ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 3, Number 3, Summer 1973

EXISTENCE THEOREMS FOR BOUNDARY VALUE PROBLEMS OF nTH ORDER ORDINARY DIFFERENTIAL EQUATIONS[1]

GENE A. KLAASEN

Consider the nth order boundary value problem

$$y^{(n)} = f(x, y, y', \cdots, y^{(n-1)}), \tag{1}$$

$$y(x_i) = y_i \quad \text{for } 1 \leqq i \leqq n, \tag{2}$$

where $n \geqq 2$, $a \leqq x_1 < \cdots < x_n < b$ and $y_i \in R$ for $1 \leqq i \leqq n$. Throughout this paper some of the following conditions will be assumed for the differential equation (1).

(A) f is continuous on $[a, b) \times R^n$.

(B) For each $a \leqq x_1 < \cdots < x_n < b$ and $y_i \in R$, $1 \leqq i \leqq n$, the boundary value problem (1), (2) has at most one solution on $[x_1, x_n]$.

(C) Solutions of initial value problems for (1) exist on $[a, b)$ and are unique.

(D) If $\{y_k(x)\}$ is a sequence of solutions of (1) which is monotone and bounded on some interval $[c, d] \subset [a, b)$ then $\lim_{k \to \infty} y_k(x)$ is a solution of (1) on $[c, d]$.

The question whether conditions (A)—(C) imply that the nth order problem (1), (2) has a unique solution for each partition $a \leqq x_1 < \cdots < x_n < b$ and each $y_i \in R$, $1 \leqq i \leqq n$, has been the topic of much recent inquiry. For the case $n = 2, 3$ this question has been resolved in the affirmative with various generalizations and improvements given. Articles [1]—[6] represent a chronological reference of these results. More recently P. Hartman [7] proved that the boundary value problem (1), (2) has a unique solution provided (A)—(D) hold and left as an open question whether (D) or the equivalent hypothesis of "local solvability of boundary value problems" could be omitted and still conclude that boundary value problems are uniquely solvable.

The purpose of this paper is to present an alternate proof of that existence theorem followed by an investigation into the question, whether condition (D) can be omitted in proving the existence of solutions to boundary value problems. As a consequence of this discussion another existence theorem is obtained by replacing condition

Received by the editors October 4, 1971 and, in revised form, November 1, 1971.
AMS (MOS) subject classifications (1970). Primary 34A10, 34B10, 34B15.

[1]This work was supported in part by a University of Tennessee Faculty Research Grant.

(D) by a condition which assumes that limits of solutions possess certain differentiability properties. Finally the author shows that if the function f of equation (1) depends only on x and y, i.e. $f(x, y, y', \cdots, y^{(n-1)}) = g(x, y)$, then conditions (A), (B) and (C) are sufficient to imply existence.

1. **Uniqueness and denseness of solutions.** In this section it is shown that solutions to "k-point" boundary value problems are unique and that they are dense in themselves.

If $2 \leqq k \leqq n$ and $a \leqq x_1 < \cdots < x_k < b$ then the following boundary value problem is called a k-point boundary problem.

$$(1) \qquad y^{(n)} = f(x, y, y', \cdots, y^{(n-1)}),$$

$$(3) \qquad y^{(i)}(x_1) = y_1{}^i, \qquad y(x_j) = y_j \quad \text{where } y_1{}^i, y_j \in R$$

for $0 \leqq i \leqq n-k$, $2 \leqq j \leqq k$.

In this section we will replace condition (C) by a slightly weaker condition.

(C$'$) At each point in $[a, b)$, solutions of initial value problems for equation (1) exist locally and are unique.

The first two theorems are modifications of Theorems 1 and 2 of article [8]. The proof of the first theorem is sufficiently different from the proof of Theorem 1 of article [8] to require exposure. However the proof of Theorem 2 is essentially like the proof of Theorem 2 of article [8] and hence is omitted.

THEOREM 1. *Let equation* (1) *be such that conditions* (A), (B) *and* (C$'$) *are satisfied. Let u and v be solutions of* (1) *on* $[x_0, c] \subset [a, b)$ *such that* $u^{(i)}(x_0) = v^{(i)}(x_0)$ *for* $0 \leqq i \leqq p-1$ *but* $u^{(p)}(x_0) > v^{(p)}(x_0)$ *where* $2 \leqq p \leqq n-1$. *Then given any* $\delta > 0$ *with* $x_0 + \delta < c$ *there is a solution y of* (1) *such that $y - u$ has p distinct zeros on* $[x_0, x_0 + \delta)$. *Furthermore given any* $\epsilon > 0$, *y can be chosen such that* $|y(x) - v(x)| < \epsilon$ *for all* $x \in [x_0, c]$.

PROOF. Let m be chosen such that $v^{(p)}(x_0) < m < u^{(p)}(x_0)$. Also, choose m sufficiently near $v^{(p)}(x_0)$ that the solution y_1 of the initial value problem for (1) with $y_1{}^{(i)}(x_0) = v^{(i)}(x_0)$ for $0 \leqq i \leqq n-1$, $i \neq p$, $y_1{}^{(p)}(x_0) = m$ exists on $[x_0, c]$. Then there is a $\delta > 0$ such that $v(x) < y_1(x) < u(x)$ on $(x_0, x_0 + \delta]$. By the property of continuous dependence of solutions on initial conditions, there is an $\epsilon_1 > 0$ such that if y_2 is the solution of (1) with

$$y_2{}^{(i)}(x_0) = y_1{}^{(i)}(x_0), \qquad 0 \leqq i \leqq n-1, i \neq p-1,$$
$$y_2{}^{(p-1)}(x_0) = y_1{}^{(p-1)}(x_0) + \epsilon_1,$$

then $v(x_0 + \delta) < y_2(x_0 + \delta) < u(x_0 + \delta)$. Furthermore since $y_1^{(p-1)}(x_0) = u^{(p-1)}(x_0) = v^{(p-1)}(x_0)$, there exists a x_{p-1} such that $x_0 < x_{p-1} < x_0 + \delta$ and $u(x_{p-1}) < y_2(x_{p-1})$. Repeating the procedure again there is an $\epsilon_2 > 0$ such that if y_3 is the solution of (1) such that

$$y_3^{(i)}(x_0) = y_2^{(i)}(x_0), \qquad 0 \leqq i \leqq n-1, i \neq p-2,$$
$$y_3^{(p-2)}(x_0) = y_2^{(p-2)}(x_0) - \epsilon_2,$$

then $v(x_0 + \delta) < y_3(x_0 + \delta) < u(x_0 + \delta)$ and $u(x_{p-1}) < y_3(x_{p-1})$. Since $y_3^{(p-2)}(x_0) < y_2^{(p-2)}(x_0) = u^{(p-2)}(x_0) = v^{(p-2)}(x_0)$ there is a x_{p-2} such that $x_0 < x_{p-2} < x_{p-1} < x_0 + \delta$ and $y_3(x_{p-2}) < v(x_{p-2}) < u(x_{p-2})$. Proceeding in this manner we obtain a solution $y \equiv y_p$ and points $x_0 < x_1 < x_2 < \cdots < x_{p-1} < x_0 + \delta$ such that

$$y_p(x_k) < v(x_k) \quad \text{if } p - k \text{ is even, and}$$
$$y_p(x_k) > u(x_k) \quad \text{if } p - k \text{ is odd.}$$

Moreover

$$y_p(x_0) = u(x_0) = v(x_0) \quad \text{and} \quad v(x_0 + \delta) < y_p(x_0 + \delta) < u(x_0 + \delta).$$

Thus $y_p - u$ has p distinct zeros on $[x_0, x_0 + \delta)$ and $y_p - v$ has $p - 1$ distinct zeros on $[x_0, x_0 + \delta)$. Moreover given any $\epsilon > 0$ if m is sufficiently close to $v^{(p)}(x_0)$ and $\epsilon_1, \cdots, \epsilon_{p-1}$ are sufficiently small then $|v(x) - y_p(x)| < \epsilon$ on $[x_0, c]$.

REMARK. One can show as a consequence of Theorem 1 that if u and v are distinct solutions of (1) and if for some $x_1 \in [a, b)$, $u^{(i)}(x_1) = v^{(i)}(x_1)$ for $i = 0, 1, \cdots, n-2$, then $u^{(n-1)}(x_1) \neq v^{(n-1)}(x_1)$ and $u(x) \neq v(x)$ on $[a, b) - \{x_1\}$. Corollary 1 of [8, p. 544] is essentially this result and can be consulted for a proof.

THEOREM 2. *Let equation* (1) *be such that conditions* (A), (B) *and* (C′) *are satisfied. Suppose u and v are distinct solutions of* (1) *such that $u - v$ has k, $2 \leqq k \leqq n-1$, distinct zeros on $[a, b)$ at $a \leqq x_1 < x_2 < \cdots < x_k < b$. Assume that $u - v$ has a zero of order h at x_1. Let γ_1 be the number of zeros of $u - v$ on (x_1, b) such that $u - v > 0$ in a deleted neighborhood of the zero. Let γ_2 be the number of zeros of $u - v$ on (x_1, b) such that $u - v < 0$ in a deleted neighborhood of the zero. Let p be the number of odd order zeros of $u - v$ on (x_1, b). Then $2\gamma_i + p + h < n$ for $i = 1, 2$.*

THEOREM 3. *Suppose equation* (1) *is such that conditions* (A), (B) *and* (C′) *are satisfied. If $1 \leqq k \leqq n-1$, $a \leqq x_1 < \cdots < x_k < b$, and if u and v are distinct solutions of* (1) *such that $u^{(i)}(x_1) = v^{(i)}(x_1)$ for $i = 0, 1, \cdots, n-k-1$ and $u(x_j) = v(x_j)$ whenever $2 \leqq j \leqq k$ then in* *the notation of Theorem* 2, $\gamma_1 = \gamma_2$, $u^{(n-k)}(x_1) \neq v^{(n-k)}(x_1)$, $u - v$ *has no other zeros on* (x_1, b) *and* $\operatorname{sgn}(u^{(n-k)}(x_1) - v^{(n-k)}(x_1)) = \operatorname{sgn}(-1)^{k-1}(u(x) - v(x))$ *for* $x_k < x < b$.

PROOF. The case that $k = 1$ is discussed in the remark following Theorem 1 and the equality $\operatorname{sgn}(u^{(n-1)}(x_1) - v^{(n-1)}(x_1)) = \operatorname{sgn}(u(x) - v(x))$ for $x_1 < x < b$ is obvious.

If $k \geqq 2$ then with the notation of the previous theorem, $h \geqq n - k$, $\gamma_1 + \gamma_2 + p \geqq k - 1$, $h + 2\gamma_1 + p < n$ and $h + 2\gamma_2 + p < n$. The first, third and fourth inequalities yield $2\gamma_1 + p < k$ and $2\gamma_2 + p < k$; hence $\gamma_1 + \gamma_2 + p < k$. This implies $\gamma_1 + \gamma_2 + p = k - 1$. From this and $h + \gamma_1 + \gamma_2 + p < n$ it follows that $h = n - k$. Thus $u^{(n-k)}(x_1) \neq v^{(n-k)}(x_1)$. But $2\gamma_1 + p < k$ and $2\gamma_2 + p < k$ imply that $\gamma_1 = \gamma_2$. Thus p and $k - 1$ are both odd or both even. Thus $\operatorname{sgn}(u^{(n-k)}(x_1) - v^{(n-k)}(x_1)) = \operatorname{sgn}(-1)^p(u(x) - v(x))$ for $x_k < x < b$ implies that $\operatorname{sgn}(u^{(n-k)}(x_1) - v^{(n-k)}(x_1)) = \operatorname{sgn}(-1)^{k-1}(u(x) - v(x))$ for $x_k < x < b$.

The above relationship of signs was pointed out to me by Professor Jackson.

COROLLARY 4. *Suppose equation* (1) *is such that conditions* (A), (B) *and* (C′) *are satisfied. If* $2 \leqq k \leqq n$, $a \leqq x_1 < \cdots < x_k < b$ *and* $y_1^i, y_j \in R$ *for* $0 \leqq i \leqq n - k$ *and* $2 \leqq j \leqq k$ *then the k-point problem* (1), (3) *has at most one solution.*

If $a \leqq x_1 < b$ and if $\bar{\alpha} \in R^n$ is denoted by $\bar{\alpha} = (\alpha_0, \cdots, \alpha_{n-1})$, then let $y(x; x_1, \bar{\alpha})$ denote the unique solution of (1) such that $y^{(i)}(x_1; x_1, \bar{\alpha}) = \alpha_i$ for $0 \leqq i \leqq n - 1$.

Suppose $2 \leqq k \leqq n$, $a \leqq x_1 < \cdots < x_k < b$, $y_1^i \in R$ for $0 \leqq i \leqq n - k$ and $y_j \in R$ for $2 \leqq j \leqq k - 1$. This last condition is omitted if $k = 2$. Then let S be the set of all solutions y of (1) satisfying

$$(4) \qquad \begin{aligned} &\text{(i)} \quad \text{if } y \in S, \quad y^{(i)}(x_1) = y_1^i \quad \text{for } 0 \leqq i \leqq n - k, \\ &\text{(ii)} \quad \text{if } y \in S \text{ and } k > 2 \quad \text{then } y(x_j) = y_j \quad \text{for } 2 \leqq j \leqq k - 1. \end{aligned}$$

Let

$$(5) \qquad B = \{y(x_k) \mid y \in S\}.$$

THEOREM 5. *Suppose equation* (1) *is such that conditions* (A), (B) *and* (C′) *are satisfied. For all* $\epsilon > 0$ *and* $\gamma \in B$, $(\gamma - \epsilon, \gamma) \cap B \neq \emptyset$ *and* $(\gamma, \gamma + \epsilon) \cap B \neq \emptyset$. *Moreover if* $y \in S$ *with* $y(x_k) = \gamma$ *and* K *is a compact subset of the interval on which* y *is defined, then* $\rho \in (\gamma - \epsilon, \gamma) \cap B$ *and* $\sigma \in (\gamma, \gamma + \epsilon) \cap B$ *can be chosen so that if*

$u, v \in S$ with $u(x_k) = \rho$ and $v(x_k) = \sigma$ then $|u - y| < \epsilon$ and $|v - y| < \epsilon$ on K.

PROOF. If $k = 2$, Theorem 3 implies that the difference between distinct solutions of S is nonzero on (x_1, b). If $y \in S$ then the sequence $\{y_\lambda\} \subseteq S$ defined by $y_\lambda^{(n-1)}(x_1) = y^{(n-1)}(x_1) + 1/\lambda$ satisfies $y_\lambda(x_2) \downarrow y(x_2)$ since initial value problems have unique solutions. Thus, for $k = 2$, $(\gamma, \gamma + \epsilon) \cap B \neq \emptyset$ for all $\gamma \in B$ and $\epsilon > 0$. By a similar argument one can conclude that $(\gamma - \epsilon, \gamma) \cap B \neq \emptyset$ for all $\gamma \in B$ and $\epsilon > 0$. That solutions can be chosen arbitrarily close to the solution determined by γ is clear from the above argument.

If $k > 2$, let $\gamma \in B$ and suppose $y(x) = y(x; x_1, \bar{\alpha})$ is the corresponding solution of (1) such that $y(x_k) = \gamma$. Without loss of generality suppose $\bar{\alpha} = (0, 0, \cdots, 0)$ and write $y(x; x_1, \bar{\alpha}) = y(x; x_1, 0, 0, \cdots, 0)$. If c is such that $x_k < c < b$ then there is a $\delta > 0$ such that if $\bar{\alpha} \in R^n$ and $|\bar{\alpha}| < \delta$ then every solution $y(x; x_1, \bar{\alpha})$ exists on $[x_1, c]$. In the remainder of this proof all the initial conditions will be restricted to have norm less than δ. If $\epsilon_{n-1} > 0$ then Corollary 4 implies that $y(x; x_1, 0, \cdots, 0, -\epsilon_{n-1}) < y(x; x_1, 0, \cdots, 0) < y(x; x_1, 0, \cdots, 0, \epsilon_{n-1})$ on (x_1, c). Since solutions depend continuously on initial conditions there is a $\kappa_1 > 0$ such that if $0 \leqq |\alpha_i| \leqq \kappa_1$ for $n - k + 1 \leqq i \leqq n - 2$ then

$$y(x; x_1, 0, \cdots, 0, \alpha_{n-k+1}, \cdots, \alpha_{n-2}, -\epsilon_{n-1}) < y(x; x_1, 0, \cdots, 0)$$
$$< y(x; x_1, 0, \cdots, 0, \alpha_{n-k+1}, \cdots, \alpha_{n-2}, \epsilon_{n-1})$$

on $[x_2, x_k]$. The continuous dependence property of solutions again implies that for each set of α_i with $0 \leqq |\alpha_i| \leqq \kappa_1$, $n - k + 1 \leqq i \leqq n - 2$, there is an α_{n-1} such that $-\epsilon_{n-1} < \alpha_{n-1} < \epsilon_{n-1}$ and $y(x_{k-1}; x_1, 0, \cdots, 0, \alpha_{n-k+1}, \cdots, \alpha_{n-2}, \alpha_{n-1}) = y(x_{k-1}; x_1, 0, \cdots, 0)$. Corollary 4 implies that α_{n-1} is uniquely determined by $\alpha_{n-k+1}, \cdots, \alpha_{n-2}$. In fact it will now be shown that α_{n-1} is a continuous function of $\alpha_{n-k+1}, \cdots, \alpha_{n-2}$. It suffices to show continuity of α_{n-1} at $\alpha_{n-k+1}, \cdots, \alpha_{n-2} = 0$. For $h > 0$ chosen, let $\epsilon_{n-1} = h$ and $\kappa_1 = \delta$ in the above argument. Then $|\alpha_i| < \delta$ for $n - k + 1 \leqq i \leqq n - 2$ implies $|\alpha_{n-1}| < h$.

From now on when we write $y(x; x_1, 0, \cdots, 0, \alpha_{n-k+1}, \cdots, \alpha_{n-1})$ we will mean that $\alpha_{n-1} = \alpha_{n-1}(\alpha_{n-k+1}, \cdots, \alpha_{n-2})$ as described above. Let $\alpha_{n-2} = \kappa_1$ be fixed. Then Corollary 4 implies that

$$y(x; x_1, 0, \cdots, 0, -\alpha_{n-2}, \alpha_{n-1}) < y(x; x_1, 0, \cdots, 0)$$
$$< y(x; x_1, 0, \cdots, 0, \alpha_{n-2}, \alpha_{n-1})$$

on (x_1, x_{k-1}). By the continuous dependence property there is a

$\kappa_2 > 0$ such that if $0 \leqq |\alpha_i| \leqq \kappa_2$ for $n - k + 1 \leqq i \leqq n - 3$ then

$$y(x; x_1, 0, \cdots, 0, \alpha_{n-k+1}, \cdots, \alpha_{n-3}, -\alpha_{n-2}, \alpha_{n-1}) < y(x; x_1, 0, \cdots, 0)$$
$$< y(x; x_1, 0, \cdots, 0, \alpha_{n-k+1}, \cdots, \alpha_{n-3}, \alpha_{n-2}, \alpha_{n-1})$$

on $[x_2, x_{k-2}]$. Thus for all $0 \leqq |\alpha_i| \leqq \kappa_2$, $n - k + 1 \leqq i \leqq n - 3$, there is an α_{n-2} such that $-\kappa_1 < \alpha_{n-2} < k_1$ and

$$y(x_{k-2}; x_1, 0, \cdots, 0, \alpha_{n-k+1}, \cdots, \alpha_{n-3}, \alpha_{n-2}, \alpha_{n-1})$$
$$= y(x_{k-2}; x_1, 0, \cdots, 0).$$

Corollary 4 implies that α_{n-2} is uniquely determined by $\alpha_{n-k+1}, \cdots, \alpha_{n-3}$ and a similar argument to the one above confirms that α_{n-2} is a continuous function of $\alpha_{n-k+1}, \cdots, \alpha_{n-3}$. If we repeat this argument $k - 4$ more times we conclude that there exist a $\kappa_{k-2} > 0$ such that for each α_{n-k+1} with $0 \leqq |\alpha_{n-k+1}| \leqq \kappa_{k-2}$ there exists $\alpha_{n-k+2} = \alpha_{n-k+2}(\alpha_{n-k+1}), \cdots, \alpha_{n-1}(\alpha_{n-k+1}, \cdots, \alpha_{n-2}) = \alpha_{n-1}$ such that each α_{n-j} is a continuous function of $\alpha_{n-k+1}, \cdots, \alpha_{n-j-1}$ for $1 \leqq j \leqq k - 2$ and $y(x_i; x_1, 0, \cdots, 0, \alpha_{n-k+1}, \cdots, \alpha_{n-1}) = y(x_i; x_1, 0, \cdots, 0)$ for $2 \leqq i \leqq k - 1$.

Thus for each α_{n-k+1} with $0 \leqq |\alpha_{n-k+1}| \leqq \kappa_{k-2}$ and $z(x) \equiv y(x; x_1, 0, \cdots, 0, \alpha_{n-k+1}, \alpha_{n-k+2}, \cdots, \alpha_{n-1})$, the difference $z(x) - y(x; x_1, 0, \cdots, 0)$ has a zero of order $n - k + 1$ at x_1 and zeros at $x_2, \cdots, x_{k-1}$. By Theorem 3, if $\alpha_{n-k+1} = 0$ then the two solutions are identical otherwise $\operatorname{sgn}[z(x_k) - y(x_k; x_1, 0, \cdots, 0)] = (-1)^k \operatorname{sgn}(\alpha_{n-k+1})$. For a given $\epsilon > 0$, we have $|z(x) - y(x; x_1, 0, \cdots, 0)| < \epsilon$ on $[x_1, c]$. Under those conditions $z(x_k) \in (\gamma - \epsilon, \gamma) \cap B$ or $z(x_k) \in (\gamma, \gamma + \epsilon) \cap B$ depending on the sign of α_{n-k+1}. Given a compact subset of $[a, b)$, $\delta > 0$ can be chosen sufficiently small that $u, v \in S$ can be determined satisfying the inequalities of the theorem.

2. **The existence theorem.** To prove the existence theorem, some results concerning convergence of sequences of solutions are required.

LEMMA 6. *Suppose equation* (1) *is such that conditions* (A), (B), (C) *and* (D) *are satisfied. Let* $2 \leqq k \leqq n$, $a \leqq x_1 < \cdots < x_k < b$ *and* y_1^i, $y_j \in R$ *for* $0 \leqq i \leqq n - k - 1$, $2 \leqq j \leqq k$. *Suppose* $\{y_\lambda\}$ *is a sequence of solutions of* (1) *on* $[a, b)$ *such that* $y_\lambda^{(i)}(x_1) = y_1^i$, $y_\lambda(x_j) = y_j$ *for* $0 \leqq i \leqq n - k - 1$ *and* $2 \leqq j \leqq k$. *If* $\{y_\lambda\}$ *is monotone and bounded on some subinterval of* $[x_1, b)$ *then* $\{y_\lambda\}$ *converges to a solution of* (1) *on* $[a, b)$, *uniformly on compact subsets of* $[a, b)$.

PROOF. Let $\phi(x) = \lim_{\lambda \to \infty} y_\lambda(x)$ for $x \in J$ where J is the interval of monotonicity and boundedness of $\{y_\lambda\}$. By restricting the size of J

if necessary assume $J = [c, d]$ and $x_i \notin [c, d]$ for $1 \leqq i \leqq k$. By property (D), ϕ is a solution of (1) on $[c, d]$. Also by Dini's theorem $\{y_\lambda\}$ converges to ϕ uniformly on $[c, d]$. Let Φ be the unique solution of (1) on $[a, b)$ satisfying $\Phi = \phi$ on $[c, d]$. To prove the theorem it suffices to show that for all compact $K = [a, r]$, $d < r < b$, $\{y_\lambda\}$ converges uniformly to Φ on K. Let $c < \tau_1 < \tau_2 < \cdots < \tau_{n-1} < d$.

Consider the case that n is odd. Let $\epsilon > 0$ be given. Using Theorem 5 choose a solution v of (1) on $[a, r]$ such that $|v - \Phi| < \epsilon$, $v(\tau_i) = \phi(\tau_i)$ for $i = 1, 2, \cdots, n-1$ and $v(d) < \phi(d)$. Condition (B) and Theorem 3 imply that $v(c) < \phi(c)$. If γ_1, γ_2 and p are defined for $\phi - v$ as in Theorem 2 then $\gamma_1 = \gamma_2$ as a result of Theorem 3. Moreover, if τ_i is such that there is a deleted neighborhood on which $\phi - v > 0$ then for λ sufficiently large y_λ crosses v twice in that neighborhood. Also for λ large $v(c) < y_\lambda(c) < \phi(c)$ and $v(d) < y_\lambda(d) < \phi(d)$. Thus for large λ, y_λ crosses v $2\gamma_2 + p + 1$ times in (c, d), but v was chosen such that $\gamma_1 + \gamma_2 + p + 1 = n - 1$ and hence y_λ crosses v $n - 1$ times in (c, d). Consequently $v(x) < y_\lambda(x)$ on $[a, c) \cap (d, r]$ for λ sufficiently large. Similarly there is a solution $w(x)$ of (1) on $[a, r]$ with $|w - \Phi| < \epsilon$ on $[a, r]$, $w(\tau_i) = \Phi(\tau_i)$ for $i = 1, 2, \cdots, n-1$, $w(c) > \phi(c)$ and $w(d) > \phi(d)$. By a similar argument for λ sufficiently large $w > y_\lambda$ on $[a, c) \cap (d, r]$. Hence for λ large, $|y_\lambda - \Phi| < \epsilon$ on $[a, r]$.

The case that n is even can be handled in a similar manner.

Corollary 7. *Let equation* (1) *be such that conditions* (A), (B), (C) *and* (D) *are satisfied. Let* S *and* B *be described by* (4) *and* (5). *Then* B *is a connected open set.*

Proof. If $\alpha < \beta$, $\alpha, \beta \in B$ and $u, v \in S$ such that $u(x_k) = \alpha$ and $v(x_k) = \beta$ then if $y \in S$ such that $\alpha < y(x_k) < \beta$, $u(x) < y(x) < v(x)$ on (x_{k-1}, x_k). Hence Lemma 6 implies that $B \cap [\alpha, \beta]$ is closed. This property of B in addition to the property of B exposed in Theorem 5 imply that B is connected and open.

Corollary 8. *Suppose equation* (1) *is such that conditions* (A), (B), (C) *and* (D) *are satisfied. If the set* S *of* (4) *is nonempty then* S *cannot be uniformly bounded above or below on any subinterval of* $[x_1, b)$.

Proof. If S $\neq 0$, Theorem 5 and Corollary 7 imply that B is open. If S is uniformly bounded on some subinterval of $[x_1, b)$ then Lemma 6 implies that B is closed and bounded which is a contradiction. Minor alterations in the argument lead to the same contradiction if S is bounded below or bounded above on some subinterval of $[x_1, b)$.

Theorem 9. *Suppose equation*(1) *is such that conditions* (A), (B), *(C) and (D) are satisfied. Let $2 \leqq k \leqq n$, $a \leqq x_1 < \cdots < x_k < b$ and $y_1{}^i, y_j \in R$ for $0 \leqq i \leqq n-k$ and $2 \leqq j \leqq k$ then the boundary value problem $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$, $y^{(i)}(x_1) = y_1{}^i$, $y(x_j) = y_j$ for $0 \leqq i \leqq n-k$ and $2 \leqq j \leqq k$ has a unique solution.*

PROOF. The uniqueness condition has been established and we will probe the existence of solutions using induction on k.

The case when $k = 2$. Let S_1 be the set of all solutions y of (1) such that $y^{(i)}(x_1) = y_1{}^i$ for $0 \leqq i \leqq n-2$. Let $S_1(x) = \{y(x) \mid y \in S_1\}$. Then Corollary 7 implies that $S_1(x_2)$ is an open interval. Suppose $\text{lub}\{S_1(x_2)\} \leqq y_2$. Let u be a solution of (1) such that $u(x_2) = y_2$. We will consider three cases as $u(x_1) > y_1{}^0$, $u(x_1) < y_1{}^0$ or $u(x_1) = y_1{}^0$. First if $u(x_1) > y_1{}^0$, then by Corollary 8, there is a solution $y_1(x) \in S_1$ such that $y_1(x) > u(x)$ for some $x \in (x_1, x_2)$. Let $x_1 < \tau_1 < \tau_2 < x_2$ such that $y_1(\tau_j) = u(\tau_j)$ for $j = 1, 2$.

Inductively suppose for $1 \leqq \lambda < [n/2]$ and $x_1 < \tau_{2\lambda-1} < \tau_{2\lambda-3} < \cdots < \tau_1 < \tau_2 < \tau_4 < \cdots < \tau_{2\lambda} < x_2$, the set S_λ of all solutions y of (1) satisfying

(i) $y^{(i)}(x_1) = y_1{}^i$ for $0 \leqq i \leqq n - 2\lambda$,

(ii) $y(\tau_j) = u(\tau_j)$ for $1 \leqq j \leqq 2\lambda - 2$, if $\lambda \neq 1$,

also satisfies the properties

(iii) If $y \in S_\lambda$ then $y(x_2) < y_2$.

(iv) There is a $y_\lambda \in S_\lambda$ such that $y(\tau_j) = u(\tau_j)$ for $1 \leqq j \leqq 2\lambda$.

Then let $S_{\lambda+1}$ be the set of all solutions y of (1) such that $y^{(i)}(x_1) = y_1{}^i$ for $0 \leqq i \leqq n - 2(\lambda+1)$ and $y(\tau_j) = u(\tau_j)$ for $1 \leqq j \leqq 2\lambda$. $y_\lambda \in S_{\lambda+1}$ and if $y \in S_{\lambda+1}$ is such that $y(x_2) \geqq y_2$ then as a consequence of Theorem 3, $y^{(n-2\lambda-1)}(x_1) > y_\lambda{}^{(n-2\lambda-1)}(x_1) = y_1{}^{n-2\lambda-1}$. Hence for each $v \in S_\lambda$ there is a $\delta > 0$ such that $y(x) > v(x)$ on $(x_1, x_1 + \delta)$. But if $\lambda > 1$, S_λ is not bounded above on any of the intervals $(x_1, \tau_{2\lambda-3})$, $(\tau_{2\lambda-2}, x_2)$ and (x_2, b). Since $v(x_2) < y_2$ for all $v \in S_\lambda$ there is a $v \in S_\lambda$ such that v crosses y in $(x_1, \tau_{2\lambda-3})$, $(\tau_{2\lambda-2}, x_2)$ and (x_2, b) in addition to the common zeros at τ_j for $1 \leqq j \leqq 2\lambda - 2$. In case $\lambda = 1$, S_λ is not bounded above on (x_1, x_2) or (x_2, b) and since $v(x_2) < y_2$ for all $v \in S_\lambda$ there is a $v \in S_\lambda$ such that $v > y$ for some x in (x_1, x_2) and some x in (x_2, b). But for $\lambda \geqq 1$ this violates Corollary 4 because y and v then have $2\lambda + 1$ zeros on (x_1, b) and a zero of multiplicity $n - 2\lambda - 1$ at x_1. Hence for all $y \in S_{\lambda+1}$, $y(x_2) < y_2$. As a consequence of Corollary 8, there is a solution $y_{\lambda+1} \in S_{\lambda+1}$ such that $y_{\lambda+1} > u$ at points in $(x_1, \tau_{2\lambda-1})$ and $(\tau_{2\lambda}, x_2)$ and hence there are points $x_1 < \tau_{2\lambda+1} < \tau_{2\lambda-1} < \cdots < \tau_{2\lambda} < \tau_{2\lambda+2} < x_2$ such that $y_{\lambda+1}(\tau_j) = u(\tau_j)$ for $1 \leqq j \leqq 2\lambda + 2$. $S_{\lambda+1}$ satisfies the analogue of conditions (iii) and (iv) which S_λ satisfied. As a consequence of this inductive argument, for $\lambda = [n/2]$, there are points $x_1 < \tau_{2[n/2]-1} < \tau_{2[n/2]-3} < \cdots < \tau_1 < \tau_2 < \tau_4 < \cdots < \tau_{2[n/2]} < x_2$ and the nonempty set $S_{[n/2]}$ of all solutions y of (1) such that $y^{(i)}(x_1) = y_1{}^i$ for $0 \leqq i \leqq n - 2[n/2]$ and $y(\tau_j) = u(\tau_j)$ for $1 \leqq j \leqq 2[n/2] - 2$. In addition $S_{[n/2]}$ satisfies the analogue of conditions (iii) and (iv); namely, for all $y \in S_{[n/2]}$, $y(x_2) < y_2$ and there is a $y_{[n/2]} \in S_{[n/2]}$ such that $y_{[n/2]}(\tau_i) = u(\tau_i)$ for $1 \leqq j \leqq 2[n/2]$. If n is even this contradicts condition (B). If n is odd then since $S_{[n/2]}$ is not bounded above in $(x_1, \tau_{2[n/2]-3})$, $(\tau_{2[n/2]-2}, x_2)$ and (x_2, b), there is a solution $y \in S_{[n/2]}$ such that y crosses u in each of these intervals. But y equals u at the $n - 3$ points $\tau_1, \tau_2, \cdots, \tau_{n-3}$. This again contradicts condition (B). Hence we conclude that $u(x_1) \not> y_1{}^0$.

The second alternative is that $u(x_1) < y_1{}^0$. Then u crosses each $y \in S_1$ in the interval (x_1, x_2). Let $y_1 \in S_1$ be fixed and $x_1 < \tau_1 < x_2$ such that $y_1(\tau_1) = u(\tau_1)$. In this case define S_2 to be the set of all solutions y of (1) such that $y^{(i)}(x_1) = y_1{}^i$ for $0 \leqq i \leqq n - 3$ and $y(\tau_1) = u(\tau_1)$. The remaining part of this argument is similar to the previous inductive argument and we conclude that $u(x_1) \not< y_1{}^0$. This leaves the only remaining possibility that $u(x_1) = y_1{}^0$ but then by Theorem 5 there is another solution $v(x)$ of (1) such that $v(x_1) > y_1{}^0$ and $v(x_2) = y_2$ which case we have already eliminated. Hence $\text{lub}\{S_1(x_2)\} > y_2$. Similarly the glb $\{S_1(x_2)\} < y_2$ and so $y_2 \in S_1(x_2)$ and the theorem is true for $k = 2$.

Suppose the theorem is true for all $k < \lambda$ where λ is some fixed integer with $2 < \lambda \leqq n$ and that $a \leqq x_1 < \cdots < x_\lambda < b$ with $y_1{}^i, y_j \in R$ for $0 \leqq i \leqq n - \lambda$ and $2 \leqq j \leqq \lambda$. Let S be the set of all solutions y of (1) such that $y^{(i)}(x_1) = y_1{}^i$ for $0 \leqq i \leqq n - \lambda$ and $y(x_j) = y_j$ for $2 \leqq j \leqq \lambda - 1$. By the induction hypothesis $S \neq \emptyset$ and by Corollary 7 $S(x_\lambda) \equiv \{y(x_\lambda) \mid y \in S\}$ is an open interval. If $y_\lambda \in S(x_\lambda)$ then we are through. Suppose for example that $\text{lub}\{S(x_\lambda)\} \leqq y_\lambda$ and let $u(x)$ be a solution of (1) such that $u^{(i)}(x_1) = y_1{}^i$ for $0 \leqq i \leqq n - \lambda$ and $u(x_j) = y_j$ for $2 \leqq j \leqq \lambda$, $j \neq \lambda - 1$. For all $y \in S$, $u - y$ has a zero of order $n - \lambda + 1$ at x_1 and zeros at $x_2, x_3, \cdots, x_{\lambda-2}$. If $u(x_{\lambda-1}) < y_{\lambda-1} = y(x_{\lambda-1})$ for all $y \in S$ then each $y \in S$ crosses u in $(x_{\lambda-1}, x_\lambda)$. S is not bounded above on (x_λ, b) and hence there is a $y_1 \in S$ which crosses u in (x_λ, b) as well. But then $y_1 - u$ has a zero of order $n - \lambda + 1$ at x_1 and $\lambda - 1$ zeros on (x_1, b) which violates Corollary 4. If $u(x_{\lambda-1}) > y_{\lambda-1} = y(x_{\lambda-1})$ then since S is not bounded above in either of $(x_{\lambda-1}, x_\lambda)$ and (x_λ, b) there is a $y_1 \in S$ such that y_1 crosses u in $(x_{\lambda-1}, x_\lambda)$ and (x_λ, b). Hence we again arrive at a contradiction. If $u(x_{\lambda-1}) = y_{\lambda-1}$ then $u \in S$ and as a consequence of Theorem 5 $\text{lub}\{S(x_\lambda)\} > y_\lambda$. Therefore it must be the case that lub$\{S(x_\lambda)\} > y_\lambda$. Similarly glb$\{S(x_\lambda)\} < y_\lambda$ and hence $y_\lambda \in S(x_\lambda)$ and the boundary value problem is solved for the case $k = \lambda$. Thus by the induction principle the theorem is true for all $2 \leqq k \leqq n$.

The existence of solutions to the n-point problem evolves from this theorem by setting $k = n$ and will be stated separately for emphasis.

THEOREM 10. *Suppose equation* (1) *is such that conditions* (A), (B), (C) *and* (D) *are satisfied. If* $a \leqq x_1 < x_2 < \cdots < x_n < b$ *and* $y_j \in R$ *for* $1 \leqq j \leqq n$ *then the boundary value problem* $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$, $y(x_j) = y_j$ *for* $1 \leqq j \leqq n$ *has a unique solution.*

COROLLARY 11. *If* $2 \leqq k \leqq n$, $a \leqq x_1 < \cdots < x_k < b$, $y_j^i \in R$ *for* $0 \leqq i \leqq \mu_j - 1$, $1 \leqq \mu_j$ *and* $\sum_{j=1}^{k} \mu_j = n$ *then the boundary value problem* $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$, $y^{(i)}(x_j) = y_j^i$ *for* $0 \leqq i \leqq \mu_j - 1$, $1 \leqq j \leqq k$, *has a unique solution on* $[x_1, x_k]$. *This result follows from the major theorem of* [9] *since Theorem* 10 *asserts that the solution set of equation* (1) *is an n parameter family.*

3. **Properties of limits of solutions.** Suppose ϕ is the limit of a monotone, bounded sequence of solutions of (1) on some compact subinterval of $[a, b)$ and suppose f is such that conditions (A), (B) and (C) are satisfied. Without assuming condition (D) we will show that ϕ satisfies a uniqueness property with respect to solutions of (1) and that ϕ has differentiability properties.

A preliminary lemma is required to develop the properties of ϕ.

LEMMA 12. *Suppose f satisfies condition* (A). *For every* $\epsilon > 0$, $1 < k \leqq n$, $[c, d] \subset [a, b)$ *and* $M > 0$ *there is a* $\delta > 0$ *such that if* $c \leqq x_1 < x_2 < \cdots < x_k \leqq d$ *with* $|x_k - x_1| < \delta$, y_1^i, $y_j \in R$ *for* $0 \leqq i \leqq n - k$, $2 \leqq j \leqq k$ *and if the unique* $n - 1$ *degree polynomial* $p(x)$, *with* $p^{(i)}(x_1) = y_1^i$ *for* $0 \leqq i \leqq n - k$, $p(x_j) = y_j$ *for* $2 \leqq j \leqq k$ *satisfies* $|p^{(i)}(x)| \leqq M$ *for* $0 \leqq i \leqq n - 1$ *and* $x_1 \leqq x \leqq x_k$ *then the boundary value problem* $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$, $y^{(i)}(x_1) = y_1^i$, $y(x_j) = y_j$ *for* $0 \leqq i \leqq n - k$, $2 \leqq j \leqq k$ *has a solution* $y \in C^n[x_1, x_k]$ *and* $|p^{(i)}(x) - y^{(i)}(x)| < \epsilon$ *on* $[x_1, x_k]$ *for* $0 \leqq i \leqq n - 1$.

The proof of this lemma is a standard application of Schauder fixed point theorem to the mapping

$$T_y(x) = \int_{x_1}^{x_k} G(x, s) f(s, y(x), y'(s), \cdots, y^{(n-1)}(s))\, ds + p(x)$$

defined on $C^{n-1}[x_1, x_k]$ where $G(x, s)$ is the Green's function for the boundary value problem $y^{(n)} = 0$, $y^i(x_1) = 0$, $y(x_j) = 0$ for $0 \leqq i \leqq n - k, 2 \leqq j \leqq k$.

The following property is shared by monotone limits of solutions as well as solutions themselves.

DEFINITION 13. A function ϕ defined on an interval $I \subset [a, b)$ is said to be a generalized solution of (1) on I if for each $a \leqq x_1 < \cdots < x_n < b$ with $[x_1, x_n] \subset I$ and any solution y of (1),

(i) if $(-1)^i[y(x_i) - \phi(x_i)] > 0$ for $1 \leqq i \leqq n$ then $-[y(x) - \phi(x)] > 0$ on $[a, x_1] \cap I$ and $(-1)^n[y(x) - \phi(x)] > 0$ on $[x_n, b) \cap I$,

(ii) if $(-1)^i[y(x_i) - \phi(x_i)] < 0$ for $1 \leqq i \leqq n$ then $-[y(x) - \phi(x)] < 0$ on $[a, x_1] \cap I$ and $(-1)^n[y(x) - \phi(x)] < 0$ on $[x_n, b) \cap I$.

Notice that condition (B) implies that a solution of (1) is a generalized solution of (1).

THEOREM 14. *Suppose equation* (1) *is such that conditions* (A), (B) *and* (C′) *are satisfied. If* $\{y_k\}$ *is a monotone, bounded sequence of solutions which converge to* ϕ *on* $[c, d] \subset [a, b)$ *then* ϕ *is a generalized solution on* $[c, d]$.

PROOF. For definiteness assume $\{y_k\}$ is an increasing sequence. In order to show that ϕ is a generalized solution of (1) on $[c, d]$, suppose $c \leqq \tau_1 < \tau_2 < \cdots < \tau_n \leqq d$ and that y is a solution of (1) on $[c, d]$ such that $(-1)^i(y(\tau_i) - \phi(\tau_i)) > 0$ for $1 \leqq i \leqq n$. The most difficult case arises if $c < \tau_1$ and $\tau_n < d$. Since $y_k \to \phi$ on $[c, d]$, there is a k_0 such that $k \geqq k_0$ implies $(-1)^i(y(\tau_i) - y_k(\tau_i)) > 0$ for $1 \leqq i \leqq n$. Hence for $k \geqq k_0$, y_k crosses y $n-1$ times in (τ_1, τ_n) and $y < y_k < \phi$ on $[c, \tau_1]$. If n is odd and $k \geqq k_0$ then $y < y_k < \phi$ on $[\tau_n, d]$. If n is even then $y > y_k$ for $k \geqq k_0$ on $[\tau_n, d]$ and hence $y \geqq \phi = \lim_{k\to\infty} y_k$ on $[\tau_n, d]$. But if $y(x_0) = \phi(x_0)$ for some $x_0 \in (\tau_n, d]$ then replace y by a solution u of (1) such that $y(\tau_i) = u(\tau_i)$ for $1 \leqq i \leqq n-1$ and $\phi(\tau_n) < u(\tau_n) < y(\tau_n)$. This can be accomplished by Theorem 5. Then $u(x_0) < y(x_0) = \phi(x_0)$ which implies that for k sufficiently large u and y_k cross n times on $[\tau_1, d]$. Since this violates condition (B), $y > \phi$ on $[\tau_n, d]$ if n is even. Hence ϕ satisfies condition (i) of the definition of generalized solution. A similar argument is required to show that ϕ satisfies condition (ii) of that definition and hence ϕ is a generalized solution of (1) on $[c, d]$.

The next result displays the differentiability properties of generalized solutions. If ϕ is defined in a right-hand neighborhood of x_0, let $D^0\phi(x_0 + 0) = \lim_{x\to x_0+} \phi(x)$ if it exists. Inductively, if $D^i\phi(x_0 + 0)$ exists for $0 \leqq i \leqq k-1$ then let

$$D^k\phi(x_0 + 0) = \lim_{x\to x_0+} \left\{ \frac{k!}{(x - x_0)^k} \left(\phi(x) - \sum_{j=0}^{k-1} \frac{D^j\phi(x_0 + 0)(x - x_0)^j}{j!} \right) \right\}$$

if it exists. In a similar fashion $D^i\phi(x_0 - 0)$ are defined by taking the appropriate left-hand limits. If for some x_0, $D^i\phi(x_0 - 0) = D^i\phi(x_0 + 0)$ then we will denote their common value by $D^i\phi(x_0)$. It should be noted that if $\phi^{(k)}(x_0)$ exists then $D^k\phi(x_0)$ exists and $\phi^{(k)}(x_0) = D^k\phi(x_0)$.

THEOREM 15. *Suppose equation* (1) *is such that conditions* (A) *and* (B) *are satisfied. If* ϕ *is a bounded generalized solution of* (1) *on an interval* $I \subset [a, b)$, *then* $D^0\phi(x+0)$ *and* $D^0\phi(x-0)$ *exist for all* $x \in I^0$ *with the appropriate one-sided limits at the extreme points of* I. *Moreover if for some* $x \in I$, $D^i\phi(x+0)$ *exists and is finite for* $0 \leqq i \leqq k-1$ *where* $k \leqq n-1$ *then* $D^k\phi(x+0)$ *exists as an extended real number. Similarly if* $D^i\phi(x-0)$ *exists for* $0 \leqq i \leqq k-1$ *then* $D^k\phi(x-0)$ *exists as an extended real number.*

PROOF. Suppose for some $x_0 \in I$, $D^0\phi(x_0+0)$ does not exist. Then there are $\alpha, \beta \in R$ such that

$$\liminf_{x\to x_0+} \phi(x) < \alpha < \beta < \limsup_{x\to x_0+} \phi(x). \tag{6}$$

With $p(x) = (\alpha+\beta)/2$ on $[x_0, b)$, $M = (\alpha+\beta)/2$ and $\epsilon = |\beta - \alpha|/2$, Lemma 12 asserts the existence of a $\delta > 0$ such that the boundary value problem

$$y^{(n)} = f(x, y, y', \cdots, y^{(n-1)}),$$
$$y(x_0) = (\alpha+\beta)/2 = y(x_0+\delta), \qquad y^{(i)}(x_0) = 0$$

for $1 \leqq i \leqq n-2$ has a solution $y(x)$ on $[x_0, x_0+\delta]$ and $\alpha < y(x) < \beta$ on $[x_0, x_0+\delta]$. This bound on y and inequality (6) imply that x_0 is an accumulation point of real numbers x for which $\phi(x) > y(x)$ and $\phi(x) < y(x)$. But this is incompatible with ϕ being a generalized solution. Thus $\lim_{x\to x_0+} \phi(x)$ exists and is finite since ϕ is bounded. A similar argument will show that $D^0\phi(x-0)$ exists for any $x \in I$ which is not a left extreme point of I.

Secondly, suppose for some $x_0 \in I$, $D^i\phi(x_0+0)$ exists for $0 \leqq i \leqq k-1$ but $D^k\phi(x_0+0)$ does not exist as an extended real number. Then there are $\alpha, \beta \in R$ such that

$$\begin{aligned}&\liminf_{x\to x_0+} \left\{\frac{k!}{(x-x_0)^k}\left(\phi(x) - \sum_{i=0}^{k-1} \frac{D^i\phi(x_0+0)(x-x_0)^i}{i!}\right)\right\} < \alpha \\ &< \beta < \limsup_{x\to x_0+} \left\{\frac{k!}{(x-x_0)^k}\left(\phi(x) - \sum_{i=0}^{k-1} \frac{D^i\phi(x_0+0)(x-x_0)^i}{i!}\right)\right\}.\end{aligned} \tag{7}$$

Let $p(x)$ be the $n-1$ degree polynomial such that $p^{(i)}(x_0) = D^i\phi(x_0+0)$ for $0 \leqq i \leqq k-1$, $p^{(k)}(x_0) = (\alpha+\beta)/2$, $p^{(j)}(x_0) = 0$ for $k+1 \leqq j \leqq n-1$. Also let $M = \max_{0 \leqq i \leqq n-1} |p^{(i)}(x)|$ for $c \leqq x \leqq d$ where $a \leqq c < x_0 < d < b$. Let $\epsilon = (\beta - \alpha)/2$. Then Lemma 12 implies the existence of a $\delta > 0$ such that the boundary value problem $y^{(i)}(x_0) = p^{(i)}(x_0)$ for $0 \leqq i \leqq n-2$, $y(x_0+\delta) = p(x_0+\delta)$ has a solution y on $[x_0, x_0+\delta]$ and $|y^{(k)}(x) - p^{(k)}(x)| < \epsilon$ on $[x_0, x_0+\delta]$. In particular $\alpha < y^{(k)}(x_0) < \beta$ and $y^{(i)}(x_0) = D^i\phi(x_0+0)$ for $0 \leqq i \leqq k-1$. Also

$$y^{(k)}(x_0) = \lim_{x \to x_0+} \left\{ \frac{k!}{(x-x_0)^k} \left(y(x) - \sum_{i=0}^{k-1} \frac{D^i\phi(x_0+0)(x-x_0)^i}{i!} \right) \right\}$$

and hence in view of inequality (7) we arrive at the same incompatibility with ϕ being a generalized solution. Thus $D^k\phi(x_0+0)$ must exist as an extended real number. The existence of $D^k\phi(x_0-0)$ can be similarly handled. This completes the proof of Theorem 15.

This result, by means of the following lemma, yields an existence theorem.

LEMMA 16. *Suppose equation* (1) *is such that conditions* (A), (B) *and* (C′) *are satisfied. If ϕ is a generalized solution of* (1) *on an interval $I \subset [a, b)$ and x_0 is an interior point of I at which $D^{(n-1)}\phi(x_0+0)$ exists and is finite then there exists a $\delta > 0$ such that ϕ is a solution of* (1) *on* $[x_0, x_0+\delta)$.

PROOF. Let $x_0 \in (c, d) \subset I$ be such that $D^{(n-1)}\phi(x_0+0)$ exists and is finite. Then $D^i\phi(x_0+0)$ are finite for all $0 \leqq i \leqq n-1$ by definition. Let $x_0 < x_n < d$ and $p(x; x_n)$ be the unique $n-1$ degree polynomial such that $p^{(i)}(x_0; x_n) = D^i\phi(x_0+0)$ for $0 \leqq i \leqq n-2$ and $p(x_n; x_n) = \phi(x_n)$. Then

$$\begin{aligned} p(x; x_n) &= p(x_0; x_n) + p'(x_0; x_n)(x-x_0) + \cdots \\ &\quad + \frac{p^{(n-1)}(x_0; x_n)(x-x_0)^{n-1}}{(n-1)!} \\ &= \sum_{i=0}^{n-2} \left(\frac{D^i\phi(x_0+0)(x-x_0)^i}{i!} \right) \\ &\quad + \frac{p^{(n-1)}(x_0; x_n)(x-x_0)^{n-1}}{(n-1)!}. \end{aligned}$$

Setting $x = x_n$ above with $p(x_n; x_n) = \phi(x_n)$ and solving for $p^{(n-1)}(x_0; x_n)$ we have

$$p^{(n-1)}(x_0; x_n) = \frac{(n-1)!}{(x_n - x_0)^{n-1}} \left\{ \phi(x_n) - \sum_{i=0}^{n-2} \frac{D^i\phi(x_0+0)(x_n - x_0)^i}{i!} \right\}$$

thus $\lim_{x_n \to x_0} p^{n-1}(x_0; x_n) = D^{n-1}\phi(x_0 + 0)$. Hence if $p(x)$ is the $n-1$ degree polynomial such that $p^{(i)}(x_0) = D^i\phi(x_0 + 0)$ for $0 \leqq i \leqq n-1$ then $\lim_{x_n \to x_0} p(x; x_n) = p(x)$ in $C^{n-1}[x_0, d]$. Thus for $\gamma > 0$ sufficiently small there exists an M such that if $x_0 < x_n \leqq x_0 + \gamma$, $|p^{(i)}(x; x_n)| \leqq M$ for $0 \leqq i \leqq n-1$ and $x_0 \leqq x \leqq x_0 + \gamma$. By Lemma 12, there exists a $\delta > 0$ such that if $x_0 < x_n < x_0 + \delta$ then the boundary value problem $y^{(n)} = f(x, y, \cdots, y^{(n-1)})$, $y^{(i)}(x_0) = p^{(i)}(x_0; x_n) = D^i\phi(x_0 + 0)$ for $0 \leqq i \leqq n-2$, $y(x_n) = p(x_n; x_n)$ has a solution y on $[x_0, x_n]$. Suppose $y(x) \neq \phi(x)$ for some x in $[x_0, x_n]$. To be specific assume $y(x_{n-1}) > \phi(x_{n-1})$ where $x_0 < x_{n-1} < x_n$. Let $y_1(x)$ be a solution of (1) such that $y_1^{(i)}(x_0) = y^{(i)}(x_0) = D^i\phi(x_0 + 0)$ for $0 \leqq i \leqq n-2$ but the difference $y_1^{(n-1)}(x_0) - y^{(n-1)}(x_0) < 0$ and sufficiently small that $y_1(x_{n-1}) > \phi(x_{n-1})$ and $y_1(x_n) < y(x_n) = \phi(x_n)$. Let $y_2(x)$ be the solution of (1) such that $y_2^{(i)}(x_0) = y_1^{(i)}(x_0)$ for $0 \leqq i \leqq n-1$, $i \neq n-2$, but the difference $y_2^{(n-2)}(x_0) - y_1^{(n-2)}(x_0) < 0$ and sufficiently small that $y_2(x_{n-1}) > \phi(x_{n-1})$ and $y_2(x_n) < \phi(x_n)$ then there exists an x_{n-2} such that $x_0 < x_{n-2} < x_{n-1}$ and $y_2(x_{n-2}) < \phi(x_{n-2})$. Define $y_k(x)$ inductively as follows. Given that $2 \leqq k \leqq n-1$ and $y_{k-1}(x)$ is a solution (1) such that $y_{k-1}^{(i)}(x_0) = y_{k-2}^{(i)}(x_0)$ for $0 \leqq i \leqq n-1$, $i \neq n-k+1$, $(-1)^k(y_{k-1}^{(n-k+1)}(x_0) - y_{k-2}^{(n-k+1)}(x_0)) > 0$ and there are $x_0 < x_{n-k+1} < \cdots < x_n$ such that $(-1)^{n+1-i}(y_{k-1}(x_i) - \phi(x_i)) > 0$ for $n-k+1 \leqq i \leqq n$; define $y_k(x)$ to be a solution of (1) such that $y_k^{(i)}(x_0) = y_{k-1}^{(i)}(x_0)$ for $0 \leqq i \leqq n-1$, $i \neq n-k$, $(-1)^{k+1}(y_k^{(n-k)}(x_0) - y_{k-1}^{(n-k)}(x_0)) > 0$ so small that $(-1)^{n+1-i}(y_k(x_i) - \phi(x_i)) > 0$ for $n-k+1 \leqq i \leqq n$. Then there exists an x_{n-k} such that $x_0 < x_{n-k} < x_{n-k+1} < \cdots < x_n$ and $(-1)^{k+1}(y_k(x_{n-k}) - \phi(x_{n-k})) > 0$. In particular for $k = n-1$ $(-1)^{n-1-i}(y_{n-1}(x_i) - \phi(x_i)) > 0$ for $1 \leqq i \leqq n$ where $x_0 < x_1 < x_2 < \cdots < x_n < b$. But then $y_{n-1}(x_0) = \phi(x_0)$ violates the hypothesis that ϕ is a generalized solution. Hence $\phi \equiv y$ on $[x_0, x_n]$.

As a consequence of this result we can conclude an existence theorem.

THEOREM 17. *If equation* (1) *is such that* (A), (B) *and* (C) *are satisfied and if for each generalized solution* ϕ *of* (1) *there exists an* $x_0 \in [a, b)$ *such that either* $D^{(n-1)}\phi(x_0 + 0)$ *or* $D^{(n-1)}\phi(x_0 - 0)$ *exists and is finite then for each* $a \leqq x_1 < \cdots < x_n < b$ *and* $y_i \in R$, $1 \leqq i \leqq n$, *the boundary value problem* (1), (2) *has a unique solution.*

PROOF. It suffices to show that property (D) is valid. But the ϕ expressed in property (D), being a generalized solution, is a solution on some subinterval of $[c, d]$ as a consequence of Lemma 16. The proof of Lemma 6 provides a technique for proving that ϕ is then a solution on $[c, d]$ and property (D) is satisfied.

Using a theorem about differentiation of arbitrary real valued functions, see [10, p. 16 ff.], one can conclude that if ϕ is a generalized solution of (1) on $[c, d] \subset [a, b)$ then $D'\phi(x + 0)$ exists and is finite for almost all $x \in [c, d]$. The author however, has not been able to extend these results to higher differentiation.

We conclude with an existence theorem for the case when f depends on x and y only.

THEOREM 18. *Suppose* $f(x, y, \cdots, y^{(n-1)}) \equiv g(x, y)$ *and conditions* (A), (B), *and* (C) *are valid, then the boundary value problem* (1), (2) *has a unique solution for each* $a \leqq x_1 < \cdots < x_n < b$ *and* $y_i \in R$, $1 \leqq i \leqq n$.

PROOF. Suppose $\{y_k\}$ is a monotone, bounded sequence of solutions which converges pointwise to a function ϕ on $[c, d] \subset [a, b)$. Let $c = x_1 < x_2 < \cdots < x_n = d$ and p_k be the unique polynomial of degree $n - 1$ such that $p_k(x_i) = y_k(x_i)$ for $1 \leqq i \leqq n$ and $k = 1, 2, 3, \cdots$. Then p_k converges uniformly to p where p is the unique polynomial of degree $n - 1$ such that $p(x_i) = \lim_{k\to\infty} y_k(x_i)$ for $1 \leqq i \leqq n$. Thus if $G(x, s)$ is the Green's function for the boundary value problem $y^{(n)} = 0, y(x_i) = 0$ for $1 \leqq i \leqq n$ then

$$y_k(x) = \int_c^d G(x, s)g(s, y_k(s))\, ds + p_k(x) \quad \text{for all } x \in [c, d].$$

Let

$$\omega_k(x) = y_k(x) - p_k(x) = \int_c^d G(x, s)g(s, y_k(s))\, ds.$$

Since $y_k(x)$ are uniformly bounded on $[c, d]$,

$$M \equiv \sup\{|g(x, y_k(x))| : c \leqq x \leqq d, k \geqq 1\}$$

exists. Aso $\partial G/\partial x$ exists and is continuous on $[c, d] \times [c, d]$ and hence there is a constant $K > 0$ $|\partial G(x, s)/\partial x| \leqq K$ for x, $s \in [c, d]$. Thus if $t \neq x$,

$$|\omega_k(x) - \omega_k(t)| \leqq \int_c^d |G(x, s) - G(t, s)|\, |g(s, y_k(s))|\, ds$$
$$\leqq MK|x - t|(d - c).$$

Hence $\omega_k(x)$ is uniformly bounded and equicontinuous on $[c, d]$. Thus a subsequence and by monotonicity the sequence $\{y_k\}$ converges uniformly to ϕ on $[c, d]$. Taking limits through the integral representation of y_k yields that ϕ is a solution on $[c, d]$ and hence property (D) is satisfied.

References

1. A. Lasota and Z. Opial, *On the existence and uniqueness of solutions of a boundary value problem for an ordinary second-order differential equation,* Colloq. Math. 18 (1967), 1-5. MR 36 #2871.

2. L. K. Jackson, *Subfunctions and second-order ordinary differential inequalities,* Advances in Math. 2(1968), 307-363. MR 37 #5462.

3. L. F. Shampine, *Existence and uniqueness for nonlinear boundary value problems,* J. Differential Equations 5 (1969), 346-351. MR 39 #534.

4. K. Schrader, *Existence theorems for second order boundary value problems,* J. Differential Equations 5 (1969), 572-584. MR 39 #532.

5. K. Schrader and P. Waltman, *An existence theorem for nonlinear boundary value problems,* Proc. Amer. Math. Soc. 21 (1969), 653-656. MR 39 #533.

6. L. Jackson and K. Schrader, *Existence and uniqueness of solutions of boundary value problems for third order differential equations,* J. Differential Equations 9 (1971), 46-54. MR 42 #4813.

7. P. Hartman, *On n-parameter families and interpolation problems for nonlinear ordinary differential equations,* Trans. Amer. Math. Soc. 154 (1971), 201-226.

8. L. Jackson and G. Klaasen, *Uniqueness of solutions of boundary value problems for ordinary differential equations,* SIAM J. Appl. Math. 19 (1970), 542-546. MR 42 #4812.

9. P. Hartman, *Unrestricted n-parameter families,* Rend. Circ. Mat. Palermo (2) 7 (1958), 123-142. MR 21 #4211.

10. F. Riesz and B. Sz.-Nagy, *Leçons d'analyse fonctionnelle,* Akad. Kiadó, Budapest, 1953; English transl., Ungar, New York, 1955. MR 15, 132; MR 17, 175.

University of Tennessee, Knoxville, Tennessee 37916