*Existence, Uniqueness, Stability, and Monotone Dependence in a Stefan Problem for the Heat Equation**

J. R. CANNON & C. DENSON HILL

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1. Introduction. Consider the following Stefan problem for the heat equation: Given the data f, φ, and b, find two functions $s = s(t)$ and $u = u(x, t)$ such that the pair (s, u) satisfies both

$$(1.1)\qquad \begin{cases} Lu \equiv u_{xx} - u_t = 0 & \text{in} \quad 0 < x < s(t), \qquad 0 < t \leqq T \\ u(0, t) = f(t) \geqq 0 & \text{and} \quad u(s(t), t) = 0 \quad \text{for} \quad 0 \leqq t \leqq T \\ u(x, 0) = \varphi(x) \geqq 0 & \text{for} \quad 0 \leqq x \leqq b, \qquad s(0) = b \geqq 0 \end{cases}$$

and the free boundary condition

$$(1.2)\qquad \dot{s}(t) = -u_x(s(t), t) \quad \text{for} \quad 0 < t \leqq T.$$

Here T is a fixed but arbitrary positive number.

In this paper we prove the global existence and uniqueness of the solution (s, u) of the problem (1.1), (1.2) and show that the free boundary $x = s(t)$ depends continuously and monotonically on the data f, φ, and b. The main tool used in our analysis is the maximum principle, both in its strong form [10], and in the form of the parabolic version of Hopf's Lemma [4]. The constructive element in our approach is based on the idea of *retarding the argument* (*c.f.*, equation (3.3)) in the free boundary condition (1.2). Although we present the proofs only in the case of the heat operator in (1.1), our technique applies in principle to the case of a linear parabolic operator of the second order with variable coefficients in which the coefficient of u is nonpositive so that the maximum principle holds.

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The assumptions required on the Stefan data are as follows:

If $b > 0$ we make the assumptions (A): f and φ are nonnegative, the combination f, φ is continuous except possibly for a finite number of bounded jumps, and there exists a positive constant N such that

$$0 \leqq \varphi(x) \leqq N(b - x) \quad \text{for} \quad 0 \leqq x \leqq b. \tag{1.3}$$

If $b = 0$, there is no φ, and we make the assumptions (B): f is continuous except possibly for a finite number of bounded jumps, and there exists positive constants l and L such that

$$lt \leqq f(t) \leqq Lt \quad \text{for} \quad 0 \leqq t \leqq T. \tag{1.4}$$

The piecewise continuity requirement on the data is made here merely for the sake of being specific. Likewise the Lipschitz assumption (1.4) can be relaxed (*c.f.*, section 7).

By a solution u of (1.1) for a given continuous $s(t)$ that is positive for $t > 0$, we mean a function $u = u(x, t)$ such that:

1°. $u_{xx}, u_t \in C$ and $u_{xx} - u_t = 0$ in $0 < x < s(t), 0 < t \leqq T$.
2°. $u \in C$ on $0 \leqq x \leqq s(t), 0 \leqq t \leqq T$, except at points of discontinuity of f or φ, and at points of continuity has the boundary values indicated in (1.1).
3°. At a point of discontinuity of f or φ, $0 \leqq \underline{\lim}\, u \leqq \overline{\lim}\, u < \infty$ as the point is approached from the interior of the region.

It is well known that such a solution u of (1.1) is unique.

By a solution (s, u) of (1.1), (1.2), we mean that $s(0) = b$ and $s(t) > 0$, $s(t) \in C^1$ for $0 < t \leqq T$, that $u(x, t)$ is the corresponding solution of (1.1), and that $u_x(s(t), t)$ exists, is continuous, and satisfies (1.2).

The existence and uniqueness of the solutions of problems similar to (1.1), (1.2) has been established by several authors ([3], [5], [8], [9]; see [6] for more references) under various hypotheses on the data. Global stability for the problem in which u_x rather than u is prescribed at $x = 0$ was established in [2]. In [9] the infinite differentiability of the free boundary $x = s(t)$ was discussed.

The advantage of our existence proof lies in the fact that it is simple and direct and that it requires minimal smoothness assumptions on the data f and φ. In particular, it achieves a global result in one step rather than require the piecing together of individual results for small time intervals. Our uniqueness result follows from a stability theorem, the proof of which is based on the ideas of [2]. Theorems 6 and 7 settle some questions raised by Friedman at the end of [5].

2. Reformulation of the boundary condition. We set $Lu = 0$ and $v = -x$ in Stoke's theorem

$$\int_D \{vLu - uMv\}\, dx\, dt = \int_{\partial D} \{(vu_x - uv_x)\, dt + uv\, dx\},$$

where M is the adjoint of L, to obtain the identity

$$\int_{\partial D} \{(u - xu_x)\, dt - xu\, dx\} = 0. \tag{2.1}$$

Let (s, u) be a solution of (1.1), (1.2). By applying (2.1) to the region D: $0 \leqq \xi \leqq s(\tau)$, $0 < \sigma \leqq \tau \leqq t \leqq T$ and letting σ tend to zero, we obtain the relation

$$s(t)^2 = b^2 + 2\int_0^t f(\tau)\, d\tau + 2\int_0^b \xi\varphi(\xi)\, d\xi - 2\int_0^{s(t)} \xi u(\xi, t)\, d\xi. \tag{2.2}$$

Conversely if (s, u) satisfies (1.1), (2.2) and u_x exists and is continuous at the boundary $x = s(t)$, then it follows from (2.1) that

$$s(t)^2 = b^2 - 2\int_0^t s(\tau)u_x(s(\tau), \tau)\, d\tau \tag{2.3}$$

and, upon differentiating (2.3), that (1.2) is satisfied. Therefore (1.1), (2.2) is apparently a more general formulation of the Stefan problem than (1.1), (1.2). In the Appendix, however, we prove

Lemma 1. Under the assumptions (A) (*or* (B)), *let u be a solution of* (1.1) *and $s(t)$ be Lipschitz continuous on $0 \leqq t \leqq T$. Then $u_x(s(t), t)$ exists and is continuous for $0 < t \leqq T$.*

Thus within the class of Lipschitz continuous boundaries the two formulations (1.1), (1.2) and (1.1) ,(2.2) of the Stefan problem are actually equivalent.

3. Existence ($b > 0$). First we need

Lemma 2. Under the assumptions (A), *let u be a solution of* (1.1) *and $s(t)$ be a monotonic nondecreasing function. Then there exists a positive constant* A *depending only on b,*

$$M = \max\,(\max_{0 \leqq t \leqq T} f(t), \max_{0 \leqq x \leqq b} \varphi(x)),$$

and the Lipschitz constant N in (1.3) *such that*

$$0 \leqq \rho^{-1}u(s(t) - \rho, t) \leqq A \tag{3.1}$$

for all $0 \leqq t \leqq T$ and $0 < \rho < b$.

Proof. Since $f, \varphi \geqq 0$ it follows from the maximum principle that $u \geqq 0$. Set

$$A \equiv \max\,\{Mb^{-1}, N\}$$

and define for each t_0 , $0 \leqq t_0 \leqq T$,

$$\omega(x, t_0) \equiv A[s(t_0) - x]. \tag{3.2}$$

Observe that

$$L\omega \equiv (\partial_x^2 - \partial_t)\omega \equiv 0$$

and that $\omega(0, t_0) = As(t_0) \geqq Ab \geqq M \geqq f$ since $0 < b = s(0) \leqq s(t_0)$. At $t = 0$,

$$\omega(x, t_0) \geqq A(b - x) \geqq N(b - x) \geqq \varphi.$$

For $0 \leqq t \leqq t_0$ we have along $x = s(t)$

$$\omega(s(t), t_0) = A[s(t_0) - s(t)] \geqq 0.$$

Hence by the maximum principle $u(x, t) \leqq \omega(x, t_0)$ in the region $0 \leqq x \leqq s(t)$, $0 \leqq t \leqq t_0$. Therefore

$$0 \leqq \rho^{-1}u(s(t_0) - \rho, t_0) \leqq \rho^{-1}A\rho = A$$

for any $0 \leqq t_0 \leqq T$, which completes the proof of Lemma 2.

For each θ, $0 < \theta < b$, we construct a family (s^θ, u^θ) of approximations to the solution of (1.1), (1.2) by *retarding the argument*

$$\dot{s}(t) = -u_x(s(t - \theta), t - \theta) \tag{3.3}$$

in the free boundary condition (1.2). Let

$$\chi^\theta = \begin{cases} 1, & 0 \leqq x \leqq b - \theta \\ 0, & b - \theta < x \leqq b \end{cases}$$

and $\varphi^\theta = \chi^\theta\varphi$. In the first interval $0 \leqq t \leqq \theta$ we set $s^\theta(t) \equiv b$ and define $u^\theta(x, t)$ to be the unique solution of (1.1) in the region $0 \leqq x \leqq s^\theta(t)$, $0 \leqq t \leqq \theta$, in which s and φ have been replaced by s^θ and φ^θ, respectively. It is easy to verify that, due to our initial choice of s^θ and φ^θ, $u_x^\theta(b, t)$ exists and is continuous for $0 \leqq t \leqq \theta$. Moreover, by Lemma 2, we have $-A \leqq u_x^\theta(b, t) \leqq 0$. Now we proceed by induction. Assume that (s^θ, u^θ) has been constructed for $0 \leqq t \leqq n\theta$, that s^θ is a C^1 function, that $u_x^\theta(s^\theta(t), t)$ exists and is continuous, that $-A \leqq u_x^\theta(s^\theta(t), t) \leqq 0$, and that

$$s^\theta(t) = b - \int_\theta^t u_x^\theta(s^\theta(\tau - \theta), \tau - \theta)\, d\tau \quad \text{for} \quad t \geqq \theta. \tag{3.4}$$

In the next step $n\theta \leqq t \leqq (n + 1)\theta$ we define $s^\theta(t)$ by (3.4) and solve (1.1) for $u^\theta(x, t)$ up one more step into the region $0 \leqq x \leqq s^\theta(t)$, $n\theta \leqq t \leqq (n + 1)\theta$. By the inductive hypothesis on u_x^θ, $s^\theta(t)$ is a C^1 function satisfying $0 \leqq \dot{s}^\theta(t) \leqq A$. Hence by Lemma 1, $u_x^\theta(s^\theta(t), t)$ exists and is continuous, and from Lemma 2, we have $-A \leqq u_x^\theta(s^\theta(t), t) \leqq 0$. Thus the approximating solutions (s^θ, u^θ) can be constructed throughout the interval $0 \leqq t \leqq T$.

We summarize the results of the above construction with

Lemma 3. *For each θ, $0 < \theta < b$, there exists a solution (s^θ, u^θ) of* (1.1) *in which $\varphi = \varphi^\theta$. The function $s^\theta(t)$ is C^1 for $0 \leqq t \leqq T$, satisfies* (3.3), *and*

$$0 \leqq \dot{s}^\theta(t) \leqq A \quad \textit{for} \quad 0 \leqq t \leqq T. \tag{3.5}$$

Theorem 1. *Under the assumptions* (A), *there exists a solution* (s, u) *to the Stefan problem* (1.1), (1.2) *when* $b > 0$. *The free boundary* $s(t) \in C^1(o, T]$, *is monotonically nondecreasing, and satisfies*

$$0 \leqq \dot{s}(t) \leqq A \quad for \quad 0 \leqq t \leqq T,$$

where A *is defined by Lemma* 2.

Proof. According to (3.5) the functions $s^\theta(t)$ form an equicontinuous uniformly bounded,

$$b \leqq s^\theta(t) \leqq b + AT,$$

family. Choose a sequence of θ's tending to zero. By Arzela's Theorem there is a subsequence, denote it by $s^\theta(t)$, that converges uniformly to a monotonic Lipschitz continuous function $s(t)$. Let $u(x, t)$ be the unique solution of (1.1) for that choice of s. Given any $\epsilon > 0$ we have, for $0 \leqq t \leqq T$, $0 \leqq x \leqq b$, and all θ's of the subsequence that are sufficiently small,

$$|s^\theta(t) - s(t)| \leqq A^{-1}\epsilon = \rho,$$
$$|\varphi^\theta(x) - \varphi(x)| \leqq N\theta \leqq \epsilon.$$

For convenience we extend the functions u and u^θ by setting them identically equal to zero outside their natural domains of definition. Then in any region $0 \leqq x \leqq \max(s^\theta(t), s(t))$, $0 \leqq t \leqq T$, the difference $u^\theta - u$ is bounded, using the maximum principle and Lemma 2, by

$$|u^\theta(x, t) - u(x, t)| \leqq \max(\epsilon, A\rho) = \epsilon.$$

Thus the corresponding subsequence u^θ converges uniformly to u.

Applying (2.1) to the region $0 \leqq \xi \leqq s^\theta(\tau)$, $0 \leqq \tau \leqq t \leqq T$, we find

$$(3.6)\quad 2\int_0^t s^\theta(\tau)\dot{s}^\theta(\tau + \theta)\, d\tau = 2\int_0^t f(\tau)\, d\tau + 2\int_0^b \xi\varphi^\theta(\xi)\, d\xi - 2\int_0^{s^\theta(t)} \xi u^\theta(\xi, t)\, d\xi.$$

The left-hand side of (3.6) may be written as

$$s^\theta(t + \theta)^2 - s^\theta(\theta)^2 - 2\int_0^t [s^\theta(\tau + \theta) - s^\theta(\tau)]\dot{s}^\theta(\tau + \theta)\, d\tau$$
$$= s^\theta(t + \theta)^2 - b^2 + O(\theta),$$

by virtue of (3.5). Hence, by taking the limit in (3.6) as the subsequence θ tends to zero, it follows from the uniform convergence of s^θ to s, φ^θ to φ, and u^θ to u that (s, u) satisfies (2.2). This means that (s, u) is a solution to the problem (1.1), (2.2). Since s is Lipschitz continuous, Lemma 1 applies. Therefore (s, u) is actually a solution to the problem (1.1), (1.2) and $s(t)$ is C^1 for $0 < t \leqq T$. Q.E.D.

4. Existence ($b = 0$). First we need

Lemma 4. *Under the assumptions* (B) *on* f, *let* (s, u) *be a solution of* (1.1), (1.2) *with* $b \geqq 0$. *Then there exists a positive constant* λ *independent of* b *such that*

$$\lambda t \leqq s(t) \quad \text{for} \quad 0 \leqq t \leqq T. \tag{4.1}$$

Proof. Consider first the case $b > 0$. Set

$$\lambda \equiv [T^{-1} \log (1 + lT)]^{1/2}$$

and

$$v(x, t) \equiv \exp \{-\lambda(x - \lambda t)\} - 1.$$

Initially $s(0) = b > 0$. Assuming the contrary of (4.1), let

$$t_0 = \inf \{t \mid \lambda t > s(t)\} < T.$$

Then clearly $\dot{s}(t_0) \leqq \lambda$. Observe that

$$Lv \equiv (\partial_x^2 - \partial_t)v = 0$$

and at $x = 0$, $v = \exp \{\lambda^2 t\} - 1 < lt \leqq f$ for $0 < t < T$. Along $x = \lambda t$, for $0 \leqq t \leqq t_0$, we have $v = 0$ and $u \geqq 0$. Hence by the strong maximum principle $v < u$ in the region $0 < x < \lambda t$, $0 < t \leqq t_0$. Since both v and u vanish at the point $(\lambda t_0, t_0)$ it follows from the parabolic version of Hopf's Lemma that

$$\dot{s}(t_0) = -u_x(\lambda t_0, t_0) > -v_x(\lambda t_0, t_0) = \lambda,$$

which is a contradiction.

Now consider the case $b = 0$. For any ϵ, $0 < \epsilon < T$, we have $l(t - \epsilon) < lt \leqq f(t)$ in the interval $\epsilon \leqq t \leqq T$. Since $s(\epsilon) > 0$ the argument given in the preceding paragraph applies. We conclude that $\lambda(t - \epsilon) \leqq s(t)$ in the interval $\epsilon \leqq t \leqq T$. Taking the limit as ϵ tends to zero, we obtain (4.1).

Next we prove

Lemma 5. *Under the assumptions* (B) *on* f, *let* u *be a solution of* (1.1) *with* $s(t)$ *nondecreasing and* $\lambda t \leqq s(t)$ *for* $0 \leqq t \leqq T$ *and* $\varphi \equiv 0$ *if* $b > 0$. *Then there exists a positive constant* B *independent of* b *such that*

$$0 \leqq \rho^{-1} u(s(t) - \rho, t) \leqq B \tag{4.2}$$

for all $0 \leqq t \leqq T$ *and* $0 < \rho < s(t)$.

Proof. Set $B = L\lambda^{-1}$ and define

$$\omega(x, t_0) \equiv B[s(t_0) - x].$$

At $t = 0$ and along $x = s(t)$ for $0 \leqq t \leqq t_0$ we have $\omega \geqq 0 = u$. Also,

$$\omega(0, t_0) = Bs(t_0) \geqq B\lambda t_0 = Lt_0 \geqq Lt \geqq f(t)$$

for $0 \leqq t \leqq t_0$. Hence by the maximum principle $u(x, t) \leqq \omega(x, t_0)$ in the region $0 \leqq x \leqq s(t)$, $0 \leqq t \leqq t_0$. It follows that

$$0 \leqq \rho^{-1}u(s(t_0) - \rho, t_0) \leqq \rho^{-1}B\rho = B$$

for any $0 \leqq t_0 \leqq T$, which completes the proof of Lemma 5.

For each b, $0 < b < b_0$, we set $\varphi^b(x) \equiv 0$ for $0 \leqq x \leqq b$. By Theorem 1 there exists a solution (s^b, u^b) of (1.1), (1.2) in which $\varphi = \varphi^b$. Examination of the proof of Lemma 1 shows that in this case $s^b(t) \in C^1$ for $0 \leqq t \leqq T$.

As a corollary to Lemmas 4 and 5 we have

Lemma 6. *Under the assumptions* (B),

$$0 \leqq \dot{s}^b(t) \leqq B \quad \text{for} \quad 0 \leqq t \leqq T. \tag{4.3}$$

Theorem 2. *Under the assumptions* (B), *there exists a solution* (s, u) *to the Stefan problem* (1.1), (1.2) *when* $b = 0$. *The free boundary* $s(t) \in C^1(0, T]$, *is monotonically nondecreasing, and satisfies* $\lambda t \leqq s(t) \leqq Bt$ *and* $0 \leqq \dot{s} \leqq B$ *for* $0 \leqq t \leqq T$.

Proof. By Lemma 6 the functions $s^b(t)$, $0 < b < b_0$, form an equicontinuous uniformly bounded,

$$0 \leqq s^b(t) \leqq b_0 + BT,$$

family. As in the proof of Theorem 1, we choose a sequence of b's tending to zero and apply Arzela's Theorem to obtain a subsequence (s^b, u^b) that converges uniformly to (s, u). From (2.2), using the uniform convergence, we obtain

$$s(t)^2 = 2\int_0^t f(\tau)\, d\tau - 2\int_0^{s(t)} \xi u(\xi, t)\, d\xi. \tag{4.4}$$

Since the limit function $s(t)$ is Lipschitz continuous and we have (4.2), Lemma 1 applies. Comparison of (4.4) with the application of (2.1) to (s, u) yields (2.3) and hence (1.2). Q.E.D.

5. Stability. In this section we derive an *a priori* estimate of the dependence of the free boundary on the Stefan data. The uniqueness of the solution to the Stefan problem is an obvious corollary of such an estimate.

Throughout this section we shall use

Lemma 7. *Let* $y(t)$ *satisfy*

$$0 \leqq y(t) \leqq p(t) + q\int_0^t \frac{y(\tau)}{(t-\tau)^{1/2}}\, d\tau, \qquad 0 \leqq t \leqq T, \tag{5.1}$$

where $q \geqq 0$ *and* $p(t)$ *is nonnegative and nondecreasing. Then*

$$0 \leqq y(t) \leqq [1 + 2qt^{1/2}]p(t)e^{\pi q^2 t} \leqq C_0 p(t) \tag{5.2}$$

with

$$C_0 = [1 + 2qT^{1/2}] \exp\{\pi q^2 T\}.$$

The proof of Lemma 7 consists in applying the technique used to solve Abel integral equations to (5.1) to derive

$$0 \leqq y(t) \leqq [1 + 2qt^{1/2}]p(t) + \pi q^2 \int_0^t y(\tau)\, d\tau,$$

and then using the standard Gronwall type estimate [1, Lemma 2, page 380].

First we shall treat the case $b > 0$.

Theorem 3. *For $i = 1, 2$, let (s_i, u_i) be solutions of* (1.1), (1.2) *corresponding to data $\{f_i, \varphi_i, b_i\}$ satisfying* (A) *and such that $0 < b_1 \leqq b_2$. Then there exists a positive constant $C = C(b_1, A, T)$ such that*

$$(5.3) \qquad |s_2(t) - s_1(t)| \leqq C\left\{(b_2 - b_1) + \int_0^t |f_2(\tau) - f_1(\tau)|\, d\tau + \int_0^{b_1} \xi\, |\varphi_2(\xi) - \varphi_1(\xi)|\, d\xi + \int_{b_1}^{b_2} \xi\varphi_2(\xi)\, d\xi\right\}$$

for $0 \leqq t \leqq T$.

Proof. Set $\alpha(t) \equiv \min(s_1(t), s_2(t))$, $\beta(t) \equiv \max(s_1(t), s_2(t))$, $\delta(t) \equiv \beta(t) - \alpha(t)$, and

$$j(t) \equiv \begin{cases} 2, & s_1(t) \leqq s_2(t) \\ 1, & s_1(t) > s_2(t). \end{cases}$$

It follows from Lemma 2 that $s_1(t)$, $s_2(t)$, and hence also $\alpha(t)$, $\beta(t)$, are Lipschitz continuous. Since $s_1(t) + s_2(t) \geqq b_1 + b_2 \geqq 2b_1$, relation (2.2) yields

$$(5.4) \qquad \delta(t) \leqq (b_2 - b_1) + \frac{1}{b_1}\int_0^t |f_2(\tau) - f_1(\tau)|\, d\tau + \frac{1}{b_1}\int_0^{b_1} \xi\, |\varphi_2(\xi) - \varphi_1(\xi)|\, d\xi + \frac{1}{b_1}\int_{b_1}^{b_2} \xi\varphi_2(\xi)\, d\xi + \frac{1}{b_1}\int_0^{\alpha(t)} \xi\, |u_2(\xi, t) - u_1(\xi, t)|\, d\xi + \frac{1}{b_1}\int_{\alpha(t)}^{\beta(t)} \xi u_{j(t)}(\xi, t)\, d\xi = (b_2 - b_1) + I_1 + I_2 + I_3 + I_4 + I_5 .$$

First we estimate I_4. According to Lemma 2, $|u_2(\alpha(t), t) - u_1(\alpha(t), t)| \leqq A\delta(t)$. Hence on $0 \leqq x \leqq \alpha(t)$, $0 \leqq t \leqq T$, $u_2 - u_1$ is dominated by $W_1 + W_2$ where W_1, W_2 are the solutions of

$$(5.5) \qquad \begin{cases} W_{1_{xx}} - W_{1_t} = 0; & 0 < x < \infty, \quad 0 < t < T \\ W_1(0, t) = |f_2(t) - f_1(t)|, & 0 \leqq t \leqq T \\ W_1(x, 0) = \begin{cases} |\varphi_2(x) - \varphi_1(x)|, & 0 \leqq x \leqq b_1 \\ 0, & b_1 < x < \infty \end{cases} \end{cases}$$

and

$$(5.6)\qquad \begin{cases} W_{2xx} - W_{2t} = 0; & 0 < x < \alpha(t), \quad 0 < t < T \\ W_2(0, t) = 0, & 0 \leqq t \leqq T \\ W_2(x, 0) = 0, & 0 \leqq x \leqq b_1 \\ W_2(\alpha(t), t) = A\ \delta(t), & 0 \leqq t \leqq T, \end{cases}$$

respectively.

Application of (2.1) to (5.5) yields

$$(5.7)\qquad \begin{aligned} \frac{1}{b_1}\int_0^{\alpha(t)} \xi W_1(\xi, t)\, d\xi &< \frac{1}{b_1}\int_0^{\infty} \xi W_1(\xi, t)\, d\xi \\ &= \frac{1}{b_1}\int_0^t |f_2(\tau) - f_1(\tau)|\, d\tau + \frac{1}{b_1}\int_0^{b_1} \xi\, |\varphi_2(\xi) - \varphi_1(\xi)|\, d\xi \\ &= I_1 + I_2 . \end{aligned}$$

The solution of (5.6) can be written in the form

$$W_2(x, t) = \int_0^t \mu(\tau)\{K_x(x, t; -\alpha(\tau), \tau) + K_x(x, t; \alpha(\tau), \tau)\}\, d\tau$$

where

$$K(x, t; \xi, \tau) \equiv \frac{1}{2\pi^{1/2}(t - \tau)^{1/2}} \exp\left\{-\frac{(x - \xi)^2}{4(t - \tau)}\right\}$$

and $\mu(t)$ is the solution of

$$(5.8)\qquad A\ \delta(t) = \tfrac{1}{2}\mu(t) + \int_0^t \mu(\tau)\{K_x(\alpha(t), t; -\alpha(\tau), \tau) + K_x(\alpha(t), t; \alpha(\tau), \tau)\}\, d\tau.$$

Since $\alpha(t)$ is Lipschitz continuous with Lipschitz constant A, straightforward estimation of the kernel in (5.8) shows that

$$(5.9)\qquad 0 \leqq |\mu(t)| \leqq 2A\ ||\delta||_t + C\int_0^t \frac{|\mu(\tau)|}{(t - \tau)^{1/2}}\, d\tau,$$

where

$$||\delta||_t = \max_{0 \leqq \tau \leqq t} |\delta(\tau)|,$$

and where C denotes here, and in what is to follow, a generic constant depending only on b_1, A, and T. Lemma 7 applied to (5.9) implies

$$|\mu(t)| \leqq C\ ||\delta||_t .$$

Hence

$$
\begin{aligned}
(5.10)\qquad & \frac{1}{b_1}\int_0^{\alpha(t)} \xi W_2(\xi, t)\, d\xi \\
&= \frac{1}{b_1}\int_0^t \mu(\tau)\left\{\int_0^{\alpha(t)} \xi[K_x(\xi, t; -\alpha(\tau), \tau) + K_x(\xi, t; \alpha(\tau), \tau)]\, d\xi\right\} d\tau \\
&= \frac{1}{b_1}\int_0^t \mu(\tau)\left\{\alpha(t)[K(\alpha(t), t; -\alpha(\tau), \tau) + K(\alpha(t), t; \alpha(\tau), \tau)]\right. \\
&\qquad \left. - \int_0^{\alpha(t)} [K(\xi, t; -\alpha(\tau), \tau) + K(\xi, t; \alpha(\tau), \tau)]\, d\xi\right\} d\tau \\
&\leqq C\int_0^t \frac{|\mu(\tau)|}{(t-\tau)^{1/2}}\, d\tau \leqq C\int_0^t \frac{\|\delta\|_\tau}{(t-\tau)^{1/2}}\, d\tau.
\end{aligned}
$$

Now we estimate I_5. When $\delta(t) = 0$, I_5 vanishes. When $\delta(t) > 0$ two cases arise: either $\delta(\tau) > 0$ for $0 \leqq \tau \leqq t$ or else there is at least one t^*, $0 \leqq t^* < t$, such that $\delta(t^*) = 0$. Let $t_0 = 0$ in the first case and $t_0 = \max\{t^* \mid 0 \leqq t^* < t, \delta(t^*) = 0\}$ in the second case. Then $j(\tau) \equiv$ const. on the interval $[t_0, t]$. In either case $u_{j(t)}$ is dominated by $Z_1 + Z_2$ where Z_1, Z_2 are the solutions of

$$
(5.11)\qquad \begin{cases} Z_{1_{xx}} - Z_{1_t} = 0, & \alpha(t) < x < \beta(t), \quad 0 < t \leqq T \\ Z_1(\alpha(t), t) = Z_1(\beta(t), t) = 0, & 0 < t \leqq T \\ Z_1(x, 0) = \varphi_2(x), & b_1 \leqq x \leqq b_2 \end{cases}
$$

and

$$
(5.12)\qquad \begin{cases} Z_{2_{xx}} - Z_{2_t} = 0, & \alpha(t) < x < \infty, \quad t_0 < t \leqq T \\ Z_2(\alpha(t), t) = A\ \delta(t), & t_0 < t \leqq T \\ Z_2(x, t_0) = 0, & \alpha(t_0) \leqq x < \infty, \end{cases}
$$

respectively.

Since $\varphi_2(x) \geqq 0$ for $b_1 \leqq x \leqq b_2$, it follows from the maximum principle that $Z_{1_x}(\alpha(t), t) \geqq 0$ and $Z_{1_x}(\beta(t), t) \leqq 0$. Hence upon applying (2.1) we have

$$
(5.13)\qquad \frac{1}{b_1}\int_{\alpha(t)}^{\beta(t)} \xi Z_1(\xi, t)\, d\xi \leqq \frac{1}{b_1}\int_{b_1}^{b_2} \xi\varphi_2(\xi)\, d\xi = I_3 .
$$

The solution of (5.12) can be written in the form

$$
(5.14)\qquad Z_2(x, t) = \int_{t_0}^t \sigma(\tau) K_x(x, t, \alpha(\tau), \tau)\, d\tau
$$

where $\sigma(t)$ is the solution of

$$
(5.15)\qquad A\ \delta(t) = -\tfrac{1}{2}\sigma(t) + \int_{t_0}^t \sigma(\tau) K_x(\alpha(t), t, \alpha(\tau), \tau)\, d\tau.
$$

It follows by a derivation analogous to (5.10) that

$$(5.16)\qquad \int_{\alpha(t)}^{\beta(t)} \xi Z_2(\xi, t)\, d\xi \leqq \int_{\alpha(t)}^{\infty} \xi Z_2(\xi, t)\, d\xi \leqq C(A, T) \int_{t_0}^{t} \frac{\|\delta\|_\tau}{(t-\tau)^{1/2}}\, d\tau$$
$$\leqq C \int_0^t \frac{\|\delta\|_\tau}{(t-\tau)^{1/2}}\, d\tau.$$

Combining the estimates (5.7), (5.10), (5.13), (5.16), we have

$$(5.17)\qquad \|\delta\|_t \leqq (b_2 - b_1) + 2I_1 + 2I_2 + 2I_3 + C \int_0^t \frac{\|\delta\|_\tau}{(t-\tau)^{1/2}}\, d\tau.$$

Application of Lemma 7 to (5.17) yields the final result

$$\|\delta\|_t \leqq C\{(b_2 - b_1) + 2I_1 + 2I_2 + 2I_3\}$$
$$\leqq C\left\{(b_2 - b_1) + \int_0^t |f_2 - f_1|\, d\tau + \int_0^{b_1} \xi\, |\varphi_2 - \varphi_1|\, d\xi + \int_{b_1}^{b_2} \xi \varphi_2\, d\xi\right\}.$$

Q.E.D.

Now we treat the case $b = 0$.

Theorem 4. *For $i = 1, 2$, let $b_i = 0$ and (s_i, u_i) be solutions of* (1.1), (1.2) *corresponding to data f_i satisfying* (B). *Then there exists a positive constant $C = C(\lambda, B, T)$ such that*

$$(5.18)\qquad |s_2(t) - s_1(t)| \leqq C\, \|f_2 - f_1\|_t$$

for $0 \leqq t \leqq T$, where $\|g\|_t \equiv \max_{0 \leqq \tau \leqq t} |g(\tau)|$.

Proof. According to Lemmas 4 and 5, $s_1(t)$ and $s_2(t)$ are Lipschitz continuous and $s_1(t) + s_2(t) \geqq 2\lambda t$. Therefore

$$\delta(t) \leqq \frac{1}{\lambda t} \int_0^t |f_2(\tau) - f_1(\tau)|\, d\tau + \frac{1}{\lambda t} \int_0^{\alpha(t)} \xi\, |u_2(\xi, t) - u_1(\xi, t)|\, d\xi$$
$$+ \frac{1}{\lambda t} \int_{\alpha(t)}^{\beta(t)} \xi u_{j(t)}(\xi, t)\, d\xi = J_1 + J_2 + J_3\,.$$

First we note that $J_1 \leqq 1/\lambda\, \|f_2 - f_1\|_t$.

In the region $0 \leqq x \leqq \alpha(t)$, $0 \leqq t \leqq T$, $|u_2 - u_1|$ is dominated by the solution to a problem analogous to (5.5) plus the solution to a problem analogous to (5.12). An easy calculation, using the method in the proof of Theorem 3 and the additional fact that $\alpha(t) \leqq Bt$, shows that

$$J_2 \leqq J_1 + C \int_0^t \frac{\|\delta\|_\tau}{(t-\tau)^{1/2}}\, d\tau,$$

where C is a generic constant depending on λ, B and T.

The estimate for J_3 follows again as in Theorem 3, using $\beta(t) \leqq Bt$. The result is

$$J_3 \leqq C \int_0^t \frac{\|\delta\|_\tau}{(t-\tau)^{1/2}}\, d\tau.$$

Combining these estimates

$$||\delta||_t \leqq \frac{2}{\lambda} ||f_2 - f_1||_t + C \int_0^t \frac{||\delta||_\tau}{(t-\tau)^{1/2}}\, d\tau,$$

and a final application of Lemma 7 provides the desired result (5.18). Q.E.D.

As a corollary to Theorems 3 and 4 we have the uniqueness

Theorem 5. *Under the assumptions* (A) *when* $b > 0$, *or the assumptions* (B) *when* $b = 0$, *the solution* (s, u) *of the Stefan problem* (1.1), (1.2) *is unique.*

6. Monotone dependence. Consider two sets $\{f_i, \varphi_i, b_i\}$, $i = 1,2$, of Stefan data. If $b_i > 0$ we require that f_i, φ_i satisfy (A), and if $b_i = 0$ we require that f_i satisfy (B). Then by Theorems 1–5 there exist unique solutions (s_i, u_i) of the Stefan problem (1.1), (1.2) corresponding to the data $\{f_i, \varphi_i, b_i\}$.

Theorem 6. *Under the above assumptions, let* $f_1 \leqq f_2$, $\varphi_1 \leqq \varphi_2$, *and* $0 \leqq b_1 \leqq b_2$. *Then, for* $0 \leqq t \leqq T$,

$$s_1(t) \leqq s_2(t). \tag{6.1}$$

Proof. First we prove that, in the case $b_1 < b_2$, $s_1(t) < s_2(t)$. Assuming the contrary, set $t_0 = \min \{t \mid s_1(t) \geqq s_2(t)\}$. Clearly $\dot{s}_1(t_0) \geqq \dot{s}_2(t_0)$ and $t_0 > 0$. Ruling out the trivial case $u_2 \equiv u_1 \equiv 0$, $s_1(t) \equiv b_1$, $s_2(t) \equiv b_2$, we have by the strong maximum principle that $u_2(s_1(t), t) > 0$ for $0 < t < t_0$. Hence $u_2 - u_1 > 0$ in the region $0 < x < s_1(t)$, $0 < t \leqq t_0$. Since $u_2 - u_1$ vanishes at the point $(s_1(t_0), t_0)$ it follows from the parabolic version of Hopf's Lemma that

$$\dot{s}_1(t_0) = -u_{1_x}(s_1(t_0), t_0) < -u_{2_x}(s_1(t_0), t_0) = \dot{s}_2(t_0),$$

which is a contradiction.

Now we treat the case $b_1 = b_2$. Let (s_2^δ, u_2^δ) be the solution of (1.1), (1.2) corresponding to the Stefan data $\{f_2, \varphi_2^\delta, b_1 + \delta\}$, where

$$\varphi_2^\delta(x) = \begin{cases} \varphi_2(x), & 0 \leqq x \leqq b_1 \\ 0, & b_1 \leqq x \leqq b_1 + \delta. \end{cases}$$

By the previous paragraph $s_1(t) < s_2^\delta(t)$ for all $\delta > 0$. If $b_1 > 0$ we use Theorem 3 to conclude that $s_2^\delta(t)$ converges uniformly to $s_2(t)$ as $\delta \searrow 0$. If $b_1 = 0$ we have, from the very construction of (s_2, u_2) as given in Theorem 2, a sequence of such s_2^δ's that converge uniformly to $s_2(t)$. Hence in either case we conclude that $s_1(t) \leqq s_2(t)$. Q.E.D.

7. Asymptotic behavior in the corner. In order not to obscure the basic simplicity of our arguments we have, throughout the first six sections, made assumptions (B) in the case $b = 0$. In this section we show how the Lipschitz assumption (1.4) on f can be relaxed and we relate the asymptotic behavior of $s(t)$ as $t \searrow 0$ to that of $f(t)$.

Theorem 7. *Let* (1.4) *in assumptions* (B) *be replaced by*

$$lt^{1+\gamma} \leqq f(t) \leqq Lt^{1+\gamma} \quad \text{for} \quad 0 \leqq t \leqq T, \tag{7.1}$$

where $\gamma \geqq 0$. *Then the results on existence, uniqueness, stability, and monotone dependence in the case* $b = 0$ *continue to be valid, provided that the norm* $||g||_t$ *in* (5.18) *is replaced by*

$$||g||_{\gamma, t} \equiv \max_{0 \leqq \tau \leqq t} \tau^{-\gamma/2} |g(\tau)|. \tag{7.2}$$

Moreover there exist constants λ, Λ *such that the asymptotic relation*

$$\lambda t^{1+\gamma/2} \leqq s(t) \leqq \Lambda t^{1+\gamma/2}, \qquad 0 \leqq t \leqq T \tag{7.3}$$

holds.

Set $\alpha = 1 + \gamma/2$, $\beta = 1 + \gamma$. We shall exhibit explicitly the solution $V^\mu(x, t)$ of the Stefan problem in which the free boundary is $s(t) \equiv \mu t^\alpha$, $\mu > 0$, and obtain upper and lower bounds on $V^\mu(0, t)$.

It is easy to verify that, provided all requisite infinite series converge, the function

$$u(x, t) \equiv \sum_{n=1}^{\infty} \frac{1}{(2n)!} \frac{\partial^n}{\partial t^n} [x - s(t)]^{2n} \tag{7.4}$$

is a solution to the heat equation satisfying $u(s(t), t) = 0$, $u_x(s(t), t) = -\dot{s}(t)$. If $s(t) = \mu t^\alpha$ we have

$$V^\mu(x, t) \equiv \sum_{n=1}^{\infty} \frac{1}{(2n)!} \frac{\partial^n}{\partial t^n} [x - \mu t^\alpha]^{2n} \tag{7.5}$$

and

$$V^\mu(0, t) = \sum_{n=1}^{\infty} \frac{\Gamma(2\alpha n + 1)}{\Gamma(2n + 1)\Gamma(2\alpha n - n + 1)} [\mu^2 t^\beta]^n, \tag{7.6}$$

since $\beta = 2\alpha - 1$.

Observe that

$$\alpha \leqq \left(\frac{2\alpha n - k}{2n - k}\right) \leqq \beta$$

for $k = 0, 1, \cdots, n - 1$ and

$$\frac{\Gamma(2\alpha n + 1)}{\Gamma(2n + 1)\Gamma(2\alpha n - n + 1)} = \frac{1}{n!} \prod_{k=0}^{n-1} \left(\frac{2\alpha n - k}{2n - k}\right).$$

Thus

$$\exp\{\alpha\mu^2 t^\beta\} - 1 \leqq V^\mu(0, t) \leqq \exp\{\beta\mu^2 t^\beta\} - 1. \tag{7.7}$$

Lemma 8. *If* (1.4) *is replaced by* (7.1) *then in Lemma* 4 *the inequality* (4.1) *can be replaced by*

(7.8) $$\lambda t^\alpha \leqq s(t) \quad \textit{for} \quad 0 \leqq t \leqq T,$$

where

$$\lambda = [\beta^{-1}T^{-\beta} \log (1 + lT^\beta)]^{1/2}.$$

Proof. Replace the comparison function $V(x, t)$ used in the proof of Lemma 4 by $V^\lambda(x, t)$. Since

$$V^\lambda(0, t) \leqq \exp \{\beta\lambda^2 t^\beta\} - 1 \leqq lt^\beta \leqq f(t)$$

for $0 \leqq t \leqq T$, the same argument based on the parabolic version of Hopf's Lemma again shows that $s(t)$ cannot cross λt^α.

Lemma 9. *If* (1.4) *is replaced by* (7.1) *then in Lemma* 5 *the relation* (4.2) *can be replaced by*

(7.9) $$0 \leqq \rho^{-1}u(s(t) - \rho, t) \leqq \Lambda t^{\gamma/2}$$

where $\Lambda = L\lambda^{-1}$.

Proof. One needs only change the comparison function to

$$\omega(x, t_0) \equiv \Lambda t_0^{\gamma/2}[s(t_0) - x]$$

and note that by Lemma 8, $\omega(0, t_0) = \Lambda t_0^{\gamma/2}s(t_0) \geqq \Lambda t_0^{\gamma/2}\lambda t_0^\alpha = Lt_0^\beta \geqq Lt^\beta \geqq f(t)$ for $0 \leqq t \leqq t_0$.

Now we complete the proof of Theorem 7. By Lemmas 8 and 9 we have $0 \leqq \dot{s}^b(t) \leqq \Lambda t^{\gamma/2} \leqq \Lambda T^{\gamma/2}$ for $0 \leqq t \leqq T$. Hence existence goes as in the proof of Theorem 2 and, moreover, the limit function $s(t)$ must satisfy $\lambda t^\alpha \leqq s(t)$, $\dot{s}(t) \leqq \Lambda t^{\gamma/2}$. Therefore

$$\lambda t^\alpha \leqq s(t) \leqq \Lambda t^{1+\gamma/2} = \Lambda t^\alpha.$$

For the proofs of uniqueness, stability, and monotone dependence we need only note that $s_1(t) + s_2(t) \geqq 2\lambda t^\alpha$, that $\alpha(t), \beta(t) \leqq \Lambda t^\alpha$, and that

$$\frac{1}{\lambda t^\alpha} \int_0^t |f_2(\tau) - f_1(\tau)|\, d\tau \leqq \frac{1}{\lambda t} \int_0^t \tau^{-\gamma/2}\, |f_2(\tau) - f_1(\tau)|\, d\tau \leqq \frac{1}{\lambda}\, ||f_2 - f_1||_{\gamma,t}$$

is monotone nondecreasing. Q.E.D.

Appendix

The proof of Lemma 1 is a modification of a result of Gevrey [7]. For the sake of completeness we give it here.

Let u be the solution of (1.1) in question with

(8.1) $$|s(t) - s(\tau)| \leqq M\, |t - \tau|.$$

Without loss of generality we may assume the following: $b > 0$, $\varphi(x) = u(x, 0)$ is C^∞ for $0 \leqq x < b$ with

(8.2) $$0 \leqq \varphi(x) \leqq N(b - x), \qquad 0 \leqq x \leqq b,$$

and, rather than $f(t) = u(0, t)$, the function $g(t) = u_x(0, t)$ is prescribed and C^∞ for $0 \leqq t \leqq T$. For otherwise we could consider any interior region $0 < \delta \leqq x \leqq s(t)$, $0 < \epsilon \leqq t \leqq T$, and use (4.2).

The proof that u_x is continuous to the boundary consists in showing that u can be written in the form

$$u(x, t) = \Phi(x, t) + \int_0^t K(x, t; 0, \tau)\mu_1(\tau)\, d\tau + \int_0^t K(x, t; s(\tau), \tau)\mu_2(\tau)\, d\tau, \tag{8.3}$$

where

$$\Phi(x, t) \equiv \int_0^b K(x, t; \xi, 0)\varphi(\xi)\, d\xi,$$

$$K(x, t; \xi, \tau) \equiv \frac{1}{2(\pi)^{1/2}(t - \tau)^{1/2}} \exp\left\{-\frac{(x - \xi)^2}{4(t - \tau)}\right\},$$

and $\mu_2(t)$ *is continuous and bounded for* $0 < t \leqq T$. Indeed by the standard jump relation [6], it follows from (8.3) that

$$\lim_{x \to s(t)-0} u_x(x, t) = \Phi_x(s(t), t) + \int_0^t K_x(s(t), t; 0, \tau)\mu_1(\tau)\, d\tau + \int_0^t K_x(s(t), t; s(\tau), \tau)\mu_2(\tau)\, d\tau + \tfrac{1}{2}\mu_2(t). \tag{8.4}$$

Using (8.1) and the fact that $s(t) > 0$ it is obvious that the first three terms on the right in (8.4) exist and are continuous functions of t for $0 < t \leqq T$. Thus the continuity of u_x up to the boundary $x = s(t)$ rests on the continuity of $\mu_2(t)$.

In order that u have the representation (8.3) it is necessary and sufficient that $\mu_1(t)$, $\mu_2(t)$ form a solution of the system of integral equations

$$g(t) = u_x(0+, t) \equiv \Phi_x(0, t) - \tfrac{1}{2}\mu_1(t) + \int_0^t K_x(0, t; s(\tau), \tau)\mu_2(\tau)\, d\tau, \tag{8.5}$$

$$0 = u(s(t) - 0, t) \equiv \Phi(s(t), t) + \int_0^t K(s(t), t; 0, \tau)\mu_1(\tau)\, d\tau + \int_0^t K(s(t), t; s(\tau), \tau)\mu_2(\tau)\, d\tau. \tag{8.6}$$

The remainder of the proof consists in converting (8.5), (8.6) into a system of Volterra integral equations of the second kind. It is well known that the solution $\mu_1(t)$, $\mu_2(t)$ of the resulting system is continuous.

First we change t to y in (8.6) and multiply through by $(t - y)^{-1/2}$. Then we integrate on y from zero to t and interchange the τ and y integrations. The result is

$$-\int_0^t \frac{\Phi(s(y), y)}{(t - y)^{1/2}}\, dy = \int_0^t k_1(t, \tau)\mu_1(\tau)\, d\tau + \int_0^t k_2(t, \tau)\mu_2(\tau)\, d\tau, \tag{8.7}$$

where

$$k_1(t, \tau) \equiv \int_\tau^t \frac{K(s(y), y; 0, \tau)}{(t - y)^{1/2}}\, dy,$$

$$k_2(t, \tau) \equiv \int_\tau^t \frac{K(s(y), y; s(\tau), \tau)}{(t - y)^{1/2}}\, dy.$$

Observe that, for $\tau \leqq y \leqq t$,

$$0 \leqq \exp\left\{-\frac{s(y)^2}{4(y - \tau)}\right\} \leqq \exp\left\{-\frac{s_0^2}{4(t - \tau)}\right\}$$

and

$$\exp\left\{-\frac{M^2}{4}(t - \tau)\right\} \leqq \exp\left\{-\frac{[s(y) - s(\tau)]^2}{4(y - \tau)}\right\} \leqq 1,$$

where

$$s_0 \equiv \min_{0 \leqq y \leqq t} s(y) > 0.$$

Since

$$\int_\tau^t \frac{dy}{(t - y)^{1/2}(y - \tau)^{1/2}} \equiv \pi, \tag{8.8}$$

it follows that, as $\tau \nearrow t$,

$$k_1(t, \tau) \to 0 \tag{8.9}$$

and

$$k_2(t, \tau) \to \frac{(\pi)^{1/2}}{2}. \tag{8.10}$$

Now we differentiate (8.7) formally with respect to t. Using (8.9) and (8.10),

$$-\frac{\partial}{\partial t}\int_0^t \frac{\Phi(s(y), y)}{(t - y)^{1/2}}\, dy = \int_0^t \frac{\partial}{\partial t} k_1(t, \tau)\mu_1(\tau)\, d\tau + \int_0^t \frac{\partial}{\partial t} k_2(t, \tau)\mu_2(\tau)\, d\tau + \frac{(\pi)^{1/2}}{2}\mu_2(t). \tag{8.11}$$

The existence, continuity, and boundedness of the various t-derivatives occurring in (8.11) will be demonstrated below.

Consider any expression of the form

$$H(t, \tau) \equiv \int_\tau^t \frac{h(y, \tau)}{(t - y)^{1/2}(y - \tau)^{1/2}}\, dy, \tag{8.12}$$

where h is continuous in y and τ for $0 \leqq y, \tau \leqq T$ and h is Lipschitz continuous in y with Lipschitz constant K independent of τ. Since

$$\frac{\partial}{\partial y}\left\{-2\tan^{-1}\left(\frac{t - y}{y - \tau}\right)^{1/2}\right\} = \frac{1}{(t - y)^{1/2}(y - \tau)^{1/2}}$$

and h is of bounded variation, we can integrate (8.12) by parts and represent it as the Stieltjes integral

$$H(t, \tau) = \pi h(\tau, \tau) + \int_\tau^t 2 \tan^{-1} \left(\frac{t-y}{y-\tau}\right)^{1/2} d_y h(y, \tau). \tag{8.13}$$

Differentiating (8.13) with respect to t, we obtain

$$\frac{\partial}{\partial t} H(t, \tau) = \int_\tau^t \frac{(y-\tau)^{1/2}}{(t-\tau)(t-y)^{1/2}} d_y h(y, \tau). \tag{8.14}$$

From the Lipschitz continuity of h it is easy to verify *via* Riemann sums that

$$\left|\frac{\partial H}{\partial t}(t, \tau)\right| \leqq \pi K. \tag{8.15}$$

Moreover, it is clear from (8.14) that $\partial H/\partial t$ is continuous for $\tau < t \leqq T$.

In order to compute the t-derivatives occurring in (8.11) and thereby justify (8.11), we need only to demonstrate that (8.14) and (8.15) apply to each of

$$\begin{aligned} H(t, \tau) &\equiv k_1(t, \tau) \\ &= \int_\tau^t \frac{1}{(t-y)^{1/2}(y-\tau)^{1/2}} \left[\frac{1}{2(\pi)^{1/2}} \exp\left\{-\frac{s(y)^2}{4(y-\tau)}\right\}\right] dy, \end{aligned} \tag{8.16}$$

$$\begin{aligned} H(t, \tau) &\equiv k_2(t, \tau) \\ &= \int_\tau^t \frac{1}{(t-y)^{1/2}(y-\tau)^{1/2}} \left[\frac{1}{2(\pi)^{1/2}} \exp\left\{-\frac{(s(y)-s(\tau))^2}{4(y-\tau)}\right\}\right] dy \end{aligned} \tag{8.17}$$

and

$$\begin{aligned} H(t, 0) &\equiv \int_0^t \frac{\Phi(s(y), y)}{(t-y)^{1/2}} dy \\ &= \int_0^t \frac{1}{(t-y)^{1/2} y^{1/2}} \left[\frac{1}{2(\pi)^{1/2}} \int_0^b \exp\left\{-\frac{(s(y)-\xi)^2}{4y}\right\} \varphi(\xi)\, d\xi\right] dy. \end{aligned} \tag{8.18}$$

In order to demonstrate the applicability of (8.14) and (8.15), we need only demonstrate the Lipschitz continuity with respect to y with Lipschitz constant independent of τ for each of the h in (8.16), (8.17) and (8.18).

Now the Lipschitz continuity of h in (8.16) can be verified directly from the mean value theorem since $s(t) \geqq s_0 > 0$. However, an indirect proof of the Lipschitz continuity of the h for both (8.16) and (8.17) is easily given. By increasing M slightly, a sequence of C^1 functions $\{s_n\}$ can be found such that $|\dot{s}_n| < M$, $s_n(0) = b$, $n = 1, \cdots$, and such that $\{s_n\}$ converges uniformly to s for $0 \leqq t \leqq T$ as n tends to infinity. To be specific, consider the case of

$$h = \left[\frac{1}{2(\pi)^{1/2}} \exp\left\{-\frac{(s(y)-s(\tau))^2}{4(y-\tau)}\right\}\right].$$

From the uniform convergence of the $\{s_n\}$ it follows that h is the uniform limit of the sequence

$$h_n = \left[\frac{1}{2(\pi)^{1/2}} \exp\left\{-\frac{(s_n(y) - s_n(\tau))^2}{4(y - \tau)}\right\}\right], \qquad n = 1, 2, \cdots,$$

over $0 \leqq y, \tau \leqq T$ as n tends to infinity. Since the $|\partial h_n/\partial y|$ is bounded by a constant independent of τ and n for $0 \leqq y, \tau \leqq T$, it follows that h is Lipschitz continuous with Lipschitz constant independent of τ.

A similar argument can be used for the case of

$$h = \frac{1}{2(\pi)^{1/2}} \int_0^b \exp\left\{-\frac{(s(y) - \xi)^2}{4y}\right\}\varphi(\xi)\, d\xi.$$

In fact the only difference is that

$$\frac{\partial h_n}{\partial y} = \frac{1}{2(\pi)^{1/2}} \int_0^b \left[-\frac{2(s_n(y) - \xi)\dot{s}_n(y)}{4y} + \frac{(s_n(y) - \xi)^2}{4y^2}\right] \cdot \exp\left\{-\frac{(s_n(y) - \xi)^2}{4y}\right\}\varphi(\xi)\, d\xi$$

is slightly more difficult to estimate. However, using (8.2), the estimate is obtained by simple quadratures. For example note that the first term in the bracket together with the exponential multiplier is a perfect differential. The second term in the brackets can be handled by integration by parts using

$$0 \leqq \varphi(\xi) \leqq N(b - s_n(y)) + N(s_n(y) - \xi).$$

Consequently (8.14) and (8.15) apply to (8.16), (8.17) and (8.18) which implies that the kernels in (8.11) are actually bounded, continuous in t for $\tau < t \leqq T$, and measurable in τ. Indeed it follows from the continuity with respect to τ of the measure $d_y h(y, \tau)$ in each of (8.16) and (8.17) that the kernels in (8.11) are continuous for $0 \leqq \tau < t \leqq T$. Since the inhomogeneous term in (8.11) is bounded and continuous for $0 < t \leqq T$, it follows that the system (8.5), (8.11) of Volterra integral equations possesses a unique continuous solution $\underline{\mu}(t) = (\mu_1(t), \mu_2(t))$. It is clear from the analysis that there can exist at most one solution of (8.5), (8.6) and that the solution must be $\underline{\mu}(t)$. It remains to be shown that $\underline{\mu}(t)$ is actually a solution of (8.5), (8.6). It is clear that the derivation of (8.11) can be reversed to the point of differentiating (8.7) with respect to t; *i.e.*,

$$\frac{\partial}{\partial t}\left\{\int_0^t k_1(t, \tau)\mu_1(\tau)\, d\tau + \int_0^t k_2(t, \tau)\mu_2(\tau)\, d\tau + \int_0^t \frac{\Phi(s(y), y)}{(t - y)^{1/2}}\, dy\right\} \equiv 0. \tag{8.19}$$

Consequently, integrating (8.19), it follows that

$$\int_0^t k_1(t, \tau)\mu_1(\tau)\, d\tau + \int_0^t k_2(t, \tau)\mu_2(\tau)\, d\tau + \int_0^t \frac{\Phi(s(y), y)}{(t - y)^{1/2}}\, dy \equiv C, \tag{8.20}$$

where C is any constant. Applying the Abel inversion operation, it follows that

$$\int_0^t K(s(t), t; 0, \tau)\mu_1(\tau)\, d\tau + \int_0^t K(s(t), t; s(\tau), \tau)\mu_2(\tau)\, d\tau + \Phi(s(t), t) \equiv C\pi^{-1}t^{-1/2}. \tag{8.21}$$

Consequently, $\underline{\mu}(t)$ is a solution of (8.5), (8.21) for every choice of C. In particular for $C = 0$, $\underline{\mu}(t)$ is a solution of (8.5), (8.6) which completes the proof of Lemma 1.

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University of Minnesota, Minneapolis
and
The Rockefeller University

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