} \frac{I_\lambda(\xi_n v)}{\xi_n^2} = -\infty.$$

Then, there is an index n_0 such that $I_\lambda(\xi_n v) < 0$ for every $n \geq n_0$. Now, since

$$\lim_{n \to \infty} \Phi(\xi_n v) = 0,$$

one has that $\xi_n v \in \Phi^{-1}(]-\infty, r[)$ definitively. In conclusion, 0_X cannot be a global minimum for the restriction of the functional I_λ to $\Phi^{-1}(]-\infty, r[)$. Hence, for every $\lambda \in]0, \lambda^*[$ the problem $(S_{a,\lambda}^{f,g})$ admits a non-trivial solution $u_\lambda \in X$.

At this point, we prove that $\|u_\lambda\| \to 0$ as $\lambda \to 0^+$ and that the function $\lambda \to I_\lambda(u_\lambda)$ is negative and decreasing in $]0, \lambda^*[$.

For our goal, let us consider $\bar{\lambda} \in]0, \lambda^*[$. Moreover, let $\bar{\gamma} > 0$ and let $\lambda \in]0, \lambda^*(\bar{\gamma})[$. The functional I_λ admits a non-trivial critical point $u_\lambda \in \Phi^{-1}(]-\infty, r[)$, where

$$r := \frac{\bar{\gamma}^2}{2(2N+3)^2}.$$

Since Φ is coercive and $u_\lambda \in \Phi^{-1}(]-\infty, r[)$ for every $\lambda \in]0, \lambda^*(\bar{\gamma})[$, there exists a positive number L such that

$$\|u_\lambda\| \leq L,$$

for every $\lambda \in]0, \lambda^*(\bar{\gamma})[$.

Therefore, since Ψ' is a compact operator, there exists a positive constant M such that

$$|\Psi(u_\lambda)| \leq \|\Psi'(u_\lambda)\|_{X^*}\|u_\lambda\| < ML^2, \tag{11}$$

for every $\lambda \in]0, \lambda^*(\bar{\gamma})[$.

Now $I'_\lambda(u_\lambda) = 0$, for every $\lambda \in]0, \lambda^*(\bar{\gamma})[$ and in particular

$$I'_\lambda(u_\lambda)(u_\lambda) = 0,$$

that is,

$$\Phi(u_\lambda) = \lambda \int_V g(x)f(u_\lambda(x))u_\lambda(x)d\mu, \tag{12}$$

for every $\lambda \in]0, \lambda^*(\bar{\gamma})[$.

Hence, by (11) and (12) it follows that

$$\lim_{\lambda \to 0^+} \Phi(u_\lambda) = 0. \tag{13}$$

Moreover, one has

$$\frac{\|u_\lambda\|^2}{2} \leq \frac{\|u_\lambda\|^2}{2} - \frac{\displaystyle\int_V a(x)u_\lambda(x)^2 d\mu}{2} = \Phi(u_\lambda), \tag{14}$$

for every $\lambda \in]0, \lambda^*(\bar{\gamma})[$. Then, conditions (13) and (14) yield

$$\lim_{\lambda \to 0^+} \|u_\lambda\| = 0.$$

Further, the map $\lambda \mapsto I_\lambda(u_\lambda)$ is negative in $]0, \lambda^*(\bar{\gamma})[$ since the restriction of the functional I_λ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

Finally, observe that

$$I_\lambda(u) = \lambda \left(\frac{\Phi(u)}{\lambda} - \Psi(u) \right),$$

for every $u \in X$ and fix $0 < \lambda_1 < \lambda_2 < \lambda^*(\bar{\gamma})$.

Moreover, put

$$m_{\lambda_1} := \left(\frac{\Phi(u_{\lambda_1})}{\lambda_1} - \Psi(u_{\lambda_1}) \right) = \inf_{u \in \Phi^{-1}(]-\infty, r[)} \left(\frac{\Phi(u)}{\lambda_1} - \Psi(u) \right),$$

and

$$m_{\lambda_2} := \left(\frac{\Phi(u_{\lambda_2})}{\lambda_2} - \Psi(u_{\lambda_2}) \right) = \inf_{u \in \Phi^{-1}(]-\infty, r[)} \left(\frac{\Phi(u)}{\lambda_2} - \Psi(u) \right).$$

Clearly, as claimed before, $m_{\lambda_i} < 0$ (for $i = 1, 2$), and $m_{\lambda_2} \leq m_{\lambda_1}$ thanks to $\lambda_1 < \lambda_2$.

Then the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $]0,\lambda^*(\bar\gamma)[$ owing to

$$I_{\lambda_2}(u_{\lambda_2}) = \lambda_2 m_{\lambda_2} \leq \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = I_{\lambda_1}(u_{\lambda_1}).$$

The proof is complete. □

Remark 3.1. We observe that condition (h_0) is technical and ensures that the solution, obtained by using Theorem 3.1, is non-trivial. Anyway, the statements of Theorem 3.1 are still true for every continuous function f that does not vanish at zero. In this last case our approach ensures the existence of one non-trivial solution, for $\lambda \in]0,\lambda^*[$, without condition (h_0). If

$$\max_{|\xi|\leq\bar\gamma} F(\xi) = 0,$$

for some $\bar\gamma > 0$, Theorem 3.1 ensures the existence of one non-trivial solution, for every $\lambda \in]0,+\infty[$.

Remark 3.2. If in addition to condition (h_0) in Theorem 3.1, the function f also satisfies

- (h_1') There exist two sequences $\{a_n\}$ and $\{b_n\}$ in $]0,\infty[$ with $b_{n+1} < a_n < b_n$, $\lim\limits_{n\to\infty} b_n = 0$ and such that $f(s) \leq 0$ for every $s \in [a_n, b_n]$;
- (h_2') Either $\sup\{s < 0 \mid f(s) > 0\} = 0$, or there is a $\delta > 0$ with $f|_{[-\delta,0]} = 0$,

then, as proved by Breckner, Rădulescu and Varga in [7], the problem $(S^{f,g}_{a,1})$ admits a sequence $\{u_n\}$ of pairwise distinct weak solutions such that $\lim\limits_{n\to\infty} \|u_n\| = 0$. In particular, $\lim\limits_{n\to\infty} \|u_n\|_\infty = 0$.

Remark 3.3. A sufficient condition that ensures hypothesis (h_0) in Theorem 3.1 is expressed by

$$\lim_{\xi\to 0^+} \frac{F(\xi)}{\xi^2} = +\infty. \tag{h_0'}$$

Further, if f is non-negative, one has

$$\sup_{\gamma>0} \frac{\gamma^2}{\max\limits_{|\xi|\leq\gamma} F(\xi)} = \sup_{\gamma>0} \frac{\gamma^2}{F(\gamma)},$$

since, in this case, $\max\limits_{|\xi|\leq\gamma} F(\xi) = F(\gamma)$, for every positive γ. Hence, Theorem 1.2 in the Introduction immediately follows from Theorem 3.1.

The following example is a direct consequence of Theorem 3.1, bearing in mind Remarks 3.1 and 2.2.

Example 3.1. For each parameter λ belonging to

$$\Lambda := \left]0, \frac{2e^{-2}}{(2N+3)^2}\right[,$$

the following Dirichlet problem

$$\begin{cases} \Delta u(x) + \lambda e^{u(x)} = u(x), & x \in V \setminus V_0, \\ u|_{V_0} = 0, \end{cases} \tag{S_λ}$$

admits at least one non-trivial strong solution. Moreover,

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = 0$$

and the function $\lambda \to I_\lambda(u_\lambda)$ is negative and decreasing in Λ.

Remark 3.4. In [11] Falconer and Hu studied the non-autonomous Dirichlet problem

$$\begin{cases} \Delta u(x) + a(x)u(x) = \lambda f(x, u(x)), & x \in V \setminus V_0, \\ u|_{V_0} = 0, \end{cases} \tag{$S^f_{a,\lambda}$}$$

where $a : V \to \mathbb{R}$ is assumed to be integrable and $f : V \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The celebrated Ambrosetti-Rabinowitz condition

(AR) *there are constants* $\nu > 2$ *and* $r \geq 0$ *such that*

$$tf(x,t) \leq \nu F(x,t) < 0,$$

for every $|t| \geq r$, *uniformly for every* $x \in V$,

is an essential request in almost all the existence theorems contained in the above cited paper. However if, for instance, f is constant for large $|t|$, assumption (AR) is violated, even though $(S^f_{a,\lambda})$ would be expected to have a non-trivial solution. The saddle point theorem copes with this case; see [11, Theorem 4.2]. We observe that Theorem 3.1 (see also Remark 3.1) obtained in this paper does not require a global growth of the non-linearity f in order to obtain the existence of one non-trivial solution as the above example shows.

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(M. Ferrara) University of Reggio Calabria, Reggio Calabria, Italy & CRIOS, University Bocconi, Milano, Italy
E-mail address: `massimiliano.ferrara@unirc.it`

(G. Molica Bisci) Faculty of Architecture, University of Reggio Calabria, Reggio Calabria, Italy
E-mail address: `gmolica@unirc.it`

(D. Repovš) Faculty of Education and Faculty of Mathematics and Physics, University of Ljubljana & Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia
E-mail address: `dusan.repovs@guest.arnes.si`