sciendo

TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/tmmp-2023-0019
Tatra Mt. Math. Publ. **84** (2023), 111–138

EXISTENCE, UNIQUENESS AND ULAM-HYERS-RASSIAS STABILITY OF DIFFERENTIAL COUPLED SYSTEMS WITH RIESZ-CAPUTO FRACTIONAL DERIVATIVE

Abdelkrim Salim[1,2] — Jamal Eddine Lazreg[1] — Mouffak Benchohra[1]

[1]Laboratory of Mathematics, Djillali Liabes University, Sidi Bel-Abbes, ALGERIA

[2]Faculty of Technology, Hassiba Benbouali University, Chlef, ALGERIA

ABSTRACT. This article deals with the existence, uniqueness and Ulam-Hyers-Rassias stability results for a class of coupled systems for implicit fractional differential equations with Riesz-Caputo fractional derivative and boundary conditions. We will employ the Banach's contraction principle as well as Schauder's fixed point theorem to demonstrate our existence results. We provide an example to illustrate the obtained results.

1. Introduction

Because of its importance in the modeling and scientific understanding of natural processes, fractional calculus has long been an essential study topic in functional space theory. Several applications in viscoelasticity and electrochemistry have been studied. Non-integer derivatives of fractional order have been utilized successfully to generalize fundamental natural principles. For more details, we recommend [1–4, 7, 14, 18, 19, 26–29].

There are several kinds of fractional derivatives, such as Riemann-Liouville fractional derivative (1847), Caputo derivative (1967), Hilfer derivative (2000), Hadamard derivative (1892) and Caputo-Fabrizio (2015), etc. The current statuses of many processes began in the past and rely on their future development, for example, stock price option. Another example is the application to

2020 Mathematics Subject Classification: 26A33, 34A08, 34A12.
Keywords: Riesz-Caputo fractional derivative, implicit problem, existence, coupled system, fixed point, Ulam stability.

the anomalous diffusion problem, where the Riesz derivative indicates nonlocality and is used to represent the dependency of diffusion concentration on path. The majority of the current work on fractional differential equations has been on Riemann-Liouville and Caputo fractional derivatives, which are one-sided fractional operators that solely represent the past or future memory effect. Fortunately, the Riesz derivative is a two-sided fractional operator that includes both left and right derivatives, allowing it to capture both past and future memory effects. This capability is useful for fractional modeling on a finite domain in particular. See [10, 11, 13] for more details.

The authors of [10] studied the existence of solution for the following boundary value problem:

$$\begin{cases} {}_0^{RC}D_{\varkappa}^{\alpha} y(\theta) = g\big(\theta, y(\theta)\big), & \theta \in \Theta := [0,\varkappa], \\ y(0) = y_0, \quad y(\varkappa) = y_{\varkappa}, \end{cases}$$

where ${}_0^{RC}D_{\varkappa}^{\alpha}$ is a Riesz-Caputo derivative of order $0 < \alpha \leq 1$, $g : \Theta \times \mathbb{R} \to \mathbb{R}$ a continuous function and $y_0 \in \mathbb{R}$. Their arguments are based on Leray-Schauder fixed point theorem, and Schauder fixed point theorem.

In [20], Li and Wang discussed the following fractional problem:

$${}_0^{RC}D_1^{\gamma} y(t) = f\big(t, y(t)\big), \quad t \in [0,1], \quad 0 < \gamma \leq 1,$$

$$y(0) = a, \quad y(1) = by(\eta),$$

where ${}_0^{RC}D_1^{\gamma}$ is the Riesz Caputo derivative, $f \in C\big([0,1]\times\ [0,+\infty),\ [0,+\infty)\big)$, $0 < \eta < 1$, $a > 0$, $0 < b < 2$. They found the positive solutions by applying the technique of monotone iterative.

Naas *et al.* [23] investigated the existence and uniqueness results of the following fractional differential equation with the Riesz-Caputo derivative:

$$\begin{cases} {}_0^{RC}D_T^{\vartheta}\varkappa(t) + \mathfrak{F}\left(t, \varkappa(t), {}_0^{RC}D_T^{\varsigma}\varkappa(t)\right) = 0, t \in \mathcal{J} := [0,T], \\ \varkappa(0) + \varkappa(T) = 0, \quad \mu\varkappa'(0) + \sigma\varkappa'(T) = 0, \end{cases}$$

where $1 < \vartheta \leq 2$ and $, 0 < \varsigma \leq 1, {}_0^{RC}D_T^{\kappa}$ is the Riesz-Caputo fractional derivative of order $\kappa \in \{\vartheta, \varsigma\}$, $\mathfrak{F} : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, is a continuous function, and μ, σ are nonnegative constants with $\mu > \sigma$. The existence and uniqueness of solutions for their problem are demonstrated with the Riesz-Caputo derivatives via Banach's, Schaefer's, and Krasnoselskii's fixed point theorems.

While determining the precise solution of differential equations is difficult or impossible in many contexts, such as nonlinear analysis and optimization, we explore approximate solutions. It should be noted that only steady approximations are allowed. For this reason, many techniques to stability analysis are used. Mathematician Ulam originally highlighted the stability problem in functional equations in a 1940 presentation at Wisconsin University. S. M. Ulam introduced the following challenge: "Under what conditions does an additive mapping exist near an approximately additive mapping ?" [34]. The following year, in [15], Hyers provided an answer to Ulam's problem for additive functions defined on Banach spaces. In 1978, Rassias [24] demonstrated the existence of unique linear mappings near approximate additive mappings, generalizing Hyers' findings. Several research articles in the literature address the Ulam stabilities of various types of differential and integral equations, see [19, 21, 31] and the references therein.

Many researchers devoted their research work to the study of various kind of Ulam stabilities for some classes of coupled systems of fractional differential equations. For details, we refer the reader to see [5, 16, 32, 35]. In the papers [8,9,12,30,31], the authors addressed the existence, stability, and uniqueness of solutions for diverse problems with fractional differential equations using various fractional derivatives and different types of conditions.

In [6], Ali *et al.* investigated the following coupled system with impulsive and $(m+2)$-point boundary conditions:

$$\begin{cases} {}_0^C\mathrm{D}_{t_j}^{\alpha}\xi(t) = \Phi\left(t, \mu(t), {}_0^C\mathrm{D}_{t_j}^{\alpha}\xi(t)\right), & t \in [0,1], t \neq t_j, j = 1,2,\ldots,m, \\ {}_0^C\mathrm{D}_{t_i}^{\beta}\mu(t) = \Psi\left(t, \xi(t), {}_0^C\mathrm{D}_{t_i}^{\beta}\mu(t)\right), & t \in [0,1], t \neq t_i, i = 1,2,\ldots,n, \\ \xi(0) = h(\xi), \quad \xi(1) = g(\xi) \quad \text{and} \quad \mu(0) = \kappa(\mu), \quad \mu(1) = f(\mu), \\ \Delta\xi\left(t_j\right) = I_j\big(\xi(t_j)\big), \quad \Delta\xi'(t_j) = \bar{I}_j\big(\xi(t_j)\big), \quad j = 1,2,\ldots,m, \\ \Delta\mu\left(t_i\right) = I_i\big(\mu(t_i)\big), \quad \Delta\mu'(t_i) = \bar{I}_i\big(\mu(t_i)\big), \quad i = 1,2,\ldots,n, \end{cases}$$

where $1 < \alpha, \beta \leq 2, \Phi, \Psi : [0,1] \times \mathrm{R} \times \mathrm{R} \to \mathrm{R}$, and $g, h; f, \kappa : C(\mathrm{J}, \mathrm{R}) \to \mathrm{R}$ are continuous functions. The authors used the Schaefer fixed point and Banach contraction theorems to obtain conditions for the existence and uniqueness of positive solutions and discussed the Hyers–Ulam stability of the concerned solutions.

Wang *et al.* [35] studied the following implicit coupled system involving Caputo fractional-order derivative:

$$\begin{cases} {}^cD^p u(t) - \alpha\big(t, y(t), {}^cD^p u(t)\big) - \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\sigma-1} g\big(s, y(s), {}^cD^p u(s)\big)ds = 0, \\ {}^cD^q y(t) - \chi\big(t, u(t), {}^cD^q y(t)\big) - \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\sigma-1} f\big(s, u(s), {}^cD^p y(s)\big)ds = 0, \\ u(t)|_{t=0} = -\, u(t)|_{t=T}\,, \; {}^cD^r u(t)|_{t=0} = -\, {}^cD^r u(t)|_{t=T}\,, \\ y(t)|_{t=0} = -\, y(t)|_{t=T}\,, \; {}^cD^\omega y(t)|_{t=0} = -\, {}^cD^\omega y(t)|_{t=T}\,, \end{cases}$$

where $1 < p, q \leq 2, 0 \leq r, \omega \leq 2, \sigma, \delta > 0$ and $t \in \mathcal{J} = [0,T], T > 0$. The functions $\alpha, \chi, g, f : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous. They studied the existence, uniqueness and various kinds of stability such as Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability, and generalized Ulam--Hyers-Rassias stability.

Motivated by the mentioned works, in this paper, we investigate the existence, uniqueness and Ulam stability of the following implicit coupled system:

$$\begin{cases} {}_0^{RC}D_{\varkappa}^{\alpha_1}x(\theta) &= \varphi_1\left(\theta, x(\theta), y(\theta),\ {}_0^{RC}D_{\varkappa}^{\alpha_1}x(\theta),\ {}_0^{RC}D_{\varkappa}^{\alpha_2}y(\theta)\right), \\ {}_0^{RC}D_{\varkappa}^{\alpha_2}y(\theta) &= \varphi_2\left(\theta, x(\theta), y(\theta),\ {}_0^{RC}D_{\varkappa}^{\alpha_1}x(\theta),\ {}_0^{RC}D_{\varkappa}^{\alpha_2}y(\theta)\right), \end{cases} \tag{1}$$

where $\theta \in \Theta := [0, \varkappa]$, with the boundary conditions

$$\begin{cases} \beta_1 x(0) + \beta_2 x(\varkappa) &= \beta_3, \\ \delta_1 y(0) + \delta_2 y(\varkappa) &= \delta_3, \end{cases} \tag{2}$$

where for $i = 1, 2$, ${}_0^{RC}D_{\varkappa}^{\alpha_i}$ represent the Riesz-Caputo derivatives of order

$$0 < \alpha_i \leq 1, \quad \delta_1, \delta_2, \delta_3, \beta_1, \beta_2, \beta_3 \in \mathbb{R},$$

where

$$\delta_1 + \delta_2 \neq 0 \quad \text{and} \quad \beta_1 + \beta_2 \neq 0, \quad \varphi_i : \Theta \times \mathbb{R}^4 \to \mathbb{R}$$

are given functions.

The following is how the current paper is arranged. In Section 2, we present certain notations and review some preliminary information on the Riesz-Caputo fractional derivative and auxiliary results. Section 3 presents, in the first part, two existence and uniqueness results to the system (1)–(2) based on the Banach contraction principle and Schauder's fixed point theorem are given. The Ulam-Hyers-Rassias Stability for our problem is discussed in the next subsection. Finally, we provide an example to demonstrate the application of our study results.

2. Preliminaries

In this section, we introduce some notations, definitions, and preliminary facts which are used throughout this paper.

We denote by $C(\Theta, \mathbb{R})$ the Banach space of all continuous functions from Θ to $\mathbb{R}$, with the norm

$$\|\xi\|_\infty = \sup\{|\xi(\theta)| : \theta \in \Theta\}.$$

Now, let us consider the Banach space

$$\mathcal{F} := C(\Theta, \mathbb{R}) \times C(\Theta, \mathbb{R}),$$

with the norm

$$\|(x, y)\|_{\mathcal{F}} = \max\left\{\|x\|_\infty, \|y\|_\infty\right\}.$$

Definition 2.1 ([17]). Let $\alpha > 0$. The left and right Riemann-Liouville fractional integrals of a function $\varphi \in C(\Theta, \mathbb{R})$ of order α are given respectively by

$$ {}_0I_\theta^\alpha \varphi(\theta) = \frac{1}{\Gamma(\alpha)} \int_0^\theta (\theta - \varrho)^{\alpha-1} \varphi(\varrho) d\varrho, $$

and

$$ {}_\theta I_\varkappa^\alpha \varphi(\theta) = \frac{1}{\Gamma(\alpha)} \int_\theta^\varkappa (\varrho - \theta)^{\alpha-1} \varphi(\varrho) d\varrho. $$

Definition 2.2 ([17]). Let $\alpha > 0$. The Riesz fractional integral of a function $\varphi \in C(\Theta, \mathbb{R})$ of order α is defined by

$$ \begin{aligned} {}_0I_\varkappa^\alpha \varphi(\theta) &= \frac{1}{\Gamma(\alpha)} \int_0^\varkappa |\theta - \varrho|^{\alpha-1} \varphi(\varrho) d\varrho \\ &= {}_0I_\theta^\alpha \varphi(\theta) + {}_\theta I_\varkappa^\alpha \varphi(\theta), \end{aligned} $$

where ${}_0I_\theta^\alpha$ and ${}_\theta I_\varkappa^\alpha$ are the left and right fractional integrals of Riemann-Liouville.

Definition 2.3 ([17]). Let $\alpha \in (n, n+1]$, $n \in \mathbb{N}$. The left and right Caputo fractional derivatives of a function $\varphi \in C^{n+1}(\Theta, \mathbb{R})$ of order α are given respectively by

$$ {}_0^C D_\theta^\alpha \varphi(\theta) = \frac{1}{\Gamma(n+1-\alpha)} \int_0^\theta (\theta - \varrho)^{n-\alpha} \varphi^{(n+1)}(\varrho) d\varrho, $$

and

$$ {}_\theta^C D_\varkappa^\alpha \varphi(\theta) = \frac{(-1)^{n+1}}{\Gamma(n+1-\alpha)} \int_\theta^\varkappa (\varrho - \theta)^{n-\alpha} \varphi^{(n+1)}(\varrho) d\varrho. $$

Definition 2.4 ([17]). Let $\alpha \in (n, n+1]$, $n \in \mathbb{N}$. The Riesz-Caputo fractional derivative of a function $\varphi \in C^{n+1}(\Theta, \mathbb{R})$ of order α is given by

$$ \begin{aligned} {}_0^{RC} D_\varkappa^\alpha \varphi(\theta) &= \frac{1}{\Gamma(n+1-\alpha)} \int_0^\varkappa |\theta - \varrho|^{n-\alpha} \varphi^{(n+1)}(\varrho) d\varrho \\ &= \frac{1}{2} \left({}_0^C D_\theta^\alpha \varphi(\theta) + (-1)^{n+1} {}_\theta^C D_\varkappa^\alpha \varphi(\theta) \right), \end{aligned} $$

where ${}_0^C D_\theta^\alpha$ is the left Caputo derivative and ${}_\theta^C D_\varkappa^\alpha$ is the right one. If we take $0 < \alpha \leq 1$ and $\varphi \in C(\Theta, \mathbb{R})$, we obtain

$$ {}_0^{RC} D_\varkappa^\alpha \varphi(\theta) = \frac{1}{2} \left({}_0^C D_\theta^\alpha \varphi(\theta) - {}_\theta^C D_\varkappa^\alpha \varphi(\theta) \right). $$

Lemma 2.5 ([17]). *If $\xi \in C^{n+1}(\Theta, \mathbb{R})$ and $\alpha \in (n, n+1]$, then we have*

$$ {}_0I_\theta^\alpha \, {}_0^C D_\theta^\alpha \xi(\theta) = \xi(\theta) - \sum_{k=0}^{n} \frac{\xi^{(k)}(0)}{k!}(\theta - 0)^k, $$

and

$$ {}_\theta I_\varkappa^\alpha \, {}_\theta^C D_\varkappa^\alpha \xi(\theta) = (-1)^{n+1} \left[\xi(\theta) - \sum_{k=0}^{n} \frac{(-1)^k \xi^{(k)}(\varkappa)}{k!}(\varkappa - \theta)^k \right]. $$

Consequently, we may have

$$ {}_0I_\varkappa^\alpha \, {}_0^{RC} D_\varkappa^\alpha \xi(\theta) = \frac{1}{2}\left({}_0I_\theta^\alpha \, {}_0^C D_\theta^\alpha \xi(\theta) + (-1)^{n+1} \, {}_\theta I_\varkappa^\alpha \, {}_\theta^C D_\varkappa^\alpha \xi(\theta) \right). $$

In particular, if $0 < \alpha \leq 1$, then we obtain

$$ {}_0I_\varkappa^\alpha \, {}_0^{RC} D_\varkappa^\alpha \xi(\theta) = \xi(\theta) - \frac{1}{2}\big(\xi(0) + \xi(\varkappa)\big). $$

2.1. Some Fixed Point Theorems

Theorem 2.6 (Banach's fixed point theorem [33]). *Let E be a Banach space and $\mathcal{H} : E \longrightarrow E$ a contraction, i.e. there exists $k \in [0, 1)$ such that*

$$ \|\mathcal{H}(\xi_1) - \mathcal{H}(\xi_2)\| \leq k\|\xi_1 - \xi_2\|, \quad \text{for all } \xi_1, \xi_2 \in \mathbb{R}. $$

Then $\mathcal{H}$ has a unique fixed point.

Theorem 2.7 (Schauder's fixed point theorem [33]). *Let E be a Banach space, D a bounded, closed, convex subset of E, and $\mathcal{H} : D \longrightarrow D$ a compact and continuous map. Then $\mathcal{H}$ has at least one fixed point in D.*

3. Main Results

3.1. Existence and Uniqueness Results

Firstly, we provide the following result in order to convert our system (1)–(2) into a coupled system of fractional integral equations.

Theorem 3.1. *Let $0 < \alpha \leq 1$, and let $\varpi : \Theta \to \mathbb{R}$ be a continuous function. The boundary value problem*

$$ {}_0^{RC} D_\varkappa^\alpha y(\theta) = \varpi(\theta), \quad \theta \in \Theta, \tag{3} $$

$$ \delta_1 y(0) + \delta_2 y(\varkappa) = \delta_3, \tag{4} $$

has a unique solution given by

$$y(\theta) = \frac{\delta_3}{\delta_1+\delta_2} - \frac{\delta_1}{(\delta_1+\delta_2)\Gamma(\alpha)}\int_0^{\varkappa} \varrho^{\alpha-1}\varpi(\varrho)d\varrho - \frac{\delta_2}{(\delta_1+\delta_2)\Gamma(\alpha)}\int_0^{\varkappa} |\varkappa-\varrho|^{\alpha-1}\varpi(\varrho)d\varrho + \frac{1}{\Gamma(\alpha)}\int_0^{\varkappa} |\theta-\varrho|^{\alpha-1}\varpi(\varrho)d\varrho, \tag{5}$$

where $\delta_1, \delta_2, \delta_3 \in \mathbb{R}$ *and* $\delta_1+\delta_2 \neq 0$.

P r o o f. From Definition 2.2, Definition 2.4, and Lemma 2.5, we have

$${}_0I^{\alpha}_{\varkappa}\, {}^{RC}_{\ 0}D^{\alpha}_{\varkappa} y(\theta) = y(\theta) - \frac{1}{2}\big(y(0)+y(\varkappa)\big),$$

which implies that

$$\begin{aligned} y(\theta) &= \frac{1}{2}\big(y(0)+y(\varkappa)\big) + {}_0I^{\alpha}_{\varkappa}\varpi(\theta), \\ &= \frac{1}{2}\big(y(0)+y(\varkappa)\big) + \frac{1}{\Gamma(\alpha)}\int_0^{\varkappa} |\theta-\varrho|^{\alpha-1}\varpi(\varrho)d\varrho \\ &= \frac{1}{2}\big(y(0)+y(\varkappa)\big) + \frac{1}{\Gamma(\alpha)}\int_0^{\theta} (\theta-\varrho)^{\alpha-1}\varpi(\varrho)d\varrho \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{\theta}^{\varkappa} (\varrho-\theta)^{\alpha-1}\varpi(\varrho)d\varrho. \end{aligned}$$

For $\theta = 0$, we have

$$y(\varkappa) = y(0) - \frac{2}{\Gamma(\alpha)}\int_0^{\varkappa} \varrho^{\alpha-1}\varpi(\varrho)d\varrho.$$

Then,

$$y(\theta) = y(0) - \frac{1}{\Gamma(\alpha)}\int_0^{\varkappa} \varrho^{\alpha-1}\varpi(\varrho)d\varrho + \frac{1}{\Gamma(\alpha)}\int_0^{\varkappa} |\theta-\varrho|^{\alpha-1}\varpi(\varrho)d\varrho. \tag{6}$$

For $\theta = \varkappa$, we have

$$y(\varkappa) = y(0) - \frac{1}{\Gamma(\alpha)}\int_0^{\varkappa} \varrho^{\alpha-1}\varpi(\varrho)d\varrho + \frac{1}{\Gamma(\alpha)}\int_0^{\varkappa} |\varkappa-\varrho|^{\alpha-1}\varpi(\varrho)d\varrho.$$

By using the condition (4), we obtain

$$\delta_3 - \delta_1 y(0) = \delta_2 y(0) - \frac{\delta_2}{\Gamma(\alpha)} \int\limits_0^{\varkappa} \varrho^{\alpha-1} \varpi(\varrho) d\varrho + \frac{\delta_2}{\Gamma(\alpha)} \int\limits_0^{\varkappa} |\varkappa - \varrho|^{\alpha-1} \varpi(\varrho) d\varrho,$$

which implies

$$y(0) = \frac{\delta_3}{\delta_1+\delta_2} + \frac{\delta_2}{(\delta_1+\delta_2)\Gamma(\alpha)} \int\limits_0^{\varkappa} \varrho^{\alpha-1} \varpi(\varrho) d\varrho - \frac{\delta_2}{(\delta_1+\delta_2)\Gamma(\alpha)} \int\limits_0^{\varkappa} |\varkappa - \varrho|^{\alpha-1} \varpi(\varrho) d\varrho.$$

Then, replacing the value of $y(0)$ in (6), we obtain (5).

Reciprocally, taking $\theta = 0$ in (5), we get

$$y(0) = \frac{\delta_3}{\delta_1+\delta_2} + \frac{\delta_2}{(\delta_1+\delta_2)\Gamma(\alpha)} \int\limits_0^{\varkappa} \varrho^{\alpha-1} \varpi(\varrho) d\varrho - \frac{\delta_2}{(\delta_1+\delta_2)\Gamma(\alpha)} \int\limits_0^{\varkappa} |\varkappa - \varrho|^{\alpha-1} \varpi(\varrho) d\varrho,$$

and taking $\theta = \varkappa$, we get

$$y(\theta) = \frac{\delta_3}{\delta_1+\delta_2} - \frac{\delta_1}{(\delta_1+\delta_2)\Gamma(\alpha)} \int\limits_0^{\varkappa} \varrho^{\alpha-1} \varpi(\varrho) d\varrho + \frac{\delta_1}{(\delta_1+\delta_2)\Gamma(\alpha)} \int\limits_0^{\varkappa} |\varkappa - \varrho|^{\alpha-1} \varpi(\varrho) d\varrho.$$

Thus, we can obtain $\delta_1 y(0) + \delta_2 y(\varkappa) = \delta_3$, which implies that (2) is verified. Also, we can easily show by Lemma 2.5 that if y verifies equation (5), then it satisfies the equation (3). □

As a consequence of Theorem 3.1, we have the following result.

Lemma 3.2. *Let $i = 1, 2$, $0 < \alpha_i \le 1$, $\delta_1, \delta_2, \delta_3, \beta_1, \beta_2, \beta_3 \in \mathbb{R}$, where $\delta_1 + \delta_2 \neq 0$ and $\beta_1 + \beta_2 \neq 0$, $\varphi_i : \Theta \times \mathbb{R}^4 \to \mathbb{R}$ be continuous functions. Then $(x, y) \in \mathcal{F}$ satisfies the coupled system (1)–(2) if and only if (x, y) is the fixed point of the operator $\aleph : \mathcal{F} \to \mathcal{F}$ defined by:*

$$\aleph(x,y)(\theta) = \big(\aleph_1(x,y)(\theta), \aleph_2(x,y)(\theta)\big), \quad \theta \in \Theta, \tag{7}$$

where $\aleph_1$ *and* $\aleph_2$ *are the operators defined for* $\theta \in \Theta$*, as follows:*

$$\aleph_1(x,y)(\theta) = \frac{\beta_3}{\beta_1+\beta_2} - \frac{\beta_1}{(\beta_1+\beta_2)\Gamma(\alpha_1)} \int_0^{\varkappa} \varrho^{\alpha_1-1}\varpi_1(\varrho)d\varrho - \frac{\beta_2}{(\beta_1+\beta_2)\Gamma(\alpha_1)} \int_0^{\varkappa} |\varkappa-\varrho|^{\alpha_1-1}\varpi_1(\varrho)d\varrho + \frac{1}{\Gamma(\alpha_1)} \int_0^{\varkappa} |\theta-\varrho|^{\alpha_1-1}\varpi_1(\varrho)d\varrho, \tag{8}$$

and

$$\aleph_2(x,y)(\theta) = \frac{\delta_3}{\delta_1+\delta_2} - \frac{\delta_1}{(\delta_1+\delta_2)\Gamma(\alpha_2)} \int_0^{\varkappa} \varrho^{\alpha_2-1}\varpi_2(\varrho)d\varrho - \frac{\delta_2}{(\delta_1+\delta_2)\Gamma(\alpha_2)} \int_0^{\varkappa} |\varkappa-\varrho|^{\alpha_2-1}\varpi_2(\varrho)d\varrho + \frac{1}{\Gamma(\alpha_2)} \int_0^{\varkappa} |\theta-\varrho|^{\alpha_2-1}\varpi_2(\varrho)d\varrho, \tag{9}$$

where for $i = 1,2$, $\varpi_i \in C(\Theta,\mathbb{R})$ *satisfy the following system of functional equations:*

$$\begin{cases} \varpi_1(\theta) = \varphi_1\big(\theta, x(\theta), y(\theta), \varpi_1(\theta), \varpi_2(\theta)\big), \\ \varpi_2(\theta) = \varphi_2\big(\theta, x(\theta), y(\theta), \varpi_1(\theta), \varpi_2(\theta)\big). \end{cases}$$

We are now in a position to prove the existence result of the problem (1)–(2) based on the Banach's contraction principle.

Let us assume the following assumptions:

(Ax1) The functions $\varphi_i : \Theta \times \mathbb{R}^4 \to \mathbb{R}; i = 1,2$, are continuous.

(Ax2) There exist constants $\psi_j, \bar{\psi}_j; j = 1, \ldots, 4$, such that

$$\psi_j, \bar{\psi}_j > 0, \quad 0 < \psi_3 < 1, \quad 0 < \bar{\psi}_4 < 1$$

and

$$|\varphi_1(\theta, x_1, y_1, w_1, z_1) - \varphi_1(\theta, x_2, y_2, w_2, z_2)| \le \psi_1|x_1-x_2| + \psi_2|y_1-y_2| + \psi_3|w_1-w_2| + \psi_4|z_1-z_2|,$$

and

$$|\varphi_2(\theta, x_1, y_1, w_1, z_1) - \varphi_2(\theta, x_2, y_2, w_2, z_2)| \le \bar{\psi}_1|x_1-x_2| + \bar{\psi}_2|y_1-y_2| + \bar{\psi}_3|w_1-w_2| + \bar{\psi}_4|z_1-z_2|,$$

for any $x_i, y_i, w_i, z_i \in \mathbb{R}$ and $\theta \in \Theta$, where $i = 1,2$.

Set

$$\mathcal{A}_1 = \frac{(|\beta_1| + |\beta_2| + |\beta_1 + \beta_2|)\varkappa^{\alpha_1}}{|\beta_1 + \beta_2|\Gamma(\alpha_1 + 1)} \left[\frac{\psi_4\bar{\psi}_1 + \psi_1(1 - \bar{\psi}_4)}{(1 - \bar{\psi}_4)(1 - \psi_3) - \psi_4\bar{\psi}_3} \right],$$

$$\mathcal{A}_2 = \frac{(|\beta_1| + |\beta_2| + |\beta_1 + \beta_2|)\varkappa^{\alpha_1}}{|\beta_1 + \beta_2|\Gamma(\alpha_1 + 1)} \left[\frac{\psi_4\bar{\psi}_2 + \psi_2(1 - \bar{\psi}_4)}{(1 - \bar{\psi}_4)(1 - \psi_3) - \psi_4\bar{\psi}_3} \right],$$

$$\mathcal{B}_1 = \frac{(|\delta_1| + |\delta_2| + |\delta_1 + \delta_2|)\varkappa^{\alpha_2}}{|\delta_1 + \delta_2|\Gamma(\alpha_2 + 1)} \left[\frac{\psi_1\bar{\psi}_3 + \bar{\psi}_1(1 - \psi_3)}{(1 - \bar{\psi}_4)(1 - \psi_3) - \psi_4\bar{\psi}_3} \right],$$

and

$$\mathcal{B}_2 = \frac{(|\delta_1| + |\delta_2| + |\delta_1 + \delta_2|)\varkappa^{\alpha_2}}{|\delta_1 + \delta_2|\Gamma(\alpha_2 + 1)} \left[\frac{\psi_2\bar{\psi}_3 + \bar{\psi}_2(1 - \psi_3)}{(1 - \bar{\psi}_4)(1 - \psi_3) - \psi_4\bar{\psi}_3} \right].$$

Theorem 3.3. *Assume that the assumptions* (Ax1)–(Ax2) *hold. If*

$$\Upsilon := \max\{\mathcal{A}_1, \mathcal{B}_1\} + \max\{\mathcal{A}_2, \mathcal{B}_2\} + \frac{\psi_4\bar{\psi}_3}{(1 - \bar{\psi}_4)(1 - \psi_3)} < 1, \qquad (10)$$

then the implicit fractional coupled system (1)–(2) *has a unique solution in* $\mathcal{F}$.

Proof. Let us demonstrate that the operator $\aleph$ defined in (7) has a unique fixed point on $\mathcal{F}$. Let $(x, y), (\bar{x}, \bar{y}) \in \mathcal{F}$ and $\theta \in \Theta$. Then we have

$$\begin{aligned}|\aleph_1(x, y)(\theta) - \aleph_1(\bar{x}, \bar{y})(\theta)| \leq{}& \frac{|\beta_1|}{|\beta_1 + \beta_2|\Gamma(\alpha_1)} \int_0^{\varkappa} \varrho^{\alpha_1 - 1} |\varpi_1(\varrho) - \bar{\varpi}_1(\varrho)| d\varrho \\ &+ \frac{|\beta_2|}{|\beta_1 + \beta_2|\Gamma(\alpha_1)} \int_0^{\varkappa} |\varkappa - \varrho|^{\alpha_1 - 1} |\varpi_1(\varrho) - \bar{\varpi}_1(\varrho)| d\varrho \\ &+ \frac{1}{\Gamma(\alpha_1)} \int_0^{\varkappa} |\theta - \varrho|^{\alpha_1 - 1} |\varpi_1(\varrho) - \bar{\varpi}_1(\varrho)| d\varrho,\end{aligned}$$

and

$$\begin{aligned}|\aleph_2(x, y)(\theta) - \aleph_2(\bar{x}, \bar{y})(\theta)| \leq{}& \frac{|\delta_1|}{|\delta_1 + \delta_2|\Gamma(\alpha_2)} \int_0^{\varkappa} \varrho^{\alpha_2 - 1} |\varpi_2(\varrho) - \bar{\varpi}_2(\varrho)| d\varrho \\ &+ \frac{|\delta_2|}{|\delta_1 + \delta_2|\Gamma(\alpha_2)} \int_0^{\varkappa} |\varkappa - \varrho|^{\alpha_2 - 1} |\varpi_2(\varrho) - \bar{\varpi}_2(\varrho)| d\varrho \\ &+ \frac{1}{\Gamma(\alpha_2)} \int_0^{\varkappa} |\theta - \varrho|^{\alpha_2 - 1} |\varpi_2(\varrho) - \bar{\varpi}_2(\varrho)| d\varrho,\end{aligned}$$

where

$$\begin{cases} \varpi_1(\theta) = \varphi_1\big(\theta, x(\theta), y(\theta), \varpi_1(\theta), \varpi_2(\theta)\big), \\ \varpi_2(\theta) = \varphi_2\big(\theta, x(\theta), y(\theta), \varpi_1(\theta), \varpi_2(\theta)\big), \end{cases}$$

and

$$\begin{cases} \bar{\varpi}_1(\theta) = \varphi_1\big(\theta, \bar{x}(\theta), \bar{y}(\theta), \bar{\varpi}_1(\theta), \bar{\varpi}_2(\theta)\big), \\ \bar{\varpi}_2(\theta) = \varphi_2\big(\theta, \bar{x}(\theta), \bar{y}(\theta), \bar{\varpi}_1(\theta), \bar{\varpi}_2(\theta)\big). \end{cases}$$

Then, by (Ax2) we find that

$$\begin{aligned} &|\varpi_1(\theta) - \bar{\varpi}_1(\theta)| \\ &= \big|\varphi_1\big(\theta, x(\theta), y(\theta), \varpi_1(\theta), \varpi_2(\theta)\big) - \varphi_1\big(\theta, \bar{x}(\theta), \bar{y}(\theta), \bar{\varpi}_1(\theta), \bar{\varpi}_2(\theta)\big)\big| \\ &\leq \psi_1|x(\theta) - \bar{x}(\theta)| + \psi_2|y(\theta) - \bar{y}(\theta)| + \psi_3|\varpi_1(\theta) - \bar{\varpi}_1(\theta)| \\ &\quad + \psi_4|\varpi_2(\theta) - \bar{\varpi}_2(\theta)|, \end{aligned}$$

which implies

$$\begin{aligned} |\varpi_1(\theta) - \bar{\varpi}_1(\theta)| \leq \frac{\psi_1}{1-\psi_3}|x(\theta) - \bar{x}(\theta)| + \frac{\psi_2}{1-\psi_3}|y(\theta) - \bar{y}(\theta)| \\ + \frac{\psi_4}{1-\psi_3}|\varpi_2(\theta) - \bar{\varpi}_2(\theta)|. \end{aligned}$$

Similarly, one can find that

$$\begin{aligned} |\varpi_2(\theta) - \bar{\varpi}_2(\theta)| \leq \frac{\bar{\psi}_1}{1-\bar{\psi}_4}|x(\theta) - \bar{x}(\theta)| + \frac{\bar{\psi}_2}{1-\bar{\psi}_4}|y(\theta) - \bar{y}(\theta)| \\ + \frac{\bar{\psi}_3}{1-\bar{\psi}_4}|\varpi_1(\theta) - \bar{\varpi}_1(\theta)|. \end{aligned}$$

Therefore,

$$\begin{aligned} |\varpi_1(\theta) - \bar{\varpi}_1(\theta)| \leq \frac{\psi_1}{1-\psi_3}|x(\theta) - \bar{x}(\theta)| + \frac{\psi_2}{1-\psi_3}|y(\theta) - \bar{y}(\theta)| \\ + \frac{\psi_4\bar{\psi}_1}{(1-\bar{\psi}_4)(1-\psi_3)}|x(\theta) - \bar{x}(\theta)| \\ + \frac{\psi_4\bar{\psi}_2}{(1-\bar{\psi}_4)(1-\psi_3)}|y(\theta) - \bar{y}(\theta)| \\ + \frac{\psi_4\bar{\psi}_3}{(1-\bar{\psi}_4)(1-\psi_3)}|\varpi_1(\theta) - \bar{\varpi}_1(\theta)|, \end{aligned}$$

which implies that

$$|\varpi_1(\theta)-\bar{\varpi}_1(\theta)| \le \frac{\psi_4\bar{\psi}_1+\psi_1(1-\bar{\psi}_4)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\|x-\bar{x}\|_\infty + \frac{\psi_4\bar{\psi}_2+\psi_2(1-\bar{\psi}_4)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\|y-\bar{y}\|_\infty.$$

By following the same approach, we can also obtain the following:

$$|\varpi_2(\theta)-\bar{\varpi}_2(\theta)| \le \frac{\psi_1\bar{\psi}_3+\bar{\psi}_1(1-\psi_3)}{(1-\bar{\psi}_4)(1-\psi_3)}|x(\theta)-\bar{x}(\theta)| + \frac{\psi_2\bar{\psi}_3+\bar{\psi}_2(1-\psi_3)}{(1-\bar{\psi}_4)(1-\psi_3)}|y(\theta)-\bar{y}(\theta)| + \frac{\psi_4\bar{\psi}_3}{(1-\bar{\psi}_4)(1-\psi_3)}|\varpi_2(\theta)-\bar{\varpi}_2(\theta)|,$$

which implies that

$$|\varpi_2(\theta)-\bar{\varpi}_2(\theta)| \le \frac{\psi_1\bar{\psi}_3+\bar{\psi}_1(1-\psi_3)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\|x-\bar{x}\|_\infty + \frac{\psi_2\bar{\psi}_3+\bar{\psi}_2(1-\psi_3)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\|y-\bar{y}\|_\infty.$$

As a consequence, we get

$$\begin{aligned}
&|\aleph_1(x,y)(\theta)-\aleph_1(\bar{x},\bar{y})(\theta)| \\
&\le \frac{|\beta_1|}{|\beta_1+\beta_2|\Gamma(\alpha_1)}\left[\frac{\psi_4\bar{\psi}_1+\psi_1(1-\bar{\psi}_4)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\|x-\bar{x}\|_\infty\right. \\
&\quad \left.+\frac{\psi_4\bar{\psi}_2+\psi_2(1-\bar{\psi}_4)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\|y-\bar{y}\|_\infty\right]\int_0^{\varkappa}\varrho^{\alpha_1-1}d\varrho \\
&\quad +\frac{|\beta_2|}{|\beta_1+\beta_2|\Gamma(\alpha_1)}\left[\frac{\psi_4\bar{\psi}_1+\psi_1(1-\bar{\psi}_4)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\|x-\bar{x}\|_\infty\right. \\
&\quad \left.+\frac{\psi_4\bar{\psi}_2+\psi_2(1-\bar{\psi}_4)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\|y-\bar{y}\|_\infty\right]\int_0^{\varkappa}|\varkappa-\varrho|^{\alpha_1-1}d\varrho \\
&\quad +\frac{1}{\Gamma(\alpha_1)}\left[\frac{\psi_4\bar{\psi}_1+\psi_1(1-\bar{\psi}_4)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\|x-\bar{x}\|_\infty\right. \\
&\quad \left.+\frac{\psi_4\bar{\psi}_2+\psi_2(1-\bar{\psi}_4)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\|y-\bar{y}\|_\infty\right]\int_0^{\varkappa}|\theta-\varrho|^{\alpha_1-1}d\varrho,
\end{aligned}$$

and

$$
\begin{aligned}
&|\aleph_2(x,y)(\theta)-\aleph_2(\bar{x},\bar{y})(\theta)|\\
&\le \frac{|\delta_1|}{|\delta_1+\delta_2|\Gamma(\alpha_2)}\left[\frac{\psi_1\bar{\psi}_3+\bar{\psi}_1(1-\psi_3)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\|x-\bar{x}\|_\infty\right.\\
&\quad\left.+\frac{\psi_2\bar{\psi}_3+\bar{\psi}_2(1-\psi_3)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\|y-\bar{y}\|_\infty\right]\int_0^{\varkappa}\varrho^{\alpha_2-1}d\varrho\\
&\quad+\frac{|\delta_2|}{|\delta_1+\delta_2|\Gamma(\alpha_2)}\left[\frac{\psi_1\bar{\psi}_3+\bar{\psi}_1(1-\psi_3)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\|x-\bar{x}\|_\infty\right.\\
&\quad\left.+\frac{\psi_2\bar{\psi}_3+\bar{\psi}_2(1-\psi_3)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\|y-\bar{y}\|_\infty\right]\int_0^{\varkappa}|\varkappa-\varrho|^{\alpha_2-1}d\varrho\\
&\quad+\frac{1}{\Gamma(\alpha_2)}\left[\frac{\psi_1\bar{\psi}_3+\bar{\psi}_1(1-\psi_3)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\|x-\bar{x}\|_\infty\right.\\
&\quad\left.+\frac{\psi_2\bar{\psi}_3+\bar{\psi}_2(1-\psi_3)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\|y-\bar{y}\|_\infty\right]\int_0^{\varkappa}|\theta-\varrho|^{\alpha_2-1}d\varrho.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\|\aleph_1(x,y)-\aleph_1(\bar{x},\bar{y})\|_\infty\\
&\le \frac{(|\beta_1|+|\beta_2|+|\beta_1+\beta_2|)\varkappa^{\alpha_1}}{|\beta_1+\beta_2|\Gamma(\alpha_1+1)}\left[\frac{\psi_4\bar{\psi}_1+\psi_1(1-\bar{\psi}_4)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\|x-\bar{x}\|_\infty\right.\\
&\quad\left.+\frac{\psi_4\bar{\psi}_2+\psi_2(1-\bar{\psi}_4)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\|y-\bar{y}\|_\infty\right]\\
&\le \mathcal{A}_1\|x-\bar{x}\|_\infty+\mathcal{A}_2\|y-\bar{y}\|_\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\aleph_2(x,y)-\aleph_2(\bar{x},\bar{y})\|_\infty\\
&\le \frac{(|\delta_1|+|\delta_2|+|\delta_1+\delta_2|)\varkappa^{\alpha_2}}{|\delta_1+\delta_2|\Gamma(\alpha_2+1)}\left[\frac{\psi_1\bar{\psi}_3+\bar{\psi}_1(1-\psi_3)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\|x-\bar{x}\|_\infty\right.\\
&\quad\left.+\frac{\psi_2\bar{\psi}_3+\bar{\psi}_2(1-\psi_3)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\|y-\bar{y}\|_\infty\right]\\
&\le \mathcal{B}_1\|x-\bar{x}\|_\infty+\mathcal{B}_2\|y-\bar{y}\|_\infty.
\end{aligned}
$$

We can deduce now that

$$\begin{aligned}\|\aleph(x,y)-\aleph(\bar{x},\bar{y})\|_{\mathcal{F}} &= \max\left\{\|\aleph_1(x,y)-\aleph_1(\bar{x},\bar{y})\|_\infty, \|\aleph_2(x,y)-\aleph_2(\bar{x},\bar{y})\|_\infty\right\}\\ &\le \left(\max\{\mathcal{A}_1,\mathcal{B}_1\}+\max\{\mathcal{A}_2,\mathcal{B}_2\}\right)\|(x,y)-(\bar{x},\bar{y})\|_{\mathcal{F}}\\ &\le \Upsilon\|(x,y)-(\bar{x},\bar{y})\|_{\mathcal{F}}.\end{aligned}$$

Consequently, by the Banach's contraction principle, the operator $\aleph$ has a unique fixed point which is solution of the problem (1)–(2). □

Remark 1. Let us put

$$\psi_j = \wp_j, \quad \bar{\psi}_j = \bar{\wp}_j, \quad \text{for } j = 1,\dots,4,$$

and

$$\wp_5(\theta) = |\varphi_1(\theta,0,0,0,0)|, \quad \bar{\wp}_5(\theta) = |\varphi_2(\theta,0,0,0,0)|,$$

then the hypothesis (Ax2) implies that

$$|\varphi_1(\theta,x,y,w,z)| \le \wp_1|x| + \wp_2|y| + \wp_3|w| + \wp_4|z| + \wp_5(\theta),$$

and

$$|\varphi_2(\theta,x,y,w,z)| \le \bar{\wp}_1|x| + \bar{\wp}_2|y| + \bar{\wp}_3|w| + \bar{\wp}_4|z| + \bar{\wp}_5(\theta),$$

for any $x,y,w,z \in \mathbb{R}$, $\theta \in \Theta$ and $\wp_5, \bar{\wp}_5 \in C(\Theta,\mathbb{R}_+)$, with

$$\wp_5{}^* = \sup_{\theta\in\Theta} \wp_5(\theta) \quad \text{and} \quad \bar{\wp}_5^* = \sup_{\theta\in\Theta} \bar{\wp}_5(\theta).$$

Theorem 3.4. *Assume that the hypotheses* (Ax1)–(Ax2) *hold. Then the implicit fractional problem* (1)–(2) *has at least one solution.*

Proof. This proof is based on the fixed point theorem of Schauder. We establish the proof in several steps.

Step 1 (The operator $\aleph$ is continuous)**.** Let $\{(x_n,y_n)\}_{n\in\mathbb{N}}$ be a sequence such that $(x_n,y_n) \longrightarrow (x,y)$ in $\mathcal{F}$. Then for each $\theta \in \Theta$, we have

$$\begin{aligned}&|\aleph_1(x_n,y_n)(\theta) - \aleph_1(x,y)(\theta)|\\ &\quad\le \frac{|\beta_1|}{|\beta_1+\beta_2|\Gamma(\alpha_1)}\int_0^{\varkappa} \varrho^{\alpha_1-1}|\varpi_{1,n}(\varrho)-\varpi_1(\varrho)|d\varrho\\ &\qquad+ \frac{|\beta_2|}{|\beta_1+\beta_2|\Gamma(\alpha_1)}\int_0^{\varkappa} |\varkappa-\varrho|^{\alpha_1-1}|\varpi_{1,n}(\varrho)-\varpi_1(\varrho)|d\varrho\\ &\qquad+ \frac{1}{\Gamma(\alpha_1)}\int_0^{\varkappa} |\theta-\varrho|^{\alpha_1-1}|\varpi_{1,n}(\varrho)-\varpi_1(\varrho)|d\varrho,\end{aligned}$$

and

$$
\begin{aligned}
&|\aleph_2(x_n,y_n)(\theta)-\aleph_2(x,y)(\theta)| \\
&\quad\leq \frac{|\delta_1|}{|\delta_1+\delta_2|\Gamma(\alpha_2)}\int_0^{\varkappa}\varrho^{\alpha_2-1}|\varpi_{2,n}(\varrho)-\varpi_2(\varrho)|d\varrho \\
&\qquad+\frac{|\delta_2|}{|\delta_1+\delta_2|\Gamma(\alpha_2)}\int_0^{\varkappa}|\varkappa-\varrho|^{\alpha_2-1}|\varpi_{2,n}(\varrho)-\varpi_2(\varrho)|d\varrho \\
&\qquad+\frac{1}{\Gamma(\alpha_2)}\int_0^{\varkappa}|\theta-\varrho|^{\alpha_2-1}|\varpi_{2,n}(\varrho)-\varpi_2(\varrho)|d\varrho,
\end{aligned}
$$

where

$$
\begin{cases}
\varpi_1(\theta) = \varphi_1\big(\theta,x(\theta),y(\theta),\varpi_1(\theta),\varpi_2(\theta)\big),\\
\varpi_2(\theta) = \varphi_2\big(\theta,x(\theta),y(\theta),\varpi_1(\theta),\varpi_2(\theta)\big),
\end{cases}
$$

and

$$
\begin{cases}
\varpi_{1,n}(\theta) = \varphi_1\big(\theta,x_n(\theta),y_n(\theta),\varpi_{1,n}(\theta),\varpi_{2,n}(\theta)\big),\\
\varpi_{2,n}(\theta) = \varphi_2\big(\theta,x_n(\theta),y_n(\theta),\varpi_{1,n}(\theta),\varpi_{2,n}(\theta)\big).
\end{cases}
$$

By (Ax2), we have

$$
\begin{aligned}
|\varpi_{1,n}(\theta)-\varpi_1(\theta)| \leq{}& \frac{\psi_4\bar{\psi}_1+\psi_1(1-\bar{\psi}_4)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\|x_n-x\|_\infty \\
&+\frac{\psi_4\bar{\psi}_2+\psi_2(1-\bar{\psi}_4)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\|y_n-y\|_\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
|\varpi_{2,n}(\theta)-\varpi_2(\theta)| \leq{}& \frac{\psi_1\bar{\psi}_3+\bar{\psi}_1(1-\psi_3)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\|x_n-x\|_\infty \\
&+\frac{\psi_2\bar{\psi}_3+\bar{\psi}_2(1-\psi_3)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\|y_n-y\|_\infty.
\end{aligned}
$$

Since $(x_n,y_n)\to(x,y)$, then

$$\varpi_{1,n}(\theta)\to\varpi_1(\theta)$$

and

$$\varpi_{2,n}(\theta)\to\varpi_2(\theta),$$

as $n\to\infty$ for each $\theta\in\Theta$, and since φ_1 and φ_2 are continuous, then we have

$$|\aleph_1(x_n,y_n)(\theta)-\aleph_1(x,y)(\theta)|\longrightarrow 0$$

and

$$|\aleph_2(x_n,y_n)(\theta)-\aleph_2(x,y)(\theta)|\longrightarrow 0,$$

as $n\to\infty$.

By applying the Lebesgue dominated convergence theorem, we get

$$|\aleph(x_n, y_n)(\theta) - \aleph(x, y)(\theta)| \longrightarrow 0 \qquad as\ n \longrightarrow \infty,$$

hence

$$\|\aleph(x_n, y_n) - \aleph(x, y)\|_{\mathcal{F}} \longrightarrow 0 \qquad as\ n \longrightarrow \infty,$$

which implies that $\aleph$ is continuous.

Define the ball $D_R = \{(x, y) \in \mathcal{F} : \|(x, y)\|_{\mathcal{F}} \leq R\}$, where

$$R \geq \frac{\lambda_3}{1 - \lambda_1 - \lambda_2},$$

where

$$\lambda_1 = \max\{\mathcal{L}_1, \mathcal{L}_2\}, \quad \lambda_2 = \max\{\mathcal{L}_3, \mathcal{L}_4\} \quad \text{and} \quad \lambda_3 = \max\{\mathcal{L}_5, \mathcal{L}_6\},$$

such that

$$\mathcal{L}_1 = \left[\frac{(|\beta_1| + |\beta_2| + |\beta_1 + \beta_2|)\varkappa^{\alpha_1}}{|\beta_1 + \beta_2|\Gamma(\alpha_1 + 1)}\right]\left[\frac{\wp_1(1 - \bar{\wp}_4) + \bar{\wp}_1\wp_4}{(1 - \wp_3)(1 - \bar{\wp}_4) - \bar{\wp}_3\wp_4}\right],$$

$$\mathcal{L}_2 = \left[\frac{(|\delta_1| + |\delta_2| + |\delta_1 + \delta_2|)\varkappa^{\alpha_2}}{|\delta_1 + \delta_2|\Gamma(\alpha_2 + 1)}\right]\left[\frac{\bar{\wp}_1(1 - \wp_3) + \wp_1\bar{\wp}_3}{(1 - \wp_3)(1 - \bar{\wp}_4) - \bar{\wp}_3\wp_4}\right],$$

$$\mathcal{L}_3 = \left[\frac{(|\beta_1| + |\beta_2| + |\beta_1 + \beta_2|)\varkappa^{\alpha_1}}{|\beta_1 + \beta_2|\Gamma(\alpha_1 + 1)}\right]\left[\frac{\wp_2(1 - \bar{\wp}_4) + \bar{\wp}_2\wp_4}{(1 - \wp_3)(1 - \bar{\wp}_4) - \bar{\wp}_3\wp_4}\right],$$

$$\mathcal{L}_4 = \left[\frac{(|\delta_1| + |\delta_2| + |\delta_1 + \delta_2|)\varkappa^{\alpha_2}}{|\delta_1 + \delta_2|\Gamma(\alpha_2 + 1)}\right]\left[\frac{\bar{\wp}_2(1 - \wp_3) + \wp_2\bar{\wp}_3}{(1 - \wp_3)(1 - \bar{\wp}_4) - \bar{\wp}_3\wp_4}\right],$$

$$\mathcal{L}_5 = \left[\frac{(|\beta_1| + |\beta_2| + |\beta_1 + \beta_2|)\varkappa^{\alpha_1}}{|\beta_1 + \beta_2|\Gamma(\alpha_1 + 1)}\right]\left[\frac{\wp_5{}^*(1 - \bar{\wp}_4) + \bar{\wp}_5^*\wp_4}{(1 - \wp_3)(1 - \bar{\wp}_4) - \bar{\wp}_3\wp_4}\right] + \frac{|\beta_3|}{|\beta_1 + \beta_2|},$$

and

$$\mathcal{L}_6 = \left[\frac{(|\delta_1| + |\delta_2| + |\delta_1 + \delta_2|)\varkappa^{\alpha_2}}{|\delta_1 + \delta_2|\Gamma(\alpha_2 + 1)}\right]\left[\frac{\bar{\wp}_5^*(1 - \wp_3) + \wp_5{}^*\bar{\wp}_3}{(1 - \wp_3)(1 - \bar{\wp}_4) - \bar{\wp}_3\wp_4}\right] + \frac{|\delta_3|}{|\delta_1 + \delta_2|}.$$

It is clear that D_R is a bounded, closed and convex subset of $\mathcal{F}$.

Step 2 ($\aleph(D_R) \subset D_R$)**.** Let $(x, y) \in D_R$ and $\theta \in \Theta$, then (8) and (9) imply that

$$\begin{aligned} |\aleph_1(x, y)(\theta)| \leq{}& \frac{|\beta_3|}{|\beta_1 + \beta_2|} + \frac{|\beta_1|}{|\beta_1 + \beta_2|\Gamma(\alpha_1)}\int_0^{\varkappa} \varrho^{\alpha_1 - 1}|\varpi_1(\varrho)|d\varrho \\ &+ \frac{|\beta_2|}{|\beta_1 + \beta_2|\Gamma(\alpha_1)}\int_0^{\varkappa} |\varkappa - \varrho|^{\alpha_1 - 1}|\varpi_1(\varrho)|d\varrho \\ &+ \frac{1}{\Gamma(\alpha_1)}\int_0^{\varkappa} |\theta - \varrho|^{\alpha_1 - 1}|\varpi_1(\varrho)|d\varrho, \end{aligned}$$

and

$$|\aleph_2(x,y)(\theta)| \leq \frac{|\delta_3|}{|\delta_1+\delta_2|} + \frac{|\delta_1|}{|\delta_1+\delta_2|\Gamma(\alpha_2)}\int_0^{\varkappa} \varrho^{\alpha_2-1}|\varpi_2(\varrho)|d\varrho$$
$$+ \frac{|\delta_2|}{|\delta_1+\delta_2|\Gamma(\alpha_2)}\int_0^{\varkappa} |\varkappa-\varrho|^{\alpha_2-1}|\varpi_2(\varrho)|d\varrho$$
$$+ \frac{1}{\Gamma(\alpha_2)}\int_0^{\varkappa} |\theta-\varrho|^{\alpha_2-1}|\varpi_2(\varrho)|d\varrho.$$

By Remark 1, we have

$$|\varpi_1(\theta)| \leq \wp_1|x(\theta)| + \wp_2|y(\theta)| + \wp_3|\varpi_1(\theta)| + \wp_4|\varpi_2(\theta)| + \wp_5(\theta),$$

which implies that

$$|\varpi_1(\theta)| \leq \frac{\wp_1}{1-\wp_3}|x(\theta)| + \frac{\wp_2}{1-\wp_3}|y(\theta)| + \frac{\wp_4}{1-\wp_3}|\varpi_2(\theta)| + \frac{{\wp_5}^*}{1-\wp_3}.$$

Similarly, one can find that

$$|\varpi_2(\theta)| \leq \frac{\bar{\wp}_1}{1-\bar{\wp}_4}|x(\theta)| + \frac{\bar{\wp}_2}{1-\bar{\wp}_4}|y(\theta)| + \frac{\bar{\wp}_3}{1-\bar{\wp}_4}|\varpi_1(\theta)| + \frac{\bar{\wp}_5^*}{1-\bar{\wp}_4}.$$

Therefore

$$|\varpi_1(\theta)| \leq \frac{\wp_1(1-\bar{\wp}_4)+\bar{\wp}_1\wp_4}{(1-\wp_3)(1-\bar{\wp}_4)-\bar{\wp}_3\wp_4}\|x\|_\infty + \frac{\wp_2(1-\bar{\wp}_4)+\bar{\wp}_2\wp_4}{(1-\wp_3)(1-\bar{\wp}_4)-\bar{\wp}_3\wp_4}\|y\|_\infty$$
$$+ \frac{{\wp_5}^*(1-\bar{\wp}_4)+\bar{\wp}_5^*\wp_4}{(1-\wp_3)(1-\bar{\wp}_4)-\bar{\wp}_3\wp_4},$$

and

$$|\varpi_2(\theta)| \leq \frac{\bar{\wp}_1(1-\wp_3)+\wp_1\bar{\wp}_3}{(1-\wp_3)(1-\bar{\wp}_4)-\bar{\wp}_3\wp_4}\|x\|_\infty + \frac{\bar{\wp}_2(1-\wp_3)+\wp_2\bar{\wp}_3}{(1-\wp_3)(1-\bar{\wp}_4)-\bar{\wp}_3\wp_4}\|y\|_\infty$$
$$+ \frac{\bar{\wp}_5^*(1-\wp_3)+{\wp_5}^*\bar{\wp}_3}{(1-\wp_3)(1-\bar{\wp}_4)-\bar{\wp}_3\wp_4}.$$

Thus, for each $\theta \in \Theta$, we have

$$|\aleph_1(x,y)(\theta)| \leq \frac{|\beta_3|}{|\beta_1+\beta_2|} + \left[\frac{|\beta_1|}{|\beta_1+\beta_2|\Gamma(\alpha_1)}\int_0^{\varkappa} \varrho^{\alpha_1-1}d\varrho\right.$$
$$+ \frac{|\beta_2|}{|\beta_1+\beta_2|\Gamma(\alpha_1)}\int_0^{\varkappa} |\varkappa-\varrho|^{\alpha_1-1}d\varrho$$
$$\left.+ \frac{1}{\Gamma(\alpha_1)}\int_0^{\varkappa} |\theta-\varrho|^{\alpha_1-1}d\varrho\right]\left[\frac{\wp_1(1-\bar{\wp}_4)+\bar{\wp}_1\wp_4}{(1-\wp_3)(1-\bar{\wp}_4)-\bar{\wp}_3\wp_4}\|x\|_\infty\right.$$
$$\left.+ \frac{\wp_2(1-\bar{\wp}_4)+\bar{\wp}_2\wp_4}{(1-\wp_3)(1-\bar{\wp}_4)-\bar{\wp}_3\wp_4}\|y\|_\infty + \frac{{\wp_5}^*(1-\bar{\wp}_4)+\bar{\wp}_5^*\wp_4}{(1-\wp_3)(1-\bar{\wp}_4)-\bar{\wp}_3\wp_4}\right],$$

and

$$|\aleph_2(x,y)(\theta)| \le \frac{|\delta_3|}{|\delta_1+\delta_2|} + \Bigg[\frac{|\delta_1|}{|\delta_1+\delta_2|\Gamma(\alpha_2)}\int_0^{\varkappa} \varrho^{\alpha_2-1} d\varrho$$

$$+ \frac{|\delta_2|}{|\delta_1+\delta_2|\Gamma(\alpha_2)}\int_0^{\varkappa} |\varkappa-\varrho|^{\alpha_2-1} d\varrho$$

$$+ \frac{1}{\Gamma(\alpha_2)}\int_0^{\varkappa} |\theta-\varrho|^{\alpha_2-1} d\varrho\Bigg]\Bigg[\frac{\bar{\wp}_1(1-\wp_3)+\wp_1\bar{\wp}_3}{(1-\wp_3)(1-\bar{\wp}_4)-\bar{\wp}_3\wp_4}\|x\|_\infty$$

$$+ \frac{\bar{\wp}_2(1-\wp_3)+\wp_2\bar{\wp}_3}{(1-\wp_3)(1-\bar{\wp}_4)-\bar{\wp}_3\wp_4}\|y\|_\infty + \frac{\bar{\wp}_5^*(1-\wp_3)+{\wp_5}^*\bar{\wp}_3}{(1-\wp_3)(1-\bar{\wp}_4)-\bar{\wp}_3\wp_4}\Bigg].$$

Therefore,

$$\|\aleph_1(x,y)\|_\infty \le \frac{|\beta_3|}{|\beta_1+\beta_2|} + \left[\frac{(|\beta_1|+|\beta_2|+|\beta_1+\beta_2|)\varkappa^{\alpha_1}}{|\beta_1+\beta_2|\Gamma(\alpha_1+1)}\right]$$

$$\times \Bigg[\frac{\wp_1(1-\bar{\wp}_4)+\bar{\wp}_1\wp_4}{(1-\wp_3)(1-\bar{\wp}_4)-\bar{\wp}_3\wp_4}\|x\|_\infty$$

$$+ \frac{\wp_2(1-\bar{\wp}_4)+\bar{\wp}_2\wp_4}{(1-\wp_3)(1-\bar{\wp}_4)-\bar{\wp}_3\wp_4}\|y\|_\infty + \frac{{\wp_5}^*(1-\bar{\wp}_4)+\bar{\wp}_5^*\wp_4}{(1-\wp_3)(1-\bar{\wp}_4)-\bar{\wp}_3\wp_4}\Bigg]$$

$$\le \lambda_1\|x\|_\infty + \lambda_2\|y\|_\infty + \lambda_3,$$

and

$$\|\aleph_2(x,y)\|_\infty \le \frac{|\delta_3|}{|\delta_1+\delta_2|} + \left[\frac{(|\delta_1|+|\delta_2|+|\delta_1+\delta_2|)\varkappa^{\alpha_2}}{|\delta_1+\delta_2|\Gamma(\alpha_2+1)}\right]$$

$$\times \Bigg[\frac{\bar{\wp}_1(1-\wp_3)+\wp_1\bar{\wp}_3}{(1-\wp_3)(1-\bar{\wp}_4)-\bar{\wp}_3\wp_4}\|x\|_\infty$$

$$+ \frac{\bar{\wp}_2(1-\wp_3)+\wp_2\bar{\wp}_3}{(1-\wp_3)(1-\bar{\wp}_4)-\bar{\wp}_3\wp_4}\|y\|_\infty + \frac{\bar{\wp}_5^*(1-\wp_3)+{\wp_5}^*\bar{\wp}_3}{(1-\wp_3)(1-\bar{\wp}_4)-\bar{\wp}_3\wp_4}\Bigg]$$

$$\le \lambda_1\|x\|_\infty + \lambda_2\|y\|_\infty + \lambda_3.$$

Finally, we obtain

$$\|\aleph(x,y)\|_{\mathcal{F}} \le (\lambda_1+\lambda_2)R + \lambda_3 \le R.$$

Hence, $\aleph(D_R) \subset D_R$.

Step 3 ($\aleph(D_R)$ is equicontinuous)**.** Let $\theta_1, \theta_2 \in \Theta$ where $\theta_1 < \theta_2$ and $y \in D_R$. Then

$$\begin{aligned}
&|\aleph_1(x,y)(\theta_2) - \aleph_1(x,y)(\theta_1)| \\
&= \left| \frac{1}{\Gamma(\alpha_1)} \int_0^{\varkappa} |\theta_2 - \varrho|^{\alpha_1 - 1} \varpi_1(\varrho) d\varrho - \frac{1}{\Gamma(\alpha_1)} \int_0^{\varkappa} |\theta_1 - \varrho|^{\alpha_1 - 1} \varpi_1(\varrho) d\varrho \right| \\
&\leq \frac{1}{\Gamma(\alpha_1)} \int_0^{\varkappa} \left| |\theta_2 - \varrho|^{\alpha_1 - 1} - |\theta_1 - \varrho|^{\alpha_1 - 1} \right| |\varpi_1(\varrho)| d\varrho \\
&\leq \frac{R}{\Gamma(\alpha_1)} \int_0^{\varkappa} \left| |\theta_2 - \varrho|^{\alpha_1 - 1} - |\theta_1 - \varrho|^{\alpha_1 - 1} \right| d\varrho,
\end{aligned}$$

and

$$\begin{aligned}
&|\aleph_2(x,y)(\theta_2) - \aleph_2(x,y)(\theta_1)| \\
&= \left| \frac{1}{\Gamma(\alpha_2)} \int_0^{\varkappa} |\theta_2 - \varrho|^{\alpha_2 - 1} \varpi_2(\varrho) d\varrho - \frac{1}{\Gamma(\alpha_2)} \int_0^{\varkappa} |\theta_1 - \varrho|^{\alpha_2 - 1} \varpi_2(\varrho) d\varrho \right| \\
&\leq \frac{1}{\Gamma(\alpha_2)} \int_0^{\varkappa} \left| |\theta_2 - \varrho|^{\alpha_2 - 1} - |\theta_1 - \varrho|^{\alpha_2 - 1} \right| |\varpi_2(\varrho)| d\varrho \\
&\leq \frac{R}{\Gamma(\alpha_2)} \int_0^{\varkappa} \left| |\theta_2 - \varrho|^{\alpha_2 - 1} - |\theta_1 - \varrho|^{\alpha_2 - 1} \right| d\varrho.
\end{aligned}$$

Then, when $\theta_1 \longrightarrow \theta_2$, the right-hand side of the inequalities tend to zero, so we conclude that the operator $\aleph$ is equicontinuous. According to the three steps and the Ascoli-Arzela theorem, we deduce that the operator $\aleph$ has at least a fixed point which is the solution of the coupled system (1)–(2).

□

3.2. Ulam-Hyers-Rassias Stability

Now, we consider the Ulam stability for system (1)–(2). For this, we take inspiration from the following papers [6, 24, 25] and the references therein. Let $(x, y) \in \mathcal{F}$, $\epsilon_1, \epsilon_2 > 0$, and $\chi_1, \chi_2 : \Theta \longrightarrow [0, \infty)$ be continuous functions. We consider the following inequalities:

$$\begin{cases}
\left| {}_0^{RC}D_{\varkappa}^{\alpha_1} x(\theta) - \varphi_1\left(\theta, x(\theta), y(\theta),\ {}_0^{RC}D_{\varkappa}^{\alpha_1} x(\theta),\ {}_0^{RC}D_{\varkappa}^{\alpha_2} y(\theta)\right) \right| \leq \epsilon_1, \theta \in \Theta, \\
\left| {}_0^{RC}D_{\varkappa}^{\alpha_2} y(\theta) - \varphi_2\left(\theta, x(\theta), y(\theta),\ {}_0^{RC}D_{\varkappa}^{\alpha_1} x(\theta),\ {}_0^{RC}D_{\varkappa}^{\alpha_2} y(\theta)\right) \right| \leq \epsilon_2, \theta \in \Theta,
\end{cases} \tag{11}$$

$$\begin{cases}\left|{}_0^{RC}D_{\varkappa}^{\alpha_1}x(\theta)-\varphi_1\left(\theta,x(\theta),y(\theta),\,{}_0^{RC}D_{\varkappa}^{\alpha_1}x(\theta),\,{}_0^{RC}D_{\varkappa}^{\alpha_2}y(\theta)\right)\right|\le\chi_1(\theta),\theta\in\Theta,\\ \left|{}_0^{RC}D_{\varkappa}^{\alpha_2}y(\theta)-\varphi_2\left(\theta,x(\theta),y(\theta),\,{}_0^{RC}D_{\varkappa}^{\alpha_1}x(\theta),\,{}_0^{RC}D_{\varkappa}^{\alpha_2}y(\theta)\right)\right|\le\chi_2(\theta),\theta\in\Theta,\end{cases} \tag{12}$$

and

$$\begin{cases}\left|{}_0^{RC}D_{\varkappa}^{\alpha_1}x(\theta)-\varphi_1\left(\theta,x(\theta),y(\theta),\,{}_0^{RC}D_{\varkappa}^{\alpha_1}x(\theta),\,{}_0^{RC}D_{\varkappa}^{\alpha_2}y(\theta)\right)\right|\le\epsilon_1\chi_1(\theta),\theta\in\Theta,\\ \left|{}_0^{RC}D_{\varkappa}^{\alpha_2}y(\theta)-\varphi_2\left(\theta,x(\theta),y(\theta),\,{}_0^{RC}D_{\varkappa}^{\alpha_1}x(\theta),\,{}_0^{RC}D_{\varkappa}^{\alpha_2}y(\theta)\right)\right|\le\epsilon_2\chi_2(\theta),\theta\in\Theta.\end{cases} \tag{13}$$

Definition 3.5. System (1)–(2) is Ulam-Hyers (U-H) stable if there exists $a_\varphi=\max\{a_{\varphi_1},a_{\varphi_2}\}>0$ such that for each $\epsilon=\max\{\epsilon_1,\epsilon_2\}>0$ and for each solution $(x,y)\in\mathcal{F}$ of inequality (11) there exists a solution $(\bar{x},\bar{y})\in\mathcal{F}$ of (1)–(2) with

$$|(x,y)(\theta)-(\bar{x},\bar{y})(\theta)|\le\epsilon a_\varphi,\qquad \theta\in\Theta.$$

Definition 3.6. System (1)–(2) is generalized Ulam-Hyers (G.U-H) stable if there exists $K_\varphi\in C(\mathbb{R}^+,\mathbb{R}^+)$ with $K_\varphi(0)=0$ such that for each $\epsilon>0$ and for each solution $(x,y)\in\mathcal{F}$ of inequality (11) there exists a solution $(\bar{x},\bar{y})\in\mathcal{F}$ of (1)–(2) with

$$|(x,y)(\theta)-(\bar{x},\bar{y})(\theta)|\le K_\varphi(\epsilon),\qquad \theta\in\Theta.$$

Definition 3.7. (1)–(2) is Ulam-Hyers-Rassias (U-H-R) stable with respect to $\chi=\max\{\chi_1,\chi_2\}\in C(\Theta,\mathbb{R}^+)$ if there exists $a_{\varphi,\chi}=\max\{a_{\varphi_1,\chi_1},a_{\varphi_2,\chi_2}\}>0$ such that for each $\epsilon=\max\{\epsilon_1,\epsilon_2\}>0$ and for each solution $(x,y)\in\mathcal{F}$ of inequality (13) there exists a solution $(\bar{x},\bar{y})\in\mathcal{F}$ of (1)–(2) with

$$|(x,y)(\theta)-(\bar{x},\bar{y})(\theta)|\le\epsilon a_{\varphi,\chi}\chi(\theta),\qquad \theta\in\Theta.$$

Definition 3.8. System (1)–(2) is generalized Ulam-Hyers-Rassias (G.U-H-R) stable with respect to $\chi=\max\{\chi_1,\chi_2\}\in C(\Theta,\mathbb{R}^+)$ if there exists $a_{\varphi,\chi}=\max\{a_{\varphi_1,\chi_1},a_{\varphi_2,\chi_2}\}>0$ such that for each solution $(x,y)\in\mathcal{F}$ of inequality (13) there exists a solution $(\bar{x},\bar{y})\in\mathcal{F}$ of (1)–(2) with

$$|(x,y)(\theta)-(\bar{x},\bar{y})(\theta)|\le a_{\varphi,\chi}\chi(\theta),\qquad \theta\in\Theta.$$

Remark 2. It is clear that:

(1) Definition 3.5 $\Longrightarrow$ Definition 3.6
(2) Definition 3.7 $\Longrightarrow$ Definition 3.8
(3) Definition 3.7 for $\chi(.)=1\Longrightarrow$ Definition 3.5

Remark 3. A function $(x,y)\in\mathcal{F}$ is a solution of inequality (13) if and only if there exist $\upsilon_{\varphi_1},\upsilon_{\varphi_2}\in\mathcal{F}$ such that

(1) $\|\upsilon_{\varphi_1}(\theta)\|\le\epsilon_1\chi_1(\theta)$ and $\|\upsilon_{\varphi_2}(\theta)\|\le\epsilon_2\chi_2(\theta)$, $\theta\in\Theta$.
(2) ${}_0^{RC}D_{\varkappa}^{\alpha_1}x(\theta)=\varphi_1\left(\theta,x(\theta),y(\theta),\,{}_0^{RC}D_{\varkappa}^{\alpha_1}x(\theta),\,{}_0^{RC}D_{\varkappa}^{\alpha_2}y(\theta)\right)+\upsilon_{\varphi_1}(\theta)$, $\theta\in\Theta$.
(3) ${}_0^{RC}D_{\varkappa}^{\alpha_2}y(\theta)=\varphi_2\left(\theta,x(\theta),y(\theta),\,{}_0^{RC}D_{\varkappa}^{\alpha_1}x(\theta),\,{}_0^{RC}D_{\varkappa}^{\alpha_2}y(\theta)\right)+\upsilon_{\varphi_2}(\theta)$, $\theta\in\Theta$.

Theorem 3.9. *Assume that in addition to* (Ax1)–(Ax2) *and* (10), *the following hypothesis holds.*

(Ax3) *There exist nondecreasing functions* $\chi_1, \chi_2 : \Theta \longrightarrow [0, \infty)$ *and* $\kappa_{\chi_1}, \kappa_{\chi_2} > 0$ *such that for each* $\theta \in \Theta$, *we have*

$$({}_0I_{\varkappa}^{\alpha_1}\chi_1)(\theta) \leq \kappa_{\chi_1}\chi_1(\theta)$$

and

$$({}_0I_{\varkappa}^{\alpha_2}\chi_2)(\theta) \leq \kappa_{\chi_2}\chi_2(\theta).$$

Then, the system (1)–(2) *is U-H-R stable with respect to* χ.

P r o o f. Let $(x, y) \in \mathcal{F}$ be a solution if inequality (13), and let us assume that $(\bar{x}, \bar{y})$ is the unique solution of the system

$$\begin{cases} {}_0^{RC}D_{\varkappa}^{\alpha_1}\bar{x}(\theta) = \varphi_1\left(\theta, \bar{x}(\theta), \bar{y}(\theta),\ {}_0^{RC}D_{\varkappa}^{\alpha_1}\bar{x}(\theta),\ {}_0^{RC}D_{\varkappa}^{\alpha_2}\bar{y}(\theta)\right), \\ {}_0^{RC}D_{\varkappa}^{\alpha_2}\bar{y}(\theta) = \varphi_2\left(\theta, \bar{x}(\theta), \bar{y}(\theta),\ {}_0^{RC}D_{\varkappa}^{\alpha_1}\bar{x}(\theta),\ {}_0^{RC}D_{\varkappa}^{\alpha_2}\bar{y}(\theta)\right), \\ \beta_1\bar{x}(0) + \beta_2\bar{x}(\varkappa) = \beta_3, \\ \delta_1\bar{y}(0) + \delta_2\bar{y}(\varkappa) = \delta_3, \\ \bar{x}(0) = x(0),\ \bar{x}(\varkappa) = x(\varkappa), \\ \bar{y}(0) = y(0),\ \bar{y}(\varkappa) = y(\varkappa). \end{cases}$$

By Theorem 3.1, we obtain for each $\theta \in \Theta$

$$\begin{aligned} \bar{x}(\theta) = \frac{\beta_3}{\beta_1 + \beta_2} &- \frac{\beta_1}{(\beta_1 + \beta_2)\Gamma(\alpha_1)} \int_0^{\varkappa} \varrho^{\alpha_1 - 1}\bar{\varpi}_1(\varrho)d\varrho \\ &- \frac{\beta_2}{(\beta_1 + \beta_2)\Gamma(\alpha_1)} \int_0^{\varkappa} |\varkappa - \varrho|^{\alpha_1 - 1}\bar{\varpi}_1(\varrho)d\varrho \\ &+ \frac{1}{\Gamma(\alpha_1)} \int_0^{\varkappa} |\theta - \varrho|^{\alpha_1 - 1}\bar{\varpi}_1(\varrho)d\varrho, \end{aligned}$$

and

$$\begin{aligned} \bar{y}(\theta) = \frac{\delta_3}{\delta_1 + \delta_2} &- \frac{\delta_1}{(\delta_1 + \delta_2)\Gamma(\alpha_2)} \int_0^{\varkappa} \varrho^{\alpha_2 - 1}\bar{\varpi}_2(\varrho)d\varrho \\ &- \frac{\delta_2}{(\delta_1 + \delta_2)\Gamma(\alpha_2)} \int_0^{\varkappa} |\varkappa - \varrho|^{\alpha_2 - 1}\bar{\varpi}_2(\varrho)d\varrho \\ &+ \frac{1}{\Gamma(\alpha_2)} \int_0^{\varkappa} |\theta - \varrho|^{\alpha_2 - 1}\bar{\varpi}_2(\varrho)d\varrho, \end{aligned}$$

where for $i = 1, 2$, $\bar{\varpi}_i \in C(\Theta, \mathbb{R})$ satisfy the following system of functional equations:

$$\begin{cases} \bar{\varpi}_1(\theta) = \varphi_1\big(\theta, \bar{x}(\theta), \bar{y}(\theta), \bar{\varpi}_1(\theta), \bar{\varpi}_2(\theta)\big), \\ \bar{\varpi}_2(\theta) = \varphi_2\big(\theta, \bar{x}(\theta), \bar{y}(\theta), \bar{\varpi}_1(\theta), \bar{\varpi}_2(\theta)\big). \end{cases}$$

$$\begin{cases} \varpi_1(\theta) = \varphi_1\big(\theta, x(\theta), y(\theta), \varpi_1(\theta), \varpi_2(\theta)\big), \\ \varpi_2(\theta) = \varphi_2\big(\theta, x(\theta), y(\theta), \varpi_1(\theta), \varpi_2(\theta)\big). \end{cases}$$

Since x is a solution of the inequality (13), by Remark 3, we have

$$\begin{cases} {}_0^{RC}D_{\varkappa}^{\alpha_1} x(\theta) = \varphi_1\left(\theta, x(\theta), y(\theta),\ {}_0^{RC}D_{\varkappa}^{\alpha_1} x(\theta),\ {}_0^{RC}D_{\varkappa}^{\alpha_2} y(\theta)\right) + \upsilon_{\varphi_1}(\theta), \\ {}_0^{RC}D_{\varkappa}^{\alpha_2} y(\theta) = \varphi_2\left(\theta, x(\theta), y(\theta),\ {}_0^{RC}D_{\varkappa}^{\alpha_1} x(\theta),\ {}_0^{RC}D_{\varkappa}^{\alpha_2} y(\theta)\right) + \upsilon_{\varphi_2}(\theta). \end{cases} \tag{14}$$

Clearly, from (14) we can obtain

$$\begin{aligned} x(\theta) = \frac{\beta_3}{\beta_1+\beta_2} &- \frac{\beta_1}{(\beta_1+\beta_2)\Gamma(\alpha_1)} \int_0^{\varkappa} \varrho^{\alpha_1-1}\big(\varpi_1(\varrho) + \upsilon_{\varphi_1}(\varrho)\big) d\varrho \\ &- \frac{\beta_2}{(\beta_1+\beta_2)\Gamma(\alpha_1)} \int_0^{\varkappa} |\varkappa-\varrho|^{\alpha_1-1}\big(\varpi_1(\varrho) + \upsilon_{\varphi_1}(\varrho)\big) d\varrho \\ &+ \frac{1}{\Gamma(\alpha_1)} \int_0^{\varkappa} |\theta-\varrho|^{\alpha_1-1}\big(\varpi_1(\varrho) + \upsilon_{\varphi_1}(\varrho)\big) d\varrho, \end{aligned}$$

and

$$\begin{aligned} y(\theta) = \frac{\delta_3}{\delta_1+\delta_2} &- \frac{\delta_1}{(\delta_1+\delta_2)\Gamma(\alpha_2)} \int_0^{\varkappa} \varrho^{\alpha_2-1}\big(\varpi_2(\varrho) + \upsilon_{\varphi_2}(\varrho)\big) d\varrho \\ &- \frac{\delta_2}{(\delta_1+\delta_2)\Gamma(\alpha_2)} \int_0^{\varkappa} |\varkappa-\varrho|^{\alpha_2-1}\big(\varpi_2(\varrho) + \upsilon_{\varphi_2}(\varrho)\big) d\varrho \\ &+ \frac{1}{\Gamma(\alpha_2)} \int_0^{\varkappa} |\theta-\varrho|^{\alpha_2-1}\big(\varpi_2(\varrho) + \upsilon_{\varphi_2}(\varrho)\big) d\varrho, \end{aligned}$$

where for $i = 1, 2$, $\varpi_i \in C(\Theta, \mathbb{R})$ satisfy the following system of functional equations:

$$\begin{cases} \varpi_1(\theta) = \varphi_1\big(\theta, x(\theta), y(\theta), \varpi_1(\theta), \varpi_2(\theta)\big), \\ \varpi_2(\theta) = \varphi_2\big(\theta, x(\theta), y(\theta), \varpi_1(\theta), \varpi_2(\theta)\big). \end{cases}$$

Hence, for each $\theta \in \Theta$ we have

$$\begin{aligned}
|x(\theta) - \bar{x}(\theta)| &\leq \frac{1}{\Gamma(\alpha_1)} \int_0^{\varkappa} |\theta - \varrho|^{\alpha_1 - 1} |\varpi_1(\varrho) - \bar{\varpi}_1(\varrho)| d\varrho + ({}_0I_{\varkappa}^{\alpha_1} |\upsilon_{\varphi_1}(\varrho)|) \\
&\leq \epsilon_1 \kappa_{\chi_1} \chi_1(\theta) + \frac{1}{\Gamma(\alpha_1)} \int_0^{\varkappa} |\theta - \varrho|^{\alpha_1 - 1} d\varrho \\
&\quad \times \left[\frac{\psi_4 \bar{\psi}_1 + \psi_1 (1 - \bar{\psi}_4)}{(1 - \bar{\psi}_4)(1 - \psi_3) - \psi_4 \bar{\psi}_3} \|x - \bar{x}\|_\infty \right. \\
&\quad \left. + \frac{\psi_4 \bar{\psi}_2 + \psi_2 (1 - \bar{\psi}_4)}{(1 - \bar{\psi}_4)(1 - \psi_3) - \psi_4 \bar{\psi}_3} \|y - \bar{y}\|_\infty \right] \\
&\leq \epsilon_1 \kappa_{\chi_1} \chi_1(\theta) + \frac{\varkappa^{\alpha_1}}{\Gamma(\alpha_1 + 1)} \left[\frac{\psi_4 \bar{\psi}_1 + \psi_1 (1 - \bar{\psi}_4)}{(1 - \bar{\psi}_4)(1 - \psi_3) - \psi_4 \bar{\psi}_3} \|x - \bar{x}\|_\infty \right. \\
&\quad \left. + \frac{\psi_4 \bar{\psi}_2 + \psi_2 (1 - \bar{\psi}_4)}{(1 - \bar{\psi}_4)(1 - \psi_3) - \psi_4 \bar{\psi}_3} \|y - \bar{y}\|_\infty \right],
\end{aligned}$$

and

$$\begin{aligned}
|y(\theta) - \bar{y}(\theta)| &\leq \frac{1}{\Gamma(\alpha_2)} \int_0^{\varkappa} |\theta - \varrho|^{\alpha_2 - 1} |\varpi_2(\varrho) - \bar{\varpi}_2(\varrho)| d\varrho + ({}_0I_{\varkappa}^{\alpha_2} |\upsilon_{\varphi_2}(\varrho)|) \\
&\leq \epsilon_2 \kappa_{\chi_2} \chi_2(\theta) + \frac{1}{\Gamma(\alpha_2)} \int_0^{\varkappa} |\theta - \varrho|^{\alpha_2 - 1} d\varrho \\
&\quad \times \left[\frac{\psi_1 \bar{\psi}_3 + \bar{\psi}_1 (1 - \psi_3)}{(1 - \bar{\psi}_4)(1 - \psi_3) - \psi_4 \bar{\psi}_3} \|x - \bar{x}\|_\infty \right. \\
&\quad \left. + \frac{\psi_2 \bar{\psi}_3 + \bar{\psi}_2 (1 - \psi_3)}{(1 - \bar{\psi}_4)(1 - \psi_3) - \psi_4 \bar{\psi}_3} \|y - \bar{y}\|_\infty \right] \\
&\leq \epsilon_2 \kappa_{\chi_2} \chi_2(\theta) + \frac{\varkappa^{\alpha_2}}{\Gamma(\alpha_2 + 1)} \left[\frac{\psi_1 \bar{\psi}_3 + \bar{\psi}_1 (1 - \psi_3)}{(1 - \bar{\psi}_4)(1 - \psi_3) - \psi_4 \bar{\psi}_3} \|x - \bar{x}\|_\infty \right. \\
&\quad \left. + \frac{\psi_2 \bar{\psi}_3 + \bar{\psi}_2 (1 - \psi_3)}{(1 - \bar{\psi}_4)(1 - \psi_3) - \psi_4 \bar{\psi}_3} \|y - \bar{y}\|_\infty \right].
\end{aligned}$$

Thus,

$$\begin{aligned}
\|x - \bar{x}\|_\infty \leq \epsilon_1 \kappa_{\chi_1} \chi_1(\theta) + \frac{\varkappa^{\alpha_1}}{\Gamma(\alpha_1 + 1)} &\left[\frac{\psi_4 \bar{\psi}_1 + \psi_1 (1 - \bar{\psi}_4)}{(1 - \bar{\psi}_4)(1 - \psi_3) - \psi_4 \bar{\psi}_3} \right. \\
&\left. + \frac{\psi_4 \bar{\psi}_2 + \psi_2 (1 - \bar{\psi}_4)}{(1 - \bar{\psi}_4)(1 - \psi_3) - \psi_4 \bar{\psi}_3} \right] \|(x, y) - (\bar{x}, \bar{y})\|_{\mathcal{F}},
\end{aligned}$$

and

$$\|y-\bar{y}\|_\infty \le \epsilon_2\kappa_{\chi_2}\chi_2(\theta) + \frac{\varkappa^{\alpha_2}}{\Gamma(\alpha_2+1)}\left[\frac{\psi_1\bar{\psi}_3+\bar{\psi}_1(1-\psi_3)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3} + \frac{\psi_2\bar{\psi}_3+\bar{\psi}_2(1-\psi_3)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\right]\|(x,y)-(\bar{x},\bar{y})\|_{\mathcal{F}}.$$

Then, we may obtain

$$\|(x,y)-(\bar{x},\bar{y})\|_{\mathcal{F}} \le \epsilon\kappa_\chi\chi(\theta) + \mathcal{M}\|(x,y)-(\bar{x},\bar{y})\|_{\mathcal{F}}, \tag{15}$$

where

$$\kappa_\chi = \max\{\kappa_{\chi_1},\kappa_{\chi_2}\},$$

$$\mathcal{M} = \max\{\mathcal{M}_1,\mathcal{M}_2\},$$

$$\mathcal{M}_1 = \frac{\varkappa^{\alpha_1}}{\Gamma(\alpha_1+1)}\left[\frac{\psi_4\bar{\psi}_1+\psi_1(1-\bar{\psi}_4)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3} + \frac{\psi_4\bar{\psi}_2+\psi_2(1-\bar{\psi}_4)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\right],$$

and

$$\mathcal{M}_2 = \frac{\varkappa^{\alpha_2}}{\Gamma(\alpha_2+1)}\left[\frac{\psi_1\bar{\psi}_3+\bar{\psi}_1(1-\psi_3)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3} + \frac{\psi_2\bar{\psi}_3+\bar{\psi}_2(1-\psi_3)}{(1-\bar{\psi}_4)(1-\psi_3)-\psi_4\bar{\psi}_3}\right].$$

From (15), we conclude that for each $\theta\in\Theta$, we have

$$\|(x,y)-(\bar{x},\bar{y})\|_{\mathcal{F}} \le \epsilon a_{\varphi,\chi}\chi(\theta),$$

where

$$a_{\varphi,\chi} = \frac{\kappa_\chi}{1-\mathcal{M}}.$$

Hence, the system (1)–(2) is U-H-R stable with respect to χ. □

Remark 4. If the conditions (Ax1)–(Ax2) and (10) are satisfied, then by Theorem 3.9 and Remark 2, it is clear that system (1)–(2) is U-H-R stable and G.U-H-R stable. And if $\chi(.)=1$, then system (1)–(2) is also G.U-H stable and U-H stable.

4. An Example

Consider the following implicit fractional system which is an example of our system (1)–(2):

$$\begin{cases} {}_0^{RC}D_1^{\frac{1}{2}}x(\theta) = \varphi_1\left(\theta, x(\theta), y(\theta),\ {}_0^{RC}D_1^{\frac{1}{2}}x(\theta),\ {}_0^{RC}D_1^{\frac{1}{2}}y(\theta)\right), \\ {}_0^{RC}D_1^{\frac{1}{2}}y(\theta) = \varphi_2\left(\theta, x(\theta), y(\theta),\ {}_0^{RC}D_1^{\frac{1}{2}}x(\theta),\ {}_0^{RC}D_1^{\frac{1}{2}}y(\theta)\right), \end{cases} \tag{16}$$

where $\theta\in\Theta := [0,1]$, with the boundary conditions

$$\begin{cases} x(0)+x(1)=0, \\ y(0)+y(1)=0, \end{cases} \tag{17}$$

where

$$\alpha_1=\alpha_2=\frac{1}{2}, \quad \beta_1=\beta_2=\delta_1=\delta_2=1 \quad \text{and} \quad \beta_3=\delta_3=0.$$

For $\theta\in[0,1], \xi_1,\xi_2,\xi_3,\xi_4\in\mathbb{R}$, set

$$\varphi_1(\theta,\xi_1,\xi_2,\xi_3,\xi_4)=\frac{\cos(\theta)(|\xi_1|+|\xi_2|+|\xi_3|+|\xi_4|)+\theta+1}{116e^{\theta}(1+|\xi_1|+|\xi_2|+|\xi_3|+|\xi_4|)},$$

and

$$\varphi_2(\theta,\xi_1,\xi_2,\xi_3,\xi_4)=\frac{(\theta^3+1)(|\xi_1|+|\xi_2|+|\xi_3|+|\xi_4|)+\theta^2+2\theta+2}{216e^{\theta}(1+|\xi_1|+|\xi_2|+|\xi_3|+|\xi_4|)}.$$

We observe that φ_1 and φ_2 are continuous functions. And, for any $\xi_j,\bar{\xi}_j\in\mathbb{R}$; $j=1,\ldots,4$, and $\theta\in[0,1]$, we have

$$\begin{aligned} &|\varphi_1(\theta,\xi_1,\xi_2,\xi_3,\xi_4)-\varphi_1(\theta,\bar{\xi}_1,\bar{\xi}_2,\bar{\xi}_3,\bar{\xi}_4)| \\ &\le \frac{1}{116e}\left[|\xi_1-\bar{\xi}_1|+|\xi_2-\bar{\xi}_2|+|\xi_3-\bar{\xi}_3|+|\xi_4-\bar{\xi}_4|\right], \end{aligned}$$

and

$$\begin{aligned} &|\varphi_2(\theta,\xi_1,\xi_2,\xi_3,\xi_4)-\varphi_2(\theta,\bar{\xi}_1,\bar{\xi}_2,\bar{\xi}_3,\bar{\xi}_4)| \\ &\le \frac{2}{216e}\left[|\xi_1-\bar{\xi}_1|+|\xi_2-\bar{\xi}_2|+|\xi_3-\bar{\xi}_3|+|\xi_4-\bar{\xi}_4|\right]. \end{aligned}$$

Then, the condition (*Ax2*) is satisfied with $\psi_j=\dfrac{1}{116e}$ and $\bar{\psi}_j=\dfrac{2}{216e}$, where $j=1,\ldots,4$. Also, we have

$$\Upsilon:=\max\{\mathcal{A}_1,\mathcal{B}_1\}+\max\{\mathcal{A}_2,\mathcal{B}_2\}+\frac{\psi_4\bar{\psi}_3}{(1-\bar{\psi}_4)(1-\psi_3)}<1.$$

Since all the conditions of Theorem 3.3 are satisfied, then the system (16)–(17) has a unique solution on Θ.

Hypothesis (Ax3) is satisfied with $\chi_1(\theta)=\chi_2(\theta)=4\sqrt{\pi}$ and $\kappa_{\chi_1}=\kappa_{\chi_2}=2$. Indeed, for each $\theta\in\Theta$, we get

$$\begin{aligned} {}_0I_1^{\alpha_1}2\sqrt{\pi} &= {}_0I_1^{\alpha_2}2\sqrt{\pi} \\ &\le \frac{\sqrt{\pi}}{\Gamma(\alpha_1)}\int_0^{\theta}(\theta-\rho)^{\alpha_1-1}d\varrho+\frac{\sqrt{\pi}}{\Gamma(\alpha_1)}\int_{\theta}^{1}(\varrho-\theta)^{\alpha_1-1}d\varrho \\ &\le 8\sqrt{\pi}. \end{aligned}$$

Consequently, Theorem 3.9 implies that the system (16)–17 is U-H-R stable.

REFERENCES

[1] ABBAS, S.—BENCHOHRA, M.—GRAEF, J. R.—HENDERSON, J.: *Implicit Differential and Integral Equations: Existence and stability*. In: *De Gruyter Series in Nonlinear Analysis and Applications, Vol. 26.* Walter de Gruyter, Berlin, 2018.

[2] ABBAS, S.—BENCHOHRA, M.—N'GUÉRÉKATA, G. M.: *Topics in Fractional Differential Equations*. In: *Developments in Mathematics Vol. 27*, Springer-Verlag, New York, 2012.

[3] ABBAS, S.—BENCHOHRA, M.—N'GUÉRÉKATA, G M.: *Advanced Fractional Differential and Integral Equations*. Mathematics Research Developments. Nova Science Publishers, Inc., New York, 2015.

[4] AHMAD, B.—ALSAEDI, A.—NTOUYAS, S.K.—TARIBOON, J.: *Hadamard-type Fractional Differential Equations, Inclusions and Inequalities*. Springer, Cham, 2017.

[5] ALI, A.—SAMET, B.—SHAH, K.—KHAN, R. A.: *Existence and stability of solution to a toppled systems of differential equations of non–integer order*, Bound. Value Prob. (2017), Paper no. 16, 13 pp.

[6] ALI, A.—SHAH, K.—JARAD, F.—GUPTA, V.—ABDELJAWAD, T.: *Existence and stability analysis to a coupled system of implicit type impulsive boundary value problems of fractional-order differential equations*, Adv. Difference Equ. **2019** (2019), Paper no. 101, 21 pp.

[7] BALEANU, D.—GÜVENÇ, Z. B.—MACHADO, J. A. T., EDS.: *New Trends in Nanotechnology and Fractional Calculus Applications*. In: Selected papers based on the presentations at the workshop new trends in science and technology (NTST 08), and the workshop fractional differentiation and its applications (FDA 09), Ankara, Türkei, November 2008. Springer, Dordrecht, 2010.

[8] BENKHETTOU, N.—SALIM, A.—AISSANI, K. —BENCHOHRA, M.—KARAPINAR, E.: *Non-instantaneous impulsive fractional integro-differential equations with state--dependent delay*, Sahand Commun. Math. Anal. **19** (2022), 93–109.

[9] BOURIAH, S.—SALIM, A.—BENCHOHRA, M.: *On nonlinear implicit neutral generalized Hilfer fractional differential equations with terminal conditions and delay*, Topol. Algebra Appl. **10** (2022), 77–93.

[10] CHEN, F.—BALEANU, D.—WU G.: *Existence results of fractional differential equations with Riesz-Caputo derivative*, Eur. Phys. J. **226** (2017), 3411–3425.

[11] CHEN, F.—CHEN, A.—WU, X.: *Anti-periodic boundary value problems with Riesz-Caputo derivative*, Adv. Difference Equ. **2019** (2019), Paper no. 119, 13 pp.

[12] DERBAZI, C.—HAMMOUCHE, H.—SALIM, A.—BENCHOHRA, M.: *Measure of noncompactness and fractional hybrid differential equations with hybrid conditions*, Differ. Equ. Appl. **14** (2022), 145–161.

[13] GU, C. Y.—WU, G. C.: *Positive solutions of fractional differential equations with the Riesz space derivative*, Appl. Math. Lett. **95** (2019), 59–64.

[14] HERIS, A.—SALIM, A.—BENCHOHRA, M.—KARAPINAR, E.: *Fractional partial random differential equations with infinite delay*, Results in Physics **37** (2022). `https://doi.org/10.1016/j.rinp.2022.105557`

[15] HYERS, D. H.: *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. USA **27** (1941), 222–224.

[16] KHAN, A.—SHAH, K.—LI, Y.—KHAN, T. S.: *Ulam type stability for a coupled systems of boundary value problems of nonlinear fractional differential equations*, J. Funct. Spaces (2017), Art. ID 3046013, 8. pp.

[17] KILBAS, A. A.—SRIVASTAVA, H. M.—TRUJILLO, J. J.: *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, Vol. 204, Elsevier Science B. V., Amsterdam, 2006.

[18] KRIM, S.—SALIM, A.—ABBAS, S.—BENCHOHRA, M.: *On implicit impulsive conformable fractional differential equations with infinite delay in b-metric spaces*, Rend. Circ. Mat. Palermo Ser. 2 (2022), 1–14.

[19] LAZREG, J. E.—BENCHOHRA, M.—SALIM, A.: *Existence and Ulam stability of k--generalized ψ-Hilfer fractional problem*, J. Innov. Appl. Math. Comput. Sci. **2** (2022), 1–13.

[20] LI, M.—WANG, Y.: *Existence and iteration of monotone positive solutions for fractional boundary value problems with Riesz-Caputo derivative*, Engineering Letters **29** (2021), 327–331.

[21] LUO, D.—LUO, Z.—QIU, H.: *Existence and Hyers-Ulam stability of solutions for a mixed fractional-order nonlinear delay difference equation with parameters*, Math. Probl. Eng. **2020** (2020), Art. ID 9372406, 12 pp.

[22] MONCH, H.: *Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces*, Nonlinear Anal. **4** (1980), 985–999.

[23] NAAS, A.—BENBACHIR, M.—ABDO, M. S.—BOUTIARA, A.: *Analysis of a fractional boundary value problem involving Riesz-Caputo fractional derivative*, Adv. Theory Nonlinear Anal. Appl. **6** (2022) no. 1, 14–27.

[24] RASSIAS, T. M.: *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), **72** (1978), 297–300.

[25] RUS, I.: *Ulam stability of ordinary differential equations in a Banach space*, Carpathian J. Math. **26** (2011), 103–107.

[26] SALIM, A.—AHMAD, B. —BENCHOHRA, M.—LAZREG, J. E.: *Boundary value problem for hybrid generalized Hilfer fractional differential equations*, Differ. Equ. Appl. **1**4 (2022), 379–391.

[27] SALIM, A.—BENCHOHRA, M.—GRAEF, J. R.—LAZREG, J. E.: *Initial value problem for hybrid ψ-Hilfer fractional implicit differential equations*, J. Fixed Point Theory Appl. **24** (2022), 14 pp.

[28] SALIM, A.—BENCHOHRA, M.—LAZREG, J. E.: *Nonlocal k-generalized ψ-Hilfer impulsive initial value problem with retarded and advanced arguments*, Appl. Anal. Optim. **6** (2022), 21–47.

[29] SALIM, A.—BENCHOHRA, M.—LAZREG, J. E.—HENDERSON, J.: *On k-generalized ψ-Hilfer boundary value problems with retardation and anticipation*, Adv. Theory of Nonlinear Anal. and Appl. **6** (2022), no. 2, 173–190.

[30] SALIM, A.—BENCHOHRA, M.—LAZREG, J. E.—KARAPINAR, E.: *On k-generalized ψ-Hilfer impulsive boundary value problem with retarded and advanced arguments*, J. Math. Ext. **15** (2021), 1–39.

[31] SALIM, A.—LAZREG, J. E.—AHMAD, B.—BENCHOHRA, M.—NIETO, J. J.: *A study on k-generalized ψ-Hilfer derivative operator*, Vietnam J. Math. (2022), 1–19.

[32] SHAH, K.—TUNC, C.: *Existence theory and stability analysis to a system of boundary value problem*, J. Taibah Univ. Sci. **11** (2017), no. 6, 1330–1342

[33] SMART, D. R.: *Fixed Point Theory*. Cambridge Univ. Press, Cambridge, 1974.

[34] ULAM, S. M.: *Problems in Modern Mathematics*, Science Editions John Wiley & Sons, Inc., New York, 1964.

[35] WANG, J.—ZADA, A.—WAHEED, H.: *Stability analysis of a coupled system of nonlinear implicit fractional anti-periodic boundary value problem*, Math Meth Appl Sci. **42** (2019), 6706–6732.

Received December 8, 2022

Abdelkrim Salim
Faculty of Technology
Hassiba Benbouali University of Chlef
P.O. Box 151 Chlef 02000
ALGERIA
E-mail: salim.abdelkrim@yahoo.com

Jamal Eddine Lazreg
Mouffak Benchohra
Laboratory of Mathematics
Djillali Liabes University of Sidi Bel-Abbes
P.O. Box 89 Sidi Bel Abbes 22000
ALGERIA
E-mail: lazregjamal@yahoo.fr
benchohra@yahoo.com