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EXISTENCE RESULTS FOR SOME DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

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Abstract

In this paper we shall establish sufficient conditions for the existence of solutions of some differential equation and its solvability in C_L, subset of the Banach space $(C[a,b], \|\cdot\|)$. The main tool used in our study is the nonexpansive operator technique.

1 Introduction

Several authors studied a special class of first order differential equations, called iterative differential equations, see for example the papers [5],[9],[10],[11],[12],[15],[16],[18],[20],[21]. These equations are important in the study of infection models and are related to the study of the motion of charged particles with retarded interaction. The general form of these equations is

$$y'(t) = f(x, y(y(t))) \tag{1.1}$$

Buica [5] studied initial value problems for (1.1) and obtained existence and existence and uniqueness results by means of fixed point techiques.
Very recently, Berinde [4] introduced the techique of nonexpansive operators in the study of first order iterative differential equations of the form (1.1) and thus extended the results from [5].
Our main aim in this paper is to use the techique of nonexpansive operators from [4] to more general iterative and non iterative first order differential equations of the form

$$y'(x) = f(x, y(x), y(\lambda x))$$

and

$$y'(x) = f(x, y(x), y(y(x)))$$

respectively, with initial condition

$$y(x_0) = y_0.$$

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2 Fixed point theory of nonexpansive mappings

We extract here the basic theory of nonexpansive mappings, from the paper [4], in order to offer the notions and results that will be needed in the next sections of the paper.
Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an *α-contraction* if there exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y), \quad \forall x, y \in X$$

In the case where $\alpha = 1$ the mapping T is said to be nonexpansive.
Let K be a nonempty subset of a real normed linear space E and $T : K \to K$ be a map. In this setting, T is nonexpansive if

$$||Tx - Ty|| \leq ||x - y||, \quad \forall x, y \in K$$

Althought the nonexpansive mappings are generalizations of α- contractions, they do not inherit properties of contractive mappings. More precisely, if K is a nonempty closed subset of a Banach space E and $T : K \to K$ is a nonexpansive mapping wich is not an α- contractions, then, as is shown by the following example, T may not have fixed points.

Example 2.1. *([10], Example 3.3, pp. 30)*
In the space $c_0(\mathbb{N})$ the isometry defined by

$$T(x_1, x_2, ...) = (1, x_1, x_2, ...)$$

maps the unit ball into its boundary but T has not fixed points.

One of the most important fixed point theorems for nonexpansive mappings, due to Browder, Ghode and Kirk, see e.g. [3], is state as follows.

Theorem 2.1. *Let K be a nonempty closed conex and bounded subset of a uniformly Banach space E. Then any nonexpansive mapping $T : K \to K$ has at least a fixed point.*

Remark 2.1. *Theorem 2.1 provides no information on the approximation of the fixed point of T is given.*

Let K be a convex subset of a normed linear space E and let $T : K \to K$ be a self-mapping. Given an $x_0 \in K$ and a real number $\lambda \in [0, 1]$, the sequence x_n defined by the formula

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, \quad n = 0, 1, 2, ...$$

is usually called *Krasnoselskij iteration* or *Krasnoselskij-Mann iteration.*
For $x_0 \in K$ the sequence x_n defined by

$$x_{n+1} = (1 - \lambda_n) \cdot x_n + \lambda_n \cdot T x_n, n = 0, 1, 2... \tag{2.1}$$

where $\lambda_n \subset [0,1]$ is a sequence of real number satisfying some appropiate condition, is called *Mann iteration.* Edelstein [8] proved that strict convexity of E sufficies for the Krasnoselskij iteration converge to a fixed point of T. The question of whether or not strict convexity can be removed has been answered in the affirmative by Ishikawa [11] by the following result.

Theorem 2.2. *([11]) Let K be a subset of a Banach E and let $T : K \to K$ be a nonexpansive mapping. For arbitrary $x_0 \in K$, consider the Mann iteration process x_n given by (2.1) under the following assumptions:*

(a) $x_n \in K$ for all positive integers n;

(b) $0 \leq \lambda_n \leq b < 1$ for all positive integers n;

(c) $\sum_{n=0}^{\infty} \lambda_n = \infty$. If x_n is bounded, then $x_n - Tx_n \to 0$ as $n \to \infty$.

The following corollaries of Theorem 2.2 will be particularly important for the application part of our paper.

Corollary 2.1. *([6]) Let K be a convex and compact subset of a Banach space E and let $T : K \to K$ be a nonexpansive mapping. If the Mann iteration process x_n satisfies assumptions (a)-(c) in Theorem 2.2 ,then x_n converges strongly to a fixed point of T.*

Proof. See Theorem 6.17 in Chidume [6]. □

Corollary 2.2. *([6]) Let K be a closed bounded convex subset of a real normed space E and $T : K \to K$ be a nonexpansive mapping. If $I - T$ maps closed bounded subset of E into closed subset of E and x_n is the Mann iteration, with λ_n satisfying assuptions (a)-(c) in Theorem 2.2, then x_n converges strongly to a fixed point of T in K.*

Proof. See Corollary 6.19 in Chidume [6]. □

3 Existence theorems and approximation of solutions of some differential equations

The following initial value problem was studied in [5]

$$\begin{cases} y'(x) = f(x, y(y(x))) \\ y(x_0) = y_0 \end{cases} \tag{**}$$

where $x_0, y_0 \in [a,b]$ and $f \in C([a,b],[a,b])$.
For $x \in [a,b]$ denote

$$C_x = \max\{x - a, b - x\},$$

and

$$(*) C_L = \{y \in C([a,b] \times [a,b] : |y(t_1) - y(t_2)| \leq L \cdot |t_1 - t_2|, \forall t_1, t_2 \in [a,b]\}; L > 0$$

For problem (**), Buică in [5] established existence and uniquess results [Theorem 1,2,4 and 5]. We formulate one of them.

Theorem 3.1. *([5]) Assume that the following conditions are satisfied for problem (**)*

(i) $f \in C([a,b] \times [a,b] \times [a,b])$

(ii) there exists $L_1 > 0$ *such that* $|f(s,u) - f(s,v)| \leq L_1 |u - v|$ *for any* $s, u, v \in [a,b]$;

(iii) if L *is the Lipschitz constant involved in (*), then*

$$M = \max\{|f(s,u)| : (s,u) \in [a,b] \times [a,b]\} \leq L$$

(iv) one of the following conditions holds:

a) $M \cdot C_{x_0} \leq C_{y_0}$;

b) $x_0 = a,\ M(b-a) \leq b - y_0, f(s,u) \geq 0,\ \forall s, u \in [a,b]$;

c) $x_0 = b,\ M(b-a) \leq y_0 - a, f(s,u) \geq 0,\ \forall s, u, \in [a,b]$.

(v) $L_1 \cdot C_{x_0} \cdot (L+1) < 1$.

Then there exists a unique solution y^* *of problem (**) in* C_L.

If condition $L_1 \cdot C_{x_0} \cdot (L+1) < 1$ is weakened to $L_1 \cdot C_{x_0} \cdot (L+1) \leq 1$ then the assertion on the existence of a unique solution of problem (**) is not true. The Theorem 3.1 was extended at Theorem 3.3 in [4].
In this paper, our aim is to obtain similar results to the ones in [4], but for the following initial value problem for a differential equation with deviating argument

$$y\prime(x) = f(x, y(x), y(\lambda x)),\ x \in [a,b] \tag{1}$$

$$y(x_0) = y_0 \tag{2}$$

where $x_0, y_0 \in [a,b]$, $\lambda \in (0,1)$ and $f \in C([a,b] \times [a,b] \times [a,b])$.
This equation is more general because it extends the result of [4] or [5].

We formulate the first existence result for solutions of differential equation (3.1) with initial condition (3.2).

Theorem 3.2. *Assume that*

(i) $f \in C([a,b] \times [a,b] \times [a,b])$

(ii) there exists $L_1 > 0$ *such that*
$|f(s,u_1,v_1) - f(s,u_2,v_2)| \leq L_1(|u_1 - u_2| + |v_1 - v_2|)$ *for any* $s, u_i, v_i \in [a,b]$, $i = 1, 2$

(iii) if L *is the Lipschits constant involved in (*), then*

$$M = \max\{|f(s,u,v)| : (s,u,v) \in [a,b] \times [a,b] \times [a,b]\} \leq L$$

(iv) One of the following conditions holds:

a) $M \cdot C_{x_0} \leq C_{y_0}$;

b) $x_0 = a,\ M(b-a) \leq b - y_0, f(s,u,v) \geq 0,\ \forall s, u, v \in [a,b]$;

c) $x_0 = b,\ M(b-a) \leq y_0 - a, f(s,u,v) \geq 0,\ \forall s, u, v \in [a,b]$.

(v) $2 \cdot L_1 \cdot C_{x_0} \leq 1$.

Then the initial value problem (3.1)+(3.2) has at least solution in C_L.

Proof. It follows from [5, Lemma 1] that C_L is a nonempty convex and compact subset of the Banach space $(C[a,b], \|\cdot\|)$ where $\|x\| = \sup_{t\in[a,b]} |x(t)|$.
Consider the integral operator $F : C_L \to C[a,b]$ defined by

$$(Fy)(t) = y_0 + \int_{x_0}^{t} f(s, y(s), y(\lambda s))ds, \quad t \in [a,b].$$

It is clear that $y \in C_L$ is a solution of initial value problem (3.1)+(3.2) if and only if y is a fixed point of F, i.e.,

$$y = Fy$$

We first prove that C_L is an invariant set with respect to F, i.e., we have $F(C_L) \subset C_L$.
If condition $(iv) - a)$ holds, then for any $y \in C_L$ and $t \in [a,b]$ we have

$$\begin{gathered}|(Fy)(t)| \le |y_0| + \left|\int_{x_0}^{t} f(s, y(s), y(\lambda s)ds\right| \le |y_0| + M \cdot |t - x_0| \le b \\ |(Fy)(t)| \ge |y_0| - \left|\int_{x_0}^{t} f(s, y(s), y(\lambda s)ds\right| \ge |y_0| - M \cdot |t - x_0| \ge \\ \ge |y_0| - M \cdot C_{x_0} \ge y_0 - C_{y_0} \ge a.\end{gathered}$$

which shows that $Fy \in [a,b]$,for any $y \in C_L$.

Now, for any $t_1, t_2 \in [a,b]$ we have

$$|(Fy)(t_1) - (Fy)(t_2)| \le \left|\int_{t_1}^{t_2} f(s, y(s), y(\lambda s))ds\right| \le M \cdot |t_1 - t_2| \le L \cdot |t_1 - t_2|$$

Thus, $Fy \in C_L, \forall y \in C_L$. In a similar way we treat the cases $(iv) - b)$ and $(iv) - c)$. Therefore $F : C_L \to C_L$ (i.e., F is a self-mapping of C_L)

We prove that F is nonexpansive operator. Let $y, z \in C_L$ and $t \in [a,b]$. Then

$$\begin{gathered}|F(y)(t) - (Fz)(t)| \le \\ \le \left|\int_{x_0}^{t} f(s, y(s), y(\lambda s)) - f(s, z(s), z(\lambda s))\right| ds \le \\ \le \int_{x_0}^{t} L_1 \left(|y(s) - z(s)| + |y(\lambda s) - z(\lambda s)|\right) ds \le \\ \le 2L_1 \cdot |t - x_0| \cdot \|y - z\| \le 2L_1 \cdot C_{x_0} \cdot \|y - z\|\end{gathered}$$

Now, by taking the maximum in last inequality, we get

$$\|Fy - Fz\| \le 2L_1 \cdot C_{x_0} \cdot \|y - z\|$$

which in viewe of condition (v), proves that F is nonexpansive operator hence continuous.
It now remains to apply the Schauder's fixed point theorem and we obtain the conclusion.

□

It is therefore the aim of this paper to show that if condition (v) hold, then we are still able to approximate a (non-unique) solution of the initial value problem (3.1)+(3.2) by means of a Krasnoselskij-Mann iteration procedure. The next theorem states the main result of this paper.

Theorem 3.3. *Assume that all condition of Theorem 3.1 are satisfied. Then the solution y^* of the initial value problem (3.1)+(3.2) can be approximated by the Krasnoselskij iteration*

$$y_{n+1}(t) = (1-\mu) \cdot y_n(t) + \mu y_0 + \mu \int_{x_0}^{t} f(s, y_n(s), y_n(\lambda s))ds, t \in [a,b], n \geq 1$$

where $\mu \in (0,1)$ and $y_1 \in C_L$ is arbitrary.

Proof. The proof is based on Corollary 1 or 2. □

Now we are applying the same technique for the iterative differential equation

$$y'(x) = f(x, y(x), y(y(x))) \tag{3.3}$$

with initial condition

$$y(x_0) = y_0 \tag{3.4}$$

where $x_0, y_0 \in [a,b]$, $f \in C([a,b] \times [a,b] \times [a,b])$ are given, wich extends the problem studied in [4]. This equation is more general than equation involved in (**). We formulate the second result on existence solutions of the initial value problem (3.3)+(3.4) in C_L.

Theorem 3.4. *Assume that*

(i) $f \in C([a,b] \times [a,b] \times [a,b])$

(ii) there exists $L_1 > 0$ such that

$|f(s,u,v) - f(s,u_2,v_2)| \leq L_1 (|u_1 - u_2| + |v_1 - v_2|)$, *for any* $s, u_i, v_i \in [a,b], i = 1,2$

(iii) if L is the Lipschits constant involved in ( ), then*

$$M = \max\{|f(s,u,v)| : (s,u,v) \in [a,b] \times [a,b] \times [a,b]\} \leq L$$

(iv) One of the following conditions holds:

a) $M \cdot C_{x_0} \leq C_{y_0}$

b) $x_0 = a, M(b-a) \leq b - y_0, f(s,u,v) \geq 0,\ \forall s,u,v \in [a,b]$

c) $x_0 = b, M(b-a) \leq y_0 - a, f(s,u,v) \geq 0,\ \forall s,u,v \in [a,b]$

(v) $L_1(2+L) \cdot C_{x_0} \leq 1$.

Then the initial value problem (3.3)+(3.4) has at least solution in C_L, which can be approximate by the Krasnoselskij iteration

$$y_{n+1}(t) = (1-\mu) \cdot y_n(t) + \mu y_0 + \mu \int_{x_0}^{t} f(s, y_n(s), y_n(y_n(s)))ds, t \in [a,b], n \geq 1$$

where $\mu \in (0,1)$ and $y_1 \in C_L$ is arbitrary.

Proof. We defined the integral operatorul $F : C_L \to C[a,b]$, by

$$(Fy)(t) = y_0 + \int_{x_0}^{t} f(s, y(s), y(y(s)))ds, \quad t \in [a,b].$$

In the same way as Theorem 3.1 we prove that C_L is an invariant set with respect to F, i.e., we have $F(C_L) \subset C_L$.

$$|(Fy)(t)| \leq |y_0| + \left| \int_{x_0}^{t} f(s, y(s), y(y(s)))ds \right| \leq |y_0| + M \cdot |t - x_0| \leq b$$

$$\begin{gathered} |(Fy)(t)| \geq |y_0| - \left| \int_{x_0}^{t} f(s, y(s), y(y(s)))ds \right| \geq |y_0| - M \cdot |t - x_0| \geq \\ \geq |y_0| - M \cdot C_{x_0} \geq y_0 - C_{y_0} \geq a \end{gathered}$$

Thus $Fy \in [a,b]$, for any $y \in C_L$.
For any $t_1, t_2 \in [a,b]$ we have:

$$|(Fy)(t) - (Fz)(t)| \leq \left| \int_{x_0}^{t} f(s, y(s), y(y(s)))ds \right| \leq M \cdot |t - t_2| \leq L \cdot |t_1 - t_2|$$

So, $Fy \in C_L$,for any $y \in C_L$. In a similar way we treat the cases $(iv) - b)$ and $(iv) - c)$.
We prove that F is nonexpansive operator. Let $y, z \in C_L$ and $t \in [a,b]$. Then

$$\begin{gathered} |(Fy)(t) - (Fz)(t)| \leq \int_{x_0}^{t} f\,|(s, y(s), y(y(s))) - f(s, z(s), z(z(s)))|\, ds \leq \\ \leq L_1 \cdot \int_{x_0}^{t} (|y(s) - z(s)| + |y(y(s)) - z(z(s))|)\, ds \leq \\ \leq L_1 \cdot \int_{x_0}^{t} (|y(s) - z(s)| + |y(y(s)) - y(z(s))| + |y(z(s)) - z(z(s))|)\, ds \leq \\ \leq L_1 \cdot \int_{x_0}^{t} (|y(s) - z(s)| + L \cdot |y(s) - z(s)| + |y(z(s)) - z(z(s))|)\, ds \end{gathered}$$

Now, by taking the maximum in last inequality, we get

$$\|Fy - Fz\| \leq L_1(2 + L) \cdot |t - x_0| \cdot \|y - z\| \leq L_1(2 + L) \cdot c_{x_0} \cdot \|y - z\|$$

which in viewe of condition (v), proves that F is nonexpansive operator hence continuous.
It now remains to apply the Schauder's fixed point theorem and we obtain the first part of conclusion and Corollary 1 or 2 get the second part of conclusion. $\square$

In the case $f(t,u,v) = f(t,v)$ the problem (3.3) is the problem (3.1) in [4].
We consider now the following iterative differential equations

$$y'(x) = f(x, y(x), y(y(x)), y(\lambda x)), \quad \lambda \in (0,1) \tag{3.5}$$

whith initial condition

$$y(x_0) = y_0 \tag{3.6}$$

This equation is more general than equation (1.1) or (3.3), because there is an additional disturbance term $y(\lambda x)$. The next result is an existence theorem for the initial value problem (3.5)+(3.6) in C_L.

Theorem 3.5. *Assume that*

(i) $f \in C([a,b] \times [a,b] \times [a,b] \times [a,b])$

(ii) there exists $L_1 > 0$ *such that*

$$(**)\, |f(s,u_1,v_1,w_1) - f(s,u_2,v_2,w_2)| \leq L_1(|u_1-u_2| + |v_1-v_2| + |w_1-w_2|), \quad \forall s, u_i, v_i, w_i \in [a,b], i = 1,2$$

(iii) if L *is the Lipschitz constant involved in (*), then*

$$M = \max\{|f(s,u,v,w)| : (s,u,v,w) \in [a,b] \times [a,b] \times [a,b] \times [a,b]\} \leq L$$

(iv) One of the following conditions holds:

a) $M \cdot C_{x_0} \leq C_{y_0}$

b) $x_0 = a, M(b-a) \leq b - y_0, f(s,u,v,w) \geq 0,\ \forall s,u,v,w \in [a,b]$

c) $x_0 = b, M(b-a) \leq y_0 - b, f(s,u,v,w) \geq 0,\ \forall s,u,v,w \in [a,b].$

(v) $L_1(3+L) \cdot C_{x_0} \leq 1$

Then the initial value problem (3.5)+(3.6) has at least solution in C_L*, which can be approximate by the Krasnoselskij iteration*

$$y_{n+1}(t) = (1-\mu) \cdot y_n(t) + \mu y_0 + \mu \int_{x_0}^{t} f(s, y_n(s), y_n(y_n(s)), y_n(\lambda s))ds, t \in [a,b], n \geq 1$$

where $\mu \in (0,1)$ *and* $y_1 \in C_L$ *is arbitrary.*

Proof. We define the integral operator $F : C_L \to C[a,b]$, by

$$(Fy)(t) = y_0 + \int_{x_0}^{t} f(s, y(s), y(y(s), y(\lambda s))ds, \quad t \in [a,b], y \in C_L$$

In the same way as Theorem 3.1 we prove that C_L is an invariant set with respect to F, i.e., we have $F(C_L) \subset C_L$.

$$|(Fy)(t)| \leq |y_0| + \left| \int_{x_0}^{t} f(s, y(s), y(y(s))y(\lambda s))ds \right| \leq |y_0| - M \cdot |t - x_0| \leq b$$

$$|(Fy)(t)| \geq |y_0| - \left| \int_{x_0}^{t} f(s, y(s), y(y(s)), y(\lambda s))ds \right| \geq |y_0| - M \cdot |t - x_0| \geq \\ \geq |y_0| - M \cdot C_{x_0} \geq a$$

Which means that $Fy \in [a,b]$,for any $y \in C_L$.

For any $t_1, t_2 \in [a,b]$ we have:

$$|(Fy)(t_1) - (Fy)(t_2)| \leq \left| \int_{t_1}^{t_2} f(s, y(s), y(y(s)), y(\lambda s))ds \right| \leq M \cdot |t - t_2| \leq L \cdot |t_1 - t_2|$$

which leads to the fact that $Fy \in C_L$,for any $y \in C_L$. In a similar way we treat the cases $(iv) - b)$ and $(iv) - c)$.

We prove that F is nonexpansive operator. Let $y, z \in C_L$ and $t \in [a,b]$. Then

$$
\begin{gathered}
|(Fy)(t)-(Fz)(t)| \leq \int_{x_0}^{t} |f(s,y(s),y(y(s)),y(\lambda s))-f(s,z(s),z(z(s)),z(\lambda s))|\, ds \leq \\
\leq L_1 \cdot \int_{x_0}^{t} (|y(s)-z(s)|+|y(y(s))-z(z(s))|+|y(\lambda s)-z(\lambda s)|)\, ds \leq \\
\leq L_1 \cdot \int_{x_0}^{t} (|y(s)-z(s)|+|y(y(s))-y(z(s))|+|y(z(s))-z(z(s))|+|y(\lambda s)-z(\lambda s)|)\, ds \leq \\
\leq L_1 \cdot \int_{x_0}^{t} (|y(s)-z(s)|+L\cdot|y(s)-z(s)|+|y(z(s))-z(z(s))|+ \\
+|y(\lambda s)-z(\lambda s)|)ds \leq L_1(3+L)\cdot|t-x_0|\cdot\|y-z\| \leq L_1(3+L)\cdot c_{x_0}\cdot\|y-z\|
\end{gathered}
$$

Now, by taking the maximum in last inequality, we get

$$\|Fy-Fz\| \leq L_1(3+L)\cdot C_{x_0}\cdot\|y-z\|$$

which in view of condition (v), proves that F is nonexpansive operator hence continuous.
It now remains to apply the Schauder's fixed point theorem and we obtain the first part of conclusion and Corollary 1 or 2 get the second part of conclusion. □

4 Examples

We conclude the paper by presenting two examples wich illustrate the generality and efficiency of our results.

Example 4.1. Consider the following initial value problem associated to an iterative differential equation with deviating argument

$$
\begin{cases} y'(x) = -\frac{1}{2} + y(y(x)) + y(\lambda x) \\ y(\frac{1}{2}) = \frac{1}{2} \end{cases} \tag{3.7}
$$

where $x \in [0,1], y \in C^1([0,1],[0,1])\,,\ \lambda \in (0,1)$. We are interested to study the solutions $y \in C^1([0,1],[0,1])$ belonging to the set

$$C_1 = \{y \in C([0,1],[0,1]) : |y(t_1)-y(t_2)| \leq |t_1-t_2|\,, \forall t_1,t_2 \in [0,1]\}\,,$$

which, in view of our notations, means that $L=1$. We have

$$a=0, b=1, x_0=\frac{1}{2} \text{ hence } C_{x_0}=\max\{x_0-a, b-x_0\}=\frac{1}{2}.$$

The function $f(x,u,v) = -\frac{1}{2}+u+v$ is Lipschitzian in the sense of (**) with respect to u and v, with Lipschitz constant $L_1 = 1$. This shows that

$$2\cdot L_1\cdot C_{x_0} = 1,$$

so the condition (v) in Theorem 3.5 is satisfied, but the condition (v) in Theorem 3.1 is not. By theorem 3.3 the solution of differential equation (3.7) can be approximated by means of the iterative method

$$y_{n+1}(t) = (1-\mu)\cdot y_n(t) + \mu y_0 + \mu \int_{x_0}^{t} \left[-\frac{1}{2} + y_n(y_n(s)) + y_n(\lambda s)\right] ds$$

where $\mu \in (0,1)$ and $y_1 \in C_1$ is arbitrary.

Example 4.2. We consider the initial value problem

$$\begin{cases} y'(x) = -\frac{1}{3} + \frac{1}{2} \cdot y(x) + \frac{1}{2} y(y(x)) \\ y\left(\frac{1}{3}\right) = \frac{1}{3} \end{cases} \tag{3.8}$$

where $x \in [0,1], y \in C^1([0,1],[0,1])$.

We show that the problem (3.8) has at least solution in

$$C_1 = \{y \in C([0,1],[0,1]) : |y(t_1) - y(t_2)| \leq |t_1 - t_2|, \forall t_1, t_2 \in [0,1]\}$$

We have $L = 1, a = 0, b = 1, x_0 = \frac{1}{3}$ and $\max\{x_0 - a, b - x_0\} = \frac{2}{3}$.
The function $f(x,u,v) = -\frac{1}{3} + \frac{u+v}{2}$ is Lipschitzian with Lipschitz constant $L_1 = \frac{1}{2}$.

Under these conditions we have $L_1(2+L) \cdot C_{x_0} = 1$, so Theorem 3.4 is applicable but the Theorem 3.1 is not. Note also that $y(x) = \frac{1}{3}, x \in [0,1]$ is a solution of the initial value problem (3.8). By Theorem 3.4 the initial value problem (3.8) has at least one solution in C_1 that can be approximate by Krasnoselskij iteration

$$y_{n+1}(t) = (1-\mu) \cdot y_n(t) + \mu y_0 + \mu \int_{x_0}^{t} \left[-\frac{1}{3} + \frac{1}{2} \cdot y_n(s) + \frac{1}{2} \cdot y_n(y_n(s)) \right] ds$$

where $\mu \in (0,1)$ and $y_1 \in C_1$ is arbitrary.
In particular cases, if $f(t,u,v) = f(t,v)$, we find the differential equation studied in [4].

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