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Theorems regarding existence and uniqueness of weak solutions to mixed boundary value problems in the linear theory of micropolar shells in statics and dynamics are proved. Convergence of FEM for the static mixed problems is established. Eigenvalue problems for micropolar shells are studied and properties of the spectrum and eigenmodes are formulated.

Introduction

Although shell theory is one of the oldest areas in continuum mechanics and applied mathematics, it is still under development. New materials and the increasing requirements of engineering practice stimulate the appearance of new non-classical versions of shell theories. In the recently developed 6-parametric or micropolar shell theory [3, 12, 16, 18], a shell is a Cosserat two-dimensional continuum (surface Σ) as introduced by the Cosserat brothers [10] over 100 years ago. A linear version of this theory is presented in [3, 12] and [14]. For infinitesimal deformations, at each point of Σ the shell kinematics is described by six scalar quantities: three of these are the components of the displacement vector $\boldsymbol{u}$ and the other three are the components of the microrotation vector $\boldsymbol{\vartheta}$. A shell particle has six degrees of freedom described by the components of $\boldsymbol{u}$ and ϑ. The vectors $\boldsymbol{u}$ and ϑ are mutually independent. For a micropolar shell, we can assign the force and couple loads acting on the shell surface. The order of the equilibrium equations in the theory is 12, so we should supplement the equations with 6 conditions on the shell edge. On the portion of the edge that is free from geometrical constraints, we should assign forces and couple distributions.

The micropolar theory is used, in particular, to describe branching shells: thin-walled bodies with complex internal structure. These include multilayered or composite plates and shells, shells with internal partitions and stringers, cellular bodies made from highly porous materials such as foams.

Mathematical studies of boundary value problems in shell theory are presented in the literature; see, for example, [2, 7, 8, 15, 19]. In large part these consider classical versions of shell theory.

In this paper we prove the existence and uniqueness of weak solutions to boundary value problems of statics and dynamics for micropolar shells. We also consider the properties of the spectrum in this theory.

Sect. 1 presents the governing relations of the linear theory of a micropolar shell. The weak setup of equilibrium problems is studied in Sect. 2. Here we introduce the setup of the problem in the energy space, and prove the theorem on the uniqueness and existence of the weak solution. In Sect. 3 we establish some properties of the spectrum of the eigenvalue problem and present the Rayleigh principle for micropolar shells. Sect. 4 treats the problems of existence and uniqueness for dynamic problems in micropolar shell theory.

1 Basic relations for micropolar shells

Following [14], we present the micropolar shell equations for small deformations. An elastic micropolar shell is represented by a deformable surface Σ possessing surface energy and other characteristics distributed over the surface. The shell deformation is described by two vector fields

$$\boldsymbol{u} = \boldsymbol{u}(q^1, q^2, t), \quad \vartheta = \vartheta(q^1, q^2, t), \tag{1}$$

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where q^1, q^2 are coordinates on Σ, t is time, $\boldsymbol{u}$ is the displacement vector, and $\boldsymbol{\vartheta}$ is the microrotation vector at a point on Σ. The position vector of a point on Σ is $\boldsymbol{r} = \boldsymbol{r}(q^1, q^2)$.

We shall require the main and dual bases on Σ, denoted by $\boldsymbol{r}_\alpha$ and $\boldsymbol{r}^\beta$, respectively. They are determined by the formulas

$$\boldsymbol{r}_\alpha = \frac{\partial \boldsymbol{r}}{\partial q^\alpha}, \quad \boldsymbol{r}_\alpha \cdot \boldsymbol{r}^\beta = \delta_\alpha^\beta, \quad \boldsymbol{n} \cdot \boldsymbol{r}^\beta = 0, \quad \alpha, \beta = 1, 2,$$

where δ_α^β is the Kronecker symbol and $\boldsymbol{n}$ is the unit normal to Σ.

The dynamical equations are

$$\nabla \cdot \mathbf{T} + \boldsymbol{q} = \rho \ddot{\boldsymbol{u}} + \rho \Theta_1 \cdot \ddot{\vartheta}, \quad \nabla \cdot \mathbf{M} + \mathbf{T}_\times + \boldsymbol{m} = \rho \boldsymbol{\Theta}_1^{\mathrm{T}} \cdot \ddot{\boldsymbol{u}} + \rho \Theta_2 \cdot \ddot{\vartheta}, \tag{2}$$

where $\mathbf{T}$ and $\mathbf{M}$ are surface stress and couple stress tensors, respectively, $\nabla \equiv \boldsymbol{r}^\alpha \frac{\partial}{\partial q^\alpha}$ is the surface nabla operator, an overdot denotes differentiation with respect to time t, ρ is the shell surface density, $\boldsymbol{\Theta}_1$ and $\boldsymbol{\Theta}_2$ are the inertia tensors, and $\boldsymbol{q}$ and $\boldsymbol{m}$ are distributed surface forces and couples, respectively. The tensor $\boldsymbol{\Theta}_2$ is symmetric with $\boldsymbol{\Theta}_2^T = \Theta_2$. Here $\mathbf{T}_\times$ denotes the vectorial invariant of the second-order tensor $\mathbf{T}$ [14].

On some portion of the boundary the kinematic conditions

$$\boldsymbol{u}\big|_{\omega_1} = \boldsymbol{u}^0(s, t), \qquad \boldsymbol{\vartheta}\big|_{\omega_3} = \boldsymbol{\vartheta}^0(s, t), \tag{3}$$

are specified. On the rest of the boundary the conditions are static:

$$\nu \cdot \mathbf{T}\big|_{\omega_2} = \boldsymbol{\varphi}(s, t), \qquad \nu \cdot \mathbf{M}\big|_{\omega_4} = \boldsymbol{\ell}(s, t). \tag{4}$$

Here $\boldsymbol{u}^0(s, t)$ and $\vartheta^0(s, t)$ are given vector functions of the length parameter s and time t, defining the displacements and microrotations of the shell edge, and $\boldsymbol{\nu}$ is the external normal vector to the shell contour $\omega \equiv \partial\Sigma$ such that $\nu \cdot \boldsymbol{n} = 0$. The functions $\varphi(s, t)$ and $\boldsymbol{\ell}(s, t)$ determine the surface stresses and stress couples on the edge. We have $\omega = \omega_1 \cup \omega_2 = \omega_3 \cup \omega_4$, where $\omega_2 = \omega \backslash \omega_1$ and $\omega_4 = \omega \backslash \omega_3$.

The linear strain measures take the form

$$\boldsymbol{\epsilon} = \nabla \boldsymbol{u} + \mathbf{A} \times \vartheta, \qquad \boldsymbol{\kappa} = \nabla \boldsymbol{\vartheta}, \tag{5}$$

where ϵ and $\boldsymbol{\kappa}$ are non-symmetric surface strain and bending strain tensors, respectively, and $\mathbf{I}$ and $\mathbf{A} = \mathbf{I} - \boldsymbol{n} \otimes \boldsymbol{n} = \boldsymbol{r}_\alpha \otimes \boldsymbol{r}^\alpha$ are the three- and two-dimensional unit tensors, respectively.

The constitutive equations for an elastic shell are represented through the strain energy density $U = U(\boldsymbol{\epsilon}, \kappa)$:

$$\mathbf{T} = \frac{\partial U}{\partial \boldsymbol{\epsilon}}, \qquad \mathbf{M} = \frac{\partial U}{\partial \boldsymbol{\kappa}}. \tag{6}$$

For an isotropic shell, U is a quadratic form with respect to the components of ϵ and $\boldsymbol{\kappa}$ [11]:

$$\begin{aligned} 2U &= \alpha_1 \mathrm{tr}^2 \epsilon + \alpha_2 \mathrm{tr}\, \tilde{\epsilon}^2 + \alpha_3 \mathrm{tr} \left(\tilde{\epsilon} \cdot \tilde{\epsilon}^T \right) + \alpha_4 \boldsymbol{n} \cdot \boldsymbol{\epsilon}^T \cdot \epsilon \cdot \boldsymbol{n} \\ &\quad + \beta_1 \mathrm{tr}^2 \kappa + \beta_2 \mathrm{tr}\, \tilde{\kappa}^2 + \beta_3 \mathrm{tr} \left(\tilde{\kappa} \cdot \tilde{\kappa}^T \right) + \beta_4 \boldsymbol{n} \cdot \kappa^T \cdot \boldsymbol{\kappa} \cdot \boldsymbol{n}, \\ \tilde{\boldsymbol{\epsilon}} &= \epsilon \cdot \mathbf{A}, \quad \tilde{\kappa} = \kappa \cdot \mathbf{A}, \end{aligned} \tag{7}$$

where α_k and β_k $(k = 1, 2, 3, 4)$ are elastic moduli. Substituting (7) into (6), we get

$$\mathbf{T} = \alpha_1 \mathbf{A} \mathrm{tr}\, \epsilon + \alpha_2 \tilde{\epsilon}^T + \alpha_3 \tilde{\boldsymbol{\epsilon}} + \alpha_4 (\epsilon \cdot \boldsymbol{n}) \boldsymbol{n}, \quad \mathbf{M} = \beta_1 \mathbf{A} \mathrm{tr}\, \boldsymbol{\kappa} + \beta_2 \tilde{\boldsymbol{\kappa}}^T + \beta_3 \tilde{\kappa} + \beta_4 (\boldsymbol{\kappa} \cdot \boldsymbol{n}) \boldsymbol{n}. \tag{8}$$

Suppose the energy density U is a positive definite function of its arguments; that is, there exists a positive constant C such that

$$U(\boldsymbol{\epsilon}, \kappa) \geq C(\|\boldsymbol{\epsilon}\|^2 + \|\boldsymbol{\kappa}\|^2). \tag{9}$$

To avoid constants and norms carrying units, we suppose that all quantities and equations have been cast in dimensionless form. The norm of the second order tensor $\mathbf{X}$ is defined by $\|\mathbf{X}\| = \left[\mathrm{tr}\,(\mathbf{X} \cdot \mathbf{X}^T)\right]^{1/2} = (X_{mn} X^{mn})^{1/2}$, where X_{mn} and X^{mn} are co- and contravariant components of $\mathbf{X}$ in a basis, respectively.

Inequality (9) implies the following inequalities for the elastic moduli:

$$
\begin{aligned}
&2\alpha_1+\alpha_2+\alpha_3>0, \quad \alpha_2+\alpha_3>0, \quad \alpha_3-\alpha_2>0, \quad \alpha_4>0,\\
&2\beta_1+\beta_2+\beta_3>0, \quad \beta_2+\beta_3>0, \quad \beta_3-\beta_2>0, \quad \beta_4>0.
\end{aligned}
\tag{10}
$$

The set of Eqs. (2)–(4), (5), and (8) constitutes a linear boundary value problem in the unknown variables $\boldsymbol{u}$ and $\boldsymbol{\vartheta}$; the equations describe the motion of a micropolar shell in the case of small deformations.

We consider the coordinates q^1, q^2 on Σ related to the lines of principal curvature. The basis vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ related to q^1, q^2 are orthogonal; they are eigenvectors of the curvature tensor

$$
\mathbf{B} \equiv -\nabla \boldsymbol{n} = -\frac{1}{R_1}\boldsymbol{e}_1\otimes\boldsymbol{e}_1 - \frac{1}{R_2}\boldsymbol{e}_2\otimes\boldsymbol{e}_2.
$$

Here R_1, R_2 are the radii of the principal curvatures.

In this basis, $\boldsymbol{u}$ and $\boldsymbol{\vartheta}$ are represented as

$$
\boldsymbol{u} = u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2 + w\boldsymbol{n}, \quad \boldsymbol{\vartheta} = \vartheta_1\boldsymbol{e}_1 + \vartheta_2\boldsymbol{e}_2 + \vartheta_3\boldsymbol{n}.
$$

The nabla operator takes the form

$$
\nabla = \frac{1}{A_1}\boldsymbol{e}_1\frac{\partial}{\partial q^1} + \frac{1}{A_2}\boldsymbol{e}_2\frac{\partial}{\partial q^2},
$$

where the A_α are the Lamé coefficients, $A_1 = (\boldsymbol{r}_1\cdot\boldsymbol{r}_1)^{1/2}$ and $A_2 = (\boldsymbol{r}_2\cdot\boldsymbol{r}_2)^{1/2}$. The strain tensors (5) are

$$
\epsilon = \varepsilon_{\alpha\beta}\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\beta + \varepsilon_\alpha\boldsymbol{e}_\alpha\otimes\boldsymbol{n}, \quad \kappa = \kappa_{\alpha\beta}\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\beta + \kappa_\alpha\boldsymbol{e}_\alpha\otimes\boldsymbol{n},
\tag{11}
$$

where

$$
\begin{aligned}
\varepsilon_{11} &= \frac{1}{A_1}\frac{\partial u_1}{\partial q^1} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial q^2}u_2 + \frac{w}{R_1}, &
\varepsilon_{22} &= \frac{1}{A_2}\frac{\partial u_2}{\partial q^2} + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial q^1}u_1 + \frac{w}{R_2},\\
\varepsilon_{12} &= \frac{1}{A_1}\frac{\partial u_2}{\partial q^1} - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial q^2}u_1 - \vartheta_3, &
\varepsilon_{21} &= \frac{1}{A_2}\frac{\partial u_1}{\partial q^2} - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial q^1}u_2 + \vartheta_3,\\
\varepsilon_1 &= \frac{1}{A_1}\frac{\partial w}{\partial q_1} + \frac{u_1}{R_1} + \vartheta_2, &
\varepsilon_2 &= \frac{1}{A_2}\frac{\partial w}{\partial q_2} + \frac{u_2}{R_2} - \vartheta_1,\\
\kappa_{11} &= \frac{1}{A_1}\frac{\partial \vartheta_1}{\partial q^1} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial q^2}\vartheta_2 + \frac{\vartheta_3}{R_1}, &
\kappa_{22} &= \frac{1}{A_2}\frac{\partial \vartheta_2}{\partial q^2} + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial q^1}\vartheta_1 + \frac{\vartheta_3}{R_2},\\
\kappa_{12} &= \frac{1}{A_1}\frac{\partial \vartheta_2}{\partial q^1} - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial q^2}\vartheta_1, &
\kappa_{21} &= \frac{1}{A_2}\frac{\partial \vartheta_1}{\partial q^2} - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial q^1}\vartheta_2,\\
\kappa_1 &= \frac{1}{A_1}\frac{\partial \vartheta_3}{\partial q_1} + \frac{\vartheta_1}{R_1}, &
\kappa_2 &= \frac{1}{A_2}\frac{\partial \vartheta_3}{\partial q_2} + \frac{\vartheta_2}{R_2}.
\end{aligned}
$$

2 Weak setup of boundary-value problems of statics

The equilibrium equations for the shell and the boundary conditions take the form

$$
\begin{aligned}
&\nabla\cdot\mathbf{T} + \boldsymbol{q} = \mathbf{0}, \quad \nabla\cdot\mathbf{M} + \mathbf{T}_\times + \boldsymbol{m} = \mathbf{0},\\
&\boldsymbol{u}\big|_{\omega_1} = \boldsymbol{u}^0(s), \quad \boldsymbol{\vartheta}\big|_{\omega_3} = \boldsymbol{\vartheta}^0(s), \quad \nu\cdot\mathbf{T}\big|_{\omega_2} = \boldsymbol{\varphi}(s), \quad \nu\cdot\mathbf{M}\big|_{\omega_4} = \boldsymbol{\ell}(s).
\end{aligned}
\tag{12}
$$

The boundary value problem (12) can be formulated as a variational problem. *Lagrange's variational principle* for an elastic micropolar shell starts with the formulation of the total potential energy functional

$$
J(\boldsymbol{u},\boldsymbol{\vartheta}) = \int_\Sigma U(\boldsymbol{\epsilon},\kappa)\,d\Sigma - A(\boldsymbol{u},\vartheta),
\tag{13}
$$

where the potential of external loads $A(\boldsymbol{u},\vartheta)$ is

$$
A(\boldsymbol{u},\boldsymbol{\vartheta}) = \int_\Sigma (\boldsymbol{q}\cdot\boldsymbol{u} + \boldsymbol{m}\cdot\vartheta)\ d\Sigma + \int_{\omega_2}\boldsymbol{\varphi}\cdot\boldsymbol{u}\,ds + \int_{\omega_4}\ell\cdot\boldsymbol{\vartheta}\,ds.
$$

The functional $J(\boldsymbol{u},\vartheta)$ is considered on the set of twice continuously differentiable fields of displacements and microrotations that satisfy (3). The pair $(\boldsymbol{u},\vartheta)$ that satisfies (12) is a stationary point of $J(\boldsymbol{u},\boldsymbol{\vartheta})$. Lagrange's stationary principle is minimal: on the equilibrium solution, the functional (13) attains its minimum.

The first variation of J is

$$\delta J(\boldsymbol{u},\vartheta) = \int_\Sigma \left(\mathbf{T}\cdot\cdot\delta\boldsymbol{\epsilon}^T + \mathbf{M}\cdot\cdot\delta\boldsymbol{\kappa}^T\right) d\Sigma - \delta A(\boldsymbol{u},\vartheta), \tag{14}$$

where

$$\delta A(\boldsymbol{u},\vartheta) = \int_\Sigma (\boldsymbol{q}\cdot\delta\boldsymbol{u} + \boldsymbol{m}\cdot\delta\boldsymbol{\vartheta})\, d\Sigma + \int_{\omega_2} \boldsymbol{\varphi}\cdot\delta\boldsymbol{u}\, ds + \int_{\omega_4} \boldsymbol{\ell}\cdot\delta\boldsymbol{\vartheta}\, ds, \tag{15}$$

and the symbol "$\cdot\cdot$" stands for the double dot product in the space of second order tensors, for example, $\mathbf{X}\cdot\cdot\mathbf{Y} = X_{mn}Y^{nm}$. The equation $\delta J(\boldsymbol{u},\vartheta) = 0$ serves as the basis for introduction of a weak (or generalized, or energy) solution of the problem (12). In spanned form, it is

$$\int_\Sigma \left(\mathbf{T}(\epsilon)\cdot\cdot\delta\boldsymbol{\epsilon}^T + \mathbf{M}(\kappa)\cdot\cdot\delta\kappa^T\right) d\Sigma - \int_\Sigma (\boldsymbol{q}\cdot\delta\boldsymbol{u} + \boldsymbol{m}\cdot\delta\boldsymbol{\vartheta})\, d\Sigma - \int_{\omega_2} \varphi\cdot\delta\boldsymbol{u}\, ds - \int_{\omega_4} \boldsymbol{\ell}\cdot\delta\vartheta\, ds = 0. \tag{16}$$

First we introduce the energy space. For simplicity, we suppose that Σ is sufficiently smooth and that the coordinate lines are the lines of principal curvature. Hence R_1, R_2 are the principal radii of curvature of Σ at a point. Suppose R_1, R_2 are continuous functions on Σ. Also suppose and that Lamé's coefficients A_1, A_2 of Σ are continuously differentiable on Σ and do not degenerate at any point. Hence there is a constant $m > 0$ such that

$$A_1 \geq m, \qquad A_2 \geq m. \tag{17}$$

In addition, suppose that in the coordinate plane q^1, q^2 the domain has a piecewise smooth boundary contour possessing the cone property. This means that there exists a finite triangle such that each point of the contour can be touched by the vertex of the triangle while the triangle lies wholly within the domain. The cone condition is necessary for the application of Sobolev's embedding theorem [1]. The existence–uniqueness theorems established below are valid for general coordinates q_1, q_2 that can degenerate at certain points of Σ, provided that these singular points can be removed via a local change of coordinates (as is possible for spherical coordinates on a sphere).

Let C^1 be the set of vector functions $\mathbf{U} = (\boldsymbol{u},\boldsymbol{\vartheta}) = (u_1, u_2, w, \vartheta_1, \vartheta_2, \vartheta_3)$ having continuously differentiable components on $\bar{\Sigma}$ that satisfy the boundary conditions

$$\boldsymbol{u}\big|_{\omega_1} = \mathbf{0}, \qquad \boldsymbol{\vartheta}\big|_{\omega_3} = \mathbf{0}. \tag{18}$$

On C^1 we introduce the energy inner product

$$(\mathbf{U}, \delta\mathbf{U})_e = \int_\Sigma \left(\mathbf{T}(\epsilon)\cdot\cdot\delta\boldsymbol{\epsilon}^T + \mathbf{M}(\kappa)\cdot\cdot\delta\kappa^T\right) d\Sigma, \tag{19}$$

where $\delta\mathbf{U} = (\delta\boldsymbol{u}, \delta\boldsymbol{\vartheta}) \in C^1$. It is clear that the form $(\cdot,\cdot)_e$ satisfies the inner product axioms on C^1.

Definition 2.1. The completion of the set C^1 in the norm induced by the scalar product (19),

$$\|\mathbf{U}\|_e = (\mathbf{U},\mathbf{U})_e^{1/2},$$

is called the energy space $\mathcal{E}$.

Lemma 2.2. *In the space $\mathcal{E}$ the energy norm is equivalent to the Sobolev norm*

$$\|\mathbf{U}\|^2_{(W^{1,2}(\Sigma))^6} \equiv \sum_{i=1}^{3} \left(\|u_i\|^2_{W^{1,2}(\Sigma)} + \|\vartheta_i\|^2_{W^{1,2}(\Sigma)} \right), \qquad \mathbf{U} = (u_1, u_2, u_3, \vartheta_1, \vartheta_2, \vartheta_3).$$

Proof. By the smoothness properties of Σ, there is a constant c_1 such that for any $\mathbf{U} \in \mathcal{E}$ we have

$$\|\mathbf{U}\|_e \leq c_1 \|\mathbf{U}\|_{(W^{1,2}(\Sigma))^6}.$$

The reverse inequality is established by applying Theorem 10.8 of [19]. The norm $\|\mathbf{U}\|_e^2$ has the structure required by that theorem:

$$\|\mathbf{U}\|_e^2 = \int_\Sigma \left[\mathcal{P}_2(\mathbf{U}) + \mathcal{P}_1(\mathbf{U})\right] A_1 A_2\, dq^1\, dq^2,$$

where

$$\mathcal{P}_2 = U(\epsilon^*, \boldsymbol{\kappa}^*)$$

and the components of $\boldsymbol{\epsilon}^*, \boldsymbol{\kappa}^*$ are the principal parts of $\boldsymbol{\epsilon}, \boldsymbol{\kappa}$:

$$\varepsilon_{11}^* = \frac{1}{A_1}\frac{\partial u_1}{\partial q^1}, \quad \varepsilon_{22}^* = \frac{1}{A_2}\frac{\partial u_2}{\partial q^2}, \qquad \varepsilon_{12}^* = \frac{1}{A_1}\frac{\partial u_2}{\partial q^1}, \quad \varepsilon_{21}^* = \frac{1}{A_2}\frac{\partial u_1}{\partial q^2},$$
$$\varepsilon_1^* = \frac{1}{A_1}\frac{\partial w}{\partial q_1}, \quad \varepsilon_2^* = \frac{1}{A_2}\frac{\partial w}{\partial q_2},$$
$$\kappa_{11}^* = \frac{1}{A_1}\frac{\partial \vartheta_1}{\partial q^1}, \quad \kappa_{22}^* = \frac{1}{A_2}\frac{\partial \vartheta_2}{\partial q^2}, \qquad \kappa_{12}^* = \frac{1}{A_1}\frac{\partial \vartheta_2}{\partial q^1}, \quad \kappa_{21}^* = \frac{1}{A_2}\frac{\partial \vartheta_1}{\partial q^2},$$
$$\kappa_1^* = \frac{1}{A_1}\frac{\partial \vartheta_3}{\partial q_1}, \quad \kappa_2^* = \frac{1}{A_2}\frac{\partial \vartheta_3}{\partial q_2}.$$

Each term of $\mathcal{P}_1$ contains no more than one derivative of u_i or ϑ_i as a factor. The idea of Theorem 10.8 of [19] is to split $\|\mathbf{U}\|_e^2$ into two parts. The first part $\int_\Sigma \mathcal{P}_2(\mathbf{U}) A_1 A_2\, dq^1\, dq^2$ contains the principal part of $\|\mathbf{U}\|_e^2$, which can be shown to be equivalent to the squared norm of some Sobolev space. In this case, it is seen that $\|\mathbf{U}\|_e^2$ is one of the equivalent norms of the subspace of $(W^{1,2}(\Sigma))^6$ consisting of elements satisfying (18). Each term of the second part $\int_\Sigma \mathcal{P}_1(\mathbf{U}) A_1 A_2\, dq^1\, dq^2$ conforms to the principal part; at least one of its factors is a component of $\mathbf{U}$ without the differentiation. To apply Theorem 10.8 we should check two conditions. In our nomenclature, the first is as follows:

$$\|\mathbf{U}\|_e^2 \ge 0 \quad \text{and} \quad \|\mathbf{U}\|_e^2 = 0 \ \text{implies}\ \mathbf{U} = \mathbf{0} \text{ in } \Sigma.$$

It is satisfied. Indeed, $\|\mathbf{U}\|_e^2 \ge 0$ and $\|\mathbf{U}\|_e^2 = 0$ implies $\mathbf{U} = \mathbf{0}$ in Σ. In our nomenclature, the second condition of the theorem reads as follows. For any sequence $\{\mathbf{U}_n\} \subset \mathcal{E}$ weakly convergent to zero in $(W^{1,2}(\Sigma))^6$ and such that $\|\mathbf{U}_n\|_e \to 0$ as $n \to \infty$, it follows that

$$\|\mathbf{U}_n\|_{(W^{1,2}(\Sigma))^6} \to 0.$$

We show this. As $\{\mathbf{U}_n\}$ converges weakly to zero in $(W^{1,2}(\Sigma))^6$, by Sobolev's embedding theorem [1] each sequence of the components of $\{\mathbf{U}_n\}$ converges strongly to zero in $L^2(\Sigma)$. It is seen that the terms of the form

$$\int_\Sigma k(q^1, q^2)\frac{\partial a_n}{\partial q^i}\, b_n A_1 A_2\, dq^1 dq^2 \quad \text{and} \quad \int_\Sigma k(q^1, q^2)\, a_n b_n A_1 A_2\, dq^1 dq^2$$

with continuous coefficients k, where a_n, b_n are the sequences of the components of $\{\mathbf{U}_n\}$, tend to zero as $n \to \infty$. Thus

$$\int_\Sigma \mathcal{P}_1(\mathbf{U}_n) A_1 A_2\, dq^1 dq^2 \to 0 \quad \text{as} \quad n \to \infty.$$

From $\|\mathbf{U}_n\|_e \to 0$ it follows that $\int_\Sigma \mathcal{P}_2(\mathbf{U}_n) A_1 A_2\, dq^1 dq^2 \to 0$ as $n \to \infty$. By the form of $\mathcal{P}_2$ and its positive definiteness as a quadratic form, all the sequences of the first derivatives of the components of $\{\mathbf{U}_n\}$ converge to zero in $L^2(\Sigma)$. Therefore $\|\mathbf{U}_n\|_{W^{1,2}(\Sigma)} \to 0$. This completes the proof. □

Let us assume there exists a vector function $\mathbf{U}^* = (\boldsymbol{u}^*, \boldsymbol{\vartheta}^*) \in (W^{1,2}(\Sigma))^6$ that takes the geometric boundary values of the problem:

$$\boldsymbol{u}^*\big|_{\omega_1} = \boldsymbol{u}^0(s), \quad \boldsymbol{\vartheta}^*\big|_{\omega_3} = \boldsymbol{\vartheta}^0(s).$$

If the geometric conditions are given on the whole boundary contour, which is smooth, then $\mathbf{U}^*$ exists if the components of $\boldsymbol{u}^0$ and $\boldsymbol{\vartheta}^0$ belong to $H^{-1/2}(\omega)$. For the mixed problem, we only can suppose the existence of $\mathbf{U}^*$.

We will seek a solution of the problem under consideration in the form

$$\tilde{\mathbf{U}} = \mathbf{U} + \mathbf{U}^*.$$

Substituting this into (16), we obtain

$$(\mathbf{U}, \delta\mathbf{U})_e = -(\mathbf{U}^*, \delta\mathbf{U})_e + \int_\Sigma (\boldsymbol{q}\cdot\delta\boldsymbol{u} + \boldsymbol{m}\cdot\delta\vartheta)\, d\Sigma + \int_{\omega_2} \boldsymbol{\varphi}\cdot\delta\boldsymbol{u}\, ds + \int_{\omega_4} \ell\cdot\delta\vartheta\, ds. \tag{20}$$

Definition 2.3. A weak (energy) solution of the mixed problem (8), (11), (12) is $\tilde{\mathbf{U}} = \mathbf{U} + \mathbf{U}^*$ such that $\mathbf{U} \in \mathcal{E}$ satisfies Eq. (20) for any $\delta\mathbf{U} \in \mathcal{E}$.

Definition 2.3 shows that we have reduced our problem to a problem with respect to $\mathbf{U}$ in the space $\mathcal{E}$. Clearly the right-hand side of (20) is a linear functional with respect to $\delta\mathbf{U} \in \mathcal{E}$. The Sobolev embedding theorem states that the embedding operators from $W^{1,2}(\Sigma)$ to $L^p(\Sigma)$ and $L^q(\omega)$ are continuous for any $p, q < \infty$. By Lemma 2.2, all terms on the right-hand side of (20) are continuous functions with respect to $\delta\mathbf{U} \in \mathcal{E}$. For example, consider one of the terms:

$$\begin{aligned}\left| \int_\Sigma (\boldsymbol{q} \cdot \delta\boldsymbol{u})\, d\Sigma \right| &\le \left(\int_\Sigma \|\boldsymbol{q}\|^{p/(p-1)}\, d\Sigma \right)^{(p-1)/p} \left(\int_\Sigma \|\delta\boldsymbol{u}\|^p\, d\Sigma \right)^{1/p} \\ &\le c_1 \|\delta\boldsymbol{u}\|_{(W^{1,2}(\Sigma))^3} \\ &\le c_2 \|\delta\boldsymbol{u}\|_e\end{aligned}$$

with constants c_k not dependent on $\delta\mathbf{U} \in \mathcal{E}$. At last $(\mathbf{U}^*, \delta\mathbf{U})_e$ is the result of formal substitution of $\mathbf{U}^*$ to the inner product $(\cdot,\cdot)_e$. As $\mathbf{U}^* = (\boldsymbol{u}^*, \boldsymbol{\vartheta}^*) \in (W^{1,2}(\Sigma))^6$ the components of $\mathbf{T}(\boldsymbol{\epsilon}(\mathbf{U}^*))$ and $\mathbf{M}(\boldsymbol{\kappa}(\mathbf{U}^*))$ belong to $L^2(\Sigma)$ and $(\mathbf{U}^*, \delta\mathbf{U})_e$ is a linear continuous functional in $\mathcal{E}$. So by the Riesz representation theorem for a linear continuous functional in a Hilbert space, there exists a uniquely defined element $\mathbf{U}^{**} \in \mathcal{E}$ such that

$$-(\mathbf{U}^*, \delta\mathbf{U})_e + \int_\Sigma (\boldsymbol{q} \cdot \delta\boldsymbol{u} + \boldsymbol{m} \cdot \delta\vartheta)\, d\Sigma + \int_{\omega_2} \boldsymbol{\varphi} \cdot \delta\boldsymbol{u}\, ds + \int_{\omega_4} \boldsymbol{\ell} \cdot \delta\vartheta\, ds = (\mathbf{U}^{**}, \delta\mathbf{U})_e.$$

Eq. (20) takes the form

$$(\mathbf{U}, \delta\mathbf{U})_e = -(\mathbf{U}^*, \delta\mathbf{U})_e + (\mathbf{U}^{**}, \delta\mathbf{U})_e. \tag{21}$$

It follows that

$$\mathbf{U} = -\mathbf{U}^* + \mathbf{U}^{**} \in \mathcal{E},$$

which is uniquely defined. Thus we have proved the existence of a weak (energy) solution of the problem under consideration. As $\mathbf{U}^*$ is not defined uniquely, we must also prove uniqueness of the weak solution. Suppose there exist two weak solutions of the problem, $\tilde{\mathbf{U}}_1$ and $\tilde{\mathbf{U}}_2$. Their difference $\tilde{\mathbf{U}}_2 - \tilde{\mathbf{U}}_1 \in \mathcal{E}$ satisfies

$$(\tilde{\mathbf{U}}_2 - \tilde{\mathbf{U}}_1, \delta\mathbf{U})_e = 0.$$

So $\tilde{\mathbf{U}}_2 = \tilde{\mathbf{U}}_1$. Thus we have proved

Theorem 2.4. *The mixed boundary value problem* (8), (11), *and* (12)*, describing shell equilibrium, has a unique weak (energy) solution* $\tilde{\mathbf{U}} \in (W^{1,2}(\Sigma))^6$.

The method of the proof of the theorem uses the ideas developed by I.I. Vorovich for nonlinear problems of shallow shells [19]. Some other methods of the proof of existence theorems in shell theory can be found in [8].

One reason we study existence of a weak solution is that the techniques used in the proof permit us to establish convergence of the Finite Element Methods (FEM) for these problems. As the components of the unknown variables that appear in the expression for J contain only first derivatives of $\boldsymbol{u}$ and $\boldsymbol{\vartheta}$, it makes sense to consider only "conforming" finite elements that belong to the energy space. So we introduce the set of finite elements $(\mathbf{u}_{hk}, \mathbf{v}_{hk})$ that belong to $\mathcal{E}$. The parameter h is the largest diameter of the support of the finite elements $(\mathbf{u}_{hk}, \mathbf{v}_{hk})$. We seek a finite element approximation to the solution $(\boldsymbol{u}_h, \boldsymbol{\vartheta}_h)$ in the form

$$\boldsymbol{u}_h = \boldsymbol{u}^* + \sum_k c_k \mathbf{u}_{hk}, \qquad \boldsymbol{\vartheta}_h = \boldsymbol{\vartheta}^* + \sum_k c_k \mathbf{v}_{hk}. \tag{22}$$

Let us write $\mathbf{U}_h = (\boldsymbol{u}_h, \boldsymbol{\vartheta}_h)$. Substituting this into (20), and taking $\delta\mathbf{U}$ first as $(\mathbf{u}_{h1}, \mathbf{v}_{h1})$, then as $(\mathbf{u}_{h2}, \mathbf{v}_{h2})$, and so on, we obtain the system of linear algebraic equations in the constants c_k which are known as the FEM equations. It is clear that we can repeat the considerations of the last theorem, but in the finite dimensional space having basis $(\mathbf{u}_{hk}, \mathbf{v}_{hk})$; hence we immediately find that the FEM algebraic system has an unique solution.

Now suppose the set of all finite elements $(\mathbf{u}_{hk}, \mathbf{v}_{hk})$ as $h \to 0$ is a complete set in $\mathcal{E}$. A standard procedure [9] permits us to assert that the sequence $\mathbf{U}_h$ of FEM approximations for the problem under consideration converges strongly to the weak solution of the problem in the norm of $(W^{1,2}(\Sigma))^6$.

Now we wish to touch on the equilibrium problem for a free shell whose boundary has no fixed point. Eq. (16) continues to hold in this situation. The set of admissible $\delta\mathbf{U}$ is $(W^{1,2}(\Sigma))^6$. The zero of the norm $\|\mathbf{U}\|_e = 0$ is the set of vector functions known as rigid displacements of Σ; these have the form

$$\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{\vartheta}_0 \times \boldsymbol{r}, \qquad \boldsymbol{\vartheta} = \boldsymbol{\vartheta}_0$$

with arbitrary constant vectors $\boldsymbol{u}_0, \boldsymbol{\vartheta}_0$ and the position vector $\boldsymbol{r}$. Taking $\delta\mathbf{U}$ from the class of rigid displacements in (16), we get two vector equations

$$\int_\Sigma \boldsymbol{q}\, d\Sigma + \int_\omega \varphi\, ds = \mathbf{0},$$

$$\int_\Sigma (\boldsymbol{q} \times \boldsymbol{r} + \boldsymbol{m})\, d\Sigma + \int_\omega \varphi \times \boldsymbol{r}\, ds + \int_\omega \boldsymbol{\ell}\, ds = \mathbf{0}.$$

These are the classical mechanical equations of the self-equilibrium of the shell as a rigid body. They are necessary for existence of a solution of the equilibrium problem for a free shell. It can be shown that under this additional condition on the external forces, the equilibrium problem for the free shell has a weak solution $\mathbf{U}$ in $(W^{1,2}(\Sigma))^6$. This solution is unique up to rigid displacements, so it takes the form

$$\mathbf{U} = (\boldsymbol{u} + \boldsymbol{u}_0 + \boldsymbol{\vartheta}_0 \times \boldsymbol{r}, \boldsymbol{\vartheta} + \boldsymbol{\vartheta}_0)$$

with arbitrary vector constants $\boldsymbol{u}_0$ and $\boldsymbol{\vartheta}_0$.

3 Eigenvalue problems

Consider the problem of finding the eigenfrequencies of a micropolar shell. *Rayleigh's variational principle* takes the following form.

On the set of functions $\boldsymbol{u}, \boldsymbol{\vartheta}$ *with boundary conditions* $\boldsymbol{u}\big|_{\omega_1} = \mathbf{0}$, $\vartheta\big|_{\omega_3} = \mathbf{0}$ *that obey the constraint*

$$K(\boldsymbol{u}^\circ, \boldsymbol{\vartheta}^\circ) \equiv \int_\Sigma \rho \left(\frac{1}{2} \boldsymbol{u}^\circ \cdot \boldsymbol{u}^\circ + \boldsymbol{u}^\circ \cdot \Theta_1 \cdot \vartheta^\circ + \frac{1}{2} \vartheta^\circ \cdot \Theta_2 \cdot \vartheta^\circ \right) d\Sigma = 1,$$

the eigenmodes of the shell are stationary points of the strain energy functional

$$E(\boldsymbol{u}^\circ, \boldsymbol{\vartheta}^\circ) = \int_\Sigma U(\epsilon^\circ, \kappa^\circ)\, d\Sigma, \tag{23}$$

where $\epsilon^\circ = \nabla \boldsymbol{u}^\circ + \mathbf{A} \times \boldsymbol{\vartheta}^\circ$ *and* $\kappa^\circ = \nabla \vartheta^\circ$.

Rayleigh's principle also includes the converse statement: on the set of functions $\boldsymbol{u}, \boldsymbol{\vartheta}$ satisfying the restrictions $\boldsymbol{u}\big|_{\omega_1} = \mathbf{0}$ and $\vartheta\big|_{\omega_3} = \mathbf{0}$, the stationary points of E are the eigenmodes. The solutions of the eigenoscillation problem arise in the dynamic problem when one seeks a solution in the form $\boldsymbol{u} = \boldsymbol{u}^\circ \mathrm{e}^{i\omega t}$, $\boldsymbol{\vartheta} = \boldsymbol{\vartheta}^\circ \mathrm{e}^{i\omega t}$. Hence $\boldsymbol{u}^\circ$ and ϑ° are the amplitudes of the oscillations of the displacements and microrotations.

Rayleigh's quotient is

$$R(\boldsymbol{u}^\circ, \vartheta^\circ) = \frac{E(\boldsymbol{u}^\circ, \boldsymbol{\vartheta}^\circ)}{K(\boldsymbol{u}^\circ, \vartheta^\circ)}.$$

The smallest eigenfrequency of the shell is equal to the minimum value of the functional R.

Let us formalize these considerations in the space $\mathcal{E}$. We will omit the superscript $^\circ$ in what follows. The equation for the eigenproblem in $\mathcal{E}$ takes the form

$$(\mathbf{U}, \delta\mathbf{U})_e = \lambda \langle \mathbf{U}, \delta\mathbf{U} \rangle, \tag{24}$$

where λ is the squared eigenfrequency of the problem, $\mathbf{U} \neq \mathbf{0}$, and

$$\langle \mathbf{U}, \delta\mathbf{U} \rangle = \int_\Sigma \rho \big(\boldsymbol{u} \cdot \delta\boldsymbol{u} + \boldsymbol{u} \cdot \Theta_1 \cdot \delta\boldsymbol{\vartheta} + \delta\boldsymbol{u} \cdot \Theta_1 \cdot \boldsymbol{\vartheta} + \boldsymbol{\vartheta} \cdot \Theta_2 \cdot \delta\boldsymbol{\vartheta} \big)\, d\Sigma.$$

Applying the Riesz representation theorem for a linear continuous functional in a Hilbert space (cf., [15] (Sect. 2.15)), we can write

$$\langle \mathbf{U}, \delta\mathbf{U} \rangle = (A\mathbf{U}, \delta\mathbf{U})_e$$

for some linear operator A in $\mathcal{E}$. So we have obtained an operator eigenvalue problem:

$$\mathbf{U} = \lambda A\mathbf{U}.$$

The operator A is continuous. By Sobolev's embedding theorem [1], the embedding operator from $W^{1,2}(\Sigma)$ to $L^2(\Sigma)$ is compact and hence A is also compact. From the symmetry of $\langle \mathbf{U}, \delta\mathbf{U} \rangle$ with respect to the arguments, it follows that A is selfadjoint. Finally, $\langle \mathbf{U}, \mathbf{U} \rangle \geq 0$ and $\langle \mathbf{U}, \mathbf{U} \rangle = 0$ implies $\mathbf{U} = \mathbf{0}$ almost everywhere in Σ, which means that A is positive definite. Now applying Theorem 2.14.2 of [15], which states certain spectral properties of the equation $\mathbf{U} = \lambda A\mathbf{U}$ when A is linear, compact, self-adjoint, and positive definite, we get the following.

1. The spectrum λ_k $(k = 0, 1, 2, \ldots)$ of the mixed problem is an infinite set of positive numbers. It is discrete and has no finite accumulation points. The smallest eigenvalue λ_0 is nonzero.
2. To each λ_k there corresponds no more than a finite set of linearly independent eigenvectors $\mathbf{U}_{kr}$ $(r = 1, \ldots m_k)$.
3. It is possible to select a set of eigenvectors $\mathbf{U}_{kr}$ that is orthonormal and complete in $\mathcal{E}$. This set is also orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle$.

4 Weak solutions for dynamical problems

Using the results for the equilibrium problems, we can introduce weak solutions for dynamics problem. For simplicity we take homogeneous boundary conditions

$$\boldsymbol{u}\Big|_{\omega_1} = \mathbf{0}, \quad \boldsymbol{\vartheta}\Big|_{\omega_3} = \mathbf{0}, \quad \nu \cdot \mathbf{T}\Big|_{\omega_2} = \mathbf{0}, \quad \boldsymbol{\nu} \cdot \mathrm{M}\Big|_{\omega_4} = \mathbf{0}$$

and supply the dynamical equations (2) with the initial conditions

$$\mathbf{U}(\boldsymbol{r}, t)\Big|_{t=0} = \mathbf{U}_0, \qquad \frac{\partial \mathbf{U}}{\partial t}\Big|_{t=0} = \mathbf{V}_0. \tag{25}$$

To obtain the equation needed for a weak formulation of the problem, we dot-multiply the first equation in (2) by $\boldsymbol{u}$ and the second equation by $\boldsymbol{\vartheta}$. After adding the results, we integrate over Σ and then over the time interval $[0, T]$. Taking $\delta\mathbf{U} = \mathbf{0}$ at time T and integrating by parts, we get

$$\int_0^T (\mathbf{U}, \delta\mathbf{U})_e \, dt = \int_0^T \left\langle \frac{\partial \mathbf{U}}{\partial t}, \frac{\partial \delta\mathbf{U}}{\partial t} \right\rangle dt + \langle \mathbf{V}_0, \delta\mathbf{U} \rangle\Big|_{t=0} + \int_0^T \int_\Sigma (\boldsymbol{q} \cdot \delta\boldsymbol{u} + \boldsymbol{m} \cdot \delta\boldsymbol{\vartheta}) \, d\Sigma \, dt. \tag{26}$$

We seek a weak solution in the space defined by the inner product

$$(\mathbf{U}, \mathbf{V})_{H[0,T]} = \int_0^T (\mathbf{U}, \mathbf{V})_e \, dt + \int_0^T \left\langle \frac{\partial \mathbf{U}}{\partial t}, \frac{\partial \mathbf{V}}{\partial t} \right\rangle dt.$$

The energy space $H(0, T)$ is defined as the completion of the set of vector functions $\mathbf{U}(\boldsymbol{r}, t)$ that are smooth on $\Sigma \times [0, T]$ and that satisfy the boundary conditions

$$\boldsymbol{u}\Big|_{\omega_1} = \mathbf{0}, \quad \boldsymbol{\vartheta}\Big|_{\omega_3} = \mathbf{0}.$$

Its subspace $H^0(0, T)$ is the completion of the subset of vector functions that vanish at $t = T$.

Now we can define the weak solution to the dynamical problem. We say that $\mathbf{U} \in H(0, T)$ is a weak solution to the dynamical problem if it satisfies (26) for any $\delta\mathbf{U} \in H^0(0, T)$ along with the initial condition

$$\mathbf{U}\Big|_{t \to 0} = \mathbf{V}_0 \quad \text{in} \quad (L^2(\Sigma))^6.$$

The proof of existence–uniqueness theorems for weak solutions to hyperbolic problems is quite traditional (cf. [17] or Sect. 4.6 of [13]) so we present only the conditions under which a weak solution to the dynamical problem exists and is unique:

$$\mathbf{U}_0 \in \mathcal{E}, \quad \mathbf{V}_0 \in (L^2(\Sigma))^6, \quad \boldsymbol{q}, \boldsymbol{m} \in L^2(\Sigma \times [0, T]).$$

This finalizes our study of the weak setup for typical boundary value problems of linear micropolar shell theory.

Conclusion

Within the framework of micropolar or 6-parametric linear shell theory, we prove the existence and uniqueness of weak solutions to boundary value equilibrium problems. The key point in the proof is the introduction of the energy functional space $\mathcal{E}$ and the proof that the norm of $\mathcal{E}$ is equivalent to the norm of the Sobolev space $(W^{1,2}(\Sigma))^6$. This result allows us to show existence and uniqueness of solutions to dynamical problems and to establish spectral properties similar to those for bounded bodies in linear elasticity.

Is is well known that the weak formulation of boundary value problems permits us to introduce various versions of the finite element method that is used in engineering calculations. Based on the proofs of existence theorems, the traditional considerations for various versions of the FEM demonstrate strong convergence of the finite element approximations to the weak solution in $\mathcal{E}$.

The micropolar version of shell theory, being in precision equivalent to the classical Kirchhoff–Love theory, is preferable for numerical calculations. That is, unlike the Kirchhoff–Love model which contains a fourth-order differential equation for the transversal displacement w, all equations of micropolar theory are of second order. In FEM practice using the Kirchhoff–Love theory, various finite element schemes for w are proposed; these schemes generally converge more slowly than those for FEM approximations in the micropolar theory.

The Mindlin–Reissner theory possesses a similar advantage regarding the order of the equations as compared with the Kirchhoff–Love theory. However, the linear version of the Mindlin–Reissner's theory contains five independent scalar variables of the components of $\boldsymbol{u}$ and $\boldsymbol{\vartheta}$ as $\boldsymbol{\vartheta} \cdot \boldsymbol{n} = 0$. A relative advantage of the micropolar shell theory is that it uses the complete 3D kinematics. This allows us to describe multi-folded shells, shells with junctions, etc., which is impossible with Mindlin–Reissner's theory.

Hence the investigation of boundary value problems in micropolar theory deserves special attention from engineers. Numerous examples of FEM calculations with the micropolar theory of shells can be found [3–6].

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