Existence, uniqueness and continuous dependence for hereditary systems (1).

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Summary. - *An hereditary system is a system whose present state is determined in some way by its past history. We formulate a class of such systems which includes functional differential equations of retarded type and many equations of neutral type as well as Volterra integral equations. Theorems of existence, uniqueness, continuation and continuous dependence are proved.*

1. - Introduction.

An hereditary system is a system whose present state is determined in some way by its past history. A functional differential equation of retarded type is an hereditary system in which the derivative $\dot{x}(t)$ of the state x at time t is specified as a function of the past values of x over some interval. A functional differential equation of neutral type is an hereditary system in which $\dot{x}(t)$ is specified as a function of the past values of x and $\dot{x}$ over some interval. A VOLTERRA integral equation is an hereditary system in which the state $x(t)$, $t \geq 0$, is specified as a function of its history over $[0, t]$. A difference equation is an hereditary system in which the state $x(t)$ is specified as a function of its past history over some finite interval.

In this paper, we formulate a class of hereditary systems which is large enough to include equations of all of the above mentioned types. The formulation includes all functional differential equations of retarded type, VOLTERRA integral equations and difference equations. The formulation does not include all functional differential equations of neutral type, the basic restriction being that the derivative $\dot{x}$ occurs linearly in the equations. We give theorems of existence, uniqueness and continuation of solutions, as well as theorems on the dependence of solutions on initial data and parameters.

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2. - Fixed point theorems.

In this section, we give a slight generalization of the SCHAUDER fixed point theorem due to KRASNOSELSKII [4] and introduce the class of uniformly compact operators which are useful for proving theorems on continuous dependence of solutions of hereditary differential equations on initial data as well as parameters. Throughout the paper, an operator is said to be compact if it is continuous and bounded sets into precompact sets.

LEMMA 2.1. - *Suppose* Γ *is a closed, bounded, convex subset of a Banach space* X. *If* $T:\Gamma\to X$ *is a contraction,* $S:\Gamma\to X$ *is compact,* $T(\Gamma)+S(\Gamma)\overset{\text{def}}{=}\{z=Tx+Sy,\ x,\ y\in\Gamma\}\subset\Gamma$, *then* $T+S$ *has a fixed point in* Γ.

PROOF. - If I is the identity mapping, the fact that T is a contraction implies $I-T$ is a homeomorphism between Γ and $(I-T)\Gamma$.

We next show that $S(\Gamma)\subset(I-T)\Gamma$. For any $y\in S(\Gamma)$, define the sequence of successive approximations $\{x_n\}$, $n=0,1,2,\dots$, $x_0\in\Gamma$ arbitrary, $x_{n+1}=y+Tx_n$, $n=0,1,2,\dots$. Each $x_n\in\Gamma$ since $T(\Gamma)+S(\Gamma)\subset\Gamma$. Furthermore, since T is a contraction, $|x_{n+1}-x_n|\leq\alpha|x_n-x_{n-1}|$, for some $0\leq\alpha<1$ and $n=1,2,\dots$. Therefore, the sequence $\{x_n\}$ forms a CAUCHY sequence which must converge to some element x in Γ. It is clear that x satisfies $(I-T)x=y$. Consequently, $S(\Gamma)\subset(I-T)\Gamma$.

Since $I-T$ is a homeomorphism between Γ and $(I-T)\Gamma$ and $S(\Gamma)\subset(I-T)\Gamma$, finding a fixed point of $T+S$ in Γ is equivalent to finding a fixed point of $(I-T)^{-1}S$ in Γ. The operator $(I-T)^{-1}S:\Gamma\to\Gamma$ is compact. If A is the convex closure of $(I-T)^{-1}S(\Gamma)$, then $A\subset\Gamma$ is compact from a theorem of MAZUR. Furthermore, $(I-T)^{-1}S(A)\subset A$ and the SCHAUDER fixed point theorem implies the existence of a fixed point in $A\subset\Gamma$. This proves the lemma.

DEFINITION 2.1. - Suppose X, Y, Z are BANACH spaces, Γ, Λ are subsets of X, Y respectively and $S:\Lambda\times\Gamma\to Z$. Let the values of S in Z be denoted by S_yx. The mapping S is said to be *uniformly* compact on $\Lambda\times\Gamma$ if for each closed, bounded subset $\Lambda_1\subset\Lambda$, $\Gamma_1\subset\Gamma$, the set $\{z=S_yx,\ (y,x)\in\Lambda_1\times\Gamma_1\}$ is relatively compact, and $S_y:\Gamma\to Z$ is continuous for each $y\in\Lambda$.

LEMMA 2.2. - *Suppose* X, Y *are Banach spaces,* Λ *is a subset of* Y *and* Γ *is a closed, bounded, convex subset of* X. *Also, suppose* $T:\Lambda\times\Gamma\to X$ *is such that* T_y *is a contraction for each* $y\in\Lambda$ *and* $S:\Lambda\times\Gamma\to X$ *is uniformly compact,* $T_y\Gamma+S_y\Gamma\subset\Gamma$ *for each* $y\in\Lambda$. *If there is a* $y_0\in\Lambda$ *such that* S_yx T_yx *are continuous at* y_0 *uniformly for* $x\in\Gamma$, *and the equation*

$$(I-T_y)x=S_yx \tag{2.1}$$

has a unique solution $x(y_0) \in \Gamma$ at y_0, then the solutions $x(y)$, $y \in \Lambda$, of (2.1) *in Γ are continuous at y_0.*

PROOF. - From Lemma 2.1, there is a solution of (2.1) for each y in Λ. Suppose $y_n \in \Lambda$, $n = 1, 2, \ldots$, approaches y_0 as $n \to \infty$ and let x_n be any solution of (2.1) corresponding to $y = y_n$, $n = 0, 1, 2, \ldots$. From the hypothesis, x_0 is the only solution of (2.1) for $y = y_0$. Since $T_y x$ is continuous at y_0 uniformly for $x \in \Gamma$, there is a sequence $\delta_n > 0$, $\delta_n \to 0$ as $n \to \infty$ such that $|T_{y_n} x - T_{y_0} x| < \delta_n$, for all $x \in \Gamma$, $n = 1, 2, \ldots$. Furthermore, since S is uniformly compact, there are a $z \in X$ and a subsequence of the y_n which we again label as y_n such that $S_{y_n} x_n \to z$ as $n \to \infty$. Therefore,

$$(I - T_{y_0})x_n = S_{y_n} x_n + T_{y_n} x_n - T_{y_0} x_n$$

$$\to z \quad \text{as} \quad n \to \infty.$$

Consequently, T_{y_0} a contraction implies x_n converges to $w = (I - T_{y_0})^{-1} z$ as $n \to \infty$. It is clear that w is a fixed point of $T_{y_0} + S_{y_0}$ and therefore $w = x_0$. Since every subsequence of the sequence x_n must have a subsequence converging to x_0, it follows that the sequence x_n converges to x_0. Finally, the sequence of fixed points x_n being arbitrary implies the conclusion of the lemma.

Even though it is not needed in the following, essentially the same proofs as above yield the following generalizations of Lemmas 2.1 and 2.2.

LEMMA 2.3. - *Suppose Γ is a closed, bounded convex subset of a Banach space X. If $T : \Gamma \to X$ is such that $I - T$ is a homeomorphism between Γ and $(I - T)\Gamma$, $S : X \to X$ is compact, $S(\Gamma) \subset (I - T)\Gamma$, then $T + S$ has a fixed, point in Γ.*

LEMMA 2.4. - *Suppose X, Y are Banach spaces, Λ is a subset of Y and Γ is a closed, bounded, convex subset of X. Also, suppose $T : \Lambda \times \Gamma \to X$ is such that $I - T_y$ is a homeomorphism between Γ and $(I - T_y)\Gamma$ for each $y \in \Lambda$, $S : \Lambda \times \Gamma \to X$ is uniformly compact, $S_y(\Gamma) \subset (I - T_y)\Gamma$ for each y in Λ. If there is a $y_0 \in \Lambda$ such that $S_y x$ and $T_y x$ are continuous at y_0 uniformly with respect to $x \in \Gamma$, and the equation* (2.1) *has a unique solution $x(y_0) \in \Gamma$ at y_0, then the solutions $x(y)$, $y \in \Lambda$ of* (2.1) *in Γ are continuous at y_0.*

3. - A general class of hereditary differential equations.

Let R denote the real line, R^n be an n-dimensional linear vector space with norm $|\cdot|$; let Ω denote the set of all compact subsets of R and let A be an element of Ω. It is convenient to assume that zero is the maximal element of A. Let $C_A = C(A, R^n)$ be the space of continuous functions mapping A into R^n with $|\varphi| = \sup_{\theta \in A} |\varphi(\theta)|$ for all φ in C_A.

Let E be a connected interval of R, α be a continuous mapping of $E \times A$ into R such that $\alpha(t, A) \in \Omega$, $\alpha(t, \theta) \leq t$, $\alpha(t, \theta) \leq \alpha(t, \zeta)$, $\alpha(t, 0) = t$ for all $t \in E$, $\theta \leq \zeta \in A$. If x is any continuous function mapping the range of α into R^n, we define an operator $\mathcal{A} : E \times x \to C_A$ by the relation

$$(\mathcal{A}_t x)(\theta) = x(\alpha(t, \theta)), \qquad \theta \in A, \qquad t \in E.$$

The triple $(A, \alpha, \mathcal{A})$ will be referred to as an *hereditary structure.*

Suppose $(A, \alpha, \mathcal{A})$ is an hereditary structure, $g : E \times C_A \to R^n$ and $f : E \times C_A \to R^n$. An hereditary differential equation is a relation of the form

$$\frac{d}{dt}[D(t)\mathcal{A}_t x] = f(t, \mathcal{A}_t x) \tag{3.1}$$

where

$$D(t)\varphi = \varphi(0) - g(t, \varphi), \qquad t \in E, \ \varphi \in C_A. \tag{3.2}$$

If $g = 0$, then (3.1) reduces to the equation

$$\frac{d}{dt} x(t) = f(t, \mathcal{A}_t x), \tag{3.3}$$

which is usually referred to as a functional differential equation of retarded type. Functional differential equations (3.3) of retarded type were first formulated in this manner by J. K. HALE and G. S. JONES in a seminar at RIAS in 1963 and was later published in an even more general form by G. S. JONES [3]. The formulation given above is useful for a much wider class of problems occurring in the applications-including certain equations of neutral type as well as VOLTERRA integral equations.

To appreciate the generality of (3.1), let us consider some more special cases. If $A = [-r, 0]$, $r \geq 0$, $\alpha(t, \theta) = t + \theta$, $\theta \in [-r, 0]$, $(\mathcal{A}_t x)(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$, then (3.3) reduces to the usual functional differential equations of retarded type

$$\frac{d}{dt} x(t) = f(t, x_t)$$

where we have employed the conventional notation $x_t(\theta) \stackrel{\text{def}}{=} \mathcal{A}_t x(\theta) = x(t + \theta)$, $-r \leq \theta \leq 0$. If this simpler notation is again employed, we see that system (3.1) includes the general class of equations

$$\frac{d}{dt}[D(t)x_t] = f(t, x_t) \tag{3.4}$$

where $D(t)$ is defined in (3.2).

If $f = 0$ system (3.4) includes the difference equation

$$x(t) - g(t,\ x(t),\ x(t-\omega_1),\ \ldots,\ x(t-\omega_k)) = h(t) \tag{3.5}$$

where the ω_j are nonnegative numbers. If a function satisfies (3.4) on some interval and has a sufficiently smooth derivative, then carrying out the differentiation in (3.4) leads to the system

$$\dot{x}(t) - g'_\varphi(t,\ x_t)\dot{x}_t - g'_t(t,\ x_t) - \dot{h}(t) = f(t,\ x_t), \tag{3.6}$$

where $\dot{x}_t(\theta) = \dot{x}(t+\theta)$, $-r \leq \theta \leq 0$. System (3.6) includes all of the equations of neutral type for which the derivative occurs linearly. HALE and MEYER [2] have considered equation (3.4) when $g(t,\ \varphi)$ is linear in φ. DRIVER [1] has considered (3.6) with $\tilde{g}'_\varphi(t,\ \varphi) = g(t,\ \varphi(-s(t)))$ and $s(t) \geq 0$. Driver has also treated cases linear in the deriviative which cannot be written in the from (3.4).

Equation (3.1) also includes VOLTERRA integral equations. To see this, let $A = [-1,\ 0]$, $\alpha(t,\ \theta) = t(1+\theta)$, $\mathcal{A}_t x(\theta) = x(t(1+\theta))$ and suppose $a : [0,\ \infty) \times [0,\ \infty) \times R^n \to R^n$, $h : [0,\ \infty) \to R^n$ are given functions. If

$$g(t,\ \varphi) = t\int_{-1}^{0} a(t,\ t(1+\theta),\ \varphi(\theta))d\theta + h(t),\ t \in [0,\ \infty),\ \varphi \in C_A, \tag{3.7}$$

$$D(t,\ \varphi) = \varphi(0) - g(t,\ \varphi),$$

then

$$\begin{aligned} D(t)\mathcal{A}_t x &= x(t) - t\int_{-1}^{0} a(t,\ t(1+\theta),\ x(t(1+\theta)))d\theta - h(t) \\ &= x(t) - \int_{0}^{t} a(t,\ s,\ x(s))ds - h(t). \end{aligned} \tag{3.8}$$

If $f \equiv 0$ and $x(0) = h(0)$, then (3.1) is equivalent to

$$x(t) = h(t) + \int_{0}^{t} a(t,\ s,\ x(s))ds. \tag{3.9}$$

The literature for equation (3.9) is very extensive and the most recent general presentation of existence, uniqueness, etc. is contained in the paper of MILLER and SELL [5], NEUSTADT [6].

We now formulate the initial value problem for (3.1). For the hereditary structure $(A,\ \alpha,\ \mathcal{A})$ and any $\sigma \in E$ let E_σ be the set of real numbers defined by

$$E_\sigma = \cup_{s \geq \sigma,\ s \in E}\ \overline{\mathrm{co}}\ \alpha(s,\ A) \cap (-\infty,\ \sigma] \tag{3.10}$$

where $\overline{co}\ G$ for any set G is the closed convex hull of G. The set E_σ contains the set of real numbers on which an initial function must be specified in order to integrate (3.1) on all of E. The set E_σ may be larger than necessary if one is only interested in the integration of (3.1) on a part of E, but we always use the above set for simplicity in notation.

Given a σ in E and a function $\varphi \in C_{E_\sigma}$, we say $x = x(\sigma, \varphi)$ is a *solution of* (3.1) *with initial value* φ *at* σ *if there is a* $\gamma > 0$ such that x is defined and continuous on $E_\sigma \cup [\sigma, \sigma + \gamma)$, x coincides with φ on E_σ, $D(t)\mathcal{C}_t x$ is continuosly differentiable on $[\sigma, \sigma + \gamma]$ and satisfies (3.1) on $[\sigma, \sigma + \gamma)$.

It is clear that x is a solution of (3.1) with initial value φ at σ if and only if x satisfies the equation

$$x(t) = \varphi(t), \qquad t \in E_\sigma, \tag{3.11}$$
$$D(t)\mathcal{C}_t x = D(\sigma)\mathcal{C}_\sigma \varphi + \int_\sigma^t f(s, \mathcal{C}_s x)ds, \qquad t \geq \sigma.$$

In the applications, it is convenient to have a different hereditary structure in the operator $D(t)$ than in the right hand side of (3.1). This more general situation is treated in the following way. Let $(A_1, \alpha_1, \mathcal{C}_1), (A_2, \alpha_2, \mathcal{C}_2)$ be hereditary structures, let $A = A_1 \times A_2$, $\alpha = (\alpha_1, \alpha_2)$, $\mathcal{C}_t x = (\mathcal{C}_{1t} x, \mathcal{C}_{2t} x)$. If g, f are as before, we can define a functional differential equation as (3.1). If $g(t, \varphi, \psi)$, $t \in E$, $\varphi \in C_{A_1}$, $\psi \in C_{A_2}$, is independent of ψ and $f(t, \varphi, \psi)$ is inde. pendent of φ, then the herditary structure in D is $(A_1, \alpha_1, \mathcal{C}_1)$ and the hereditary structure in f is $(A_2, \alpha_2, \mathcal{C}_1)$. No change in the statement or proofs of the theorems below is required for this more general situation.

4. - Existence of solutions.

In this section, we give sufficient conditions on g and f to ensure the existence of a solution of the initial value problem for (3.1).

DEFINITION 4.1. - Suppose A is a compact subset of R, $0 \in A$, θ in A implies $\theta \leq 0$ and suppose U is an open subset of $R \times C_A$. For any $(t, \varphi) \in U$, any $\mu \geq 0$, $s \geq 0$, let $Q(t, \varphi, \mu, s) = \{\psi \in C_A : (t, \psi) \in U, |\psi - \varphi| \leq \mu, \psi(\theta) = \varphi(\theta), \theta < -s, \theta \in A\}$. We say a continuous function $g : U \to R^n$ is *nonatomic at zero* if for any $(t, \varphi) \in U$, there exist $s_0 = s_0(t, \varphi) > 0$, $\mu_0 = \mu_0(t, \varphi) > 0$ continuous in t, φ and a scalar function $\rho(t, \varphi, \mu, s)$ defined and continuous for $(t, \varphi) \in U$, $0 \leq s \leq s_0$, $0 \leq \mu \leq \mu_0$, nondecreasing in μ, s such that $\rho(t, \varphi, \mu_0, s_0) < 1$ and

$$|g(t, \psi) - g(t, \varphi)| \leq \rho(t, \varphi, \mu, s) |\psi - \varphi| \tag{4.1}$$

for t in R, $\psi \in Q(t, \varphi, \mu, s)$ and all $0 \leq s \leq s_0$, $0 \leq \mu \leq \mu_0$.

If W is any subset of U such that s_0, μ_0 can be taken independent of (t, φ) in W for which $\rho(W, \mu_0, s_0) = \sup_W \rho(t, \varphi, \mu_0, s_0) < 1$, then we say g is *uniformly nonatomic at zero on W.*

If $g(t, \varphi)$ is linear in φ and has the representation

$$g(t, \varphi) = \int_A [d_\theta \eta(t, \theta] \varphi(\theta)$$

for all $\varphi \in C_A$, then g being nonatomic at zero is an expression of some continuity of $g(t, \varphi)$ in φ as well as the fact that the jump in $\eta(t, \theta)$ at $\theta = 0$ is < 1. If the corresponding $\rho(t, \varphi, \mu, s)$ vanishes for $s = 0$, then the measure generated by $\eta(t, \theta)$ is nonatomic at zero. This is the motivation for the terminology. In this case, any $s_0 \in A$ sufficiently small will satisfy the properties in the definition and the function $\rho(t, \varphi, \mu, s)$ is independent of μ, φ.

If there is an $\varepsilon < 0$ such that $g : E \times C_A \to R^n$ depends only upon the values of $\varphi(\theta)$ for $\theta \in A$, $\theta \leq \varepsilon < 0$, then g is nonatomic at zero with $\rho(t, \varphi, \mu, s) = 0$ for all t, φ, μ, s, $\varepsilon < -s \leq 0$. In particular, $g(t, \varphi) = a(t, \varphi(\alpha(t) - t))$, where $\alpha(t) - t \in A$, $\alpha(t) - t \leq \varepsilon < 0$, $t \in E$ is nonatomic at zero.

If $g(t, \varphi) = \varphi^2(0)$, then $|g(t, \psi) - g(t, \varphi| \leq (2|\varphi(0)| + \mu)|\varphi(0) - \psi(0)|$ if $|\psi - \varphi| \leq \mu$. Therefore, if $V = \{\varphi \in C : |\varphi| \leq \nu\}$, $2\nu + \mu_0 < 1$ and $U = (-\infty, \infty) \times V$, then g is nonatomic at zero relative to the set U. In fact, one takes $\rho(t, \varphi, \mu, s) = 2\nu + \mu$ for all t, φ, s, $0 \leq \mu \leq \mu_0$. Because of examples of this type, the term nonatomic at zero is being abused.

Suppose $A = [-1, 0]$, V is an open set in R^n, $E = (-\infty, \infty)$, $a : E \times E \times V \to R^n$ and for every compact subset K of V there is a function $p_K(t, u)$, $(t, u) \in E \times E$ such that

$$|a(t, u, x) - a(t, u, y)| \leq p_K(t, u)|x - y| \tag{4.2}$$

for (t, u, x), $(t, u, y) \in E \times E \times K$. If $g(t, \varphi)$ is defined as in (3.7), then

$$|g(t, \varphi) - g(t, \psi)| \leq \rho(t, \varphi, \mu, s)|\varphi - \psi|$$

for all $t \in E$, $\psi \in Q(t, \varphi, \mu, s)$, where

$$\rho(t, \varphi, \mu, s) = \int_{t(1-s)}^{t} p_K(t, u)du$$

$$K = \{x \in R^n : |x - \varphi(\theta)| \leq \mu, \ -s \leq \theta \leq 0\}.$$

Any conditions on the function a which will ensure that the function $\rho(t, \varphi, \mu, s)$ is continuous in all variables implies that the function g in (3.7) is nonatomic at zero. See MILLER and SELL [5] for conditions on a which will imply this latter property.

We will have many occasions to use the following machinery in slightly different forms so that it is convenient to elevate it to a lemma.

For $\sigma \in R$, $b > \sigma$, let I be either the closed interval $[\sigma, b]$ or the open interval $[\sigma, b)$. For $b = \sigma$ we take I as the empty set. Let $x \in C_{E_\sigma \cup I}$ with $x(\theta) = \varphi(\theta)$ for θ in E_σ. Suppose $(A, \alpha, \mathcal{A})$ is an hereditary structure, U an open set in $R \times C_A$ and the closure W of the set $\{(t, \mathcal{A}_t x) : t \in I\}$ as well as an ε-neighborhood $V_\varepsilon(W)$ of W are contained in U. Let $\zeta \in I$. For any real numbers γ, $\delta > 0$, define $I_\gamma(\zeta) = \{t : \zeta \leq t \leq \zeta + \gamma\}$, $F(\gamma, \delta, \zeta, \mathcal{A}_\zeta x) = \{(t, \psi) \in R \times C_A : t \in I_\gamma(\zeta), |\psi - \mathcal{A}_\zeta x| \leq \delta\}$ and $\mathcal{S}(\gamma, \delta, \zeta) = \{y \in C_{E_\zeta \cup I_\gamma(\zeta)} : y(t) = 0, t \in E_\zeta, |y(t)| \leq \delta, t \in I_\gamma(\zeta)\}$. Let $F(\gamma, \delta) = F(\gamma, \delta, W) = \cup\{F(\gamma, \delta, \zeta, \mathcal{A}_\zeta x) : \zeta \in I\}$. Finally, for $x \in C_{E_\zeta \cup I}$, define $\tilde{x}_\zeta \in C_{E_\zeta \cup I_\gamma(\zeta)}$ by $\tilde{x}_\zeta(t) = x(t)$ for t in E_ζ and $\tilde{x}_\zeta(t) = x(\zeta)$, t in $I_\gamma(\zeta)$.

LEMMA 4.1. - *Using the above notation, let* $f, g : U \to R^n$ *be continuous,* $|f| \leq M$ *on* $V_\varepsilon(W)$ *and g uniformly nonatomic at zero on* $V_\varepsilon(W)$. *Then there are positive real numbers* γ_0, δ_0, ν *and* γ, δ, $0 < \gamma \leq \gamma_0$, $0 < \delta < \delta_0/2$ *such that*

(i) (a) $M\gamma < \nu\delta/2$ *or* (b) $M\gamma < \nu(1 - \nu)\delta/2$

(ii) $|\mathcal{A}_t \tilde{x}_\zeta - \mathcal{A}_\zeta x| < \delta_0/2$, $t \in I_\gamma(\zeta)$, $\zeta \in I$ for W *compact and* $\alpha(b, \theta) < b$ *for* $\theta < 0$.

(iii) $|g(t, \mathcal{A}_t \tilde{x}_\zeta) - g(\zeta, \mathcal{A}_\zeta x)| < \nu\delta/2$, $t \in I_\gamma(\zeta)$, $\zeta \in I$ *for W compact and* $\alpha(b, \theta) < b$ *for* $\theta < 0$.

(iv) $F(\gamma_0, \delta_0) \subset U$

(v) $\rho(t, \psi, \delta_0, \gamma_0) \leq 1 - \nu$ *for* $(t, \psi) \in F(\gamma_0, \delta_0)$

(vi) $(t, \mathcal{A}_t(y + \tilde{x}_\zeta)) \in F(\gamma_0, \delta_0)$ *for* $t \in I_\gamma(\zeta)$, $\zeta \in I$, $\alpha(b, \theta) < b$ *for* $\theta < 0$, $y \in \mathcal{S}(\gamma, \delta, \zeta)$ *and* $\alpha(b, \theta) < b$ *for* $\theta < 0$.

In particular, if W is compact, $\alpha(b, \theta) < b$ *for* $\theta < 0$, *g, f only continuons, and g nonatomic at zero on U are sufficient for* (i)-(vi) *to hold for some* $V_\varepsilon(w)$.

PROOF. - Items (i), (iv), (v) are immediate from the definitions and the hypothesis. If W is compact, then the hypothesis on α in (ii) implies that x can be extended to a continuous function on $E_\sigma \cup [\sigma, b]$. The result in (ii) as well as (iii) and (vi) are now immediate. If W is compact, and g, f satisfy the conditions stated in the last part of the lemma, then g is uniformly nonatomic at zero on W and $|f| \leq M$ on some ε-neighborhood $V_\varepsilon(W)$ of W. This proves the lemma.

REMARK. - In the application of the lemma W is a point set in the existence theorem and is either compact or closed and bounded in the continuation theorems.

THEOREM 4.1. - *Suppose $(A, \alpha, \mathcal{A})$ is an hereditary structure and U is an open set in $R \times C_A$. If $g, f: U \to R^n$ are continuous and g is nonatomic at zero, then for any $\sigma \in R$, $\varphi \in C_{E_\sigma}$, $(\sigma, \mathcal{A}_\sigma\varphi) \in U$, there exists a solution of (3.1) with initial value φ at σ defined and continuous on some set $E_\sigma \cup [\sigma, b]$, $b > \sigma$.*

PROOF. - Take $W = \{(\sigma, \mathcal{A}_\sigma\varphi)\}$. W compact and the hypothesis imply that the conclusions of Lemma 4.1 hold for some numbers γ_0, $\delta_0 > 0$, $0 < \gamma \leq \gamma_0$, $0 < \delta < \delta_0/2$. In this case, $\zeta = \sigma$, $\tilde{\varphi}(t) = \varphi(t)$, $t \in E_\sigma$, and $\tilde{\varphi}(t) = \varphi(t) = \varphi(0)$, $t \in I_\gamma = I_\gamma(\sigma)$, $\tilde{x}_\zeta = \tilde{\varphi}$, I is the empty set.

Consider the transformation T, S taking $\mathcal{S} = \mathcal{S}(\gamma, \delta, \sigma)$ into $C_{E_\sigma \cup I_\gamma}$ defined by

$$(4.4)\quad \begin{aligned} &\text{(a)} \quad \begin{cases} (Ty)(t) = 0, & t \in E_\sigma \\ (Ty)(t) = g(t, \mathcal{A}_t(y + \tilde{\varphi})) - g(\sigma, \mathcal{A}_\sigma\varphi), & t \in I_\gamma \end{cases} \\ &\text{(b)} \quad \begin{cases} (Sy)(t) = 0, & t \in E_\sigma \\ (Sy)(t) = \int_\sigma^t f(s, \mathcal{A}_s(y + \tilde{\varphi}))ds, & t \in I_\gamma . \end{cases} \end{aligned}$$

Recall that the solutions of (3.1) on $(\sigma, \sigma + \gamma)$ with initial value φ at σ coincide with the solutions of (3.11). Therefore, if y^* is a fixed point of $T + S$ in $\mathcal{S}$, then $x^* = y^* + \tilde{\varphi}$ is a solution of (3.1) on $(\sigma, \sigma + \gamma)$ with initial value φ at σ. Conversely, if x^* is a solution of (3.11) with $x^* - \tilde{\varphi} \in \mathcal{S}$, then $x^* - \varphi$ is a fixed point of $S + T$.

We now show that $S + T$ has a fixed point in $\mathcal{S}$. From the definition, $Ty(t) + Sz(t) = 0$ for $t \in E_\sigma$ for all y, $z \in \mathcal{S}$. Also, for any y, z in $\mathcal{S}$, $t \in I_\gamma$ relation (4.4) and Lemma 4.1 imply

$$\begin{aligned} |Ty(t) + Sz(t)| &\leq |g(t, \mathcal{A}_t(y + \tilde{\varphi})) - g(t, \mathcal{A}_t\tilde{\varphi})| + |g(t, \mathcal{A}_t\tilde{\varphi}) - g(\sigma, \mathcal{A}_\sigma\varphi)| \\ &\quad + \int_\sigma^t |f(s, \mathcal{A}_s(z + \tilde{\varphi}))| ds \\ &\leq \rho(t, \mathcal{A}_t\tilde{\varphi}_t, \delta, \gamma)\delta + \frac{\nu\delta}{2} + \frac{\nu\delta}{2} \\ &\leq (1 - \nu)\delta + \nu\delta = \delta. \end{aligned}$$

Therefore, $T + S: \mathcal{S} \to \mathcal{S}$, $T(\mathcal{S}) + S(\mathcal{S}) \subset \mathcal{S}$. It is not difficult to show that S is continuous. Moreover, $S(\mathcal{S}) \subset \mathcal{S}$ and S is compact since $|Sy(t) - Sy(t')| \leq$ $\leq M|t - t'|$ for all t, t' in I_γ. Also, for y, z in $\mathcal{S}$, by Lemma 4.1,

$$\begin{aligned} |Ty(t) - Tz(t)| &= |g(t, \mathcal{A}_t(y + \tilde{\varphi})) - g(t, \mathcal{A}_t(z + \tilde{\varphi}))| \\ &\leq \rho(t, \mathcal{A}_t(y + \tilde{\varphi}), \delta, \gamma) |\mathcal{A}_t y - \mathcal{A}_t z| \\ &\leq (1 - \nu) |y - z| \end{aligned}$$

and T is a contraction on $\mathcal{S}$. Therefore, Lemma 2.1 implies the existence of a fixed point of $T + S$ in $\mathcal{S}$. This completes the proof of the theorem.

5. - Continuation of solutions and uniqueness.

If x is a solution of (3.1) on $E_\sigma \cup [\sigma, a)$, $a > \sigma$, we say x is a *continuation* of x if there is a $b > a$ such that $\hat{x}$ is defined on $E_\sigma \cup [\sigma, b)$, coincides with x on $E_\sigma \cup [\sigma, a)$ and satisfies (3.1) on (σ, b). A solution x is noncontinuable if no such continuation exists; that is, $E_\sigma \cup [\sigma, a)$ is the maximal interval of existence of the solution x. If the conditions of Theorem 4.1 are satisfied, then there is a solution of (3.1) on $E_\sigma \cup [\sigma, a)$ for some $a > \sigma$. ZORN's lemma implies the existence of a noncontinuable solution of (3.1). It is also true that the maximal interval of existence is open.

THEOREM 5.1. - *Under the same hypothesis as Theorem* 4.1, *if* $x(\sigma, \varphi)$ *is a noncontinuable solution of* (3.1) *on* $E_\sigma \cup [\sigma, b)$, φ *in* C_{E_σ}, $(\sigma, \mathcal{A}_\sigma\varphi) \in U$, $\alpha(b\ \theta) < b$ for $\theta < 0$, *then either*

(a) $(t_k, \mathcal{A}_{t_k}x)$ *for some sequence* $t_k \to b^-$ *as* $k \to \infty$ *tends to the boundary* ∂U *of* U *if the closure* W *of* $\{(t, \mathcal{A}_t x) : \sigma \leq t < b\}$ *is compact, or,*

(b) W *is not compact in which case* $b = \infty$ *or the closure of* $G = \{\mathcal{A}_t x : \sigma \leq t < b\}$ *is not compact.*

PROOF. - (b) follows readily from (a). We show the validity of (a) by contradiction.

Let W be compact. Then $b < \infty$. Suppose W is properly contained in U, and (a) is not true. By Lemma 4.1, there are positive numbers, γ_0, δ_0, γ, δ and ν, $0 < \gamma \leq \gamma_0$, $0 < \delta \leq \delta_0/2$ for which items (i) (b), (ii)-(vi) of that lemma hold. Define T as in the proof of Theorem 4.1 with σ replaced by $b - \gamma$, φ by the restriction $x_{b-\gamma}$ of x on $E_{b-\gamma}$, $\mathcal{A}_t(y + \tilde{\varphi})$ by $\mathcal{A}_t(y + \tilde{x}_{b-\gamma})$, t in $I_\gamma(b - \gamma)$ and y in $\mathcal{S} = \mathcal{S}(\gamma, \delta, b - \gamma)$. T is a contraction on $\mathcal{S}$. If

$$z(t) = 0, \qquad t \in E_{b-\gamma} \tag{5.1}$$

$$z(t) = \int_{b-\gamma}^{t} f(s, \mathcal{A}_s x) ds, \qquad b - \gamma \leq t \leq b,$$

then z is defined on $E_{b-\gamma} \cup [b-\gamma, b]$ since $z(b) = \lim_{t \to b^-} z(t)$ exists and $z \in \mathcal{S}(\gamma, \delta, b-\gamma)$. The choice of γ shows that $z \in (I-T)\mathcal{S}$ and there is a unique y in $\mathcal{S}$ such that $(I-T)y = z$. The uniqueness of y implies $y(t) = x(t) - \tilde{x}_{b-\gamma}(t)$ for t in $[b-\gamma, b)$. Hence $\lim_{t \to b^-} x(t)$ exists and $(b, \mathcal{Q}_b x) \in U$. By the existence theorem, x is continuable which is a contradiction. (a) must therefore hold and the theorem is proved.

We want to get information as in (a) of the above theorem in terms of sufficient conditions on g and f.

THEOREM 5.2. - *Suppose $p < 0$ is such that $g(t, \varphi)$ depends only on values of $\varphi(\theta)$ for $\theta \leq p < 0$ and f maps closed bounded sets in U into bounded sets in R^n. If $\sigma \in R$, $\varphi \in E_\sigma$, $(\sigma, \mathcal{Q}_\sigma \varphi) \in U$ and x si a noncontinuable solution $x(\sigma, \varphi)$ of* (3.1) *defined on its maximal domain of existence $E_\sigma \cup [\sigma, b)$, then, for every closed bounded subset W of U, there is a $\zeta \in [\sigma, b)$ such that $(\zeta, \mathcal{Q}_\zeta x)$ is not in W.*

PROOF. - No loss in generality occurs in assuming $(\sigma, \mathcal{Q}_\sigma \varphi)$ is in the arbitrary closed bounded set W in U. The case $b = \infty$ is trivial. Suppose $b < \infty$, $x(\sigma, \varphi)$ is such that $(t, \mathcal{Q}_t x) \in W$ for all $\sigma \leq t < b$ and $|f| \leq M$ on W. The function $\int_\sigma^t f(s, \mathcal{Q}_s x) ds$ is therefore uniformly continuous for t in $[\sigma, b)$. Also the function $g(t, \mathcal{Q}_t x)$ is uniformly continuous for t in $[\sigma, b)$ since $g(t, \varphi)$ depends only on values of $\varphi(\theta)$ for $\theta \leq p < 0$. Therefore, $x(t)$ is uniformly continuous on $[\sigma, b)$ and can be extended to a continuous function on $[\sigma, b]$. Since $(b, x_b) \in U$, x can be continued as a solution of (3.1) beyond $t = b$. This is a contradiction and proves the theorem.

The strong hypothesis was made on g in theorem 5.2 te ensure that $g(t, \mathcal{Q}_t x)$ is uniformly continuous on $[\sigma, b)$. Any other condition on g which implies this property will yield the same conclusion as in theorem 5.2. When g arises from a VOLTERRA integral equation, it is not too dificult to give conditions so that $g(t, \mathcal{Q}_t x)$ is uniformly continuous for t in $[\sigma, b)$. See MILLER and SELL [5].

THEOREM 5.3. - *Suppose $(A, \alpha, \mathcal{Q})$ is an hereditary structure and U is an open set in $R \times C_A$. If $g : U \to R^n$ is continuous, nonatomic at zero, $f : U \to R^n$ is continuous and $f(t, \varphi)$ is Lipschitzian with respect to φ in each compact set in U, then for any $\sigma \in R$, $\varphi \in C_{E_\sigma}$, $(\sigma, \mathcal{Q}_\sigma \varphi) \in U$, there is a unique solution of* (3.1) *with initial value φ at σ.*

PROOF. - The proof is essentially the same as the proof for ordinary differential equations.

6. - Continuous dependence of solutions.

In this section, we give sufficient conditions on f, g in (3.1) to ensure that the solution depends continuously upon the initial function as well as parameters.

DEFINITION 6.1. - If A is a compact subset of R, U is an open subset of $R \times C_A$, Λ is a subset of some BANACH space and $g : U \times \Lambda \to R^n$ with values $g_\lambda(t, \varphi)$, $\lambda \in \Lambda$, $(t, \varphi) \in U$, we say the family $\{g_\lambda\}$, $\lambda \in \Lambda$, of functions taking $U \to R^n$ is *equi-nonatomic at zero* provided there are functions $s_0(t, \varphi) > 0$, $\mu_0(t, \varphi) > 0$, $\rho(t, \varphi, \mu, s) < 1$ as in Definition 4.1 such that each g_λ, $\lambda \in \Lambda$ satisfies (4.1) for this $\rho(t, \varphi, \mu, s)$, $0 \leq \mu \leq \mu_0$, $0 \leq s \leq s_0$.

If there is an $\varepsilon < 0$ such that each g_λ, $\lambda \in \Lambda$, depends only upon values of $\varphi(\theta)$ for $\theta \in A$, $\theta \leq \varepsilon < 0$, then the family $\{g_\lambda\}$ is equi-nonatomic at zero. Also, if

$$g_\lambda(t, \varphi) = \int_A [d_\theta \eta_\lambda(t, \theta)]\varphi(\theta)$$

and there is an $s_0 > 0$ and $\rho < 1$ such that for $A(s_0) = \{\theta \in A : -s_0 \leq \theta \leq 0\}$

$$\left| \int_{A(s_0)} [d_\theta \eta_\lambda(t, \theta)]\varphi(\theta) \right| \leq \rho \, |\varphi|$$

for all $\varphi \in C_A$, $\lambda \in \Lambda$, then the family $\{g_\lambda\}$ is equi-nonatomic at zero.

THEOREM 6.1. - *Suppose $(A, \alpha, \mathcal{A})$ is an hereditary structure, U is an open set in $R \times C_A$, $g_k : U \to R^n$, $k = 0, 1, 2, \ldots$, are continuous functions, equi-nonatomic at zero, g_0 is uniformly continuous on closed bounded subsets of U, $g_k \to g_0$ as $k \to \infty$ uniformly on closed bounded subsets of U, $f_k : U \to R^n$, $k = 0, 1, \ldots$, are continuous and $f_k(s, \psi) \to f_0(s, \varphi)$ as $k \to \infty$, $\psi \to \varphi$ for all $(s, \varphi) \in U$. Also, for any compact W in U, there is an open neighborhood $V(W)$ of W and a constant M such that*

$$(6.1) \qquad |f_k(t, \psi)| \leq M, \qquad (t, \psi) \in V(W), \qquad k = 0, 1, 2, \ldots$$

Finally, let $\sigma \in R$, $\varphi_k \in C_{E_\sigma}$, $(\sigma, \mathcal{A}_\sigma \varphi_k) \in U$, $k = 0, 1, 2, \ldots$, $\varphi_k \to \varphi_0$ as $k \to \infty$ and suppose $x_k = x_k(\sigma, \varphi_k)$, $k = 0, 1, 2, \ldots$, is a solution of

$$(6.2) \qquad \frac{d}{dt}[D_k(t)\mathcal{A}_t x] = f_k(t, \mathcal{A}_t x), \qquad t \geq \sigma,$$

$$D_k(t)\psi = \psi(0) - g_k(t, \psi)$$

with initial value φ_k at σ. If x_0 is defined on $E_\sigma \cup [\sigma, b]$ and is unique, then there is an integer k_0 such that the x_k, $k \geq k_0$, can be defined on $E_\sigma \cup [\sigma, b]$ and $x_k(t) \to x_0(t)$ uniformly on $E_\sigma \cup [\sigma, b]$.

PROOF. - The set $W = \{(t, \mathcal{A}_t x_0),\ \sigma \leq t \leq b\}$ is a compact subset of U. Since the family of functions g_k, $k \geq 0$, is equi-nonatomic at zero, there are continuous scalar functions $s_0(t, \varphi) > 0$, $\mu_0(t, \varphi) > 0$, $\rho(t, \varphi, \mu, s) < 1$, $s_0(t, \varphi) \geq s \geq 0$, $0 \leq \mu \leq \mu_0(t, \varphi)$, $(t, \varphi) \in U$, such that each g_k satisfies (4.1) for this $\rho(t, \varphi, \mu, s)$. Since W is compact, there are $\bar{s}_0 > 0$, $\bar{\mu}_0 > 0$, $\nu > 0$ such that $0 < s_0 < s_0(t, \varphi)$, $0 < \bar{\mu}_0 < \mu_0(t, \varphi)$, $\rho(t, \varphi, \mu, s) < 1 - \nu$, $s_0(t, \varphi) \geq s \geq 0$, $0 \leq \mu \leq \mu_0(t, \varphi)$, $(t, \varphi) \in W$. By hypothesis, for any $(t, \varphi) \in W$ and any $\varepsilon > 0$, there is a $d(t, \varphi, \varepsilon) > 0$ continuous in (t, φ) such that $|s_0(t, \varphi) - s_0(\bar{t}, \psi)| < \varepsilon$, $|\mu_0(t, \varphi) - \mu_0(\bar{t}, \psi)| < \varepsilon$, $\rho(t, \varphi, \mu, s) - \rho(\bar{t}, \psi, \mu, s)| < \varepsilon$ for $|t - \bar{t}| < d(t, \varphi, \varepsilon)$, $|\varphi - \psi| < d(t, \varphi, \varepsilon)$. Therefore, for ε sufficiently small, $\bar{s}_0(\bar{t}, \psi) > \bar{s}_0$, $\mu_0(\bar{t}, \psi) > \bar{\mu}_0$, $\rho(\bar{t}, \psi, \mu, s) < 1 - \nu$ for $|t - \bar{t}| < d(t, \varphi, \varepsilon)$, $|\psi - \varphi| < d(t, \varphi, \varepsilon)$, $(t, \varphi) \in W$, $0 \leq s \leq \bar{s}_0$, $0 \leq \mu \leq \bar{\mu}_0$. Since W is compact, there is a $d_0 > 0$ such that these same inequalities hold for $|t - \bar{t}| < d_0$, $|\psi - \varphi| < d_0$, $(t, \varphi) \in W$. From the hypothesis on the f_k, there is an open neighborhood $V = V(W)$ and a $M > 0$ such that $|f_k(t, \psi)| \leq M$, $(t, \psi) \in V$, $k = 0, 1, 2, \ldots$. Choose $d_0 > 0$ so small that $(\bar{t}, \psi) \in V$ if $|t - \bar{t}| < d_0$, $|\psi - \varphi| < d_0$, $(t, \varphi) \in W$.

For any $\sigma \in R$, $\varphi \in C_A$ and any real numbers $\gamma > 0$, $\delta > 0$, define $F(\gamma, \delta) = F(\gamma, \delta, \sigma, \varphi)$ as in the proof of Theorem 4.1. From the above construction of V, there is an open neighborhood $V_1 \subset V$ of W and a $\gamma_0 > 0$, $\delta_0 > 0$ such that $F(\gamma_0, \delta_0, \sigma, \varphi) \subset V$ for any $(\sigma, \varphi) \in V_1$. Choose V_1 so that this is true and for any $\sigma \in [\tau, \infty)$ and any real numbers $\bar{\gamma}$, $\bar{\delta}$ define $\mathcal{S}(\bar{\gamma}, \bar{\delta}, \sigma)$ as in Lemma 4.1 and let $\tilde{\varphi}_k \in C_{E_\sigma \cup I_{\bar{\gamma}}}$ be defined by $\tilde{\varphi}_k(t) = \varphi_k(t)$, $t \in E_\sigma$, $\tilde{\varphi}_k(t) = \varphi_k(\sigma)$, $t \in I_{\bar{\gamma}}$. Suppose $2\bar{\delta} < \delta_0$ and choose $\bar{\gamma} \leq \gamma_0$ so that $|\mathcal{A}_t\tilde{\varphi}_0 - \mathcal{A}_\sigma\varphi_0| < \delta_0/2$, $|g_0(t, \mathcal{A}_t\tilde{\varphi}_0) - g_0(\sigma, \mathcal{A}_\sigma\varphi_0)| < \nu\bar{\delta}/2$, $t \in I_{\bar{\gamma}}$, $M\bar{\gamma} < \nu\bar{\delta}/2$. Since the φ_k, $k \geq 0$, are a compact set of C_{E_σ}, the $\tilde{\varphi}_k$, $k \geq 0$, form a compact set in $C_{E_\sigma \cup I_{\bar{\gamma}}}$, $\tilde{\varphi}_k \to \tilde{\varphi}_0$ as $k \to \infty$. Therefore, there is a $k_0 \geq 0$ such that $|\mathcal{A}_t\tilde{\varphi}_k - \mathcal{A}_\sigma\varphi_k| < \delta_0/2$, $t \in I_\gamma$, $k \geq k_0$. Thus, $(t, \mathcal{A}_t(y + \tilde{\varphi}_k)) \in F(\gamma_0, \delta_0)$ for $t \in I_\gamma$, $y \in \mathcal{S}(\bar{\gamma}, \bar{\delta}, \sigma)$. Since

$$
\begin{aligned}
|g_k(t, \mathcal{A}_t\tilde{\varphi}_k) - g_k(\sigma, \mathcal{A}_\sigma\varphi_k)| &\leq |g_k(t, \mathcal{A}_t\tilde{\varphi}_k) - g_0(t, \mathcal{A}_t\tilde{\varphi}_k)| + |g_0(t, \mathcal{A}_t\tilde{\varphi}_k) - g_0(t, \mathcal{A}_t\tilde{\varphi}_0| \\
&+ |g_0(t, \mathcal{A}_t\tilde{\varphi}_0) - g_0(\sigma, \mathcal{A}_\sigma\varphi_0)| + |g_0(\sigma, \mathcal{A}_\sigma\varphi_0) - g_0(\sigma, \mathcal{A}_\sigma\varphi_k)| \\
&+ |g_0(\sigma, \mathcal{A}_\sigma\varphi_k) - g_k(\sigma, \mathcal{A}_\sigma\varphi_k)|,
\end{aligned}
$$

the set $\tilde{\varphi}_k$, $k \geq 0$ is compact, $\tilde{\varphi}_k \to \tilde{\varphi}_0$ as $k \to \infty$, and $g_k(t, \varphi) \to g_0(t, \varphi)$ uniformly on compact sets, it follows that k_0 can also be chosen so that $|g_k(t, \mathcal{A}_t\tilde{\varphi}_k) - g_k(\sigma, \mathcal{A}_\sigma\varphi_k)| < \nu\bar{\delta}/2$, $t \in I_{\bar{\gamma}}$, $k \geq k_0$.

Now, define the operators T_k, S_k, $k = 0, 1, 2, \ldots$, taking $\mathcal{S}(\bar{\gamma}, \bar{\delta}, \sigma)$ into $C_{E_\sigma \cup I_{\bar{\gamma}}}$ by the relations

$$(T_k y)(t) = 0, \qquad t \in E_\sigma$$

$$(T_k y)(t) = g_k(t, \mathcal{C}_t(y + \tilde{\varphi}_k)) - g_k(\sigma, \mathcal{C}_\sigma \varphi_k), \qquad t \in I_{\bar{\gamma}}$$

and

$$(S_k y)(t) = 0, \qquad t \in E_\sigma,$$

$$(S_k y)(t) = \int_\sigma^t f_k(s, \mathcal{C}_s(y + \tilde{\varphi}_k))ds, \qquad t \in I_{\bar{\gamma}}$$

Since

$$|g_k(t, \mathcal{C}_t(y + \tilde{\varphi}_k)) - g_0(t, \mathcal{C}_t(y + \tilde{\varphi}_0)) + g_0(\sigma, \mathcal{C}_\sigma \varphi_0) - g_k(\sigma, \mathcal{C}_\sigma \varphi_k)|$$

$$\leq |g_k(t, \mathcal{C}_t(y + \tilde{\varphi}_k)) - g_0(t, \mathcal{C}_t(y + \tilde{\varphi}_k))| + |g_0(t, \mathcal{C}_t(y + \tilde{\varphi}_k)) - g_0(t, \mathcal{C}_t(y + \tilde{\varphi}_0))|$$

$$+ |g_0(\sigma, \mathcal{C}_\sigma \varphi_0) - g_0(\sigma, \mathcal{C}_\sigma \varphi_k)| + |g_0(\sigma, \mathcal{C}_\sigma \varphi_k) - g_k(\sigma, \mathcal{C}_\sigma \varphi_k)|$$

the $\tilde{\varphi}_k \to \tilde{\varphi}_0$ as $k \to \infty$, $g_k(t, \varphi) \to g_0(t, \varphi)$ uniformly on closed, bounded subsets of U and $g_0(t, \varphi)$ is uniformly continuous on closed, bounded subsets of U, it follows that $T_k y \to T_0 y$ as $k \to \infty$ uniformly for $y \in \mathcal{S}(\bar{\gamma}, \bar{\delta}, \sigma)$. Since $\varphi_k \to \tilde{\varphi}_0$ $f_k(t, \psi) \to f_0(t, \varphi)$ as $k \to \infty$, $\psi \to \varphi$ and the f_k are uniformly bounded on V, it follows from the LEBESQUE dominated convergence theorem that $S_k z \to S_0 y$ as $k \to \infty$, $z \to y$ for each $y \in \mathcal{S}(\bar{\gamma}, \bar{\delta}, \sigma)$. As in the proof of Theorem 4.1, the operators T_k are contractions and the S_k are uniformly compact with $T_k + S_k : \mathcal{S}(\bar{\gamma}, \bar{\delta}, \sigma) \to \mathcal{S}(\bar{\gamma}, \bar{\delta}, \sigma)$. Lemma 2.2, therefore, implies the existence of solutions $x_k(t)$ of (6.2) on $E_\sigma \cup I_{\bar{\gamma}}$ and $x_k(t) \to x_0(t)$ as $k \to \infty$ uniformly on $E_\sigma \cup I_{\bar{\gamma}}$. Due to the compactness of the set $\{(t, \mathcal{C}_t x_0) : t \in [\sigma, b]\}$, one completes the proof by successively stepping intervals of length $\bar{\gamma}$.

The above theorem on continuous dependence is satisfactory for many types of equations of neutral type, but is too restrictive for VOLTERRA integral equations. The next result will be applicable to VOLTERRA equations and requires the following

DEFINITION 6.2. - Suppose $(A, \alpha, \mathcal{C})$ is an hereditary structure and U is an open subset of $R \times C_A$. Suppose $g : U \to R^n$ is a continuous function and φ is an arbitrary element in C_{E_σ} with $(\sigma, \mathcal{C}_\sigma \varphi) \in U$. Let $I_\gamma = [\sigma, \sigma + \gamma]$, $B_\delta(\varphi) = \{z \in C_{E_\sigma \cup I_\gamma} : z(t) = \varphi(t),\ t \in E_\sigma,\ |z(t) - \varphi(\sigma)| \leq \delta,\ t \in I_\gamma\}$. Suppose δ, γ are chosen so that $(t, \mathcal{A}_t z) \in U$ for $t \in I_\gamma$, $z \in B_\delta(\varphi)$. For $(t, z) \in I_\gamma \times B_\delta(\varphi)$, the function $\tilde{g}: B_\delta(\varphi) \to C_{I_\gamma}$ defined by $\tilde{g}(z)(t) = g(t, \mathcal{A}_t z)$, $t \in I_\gamma$, $z \in B_\delta(\varphi)$, is a continuous map. We say g is a *compact mapping at* φ if $\tilde{g}$ takes every such $B_\delta(\varphi)$ into a relatively compact subset of C_{I_γ}. More precisely, the set $\tilde{g}B_\delta(\varphi)$ is bounded and for any $\delta > 0$, φ as above and $\varepsilon > 0$, there is a $d(\varepsilon, \delta, \varphi) > 0$ such that

$$|g(t, \mathcal{A}_t z) - g(t', \mathcal{A}_{t'} z)| < \varepsilon,$$

if $|t - t'| < d(\varepsilon, \delta, \varphi)$, $t, t' \in I_\gamma$, $z \in B_\delta(\varphi)$. If, in addition, $d(\varepsilon, \delta, \varphi)$ is continuous in $\varepsilon, \delta, \varphi$, we say simply that g is a compact mapping.

THEOREM 6.2. - *Suppose* $(A, \alpha, \mathcal{A})$ *is an hereditary structure,* U *is an open set in* $R \times C_A$, $g_k : U \to R^n$, $k = 0, 1, 2, \ldots$, *is continuous and compact,* g_0 *is nonatomic at zero and uniformly continuous on closed bounded subsets of* U, $g_k \to g_0$ *as* $k \to \infty$ *uniformly on closed bounded sets of* U, *and the* f_k *satisfy the conditions of Theorem* 6.1. *Also, let* $\sigma \in R$, $\varphi_k \in C_{E_\sigma}$, $(\sigma, \mathcal{A}_\sigma \varphi_k) \in U$, $k = 0, 1, 2, \ldots$, $\varphi_k \to \varphi_0$ *as* $k \to \infty$ *and let* $x_0 = x_0(\sigma, \varphi_0)$ *be a solution of* (6.2) *for* $k = 0$ *with initial value* φ_0 *at* σ. *If* x_0 *is defined on* $E_\sigma \cup [\sigma, b]$ *and is unique, then there is an integer* k_0 *and a solution* $x_k = x_k(\sigma, \varphi_k)$, $k \geq k_0$, *of* (6.2) *with initial value* φ_k *at* σ *defined on* $E_\sigma \cup [\sigma, b]$ *and* $x_k(t) \to x_0(t)$ *uniformly on* $E_\sigma \cup [\sigma, b]$.

REMARK. - In the proof, it will be clear that the g_k for $k > 0$ need only be compact at φ_k.

PROOF. - The set $W = \{(t, \mathcal{A}_t x_0), \sigma \leq t \leq b\}$ is a compact subset of U. Using g_0 and the f_k, $k \geq 0$, rather than the g_k, f_k, $k \geq 0$, as in the proof of Theorem 6.1, one can construct open neighborhoods $V_1 \subset V$ of W and find a $\gamma_0 > 0$, $\delta_0 > 0$ so that $F(\gamma_0, \delta_0, \sigma, \varphi) \subset V$ for any $(\sigma, \varphi) \in V_1$. For any $\sigma \in R$ and any real numbers $\bar{\gamma}$, $\bar{\delta}$, define $\mathcal{S}(\bar{\gamma}, \bar{\delta}, \sigma)$ as in Lemma 4.1 and let $\tilde{\varphi}_k(t) = \varphi_k(t)$, $t \in E_\sigma$, $\tilde{\varphi}_k(t) = \varphi_k(\sigma)$, $t \in I_{\bar{\gamma}}$. Suppose $2\bar{\delta} < \bar{\delta}_0$ and choose $\bar{\gamma} \leq \gamma_0$ so that $|\mathcal{A}_t \tilde{\varphi}_0 - \mathcal{A}_\sigma \varphi_0| < \delta_0/2$, $|g(t, \mathcal{A}_t \tilde{\varphi}_0) - g(\sigma, \mathcal{A}_\sigma \varphi_0| < \nu \bar{\delta}/2, t \in I_{\bar{\gamma}}$, $M\bar{\gamma} < \nu\bar{\delta}/4$. If $\varphi_k \to \varphi_0$, then there is a $k_0 \geq 0$ such that $|\mathcal{A}_t \tilde{\varphi}_k - \mathcal{A}_\sigma \varphi_k| < \delta_0/2$. Thus, $(t, \mathcal{A}_t(y + \tilde{\varphi}_k)) \in F(\gamma_0, \delta_0)$ for $t \in I_{\bar{\gamma}}$, $y \in \mathcal{S}(\bar{\gamma}, \bar{\delta}, \sigma)$. Also, the hypotheses on the g_k imply that k_0 may also be chosen so that $|g_k(t, \psi) - g_0(t, \psi)| < \bar{\delta}/8$ for $(t, \psi) \in V_1$, $k \geq k_0$.

Define the operators T_k, S_k taking $\mathcal{S}(\bar{\gamma}, \bar{\delta}, \sigma)$ into $C_{E_\sigma \cup I_{\bar{\gamma}}}$ by the relations

$$(T_k y)(t) = 0 \qquad t \in E_\sigma,$$

$$(T_k y)(t) = g_0(t, \mathcal{A}_t(y + \tilde{\varphi})) - g_0(\sigma, \mathcal{A}_\sigma \varphi_k), \qquad t \in I_{\bar{\gamma}},$$

$$S_k y = \tilde{S}_k y + \tilde{\tilde{S}}_k y,$$

$$(\tilde{S}_k y)(t) = 0, \qquad t \in E_\sigma,$$

$$(\tilde{S}_k y)(t) = \int_\sigma^t f_k(s, \mathcal{C}_s(y + \tilde{\varphi}_k))ds, \; t \in I_{\bar{\gamma}}^-,$$

$$(\tilde{\tilde{S}}_k y)(t) = 0, \qquad t \in E_\sigma$$

$$(\tilde{\tilde{S}}_k y)(t) = g_k(t, \mathcal{C}_t(y + \tilde{\varphi}_k)) - g_0(t, \mathcal{C}_t(y + \tilde{\varphi}_k)) - g_k(\sigma, \mathcal{C}_\sigma \varphi_k) + g_0(\sigma, \mathcal{C}_\sigma \varphi_k), \qquad t \in I_{\bar{\gamma}}^-$$

As in the proof of Theorem 6.1, the hypothesis on g_0 implies that each of the operators T_k is a contraction for $k \geq k_0$, $T_k y \to T_0 y$ as $k \to \infty$ uniformly for $y \in \mathcal{S}(\bar{\gamma}, \bar{\delta}, \sigma)$. Also, the $\tilde{S}_k$ are uniformly compact with $\tilde{S}_k \to S_0$ as $k \to \infty$. By hypothesis, each $\tilde{\tilde{S}}$ is compact. To prove uniform compactness, observe first that $|\tilde{\tilde{S}}_k y| \leq \nu\bar{\delta}/4$ for all $k \geq k_0$. Furthermore, for any $\varepsilon > 0$, the hypotheses on the g_k imply there is a $k_1 = k_1(\varepsilon) \geq k_0$ such that

$$| g_k(t, \mathcal{C}_t(y + \tilde{\varphi}_k)) - g_0(t, \mathcal{C}_t(y + \tilde{\varphi}_k)) | < \varepsilon/3$$

for $k \geq k_1$. Since g_0 is compact, and the set φ_k, $k \geq 0$ is a compact subset of E_σ, for this same ε, there is a $d = d(\varepsilon, \bar{\delta}) > 0$, independent of k, such that

$$| g_0(t, \mathcal{C}_t(y + \tilde{\varphi}_k)) - g_0(t', \mathcal{C}_{t'}(y + \tilde{\varphi}_k) | < \varepsilon/3$$

for all $t, t' \in I_{\bar{\gamma}}^-$, $|t - t'| \leq d$, $k \geq 0$. Consequently

$$| g_k(t, \mathcal{C}_t(y + \tilde{\varphi}_k)) - g_k(t', \mathcal{C}_{t'}(y + \tilde{\varphi}_k)) | < \varepsilon$$

for $k \geq k_1(\varepsilon)$, $|t - t'| \leq d$, $t, t' \in I_{\bar{\gamma}}^-$. Since each g_k is assumed to be compact, it follows that we can further restrict d so that the above inequality holds for all $k \geq k_0$. This proves the uniform compactness of the operators $\tilde{\tilde{S}}_k$. It is clear that $\tilde{\tilde{S}}_k \to 0$ as $k \to \infty$. Finally, the constants have been chosen in such a way that $T_k + S_k : \mathcal{S}(\bar{\gamma}, \bar{\delta}, \sigma) \to \mathcal{S}(\bar{\gamma}, \bar{\delta}, \sigma)$ and Lemma 2.2 implies the existence of solution $x_k(t)$ of (6.2) on $E_\sigma \cup I_{\bar{\gamma}}^-$ as well as the fact that $x_k(t) \to x_0(t)$ uniformly on $E_\sigma \cup I_{\bar{\gamma}}^-$. The compactness of W permits one to successively step intervals of length $\bar{\gamma}$ until the interval $[\sigma, b]$ is covered. This proves the theorem.

Theorem 5 of MILLER and SELL [5] on VOLTERRA integral equations is a special case of Theorem 6.2 in two respects. First, Theorem 6.2 involves systems much more general than VOLTERRA integral equations. Secondly, the nonatomic property at zero is only imposed on g_0 and uniqueness is only assumed for equation (6.2) with $k = 0$. If $f_0 \equiv 0$, the nonatomic property at zero of g_0 implies uniqueness of the solution of (6.2) for $k = 0$.

THEOREM 6.3. - *Suppose* $(A, \alpha, \mathcal{A})$ *is an hereditary structure,* U *is an open set in* $R \times C_A$, λ_0 *is a real or complex number,* Λ *is a neighborhood of* λ_0, $g: U \times \Lambda \to R^n$, $g(t, \psi, \lambda)$ *is continuous in* (t, ψ, λ) *at* $\lambda = \lambda_0$, g *is equi-nonatomic at zero,* $g(\cdot, \cdot, \lambda_0)$ *is uniformly continuous on closed bounded subsets of* U, $g(t, \psi, \lambda) \to g(t, \psi, \lambda_0)$ *uniformly on closed bounded subsets of* U, $f: U \times \Lambda \to R^n$, $f(t, \psi, \lambda)$ *is continuous in* (t, ψ, λ) *at* $\lambda = \lambda_0$, *the continuity in* ψ *being uniform with respect to* $\lambda \in \Lambda$. *For any* $\sigma \in R$, $\varphi \in E_\sigma$, $(\sigma, \mathcal{A}_\sigma\varphi) \in U$, $\lambda \in \Lambda$, *let* $x(\sigma, \varphi, \lambda)$ *be a solution of the equation*

$$(6.3) \qquad \frac{d}{dt}[D(t, \lambda)\mathcal{A}_t x] = f(t, \mathcal{A}_t x, \lambda)$$

$$D(t, \lambda)\psi = \psi(0) - g(t, \psi, \lambda)$$

with initial value φ *at* σ. *If the solution* $x(\sigma, \varphi, \lambda_0)$ *of* (6.3) *is unique and is defined on* $E_\sigma \cup [\sigma, b]$, *then there is a* $\zeta > 0$ *such that* (6.3) *has a solution* $x(\sigma, \bar{\varphi}, \lambda)$ *defined on* $E_\sigma \cup [\sigma, b]$ *for* $|\bar{\varphi} - \varphi| < \zeta$, $|\lambda - \lambda_0| < \zeta$, *and* $x(\sigma, \bar{\varphi}, \lambda)(t)$ *is continuous in* $(t, \bar{\varphi}, \lambda)$ *at* (t, φ, λ_0), $t \in [\sigma, b]$.

PROOF. - Theorem 6.1 implies the existence of the ζ in the statement of Theorem 6.3 and the continuity of $x(\sigma, \bar{\varphi}, \lambda)(t)$ in $(\bar{\varphi}, \lambda)$ at (φ, λ_0) uniformly with respect to t. Since $x(\sigma, \bar{\varphi}, \lambda)$ is a continuous function of t for $t \in [\sigma, b]$, the conclusion of Theorem 6.3 follows.

An analogous result using Theorem 6.2 rather than Theorem 6.1 could also be stated. The next result deals with the continuity of the solutions in the initial time σ.

THEOREM 6.4. - *Suppose* $(A, \alpha, \mathcal{A})$ *is an hereditary structure with* $\alpha(t, \theta) = t + \theta$, $\theta \in A$. *If the conditions of Theorem* 6.3 *are satisfied, then there is a* $\zeta > 0$ *such that* (6.3) *has a solution* $x(\bar{\sigma}, \bar{\varphi}, \lambda)$ *defined on* $E_{\bar{\sigma}} \cup [\bar{\sigma}, b]$ *for* $|\bar{\sigma} - \sigma| < \zeta$, $|\varphi - \bar{\varphi}| < \zeta$, $|\lambda - \lambda_0| < \zeta$, *and* $\mathcal{A}_t x(\bar{\sigma}, \bar{\varphi}, \lambda)$ *is continuous in* $(t, \bar{\sigma}, \bar{\varphi}, \lambda)$ *at* λ_0.

PROOF. - The special form $\alpha(t, \theta) = t + \theta$, $\theta \in A$, permits one to repeat the proof of the basic existence theorem by getting a fixed point of a set which is independent of σ. In fact, for any $\bar{\gamma} \geq 0$, $\bar{\delta} \geq 0$, let

$$\mathcal{S}(\bar{\gamma}, \bar{\delta}) = \{ y \in C_{A \cup [0, \bar{\gamma}]} : y(t) = 0, \ t \in A, \ |y(t)| \leq \bar{\delta}, \ t \in [0, \bar{\gamma}] \}.$$

For any $\sigma \in R$, $\varphi \in C_{E_\sigma}$, $(\sigma, \mathcal{A}_\sigma\varphi) \in C_A$, define $\tilde{\varphi} \in C_{A \cup [0, \bar{\gamma}]}$ by $\tilde{\varphi}(t) = \varphi(\sigma + t)$, $t \in A$, $\tilde{\varphi}(t) = \varphi(\sigma)$, $t \in [0, \bar{\gamma}]$ and choose $\gamma \geq 0$, $\delta \geq 0$ as in Lemma 4.1. Define the transformations T, S, taking $\mathcal{S}(\bar{\gamma}, \bar{\delta})$ into $C_{A \cup [0, \bar{\gamma}]}$ by

$$(Tz)(t) = 0, \qquad t \in A,$$

$$(Tz)(t) = g(t + \sigma,\ \mathcal{A}_t(z + \tilde{\varphi})) - g(\sigma,\ \mathcal{A}_\sigma\varphi), \qquad t \in [0,\ \bar{\gamma}]$$

and

$$(Sz)(t) = 0, \qquad t \in A,$$

$$(Sz)(t) = \int_0^t f(s + \sigma,\ \mathcal{A}_s(z + \tilde{\varphi}))ds.$$

Suppose z^* is a fixed point of $T + S$ in $\mathcal{S}(\bar{\gamma},\ \bar{\delta})$ and let $x^*(t + \sigma) = z^*(t) + \tilde{\varphi}(t)$ for $t \in A \cup [0,\ \bar{\gamma}]$. Since $\alpha(t,\ \theta) = \alpha(t + \theta)$ for all $\theta \in A$

$$x^*(t + \sigma + \theta) = z^*(t + \theta) + \tilde{\varphi}(t + \theta),\ \theta \in A,\ t \in [0,\ \bar{\gamma}],$$

implies that $\mathcal{A}_{t+\sigma}x^* = \mathcal{A}_t z^* + \mathcal{A}_t\tilde{\varphi}$. It is now clear that x^* is a solution of (6.3) with initial value φ at σ. Conversely, any solution of (6.3) such that $x^*(\cdot + \sigma) - \tilde{\varphi}(\cdot) \in \mathcal{S}(\gamma,\ \bar{\delta})$ must be a fixed point of $S + T$.

Theorem 6.1 can now be generalized to take into account variations in σ; namely, one can also allow in Theorem 6.1 a sequence $\sigma_k \in R$ converging to σ_0 as $k \to \infty$. The proof of Theorem 6.4 is now the same as the proof of Theorem 6.3.

7. - Extension of the concept of a differential equation.

In Section 3, we defined an hereditary differential equation for continuous functions $f : E \times C_A \to R^n$. On the other hand, it was then shown that the initial value problem was equivalent to

$$(7.1) \qquad x(t) = \varphi(t) \qquad t \in E_\sigma$$

$$D(t)\mathcal{A}_t x = D(\sigma)\mathcal{A}_\sigma\varphi + \int_\sigma^t f(s,\ \mathcal{A}_s x)ds, \qquad t \geq \sigma.$$

This equation clearly will be meaningful for a more general class of functions f if it is not required that $D(t)\mathcal{A}_t x$ have a continuous first derivative. The purpose of this section is to generalize the well known concept of CARATHÉODORY conditions for ordinary differential equations so as to apply to (7.1).

Suppose A is a compact subset of $(-\infty,\ \infty)$ and U is an open subset of $R \times C_A$. A function $f : U \to R^n$ is said to satisfy the *Carathéodory condition on U* if $f(t,\ \varphi)$ is measurable in t for each fixed φ, continuous in φ for each fixed t, and for any fixed $(t,\ \varphi) \in U$, there is a neighborhood $V(t,\ \varphi)$ of $(t,\ \varphi)$ and a LEBESGUE integrable function m such that

$$(7.2) \qquad |f(s, \psi)| \leq m(s), \qquad (s, \psi) \in V(t, \varphi).$$

If $f: U \to R^n$ is continuous, it is easy to see that f satisfies the CARATHÉODORY condition of U. Therefore, the theory of (7.1) for f in this more general class of functions generalizes the previous theory.

If f satisfies the CARATHÉODORY condition on a set U, $\sigma \in R$ $\varphi \in C_{E_\sigma}$, $(\sigma, \mathcal{E}_\sigma\varphi) \in U$, we say a function $x = x(\sigma, \varphi)$ is a *solution of* (7.1) *with initial value* φ *at* σ if there exists a $\gamma > 0$ such that $x \in C_{E_\sigma \cup [\sigma, \sigma+\gamma]}$, $x(t) = \varphi(t)$, $t \in E_\sigma$ and $D(t)\mathcal{E}_t x$ satisfies (6.2) almost everywhere for $t \in [\sigma, \sigma + \gamma]$.

Using essentially the same arguments as in the previous sections, one can extend all of the results to the case where f satisfies the CARATHÉODORY condition. Of course, in the theorems corresponding to Theorems 6.1 and 6.2 on continuous dependence, all f_k should satisfy the CARATHÉODORY conditions, and condition (6.1) should be replaced by the following: For any compact set W in U, there is an open neighborhood $V(W)$ of W and a LEBESGUE integrable function M such that the sequence of functions f_k, $k = 0, 1, 2, \ldots$, satisfy

$$|f_k(s, \psi)| \leq M(s), \qquad (s, \psi) \in V(W)$$

$$k = 0, 1, 2, \ldots .$$

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