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EXISTENCE THEOREM IN THE LINEAR THEORY OF MULTIPOLAR ELASTICITY

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1. Introduction

The theory of media with microstructure was developed in various papers (see e.g., [1–5]). In [7], I. Hlaváček and J. Nečas established an existence theorem in the classical linear theory of elasticity and in [8], I. Hlaváček and M. Hlaváček proved the existence of the solution in the linear theory of elasticity with couple stresses.

In this paper, we extend the results from [7], [8] to the linear theory of multipolar elasticity for the inhomogeneous anisotropic bodies.

2. Basic Equations

Let Ω be a Lipschitz region in the three-dimensional Euclidean space, occupied by an inhomogeneous and anisotropic multipolar continuum, whose boundary is Γ. The basic equations in the theory of elasticity of multipolar bodies were established by Green and Rivlin in [1], [2]. In the linear theory of multipolar elasticity these equations are:

– the equations of equilibrium

$$\sigma_{ji,j} + f_i = 0\,, \tag{2.1}$$

$$\bar{\sigma}_{ij_1\ldots j_\beta} = \sigma_{kij_1\ldots j_\beta,k} + f_{ij_1\ldots j_\beta} \quad (\beta = 1, 2, \ldots, \nu)\,, \tag{2.2}$$

$$\sigma'_{im} = \sigma'_{mi} = \sigma_{im} - \bar{\sigma}_{im}\,, \tag{2.3}$$

– the geometrical equations

$$(2.4)\qquad \begin{aligned} 2\varepsilon_{ij} &= \mu_{i,j} + \mu_{j,i}\,, \\ \mathrm{d}_{ij} &= \mu_{j,i} + \mu_{ij}\,, \\ d_{ij_1j_2\,..j_\beta} &= u_{ij_1j_2\ldots j_\beta} \quad (\beta = 2, 3, \ldots, \nu)\,, \\ e_{ij_1j_2\ldots j_\beta k} &= \mu_{ij_1j_2\ldots j_\beta,k} \quad (\beta = 1, 2, 3, \ldots, \nu)\,, \end{aligned}$$

– the constitutive equations

$$(2.5)\qquad \sigma'_{ij} = \frac{\partial\mathscr{A}}{\partial\varepsilon_{ij}}\,, \quad \sigma_{kij_1\ldots j_\beta} = \frac{\partial\mathscr{A}}{\partial e_{ij_1\ldots j_\beta k}} \quad (\beta = 1, 2, \ldots, \nu)$$

$$\bar{\sigma}_{ij} = \frac{\partial\mathscr{A}}{\partial d_{ij}}\,, \quad \bar{\sigma}_{ij_1j_2\ldots j_\beta} = \frac{\partial\mathscr{A}}{\partial d_{ij_1j_2\ldots j_\beta}} \quad (\beta = 2, 3, \ldots, \nu)\,.$$

We take the conditions on the boundary Γ in the form:

$$(2.6)\qquad u_i = \mathring{u}_i\,, \quad u_{ij_1\ldots j_\beta} = \mathring{u}_{ij_1\ldots j_\beta} \ (\beta = 1, 2, \ldots, \nu) \quad \text{on} \quad \Gamma_U$$

$$t_i \equiv \sigma_{ji}n_j = \mathring{t}_i\,, \quad t_{ij_1\ldots j_\beta} \equiv \sigma_{kij_1\ldots j_\beta}n_k = \mathring{t}_{ij_1\ldots j_\beta} \quad (\beta = 1, 2, \ldots, \nu) \quad \text{on} \quad \Gamma_T$$

where Γ_U, Γ_T represent a disjoint decomposition of Γ, $\Gamma = \Gamma_U \cup \Gamma_T \cup N$ (N has the surface measure zero).

In the relations (2.4) ε_{ij}, $d_{ij_1\ldots j_\beta}$, $e_{ij_1\ldots j_\beta k}$ $(\beta = 1, 2, \ldots, \nu)$ are the classical strain tensor and the multipolar strain tensors, $\mathscr{A}$ is the elastic energy per unit volume and the other notation in the above relations follows that from [2].

For the linear theory of elasticity, the internal energy has the following form:

$$(2.7)\qquad \begin{aligned} \mathscr{A}(\varepsilon_{ij}, d_{ij_1\ldots j_\beta}, e_{ij_1\ldots j_\beta j}) &= \tfrac{1}{2}A_{ijkm}\varepsilon_{ij}\varepsilon_{km} + \tfrac{1}{2}B_{ijkm}d_{ij}d_{km} + \\ &+ \tfrac{1}{2}\sum_{\alpha=2}^{\nu}\sum_{\gamma=2}^{\nu} C_{ij_1j_2\ldots j_\alpha kh_1h_2\ldots h_\gamma} d_{ij_1j_2\,.\,.j_\alpha} d_{kh_1h_2\ldots h_\gamma} + \\ &+ \tfrac{1}{2}\sum_{\alpha=1}^{\nu}\sum_{\gamma=1}^{\nu} D_{ij_1\ldots j_\alpha jkh_1\ldots h_\gamma h} e_{ij_1\ldots j_\alpha j} e_{kh_1\ldots h_\gamma h} + E_{ijkm}\varepsilon_{ij}d_{km} + \\ &+ \sum_{\gamma=2}^{\nu} F_{ijkh_1h_2\ldots h_\gamma}\varepsilon_{ij}d_{kh_1h_2\ldots h_\gamma} + \sum_{\gamma=1}^{\nu} G_{ijkh_1\ldots h_\gamma h}\varepsilon_{ij}e_{kh_1\ldots h_\gamma h} + \\ &+ \sum_{\gamma=2}^{\nu} H_{ijkh_1h_2\ldots h_\gamma}d_{ij}d_{kh_1h_2\ldots h_\gamma} + \sum_{\gamma=1}^{\nu} I_{ijkh_1\ldots h_\gamma h}d_{ij}e_{kh_1\ldots h_\gamma h} + \\ &+ \sum_{\alpha=2}^{\nu}\sum_{\gamma=1}^{\nu} J_{ij_1j_2\ldots j_\alpha kh_1\ldots h_\gamma h} d_{ij_1j_2\ldots j_\alpha} e_{kh_1\ldots h_\gamma h}\,, \end{aligned}$$

where

$$A_{ijkm} = A_{kmij} = A_{jikm}\,, \quad B_{ijkm} = B_{kmij}\,, \quad E_{ijkm} = E_{jikm}\,, \tag{2.8}$$

$$C_{ij_1j_2\ldots j_\alpha kh_1h_2\ldots h_\gamma} = C_{kh_1h_2\ldots h_\gamma ij_1j_2\ldots j_\alpha}\,, \quad F_{ijkh_1h_2\ldots h_\gamma} = F_{jikh_1h_2\ldots h_\gamma} \quad (\alpha, \gamma = 2, \ldots, \nu)\,,$$

$$D_{ij_1\ldots j_\alpha jkh_1\ldots h_\gamma h} = D_{kh_1\ldots h_\gamma hij_1\ldots j_\alpha j}\,, \quad G_{ijkh_1\ldots h_\gamma h} = G_{jikh_1\ldots h_\gamma h} \quad (\alpha, \gamma = 1, 2, \ldots, \nu)$$

and A_{ijkm}, B_{ijkm}, $C_{ij_1j_2\ldots j_\alpha kh_1h_2\ldots h_\gamma}$, $D_{ij_1\ldots j_\alpha jkh_1\ldots h_\gamma h}$, E_{ijkm}, $F_{ijkh_1h_2\ldots h_\gamma}$, $G_{ijkh_1\ldots h_\gamma h}$, $H_{ijkh_1h_2\ldots h_\gamma}$, $I_{ijkh_1\ldots h_\gamma h}$, $J_{ij_1j_2\ldots j_\alpha kh_1\ldots h_\gamma h}$ are bounded and measurable functions in $\bar{\Omega} = \Omega \cup \Gamma$.

We suppose that the form (2.7) is uniformly positive definite in the tensors ε_{ij} and $e_{ij_1\ldots j_\beta j}$, i.e.

$$\mathscr{A}(\varepsilon_{ij}, d_{ij_1\ldots j_\beta}, e_{ij_1\ldots j_\beta j}) \geqq c \sum_{i,j_1,\ldots,j_\nu,j=1}^{3} (\varepsilon_{ij}^2 + \sum_{\beta=1}^{\nu} e_{ij_1\ldots j_\beta j}^2) \quad (\beta = 1, 2, \ldots, \nu) \tag{2.9}$$

for each point $x \in \bar{\Omega}$ and c is a positive constant.

3. Existence and uniqueness of the weak solution

In this section we shall apply the method given in [7], [8] to prove the existence and uniqueness of a weak solution of the mixed boundary-value problem in the linear elasticity of multipolar bodies. $L_2(\Omega)$ denotes the space of real functions square-integrable in Ω in the Lebesgue sense and $W_2^{(k)}(\Omega)$ denotes the subspace of $L_2(\Omega)$ which contains the functions whose derivatives up to the order k in the sense of distributions are in $L_2(\Omega)$.

Next, W is the Cartesian product $\prod_{s=1}^{m} W_2^{(\varkappa_s)}(\Omega)$, $s = 1, 2, \ldots, m$, where $m, \varkappa_s$ are positive integers.

In our case we have

$$m = \tfrac{1}{2}(3(3^{\nu+1} - 1))\,, \quad s = 1, 2, \ldots, m\,, \quad \varkappa_1 = \varkappa_2 = \ldots = \varkappa_m = 1\,,$$

so that

$$W = \prod_{s=1}^{m} W_2^{(1)}(\Omega)\,.$$

If we denote

$$u = \{u_i, u_{ij_1\ldots j_\beta}\} \quad (\beta = 1, 2, \ldots, \nu) \tag{3.1}$$

then

$$W = \{u \mid u_i, u_{ij_1\ldots j_\beta} \in W_2^{(1)}(\Omega),\ \beta = 1, 2, \ldots, \nu\} \tag{3.2}$$

with the norm

$$(3.3) \qquad \|u\|_W^2 = \sum_{i,j_1,j_2,\ldots,j_\nu=1}^{3} \left(\|u_i\|^2_{W^{(1)}{}_2(\Omega)} + \sum_{\beta=1}^{\nu} \|u_{ij_1j_2\ldots j_\beta}\|^2_{W^{(1)}{}_2(\Omega)}\right),$$

where

$$\|v\|^2_{W^{(k)}{}_2(\Omega)} = (v, v)_k = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha v D^\alpha v \,\mathrm{d}x\,, \quad D^\alpha \equiv \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2}\,\partial x_3^{\alpha_3}}\,.$$

Let $\mathscr{D}(\Omega)$ be the space of real functions with compact support in Ω which are infinitely differentiable. Let $\mathring{W}_2^{(k)}(\Omega)$ be the closure of $\mathscr{D}(\Omega)$ in $W_2^{(k)}(\Omega)$ and $\mathring{W} = \prod_{s=1}^{m} \mathring{W}_2^{(\varkappa_s)}(\Omega)$.

If V denotes a closed subspace of W such that $\mathring{W} \subset V \subset W$, then we obtain

$$(3.4) \qquad V = \{u \mid u \in W,\ u_i = u_{ij_1\ldots j_\beta} = 0 \quad \text{on} \quad \Gamma_U\,, \quad \beta = 1, 2, \ldots, \nu\}\,.$$

According to [6], we define a bilinear form $A(v, u)$ on $W \times W$ by

$$\begin{aligned}
(3.5) \quad A(v, u) = \int_\Omega [&A_{ijkm}\varepsilon_{ij}(v)\,\varepsilon_{km}(u) + B_{ijkm}d_{ij}(v)\,d_{km}(u) + \\
&+ \sum_{\alpha=2}^{\nu}\sum_{\beta=2}^{\nu} C_{ij_1j_2\ldots j_\alpha kh_1h_2\ldots h_\beta} d_{ij_1j_2\ldots j_\alpha}(v)\,d_{kh_1h_2\ldots h_\beta}(u) + \\
&+ \sum_{\alpha=1}^{\nu}\sum_{\beta=1}^{\nu} D_{ij_1\ldots j_\alpha jkh_1\ldots h_\beta h} e_{ij_1\ldots j_\alpha j}(v)\,e_{kh_1\ldots h_\beta h}(u) + \\
&+ E_{ijkm}(\varepsilon_{ij}(v)\,d_{km}(u) + \varepsilon_{ij}(u)\,d_{km}(v)) + \sum_{\beta=2}^{\nu} F_{ijkh_1h_2\ldots h_\beta}(\varepsilon_{ij}(v)\,d_{kh_1h_2\ldots h_\beta}(u) + \\
&+ \varepsilon_{ij}(u)\,d_{kh_1h_2\ldots h_\beta}(v)) + \sum_{\beta=1}^{\nu} G_{ijkh_1\ldots h_\beta h}(\varepsilon_{ij}(v)\,e_{kh1\ldots h_\beta h}(u) + \varepsilon_{ij}(u)\,e_{kh_1\ldots h_\beta h}(v)) + \\
&+ \sum_{\beta=2}^{\nu} H_{ijkh_1h_2\ldots h_\beta}(d_{ij}(v)\,d_{kh_1h_2\ldots h_\beta}(u) + d_{ij}(u)\,d_{kh_1h_2\ldots h_\beta}(v)) + \\
&+ \sum_{\beta=1}^{\nu} I_{ijkh_1\ldots h_\beta h}(d_{ij}(v)\,e_{kh_1\ldots h_\beta h}(u) + d_{ij}(u)\,e_{kh_1\ldots h_\beta h}(v)) + \\
&+ \sum_{\alpha=2}^{\nu}\sum_{\beta=1}^{\nu} J_{ij_1j_2\ldots j_\alpha kh_1\ldots h_\beta h}(d_{ij_1j_2\ldots j_\alpha}(v)\,e_{kh_1\ldots h_\beta h}(u) + d_{ij_1j_2\ldots j_\alpha}(u)\,e_{kh_1\ldots h_\beta h}(v))]\,\mathrm{d}x
\end{aligned}$$

and the functionals

$$(3.6) \qquad f(v) = \int_\Omega (f_i v_i + \sum_{\beta=1}^{\nu} f_{ij_1\ldots j_\beta} v_{ij_1\ldots j_\beta})\,\mathrm{d}x\,,$$

$$g(v) = \int_{\Gamma_T} (\mathring{t}_i v_i + \sum_{\beta=1}^{\nu} \mathring{t}_{ij_1\ldots j_\beta} v_{ij_1\ldots j_\beta})\,\mathrm{d}\Gamma\,,$$

for each $v \in W$.

By virtue of (2.7), (2.8), we can write

$$\int_\Omega 2\mathscr{A}(\varepsilon_{ij}, d_{ij_1\ldots j_\beta}, e_{ij_1\ldots j_\beta j})\,dx = A(u, u)\,, \quad A(u, v) = A(v, u)\,. \tag{3.7}$$

From (2.9) and $(3.7)_1$ we obtain

$$A(v, v) \geqq 2c \sum_{i,j_1,\ldots,j_\nu,j=1}^{3} \int_\Omega \left(\varepsilon_{ij}^2 + \sum_{\beta=1}^{\nu} e_{ij_1\ldots j_\beta j}^2\right) \mathrm{d}x\,. \tag{3.8}$$

Let $\mathring{u} = \{\mathring{u}_i, \mathring{u}_{ij_1\ldots j_\beta}\} \in W$, $(\beta = 1, 2, \ldots, \nu)$ and $\mathring{t}_i, \mathring{t}_{ij_1\ldots j_\beta} \in L_2(\Gamma_T)$, $(\beta = 1, 2, \ldots, \nu)$ so that the boundary conditions (2.6) are met in the sense of traces and in the sense of $L_2(\Gamma_T)$, respectively. We say that $u \in W$ is a weak solution of the mixed boundary-value problem, if

$$u - \mathring{u} \in V \tag{3.9}$$

$$A(u, v) = f(v) + g(v) \tag{3.10}$$

for each $v \in V$.

The operators $N_l v = \sum_{r=1}^{m} \sum_{|\alpha| \leqq \varkappa_r} n_{lr\alpha} D^\alpha v_r$ $(l = 1, 2, \ldots, h)$, [6], $h = 3m - 3$, mapping W into $L_2(\Omega)$ are taken as follows:

$$\begin{aligned}
&N_1 v = v_{1,1}\,, && N_2 v = \tfrac{1}{2}(v_{2,1} + v_{1,2})\,, \quad N_3 v = \tfrac{1}{2}(v_{3,1} + v_{1,3})\,,\\
&N_4 v, \ldots\,, && N_m v = v_{ij_1j_2\ldots j_\beta,1}\,;\\
&N_{m+1} v = v_{2,2}\,, && N_{m+2} v = \tfrac{1}{2}(v_{3,2} + v_{2,3})\,,\\
&N_{m+3} v, \ldots\,, && N_{2m-1} v = v_{ij_1j_2\ldots j_\beta,2}\,;\\
&N_{2m} v = v_{3,3}\,, && N_{2m+1} v, \ldots, N_{3m-3} v = v_{ij_1j_2\ldots j_\beta,3}\,,
\end{aligned} \tag{3.11}$$

where $\beta = 1, 2, 3, \ldots, \nu$, and $i, j_1, j_2, \ldots, j_\nu = 1, 2, 3$. We have

$$\sum_{l=1}^{h} \|N_l v\|_{L_2(\Omega)}^2 = \sum_{i,j_1,\ldots,j_\nu,j=1}^{3} \int_\Omega \left(\varepsilon_{ij}^2 + \sum_{\beta=1}^{\nu} e_{ij_1\ldots j_\beta j}^2\right) \mathrm{d}x\,, \tag{3.12}$$

so that (3.8) implies

$$A(v, v) \geqq 2c \sum_{l=1}^{h} \|N_l v\|_{L_2(\Omega)}^2\,. \tag{3.13}$$

From (3.11) it follows that the matrix $N_{ls}\xi = \sum_{|\alpha| = \varkappa_s} n_{ls\alpha}\xi^\alpha$ has the rank m for each $\xi \neq 0$, $\xi \in C_3$.

By virtue of Theorem 3.1 from [6], we can conclude that the system of operators $N_l v$ $(l = 1, 2, 3, \ldots, h)$ is coercive on W.

For $\mathscr{P}$, we obtain

$$\mathscr{P} = \{v \mid v \in V,\ v_i = a_i + b_{ij}x_j,\ v_{ij} = b_{ij},\ v_{ij_1j_2\ldots j_\beta} = 0,\ \beta = 2, 3, \ldots, \nu\} \tag{3.14}$$

where a_i, $b_{ij} = -b_{ji}$ are constants.

The definition (3.4) of V implies $a_i = b_{ij} = 0$, hence $\mathscr{P} = \{0\}$.

All assumptions of Theorems 2.1 and 2.2 from [6] are satisfied. According to these theorems, we have

$$A(v, v) \geqq c_1 \|v\|_V^2\,, \tag{3.15}$$

for each $v \in V$, and the mixed boundary-value problem has one and only one weak solution $u \in W$, continuously dependent on the given data.

Bibliography

[1] *A. E. Green, R. S. Rivlin:* Simple Force and Stress Multipoles. Arch. Rat. Mech. Anal., 16, p. 325 (1964).

[2] *A. E. Green:* Multipolar Continuum Mechanics. Arch. Rat. Mech. Anal., 17, p. 113 (1964).

[3] *A. C. Eringen, E. S. Suhubi:* Nonlinear Theory of Simple Microelastic Solid. I and II, Int. J. Eng. Sci., 2, 189, 384 (1964).

[4] *R. D. Mindlin:* Microstructure in Linear Elasticity. Arch. Rat. Mech. Anal., 16, p. 51 (1964).

[5] *Cz. Wozniak:* Thermoelasticity of Bodies with Microstructure. Archivum Mech. Stos., XIX, 3, p. 335 (1967).

[6] *I. Hlaváček, J. Nečas:* On Inequalities of Korn's Type. I. Boundary-Value Problems for Elliptic Systems of Partial Differential Equations. Arch. Rat. Mech. Anal., 36, p. 305 (1970).

[7] *I. Hlaváček, J. Nečas:* On Inequalities of Korn's Type. II. Applications to Linear Elasticity. Arch. Rat. Mech. Anal., 36, 312 (1970).

[8] *I. Hlaváček, M. Hlaváček:* On the Existence and Uniqueness of Solution and Some Variational Principles in Linear Theories of Elasticity with Couple-Streses, II. Mindlin's Elasticity with Microstructure and the First Strain-Gradient Theory. Aplikace Matematiky, 14, p. 411 (1969).

[9] *J. Nečas:* Les méthodes directes en théorie des équations elliptiques. Academia, Prague, 1967.

Souhrn

EXISTENČNÍ VĚTY V LINEÁRNÍ TEORII MULTIPOLÁRNÍ PRUŽNOSTI

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Tento článek je věnován kombinované úloze lineární pružnosti pro multipolární tělesa. Užívá se některých výsledků z teorie lineárních eliptických parciálních diferenciálních rovnic. Pro odpovídající bilineární formu vnitřní energie je odvozena V-elipticita a je dokázána existence a unicita řešení, jakož i spojitá závislost řešení na okrajových podmínkách.

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