Existence Theory for a Class of Nonlinear Random Functional Integral Equations (*).

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Summary. – *The aim of the present paper is to study a random equation of the general form* $x(t, \omega) = (Ux)(t, \omega)$, $t \in R_+$ *and its special case a nonlinear random functional integral equation given by*

$$x(t, \omega) = F\left(t, \int_0^{g_1(t)} f_1(t, s, x(s, \omega), \omega)\, ds, \ldots, \int_0^{g_m(t)} f_m(t, s, x(s, \omega), \omega)\, ds, x(h_1(t), \omega), \ldots, x(h_p(t), \omega), \omega\right).$$

The existence and uniqueness of a random solution, a second-order stochastic process, of the equations is considered.

1. – Introduction.

Random equations of various types have been considered recently by many scientists, for example [1], [10] and [11]. One of the most common types of random equations arising naturally in the study of physical, biological and chemical phenomena are the random or stochastic integral equations, [1-2], [5-9], [11-12].

In this paper, we shall first study a random equation of the general form

$$x(t, \omega) = (Ux)(t, \omega) . \tag{1.1}$$

The particular cases of equation (1.1) are the Volterra and Fredholm random integral equations, the random functional integral equations and others. Also the random differential equations the random functional differential equations and in particular the random differential equations with a deviated argument of the neutral type can be reduced to the random equation of the form (1.1).

The general method of proof of the existence theorems of the solution of equation (1.1) will be to appeal to the comparison method. This method based on the convergence of successive approximations produced by a comparison operator Λ associated with the operator U. The abstract form of comparison method was introduced by WAŻEWSKI [13] in the case of deterministic equation.

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The purpose of this paper is to analyse the class of linear comparison operators A associated with the operator U and having the sufficient properties to ensure the existence, uniqueness and convergence of successive approximations for the special case of equation (1.1). Specifically, we shall study a nonlinear random functional integral equation given by

$$x(t,\omega)=F\Big(t,\int_0^{g_1(t)} f_1\big(t,s,x(t,\omega),\omega\big)\,ds,\ldots \\ \ldots,\int_0^{g_m(t)} f_m\big(t,s,x(t,\omega),\omega\big)\,ds,\, x\big(h_1(t),\omega\big),\ldots,x\big(h_p(t),\omega\big),\omega\Big), \tag{1.2}$$

where

(i) $t\in R_+\overset{\mathrm{df}}{=}[0,+\infty)$, and $\omega\in\Omega$, the supporting set of a complete probability measure space $(\Omega,\mathcal{F},P)$;

(ii) $x(t,\omega)$ is the unknown random function defined on R_+ with values in R;

(iii) $F(t,u_1,\ldots,u_m,x_1,\ldots,x_p,\omega)$ is map from $R_+\times R^{m+p}\times\Omega$ into R;

(iv) $f_j(t,s,x,\omega)$, $j=1,\ldots,m$, the stochastic kernels, are a map from $R_+\times R_+\times\times R\times\Omega$ into R;

(v) $g_j(t)$, $h_i(t)$, $j=1,\ldots,m$, $i=1,\ldots,p$, are non-negative scalar functions defined on R_+.

Further assumptions concerning the functions in equation (1.2) will be stated in section 2.

The equation (1.2) is generalization of equations studied by TURO [11] (if $m=p=1$ and the interval of integration is compact) and MILTON and TSOKOS [8] $\big($if $F(t,u_1,\ldots,u_m,x_1,\ldots,x_p,\omega)=h(t,\omega)+u_1+\ldots+u_m$, $f_j(t,s,x,\omega)=k_j(t-s,\omega)\varphi_j(x)$ and $g_j(t)=t\big)$.

The particular case of equation (1.2) $\big($if $F(t,u_1,\ldots,u_m,x_1,\ldots,x_p,\omega)=h(t,x_1)+u_1$ and $g_1(t)=t$, $h_1(t)=t\big)$ is equation considered by TSOKOS and PADGETT [11], MILTON and TSOKOS [6-7], HARDIMAN and TSOKOS [2], and also $\big($if moreover $F=h(t,\omega)+u_1\big)$ LEE and PADGETT [5].

Equation (1.2) is the stochastic analog of the deterministic equation studied recently by KWAPISZ and TURO [3-4].

The random neutral-differential equation

$$y'(t,\omega)=F\Big(t,y\big(g_1(t),\omega\big),\ldots,y\big(g_m(t),\omega\big),\, y'\big(h_1(t),\omega\big),\ldots \\ \ldots,y'\big(h_p(t),\omega\big),\omega\Big)$$

can be reduced to the particular case $\big($if $f_j(t,s,x,\omega)=x\big)$ of equation (1.2).

2. – Preliminaries.

Let $L_2(\Omega) = L_2(\Omega, \mathcal{F}, P)$ denote the space of all functions $z(\omega)$ from Ω into R such that

$$\int_\Omega |z(\omega)|^2 P(d\omega) < \infty .$$

That is, $L_2(\Omega)$ is the space all second-order real valued variables. For convenience, we write

$$\|z(\omega)\|_{L_2} = \left\{\int_\Omega |z(\omega)|^2 P(d\omega)\right\}^{\frac{1}{2}}, \qquad z(\omega) \in L_2(\Omega) .$$

We require for the formulation of the random equation (1.2) the following assumptions:

(i) $F\big(t, u_1(t, \omega), \ldots, u_m(t, \omega), x_1(t, \omega), \ldots, x_p(t, \omega), \omega\big)$ must for each $t \in R_+$ belong to $L_2(\Omega)$ and $F(t, u_1, \ldots, u_m, x_1, \ldots, x_p, \omega)$ is an $L_2(\Omega)$-continuous in $t \in R_+$ for each $u_j, x_i \in R$, $j = 1, \ldots, m$, $i = 1, \ldots, p$;

(ii) $f_j\big(t, s, x(s, \omega), \omega\big)$, $j = 1, \ldots, m$, are the continuous maps from $\Delta = \{(t, s)\colon 0 \leqslant s \leqslant t < \infty\}$ into $L_2(\Omega)$;

(iii) the non-negative scalar functions $g_j(t)$ and $h_i(t)$ are continuous on R_+ and $g_j(t) \leqslant t$, $h_i(t) \leqslant t$, $t \in R_+$, $j = 1, \ldots, m$, $i = 1, \ldots, p$.

DEFINITION 2.1. – We call $x(t, \omega)$ a random solution of the random equations (1.1) or (1.2) if for each $t \in R_+$, $x(t, \omega)$ is an element of $L_2(\Omega)$ and satisfies the equations (1.1) or (1.2) P-a.e.

DEFINITION 2.2. – We shall denote by $C\big(R_+, L_2(\Omega)\big)$ space of all continuous maps $x(t, \omega)$ from R_+ into $L_2(\Omega)$ with the topology of uniform convergence on compacta.

Note that it can be shown [14] that the space $C\big(R_+, L_2(\Omega)\big)$ is a locally convex space whose topology is defined by countable family of semi-norms given by

$$\|x(t, \omega)\|_n = \sup_{0 \leqslant t \leqslant n} \|x(t, \omega)\|_{L_2}, \qquad n = 1, 2, \ldots$$

DEFINITION 2.3. – A sequence $\{x_k(t, \omega)\}$ of elements of space $C\big(R_+, L_2(\Omega)\big)$ will be called a Cauchy sequence if for every $\varepsilon > 0$ and n there exists an N such that for $k > N$ and $l > N$ we have

$$\|x_k(t, \omega) - x_l(t, \omega)\|_n < \varepsilon .$$

It is clear that the space $C(R_+, L_2(\Omega))$ is complete, that is, every Cauchy sequence of its elements has a limit in $C(R_+, L_2(\Omega))$..

By $C_0(R_+, R_+)$ we denote the class of all R^+-valued functions upper semicontinuous on R_+.

3. – Existence theorem for random equation (1.1).

We introduce

ASSUMPTION A. – Suppose that

1) there exists an operator $\Lambda\colon C_0(R_+, R_+) \to C_0(R_+, R_+)$, which has the following properties:

a) if $u \in C(R_+, R_+)$ and $v = \Lambda u$, then $v \in C(R_+, R_+)$, ($C(R_+, R_+)$ denote the class of all non-negative continuous functions on R_+),

b) if $u, v \in C(R_+, R_+)$ and $u \leqslant v$, then $\Lambda u \leqslant \Lambda v$,

c) if $u_n \in C(R_+, R_+)$, $u_{n+1} \leqslant u_n$, $n = 0, 1, \ldots$, $u_n \to u$, then $\Lambda u_n \to \Lambda u$;

2) the operator $U\colon C(R_+, L_2(\Omega)) \to C(R_+ L_2(\Omega))$ fulfils the condition

$$\|(Ux)(t, \omega) - (U\bar{x})(t, \omega)\|_{L_2} \leqslant \Lambda(\|x(t, \omega) - \bar{x}(t, \omega)\|_{L_2}), \tag{3.1}$$

for $x(t, \omega), \bar{x}(t, \omega) \in C(R_+, L_2(\Omega))$.

ASSUMPTION B(r). – Suppose that

1) for a given function $r \in C(R_+, R_+)$ there exists a solution $u_0 \in C(R_+, R_+)$ of the inequality

$$\Lambda u + r \leqslant u ;$$

2) the function $u = 0$ is the unique solution of the inequality

$$u \leqslant \Lambda u$$

in the class $C_0(R_+, R_+, u_0) \overset{\text{df}}{=} \{u\colon u \in C_0(R_+, R_+),\ \|u\|^* < \infty\}$, where

$$\|u\|^* \overset{\text{df}}{=} \inf\{c\colon u \leqslant cu_0,\ c \in R^+\} .$$

We construct a sequence as follows:

$$u_{n+1} = \Lambda u_n , \qquad n = 0, 1, \ldots , \tag{3.2}$$

where u_0 is as introduced in Assumption B(r).

Similarly as in [3], by induction and Dini's theorem we can prove the following

LEMMA 3.1. – If Assumption B(r) and the condition 1) of Assumption A are satisfied, then

$$0 \leqslant u_{n+1} \leqslant u_n\,, \qquad n = 0, 1, \ldots, \text{ and } u_n \rightrightarrows 0\,,$$

where the sign $\rightrightarrows$ denotes uniform convergence in any compact subset of R_+.

Now, we define the sequence of successive approximations $\{x_n(t, \omega)\}$ by

$$x_{n+1}(t, \omega) = (Ux_n)(t, \omega)\,, \qquad n = 0, 1, \ldots, \; t \in R_+\,, \tag{3.3}$$

where $x_0(t, \omega)$ is an arbitrarily fixed element of $C(R_+, L_2(\Omega))$.

THEOREM 3.1. – If Assumptions A and B(r) are satisfied for

$$r(t) \overset{\text{df}}{=} \|(Ux_0)(t, \omega) - x_0(t, \omega)\|_{L_2}\,,$$

then there exists a random solution $\bar{x}(t, \omega) \in C(R_+, L_2(\Omega))$ of equation (1.1), and the following estimations

$$\|\bar{x}(t, \omega) - x_n(t, \omega)\|_{L_2} \leqslant u_n(t)\,, \qquad n = 0, 1, \ldots, \; t \in R_+\,, \tag{3.4}$$

hold true.

The solution $\bar{x}(t, \omega)$ of (1.1) is unique in the class

$$C(R_+, L_2(\Omega), u_0) \overset{\text{df}}{=} \{x(t, \omega): x(t, \omega) \in C(R_+, L_2(\Omega)),\; \|x(t, \omega) - x_0(t, \omega)\|_{L_2} \in C_0(R_+, R_+, u_0)\}$$

where $C_0(R_+, R_+, u_0)$ is defined in Assumption $B(r)$.

PROOF. – The following estimation

$$\|x_{n+k}(t, \omega) - x_n(t, \omega)\|_{L_2} \leqslant u_n(t)\,, \qquad n, k = 0, 1, \ldots, \; t \in R_+\,, \tag{3.5}$$

is easily obtained by induction. Hence and from $u_n \rightrightarrows 0$, $n \to \infty$ (see Lemma 3.1) it follows that $\{x_n(t, \omega)\}$ is the Cauchy sequence (see Definition 2.3) in $C(R_+, L_2(\Omega))$. Since $C(R_+, L_2(\Omega))$ is complete space, there exists an $\bar{x}(t, \omega) \in C(R_+, L_2(\Omega))$ such that $x_n(t, \omega) \to \bar{x}(t, \omega)$. If $k \to \infty$, then (3.5) yields estimation (3.4). By the estimation

$$\begin{aligned}\|\bar{x}(t, \omega) - (U\bar{x})(t, \omega)\|_{L_2} &\leqslant \|\bar{x}(t, \omega) - x_n(t, \omega)\|_{L_2} + \|(Ux_{n-1})(t, \omega) - (U\bar{x})(t, \omega)\|_{L_2} \\ &\leqslant u_n(t) + Au_{n-1} = 2u_n(t)\,, \qquad n = 0, 1, \ldots, \; t \in R_+\,,\end{aligned}$$

it follows that the random function $\bar{x}(t, \omega)$ satisfies equation (1.1).

To prove uniqueness, suppose $\bar{x}(t,\omega)$ and $\tilde{x}(t,\omega)$ are two solutions belonging to $C(R_+, L_2(\Omega), u_0)$. It is easy to prove that $\|\bar{x}(t,\omega)-\tilde{x}(t,\omega)\|_{L_2}\in C_0(R_+, R_+\, u_0)$ and

$$\|\bar{x}(t,\omega)-\tilde{x}(t,\omega)\|_{L_2}\leqslant \Lambda\big(\|\bar{x}(t,\omega)-\tilde{x}(t,\omega)\|_{L_2}\big)\,.$$

Hence and from Assumption B(r) it follows that $\|\bar{x}(t,\omega)-\tilde{x}(t,\omega)\|_{L_2}=0$. Thus the proof of theorem is complete.

4. – Lemma and some remarks.

It follows from the above general considerations that the fundamental idea in proving of the existence and uniqueness of a solution of random equation (1.1) or its special cases is associate the operator U to an operator Λ satisfying the inequality (3.1) and such that the Assumption B(r) is fulfilled.

Now we consider the comparison operator Λ defined by

$$\Lambda u = Ku + Lu\,, \tag{4.1}$$

where

$$(Ku)(t) \overset{\text{df}}{=} \sum_{j=1}^{m} k_j(t)\int_0^{g_j(t)} u(s)\,ds\,,$$

$$(Lu)(t) \overset{\text{df}}{=} \sum_{i=1}^{p} l_i(t)\,u\big(h_i(t)\big)\,,$$

and $k_j, l_i, g_j, h_i\in C(R_+, R_+)$, $g_j(t)\leqslant t$, $h_i(t)\leqslant t$, $t\in R_+$, $j=1,\dots,m$, $i=1,\dots,p$.

REMARK 4.1. – By using Banach fixed point theorem it is to prove that Assumption B(r) for any $r\in C(R_+, R_+)$ is fulfilled for Λ defined by (4.1) provided

$$\sum_{j=1}^{m} k_j(t)\,g_j(t) + \sum_{i=1}^{p} l_i(t) < 1\,,\qquad t\in R^+\,. \tag{4.2}$$

It is the aim to give conditions weaker than this one.

Define $L^n \overset{\text{df}}{=} LL^{n-1}$, $n=1,2,\dots$, $L^0\overset{\text{df}}{=} I$, where I denotes the identity operator in $C(R_+, R_+)$.

From the definition of the operator L it follows that

$$(L^n u)(t) = \sum_{i_1,\dots,i_n=1}^{p} l_n^{i_1,\dots,i_n}(t)\,u\big(h_n^{i_1,\dots,i_n}(t)\big)\,,$$

where

$$h_1^i(t) \stackrel{\mathrm{df}}{=} h_i(t)\,, \qquad h_{n+1}^{i_1,\dots,i_{n+1}}(t) \stackrel{\mathrm{df}}{=} h_n^{i_1,\dots,i_n}\big(h_{i_{n+1}}(t)\big)\,,$$

$$l_1^i(t) \stackrel{\mathrm{df}}{=} l_i(t)\,, \qquad l_{n+1}^{i_1,\dots,i_{n+1}}(t) \stackrel{\mathrm{df}}{=} l_{i_{n+1}}(t)\, l_n^{i_1,\dots,i_n}\big(h_{i_{n+1}}(t)\big)\,, \qquad i, i_n = 1, \dots, p,\ n = 0, 1, \dots .$$

Put

$$Su \stackrel{\mathrm{df}}{=} \sum_{n=0}^{\infty} L^n u$$

with the point weise convergence of the series in R_+.

LEMMA 4.1. – Assume that

(i) $k_j, l_i, g_j, h_i, r \in C(R_+, R_+)$ and $g_j(t), h_i(t) \in [0, t]$,

$t \in R_+$, $j = 1, \dots, m$, $i = 1, \dots, p$ (the case $m, p = +\infty$ is possible);

(ii) $s = Sr < \infty$, $\bar{s} = Sk < \infty$,

where $k(t) \stackrel{\mathrm{df}}{=} \sum_{j=1}^{m} k_j(t) g_j(t)$ (if $m = +\infty$ we assume that this sum is finite);

(iii) $s, \bar{s} \in C(R_+, R_+)$ and $\sup \bar{s}(t)/t < \infty$.

Then

(*a*) there exists $u_0 \in C(R_+, R_+)$ which is a unique solution of equation

$$u = SKu + Sr \tag{4.3}$$

in the class $L_{\mathrm{loc}}(R_+, R_+)$ locally integrable functions on R_+;

(*b*) the function u_0 is the unique solution of the equation

$$u = Ku + Lu + r$$

in the class $L_{\mathrm{loc}}(R_+, R_+, u_0) \stackrel{\mathrm{df}}{=} \{u\colon u \in L_{\mathrm{loc}}(R_+, R_+),\ \|u\|^* < \infty\}$, where the norm $\|\cdot\|^*$ is defined in Assumption B(r);

(*c*) the function $u = 0$ is the unique solution of the inequality

$$u \leqslant Ku + Lu$$

in the class $L_{\mathrm{loc}}(R_+, R_+, u_0)$.

PROOF. – We prove (*a*). We note that if $u \in L_{\mathrm{loc}}(R_+, R_+)$ and is the solution of equation (4.3) then $u \in (R_+, R_+)$. Thus we shall prove that equation (4.3) has a unique solution in $C(R_+, R_+)$. We shall obtain a solution first on an arbitrary closed, bounded interval $[0, n]$. Let $C([0, n], R)$ be the space of all continuous functions on $[0, n]$, where we introduce a norm $\|\cdot\|_0$ in the following way:

$$\|u\|_0 \stackrel{\mathrm{df}}{=} \sup_{t\in[0,n]} e^{-\lambda t}|u(t)|,$$

where $\lambda > \bar{\lambda} \stackrel{\mathrm{df}}{=} \sup \bar{s}(t)/t$.

Now we prove that operator SK is a contraction in $C([0, n], R)$ i.e. $\|SK\| < 1$. Indeed, from the inequality $e^{\alpha t} - 1 \leqslant \alpha e^t$ for $\alpha \in [0, 1]$, $t \in R_+$, we have

$$\|SKu\|_0 \leqslant \sup_{t\in[0,n]} e^{-\lambda t} \sum_{j=1}^{m} \sum_{n=0}^{\infty} \sum_{i_1,\ldots,i_n=1}^{p} l_n^{i_1,\ldots,i_n}(t)\, k_j\big(h_n^{i_1,\ldots,i_n}(t)\big)\cdot$$

$$\cdot \int_0^{g_j(h_n^{i_1\ldots,i_n}(t))} e^{\lambda s} \sup_{s\in[0,n]} e^{-\lambda s}|u(s)|\,ds \leqslant \frac{1}{\lambda}\frac{\bar{s}(t)}{t}\|u\|_0 \leqslant \frac{\bar{\lambda}}{\lambda}\|u\|_0 .$$

Hence it follows that $\|SK\| < 1$. Now from Banach fixed point theorem it follows that equation (4.3) has a unique solution $u_0 \in C([0, n], R_+)$. Since n is arbitrary, u_0 is a unique solution of equation (4.3) on R_+.

The remainder of the proof is similar to that of Lemma 6 [3] and is omitted.

REMARK 4.2. – If $m = 1$, $p = 1$, $k(t) \stackrel{\mathrm{df}}{=} k_1(t)$, $l(t) \stackrel{\mathrm{df}}{=} l_1(t)$, $g(t) \stackrel{\mathrm{df}}{=} g_1(t)$, and $h(t) \stackrel{\mathrm{df}}{=} h_1(t)$, $t \in R_+$, then assumption (ii) of Lemma 4.1 is of the form [12]

$$s(t) = \sum_{n=0}^{\infty} l_n(t)\, r\big(h_n(t)\big) < \infty ,$$

$$\bar{s}(t) = \sum_{n=0}^{\infty} l_n(t)\, k\big(h_n(t)\big)\, g\big(h_n(t)\big) < \infty , \qquad t \in R_+ ,$$

where

$$h_0(t) \stackrel{\mathrm{df}}{=} t , \qquad h_{n+1}(t) \stackrel{\mathrm{df}}{=} h\big(h_n(t)\big) , \qquad n = 0, 1, \ldots, \ t \in R_+ ,$$

$$l_0(t) \stackrel{\mathrm{df}}{=} 1 , \qquad l_{n+1}(t) \stackrel{\mathrm{df}}{=} \prod_{k=0}^{n} l\big(h_k(t)\big) , \qquad n = 0, 1, \ldots, \ t \in R_+ .$$

REMARK 4.3. – Now we give some effective conditions under which assumption (ii) of Lemma 4.1 is fulfilled.

a) If we assume that

$$(4.4) \qquad \begin{cases} k_j(t) \leqslant \bar{k}_j = \text{const} , & l_i(t) \leqslant \bar{l}_i = \text{const} , \quad g_j(t) \leqslant \bar{g}_j t , \\ h_i(t) \leqslant \bar{h}_i t , & \bar{g}_j,\ \bar{h}_i \in [0, 1] , \quad j = 1, \ldots, m, \ i = 1, \ldots, p, \ t \in R_+ , \end{cases}$$

and $r(t) \leq \bar{r}t$, $t \in R_+$, for some $\bar{r} \in R_+$, then assumption (ii) of Lemma 4.1 is satisfied provided $\sum_{i=1}^{p} \bar{l}_i \bar{h}_i < 1$.

b) If $k_j(t) \leq \bar{k}_j$, $l_i(t) \leq \bar{l}_i t$, $g_j(t) \leq \bar{g}_j t$, $h_i(t) \leq \bar{h}_i t$, $r(t) \leq \bar{r}t$, $\bar{k}_j, \bar{l}_i, \bar{r} \in R_+$ and $\bar{g}_j \in [0, 1]$, $\bar{h}_i \in [0, 1)$, $t \in R_+$, then assumption (ii) of Lemma 4.1 is satisfied.

c) Finally, if we suppose (4.4) and $r(t) \leq \bar{r}t^q$, $t \in R_+$, for some $\bar{r}, q \in R_+$, then (ii) of Lemma 4.1 is satisfied provided $\sum_{i=1}^{p} \bar{l}_i \bar{h}_i^q < 1$.

5. – Existence theorem of a random solution to equation (1.2).

We introduce the following

ASSUMPTION C. – We assume, in relation to equation (1.2), that there exist functions $k_j^*, \tilde{k}_j, l_i \in C(R_+, R_+)$, such that

$$\begin{aligned}\|F(t, u_1(t, \omega), \ldots, u_m(t, \omega), x_1(t, \omega), \ldots, x_p(t, \omega), \omega) \\ - F(t, \bar{u}_1(t, \omega), \ldots, \bar{u}_m(t, \omega), \bar{x}_1(t, \omega), \ldots, \bar{x}_p(t, \omega), \omega)\|_{L_2} \leq \\ \leq \sum_{j=1}^{m} k_j^*(t)\|u_j(t, \omega) - \bar{u}_j(t, \omega)\|_{L_2} + \sum_{i=1}^{p} l_i(t)\|x_i(t, \omega) - \bar{x}_i(t, \omega)\|_{L_2},\end{aligned}$$

$$\|f_j(t, s, x(t, \omega), \omega) - f_j(t, s, \bar{x}(t, \omega), \omega)\|_{L_2} \leq \tilde{k}_j(t)\|x(t, \omega) - \bar{x}(t, \omega)\|_{L_2};$$

for $u_j(t, \omega)$, $\bar{u}_j(t, \omega)$, $x_i(t, \omega)$, $\bar{x}_i(t, \omega)$, $x(t, \omega)$, $\bar{x}(t, \omega) \in L_2(\Omega)$, $t \in R^+$, $j = 1, \ldots, m$, $i = 1, \ldots, p$.

From Theorem 3.1 and Lemma 4.1 follows

THEOREM 5.1. – Consider the random integral equation (1.2) subject to the following conditions:

(i) Assumption C is satisfied;

(ii) assumptions (ii) and (iii) of Lemma 4.1 are satisfied with k_j and r defined by

$$k_j(t) = k_j^*(t)\tilde{k}_j(t), \qquad r(t) = \|(Ux_0)(t, \omega) - x_0(t, \omega)\|_{L_2}, \qquad t \in R_+, \tag{5.1}$$

where the operator U is defined by the right-hand side of equation (1.2).

Then there exists a random solution $\bar{x}(t, \omega) \in C(R_+, L_2(\Omega))$ of equation (1.2) such that

$$\|\bar{x}(t, \omega) - x_n(t, \omega)\|_{L_2} \leq u_n(t), \qquad n = 0, 1, \ldots, \ t \in R_+,$$

where $\{u_n(t)\}$ is defined by (3.2) with Λ defined by (4.1). The solution $\bar{x}(t, \omega)$ is unique in the class $L_{\text{loc}}(R_+, L_2(\Omega), u_0) \stackrel{\text{df}}{=} \{x(t, \omega): x(t, \omega) \in L_{\text{loc}}(R_+, L_2(\Omega)), \|x(t, \omega) - x_0(t, \omega)\|_{L_2} \in L_{\text{loc}}(R_+, R_+, u_0)\}$, where $L_{\text{loc}}(R_+, L_2(\Omega))$ is the class of all locally integrable functions defined on R_+ with range $L_2(\Omega)$, and $L_{\text{loc}}(R_+, R_+, u_0)$ is defined in Lemma 4.1.

PROOF. – The existence of the solution is implied by Lemmas 4.1 and 3.1 (see the proof of Theorem 3.1).

To prove the uniqueness we suppose that $\tilde{x}(t, \omega) \in L_{\text{loc}}(R_+, L_2(\Omega), u_0)$ is a random solution of (1.2) different from $\bar{x}(t, \omega)$. Then we easily infer that $\bar{u}(t) = \|\bar{x}(t, \omega) - \tilde{x}(t, \omega)\|_{L_2} \in (R_+, R_+, u_0)$ and $\bar{u} \leqslant K\bar{u} + L\bar{u}$. Hence and from (*c*) of Lemma 4.1 we conclude that $\|\bar{x}(t, \omega) - \tilde{x}(t, \omega)\|_{L_2} = 0$. Thus the theorem is proved.

Combining the Assumption C with one of conditions *a*), *b*) and *c*) of Remark 4.3 we find another existence theorem for equation (1.2) in which the assumption (ii) from Theorem 5.1 is replaced by a more effective one. For example the following theorem, which follows from part *c*) of Remark 4.3 and Theorem 5.1, show also that condition (4.2) is more restrictive than assumptions of Theorem 5.1.

THEOREM 5.2. – If Assumption C and condition (4.4) with k_i and r defined by (5.1) are satisfied and if $r(t) \leqslant \bar{r}t^q$, $t \in R_+$, for some $q, \bar{r} \in R_+$, then the assertion of Theorem 5.1 holds provided

$$\sum_{i=1}^{p} \bar{l}_i \bar{h}_i^q < 1 \,. \tag{5.2}$$

REMARK 5.1. – The following example of the random functional equation

$$x(t, \omega) = \sum_{i=1}^{\infty} x\left(\frac{t}{2^i}, \omega\right), \qquad t \in R_+,\ \omega \in \Omega\,, \tag{5.3}$$

shows that condition (5.2) is essential. For this equation condition (5.2) has the form $\sum_{i=1}^{\infty} (1/2_i) q < 1$ and is hold provided $q > 1$. In view of Theorem 5.2 there exists unique solution $x(t, \omega) = 0$ of equation (5.3) in the class of functions satisfying the condition $\|x(t, \omega)\|_{L_2} \leqslant \text{const} \cdot t^q$, $t \in R_+$, but for $q = 1$ condition (5.2) is not fulfilled and for this case each function $x(t, \omega) = a(\omega) t$ is a solution of equation (5.3.).

REFERENCES

[1] A. T. BHARUCHA-REID, *Random integral equations*, Academic Press, New York, 1972.

[2] S. T. HARDIMAN - C. P. TSOKOS, *On the Uryson type of stachastic integral equations*, Proc. Cambridge Philos. Soc., **76** (1974), pp. 297-305.

[3] M. KAWAPISZ - J. TURO, *Existence, uniqueness and successive approximations for a class of integral-functional equations*, Aequationes Math., **14** (1976), pp. 303-323.

[4] M. KAWPISZ - J. TURO, *On the existence and convergence of successive approximations for some functional equations in a Banach space*, Journal of Diff. Equations, **16** (1974), pp. 298-318.

[5] A. C. H. LEE - W. J. PADGETT, *On a heavily nonlinear stochastic integral equation*, Utilitas Math., **9** (1976), pp. 123-138.

[6] J. S. MILTON - C. P. TSOKOS, *On a non-linear perturbet stochastic integral equation*, J. Math. Phys. Sci., **5** (1971), pp. 361-374.

[7] J. S. MILTON - C. P. TSOKOS, *On a random solution of a non-linear perturbet stochastic integral equation of the Volterra type*, Bull. Austral. Math. Soc., **9** (1973), pp. 227-237.

[8] J. S. MILTON - C. P. TSOKOS, *On a class of nonlinear stochastic integral equations*, Math. Nachr., **60** (1974), pp. 71-78.

[9] A. N. V. RAO - C. P. TSOKOS, *On a class of stochastic functional integral equations*, Colloq. Math., **35** (1976), pp. 141-146.

[10] T. T. SOONG, *Random differential equations in science and engineering*, Academic Press, New York, 1973.

[11] C. P. TSOKOS - W. J. PADGETT, *Random integral equations with applications to life sciences and engineering*, Academic Press, New York, 1974.

[12] J. TURO, *On a nonlinear random functional integral equation* (to appear).

[13] T. WAZEWSKI, *Sur une procédé de prouver la convergence des approximations successive sans utilisation des séries de comparaison*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astr. et Phys., **8**, no. 1 (1960), pp. 45-52.

[14] K. YOSIDA, *Functional analysis*, Springer-Verlag, Berlin, 1965.