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Exit identities for Lévy processes observed at Poisson arrival times

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For a spectrally one-sided Lévy process, we extend various two-sided exit identities to the situation when the process is only observed at arrival epochs of an independent Poisson process. In addition, we consider exit problems of this type for processes reflected either from above or from below. The resulting Laplace transforms of the main quantities of interest are in terms of scale functions and turn out to be simple analogues of the classical formulas.

Keywords: Cramér–Lundberg risk model; dividends; exit problem; reflection; spectrally negative Levy process

1. Introduction

Consider a spectrally-negative Lévy process X, that is, a Lévy process with only negative jumps, and which is not a.s. a non-increasing process. Let

$$\psi(\theta) := \log \mathbb{E}\mathrm{e}^{\theta X(1)}, \qquad \theta \geq 0,$$

be its Laplace exponent. Denote the law of X with $X(0) = x \geq 0$ by $\mathbb{P}_x$ and the corresponding expectation by $\mathbb{E}_x$. For a fixed $a \geq 0$ define the first passage times

$$\tau_0^- := \inf\{t \geq 0\colon\ X(t) < 0\}, \qquad \tau_a^+ := \inf\{t \geq 0\colon\ X(t) > a\}.$$

Furthermore, let T_i be the arrival times of an independent Poisson process of rate $\lambda > 0$, and define the following stopping times:

$$T_0^- := \min\{T_i\colon\ X(T_i) < 0\}, \qquad T_a^+ := \min\{T_i\colon\ X(T_i) > a\}.$$

The latter times can be seen as first passage times when X is observed at Poisson arrival times (by convention $\inf\varnothing = \min\varnothing = \infty$).

These quantities have useful interpretations in various fields of applied probability. For instance, if X serves as a model for the surplus process of an insurance portfolio over time, then $\mathbb{P}_x(\tau_0^- < \infty)$ is the probability of ruin of the portfolio with initial capital x. Likewise, $\mathbb{P}_x(T_0^- < \infty)$ is the probability that ruin occurs and is detected, given that the process can only be monitored at discrete points in time modeled by an independent Poisson process. It is not hard to show that $T_0^- \downarrow \tau_0^-$ a.s. (and similarly $T_a^+ \downarrow \tau_a^+$ a.s.) as the observation rate λ tends to ∞, which may be used to retrieve the classical exit identities. Quantities related to T_0^- have been studied for a compound Poisson risk model in [2], and in [18] a simple formula for $\mathbb{P}_x(T_0^- < \infty)$ was established for general spectrally-negative Lévy processes X. Ruin-related quantities under surplus-dependent observation rates were studied in [6] and for recent results on observation rates that change according to environmental conditions, we refer to [4]. Poissonian observation is also relevant in queueing contexts, see, for example, [10].

In practice one may interpret continuous and Poissonian observation as endogenous and exogenous monitoring of the process of interest, respectively. For example, in an insurance context τ_0^- may be understood as the time of ruin (observed by the insurance company), whereas T_a^+ may be considered as the first time when shareholders receive dividends (they look at the company at discrete, here random, times). Hence, the shareholders receive dividends if the event $\{T_a^+ < \tau_0^-\}$ occurs. Another example comes from reliability theory [20], where one considers a degradation process and assumes that τ_0^- is the time of failure and $T_a^- := \min\{T_i\colon\ X(T_i) < a\}$ for some $a > 0$ is the time at which the process is observed in its critical state necessitating replacement. Hence, the event $\{T_a^- < \tau_0^-\}$ signifies preventive replacement before failure. Finally, some identities involving both continuous and Poissonian observation lead to transforms of certain occupation times, see Remark 3.2.

In this paper, we establish formulas for two-sided exit probabilities under Poissonian observation, as well as formulas for the joint transform of the exit time and the corresponding overshoot on the event of interest. In addition, we consider reflected processes and provide the joint transforms including the total amount of output (dividends) or input (required capital to remain solvent) up to the exit time. Note that the formulas also hold for spectrally-positive Lévy processes by simply exchanging the roles of the involved quantities. It turns out that the resulting formulas have a rather slim form in terms of first and second scale functions, and along the way it also proves useful to define a third scale function. The form of the expressions allows to interpret them as natural analogues of the respective counterparts under continuous observation. Finally, we note that discrete observation allows for a wide range of cases, and our list of exit identities is not exhaustive. We only consider the basic cases, that is, the ones where the corresponding events stay non-trivial if Poissonian observation is replaced by a continuous one, which, for example, excludes $\{T_a^- < \tau_0^-\}$ mentioned above.

The remainder of this paper is organized as follows. Section 2 recalls some relevant exit identities under continuous observation. Section 3 contains the main results, and the proofs are given in Section 4. Finally, Section 5 gives an illustration of some identities for the case of Cramér–Lundberg risk model with exponential claims.

Throughout this work, we use e_u to denote an exponentially distributed r.v. with rate $u > 0$, which is independent of everything else.

2. Standard exit theory

The most basic identity states that

$$\mathbb{P}_0\bigl(\tau_a^+ < \infty\bigr) = \mathrm{e}^{-\Phi a}, \qquad a \geq 0, \tag{1}$$

where $\Phi \geq 0$ is the right-most non-negative solution of $\psi(\theta) = 0$. Let us recall two fundamental functions which enter various exit identities. The (first) scale function $W(x)$ is a non-negative function, with $W(x) = 0$ for $x < 0$, continuous on $[0, \infty)$, positive for positive x, and characterized by the transform

$$\int_0^\infty e^{-\theta x} W(x)\,dx = 1/\psi(\theta), \qquad \theta > \Phi.$$

It enters the basic two-sided exit identity for $a > 0$ through

$$\mathbb{P}_x\big(\tau_a^+ < \tau_0^-\big) = W(x)/W(a), \qquad x \leq a, \tag{2}$$

see, for example, [17].

The so-called second scale function is defined by

$$Z(x,\theta) := e^{\theta x}\left(1 - \psi(\theta)\int_0^x e^{-\theta y} W(y)\,dy\right), \qquad x \geq 0 \tag{3}$$

and $Z(x,\theta) := e^{\theta x}$ for $x < 0$. Note that for $\theta = 0$, $Z(x,\theta)$ reduces to $Z(x)$ as defined in [17], Chapter 2. It is convenient to define Z as a function of two arguments, which allows to provide more general formulas. We refer to [15] for this definition and the following formulas in a more general setting of Markov additive processes. Note that for $\theta > \Phi$ we can rewrite $Z(x,\theta)$ in the form

$$Z(x,\theta) = \psi(\theta)\int_0^\infty e^{-\theta y} W(x+y)\,dy, \qquad x \geq 0, \theta > \Phi. \tag{4}$$

It is known that for $x \leq a$ one has

$$\mathbb{E}_x\big(e^{\theta X(\tau_0^-)}; \tau_0^- < \tau_a^+\big) = Z(x,\theta) - W(x)\frac{Z(a,\theta)}{W(a)}. \tag{5}$$

Moreover, for $\theta > \Phi$ we have

$$\lim_{a\to\infty} Z(a,\theta)/W(a) = \psi(\theta)/(\theta - \Phi) \tag{6}$$

and so we also find that

$$\mathbb{E}_x\big(e^{\theta X(\tau_0^-)}, \tau_0^- < \infty\big) = Z(x,\theta) - W(x)\frac{\psi(\theta)}{\theta - \Phi}. \tag{7}$$

Importantly, all the above results hold for an exponentially killed process X (cf. [14]): For a killing rate $q > 0$, we write $\psi_q(\theta) = \psi(\theta) - q$, then $\Phi_q > 0$ is the positive solution to $\psi_q(\theta) = 0$, and $W_q(x)$ is defined by

$$\int_0^\infty e^{-\theta x} W_q(x)\,dx = 1/\psi_q(\theta), \qquad \theta > \Phi_q.$$

With $Z_q(x,\theta)$ defined through $W_q(x)$ and $\psi_q(\theta)$, formula (5) in the case of killing reads

$$\mathbb{E}_x\big(\mathrm{e}^{-q\tau_0^- +\theta X(\tau_0^-)}; \tau_0^- < \tau_a^+\big) = \mathbb{E}_x\big(\mathrm{e}^{\theta X(\tau_0^-)}; \tau_0^- < \tau_a^+, \tau_0^- < e_q\big)$$
$$= Z_q(x,\theta) - W_q(x)\frac{Z_q(a,\theta)}{W_q(a)},$$

and hence the information on the time of the exit is easily added. The same adaptations hold for the other exit identities above. For the sake of readability, we will often drop the index q in the sequel, if it does not cause confusion. In this case, Φ_λ, $W_\lambda(x)$, $Z_\lambda(x,\theta)$ should be interpreted as $\Phi_{\lambda+q}$, $W_{\lambda+q}(x)$, $Z_{\lambda+q}(x,\theta)$, respectively, that is, they correspond to the process killed at rate q and then additionally killed at rate λ.

Finally, we will need the following identities which can readily be obtained from the known formulas for potential densities of X killed upon exiting a certain interval, see, for example, [11] or [17], Chapter 8.4:

$$\mathbb{P}\big(X(e_\lambda) \in \mathrm{d}x\big) = \lambda\big(\mathrm{e}^{-\Phi_\lambda x}/\psi'(\Phi_\lambda) - W_\lambda(-x)\big)\,\mathrm{d}x, \tag{8}$$

$$\mathbb{P}\big(X(e_\lambda) \in \mathrm{d}x, e_\lambda < \tau_a^+\big) = \lambda\big(\mathrm{e}^{-\Phi_\lambda a} W_\lambda(a-x) - W_\lambda(-x)\big)\,\mathrm{d}x, \tag{9}$$

$$\mathbb{P}_a\big(X(e_\lambda) \in \mathrm{d}x, e_\lambda < \tau_0^-\big) = \lambda\big(\mathrm{e}^{-\Phi_\lambda x} W_\lambda(a) - W_\lambda(a-x)\big)\,\mathrm{d}x, \tag{10}$$

where $a > 0$ and the killing rate $q \ge 0$ is implicit.

3. Results

One of the first general results concerning T_0^- was obtained in [18], where it was shown for $q=0$ and $\mathbb{E}X(1) > 0$ that

$$\mathbb{P}_x\big(T_0^- = \infty\big) = \psi'(0)\frac{\Phi_\lambda}{\lambda} Z(x,\Phi_\lambda), \qquad x \ge 0, \tag{11}$$

cf. (4). This leads to a strikingly simple identity for $x=0$: $\mathbb{P}(T_0^- = \infty) = \psi'(0)\Phi_\lambda/\lambda$. A more general result was recently obtained in [4] for an arbitrary killing rate $q \ge 0$:

$$\mathbb{P}_x\big(\tau_a^+ < T_0^-\big) = \frac{Z(x,\Phi_\lambda)}{Z(a,\Phi_\lambda)}, \qquad x \in [0,a]. \tag{12}$$

It is easy to see that both these results also hold for $x < 0$. Note the resemblance of (2) and (12), and, moreover,

$$\frac{Z(x,\Phi_\lambda)}{Z(a,\Phi_\lambda)} = \mathbb{P}_x\big(\tau_a^+ < T_0^-\big) \longrightarrow \mathbb{P}_x\big(\tau_a^+ < \tau_0^-\big) = \frac{W(x)}{W(a)} \qquad \text{as } \lambda \to \infty, \tag{13}$$

because in the limit $\lambda \to \infty$ the Poisson observation results in continuous observation of the process a.s. (although it is hard to see the convergence of the ratio directly).

We now present various exit identities for continuous and Poisson observations extending the standard exit theory.

Theorem 3.1. *For $a, \theta \geq 0$, $x \leq a$ and implicit killing rate $q \geq 0$, we have*

$$\mathbb{E}_x\left(e^{\theta X(T_0^-)}; T_0^- < \infty\right) = \frac{\lambda}{\lambda - \psi(\theta)}\left(Z(x,\theta) - Z(x,\Phi_\lambda)\frac{\psi(\theta)(\Phi_\lambda - \Phi)}{\lambda(\theta - \Phi)}\right), \quad (14)$$

$$\mathbb{E}_x\left(e^{\theta X(T_0^-)}; T_0^- < \tau_a^+\right) = \frac{\lambda}{\lambda - \psi(\theta)}\left(Z(x,\theta) - Z(x,\Phi_\lambda)\frac{Z(a,\theta)}{Z(a,\Phi_\lambda)}\right), \quad (15)$$

$$\mathbb{E}_x\left(e^{-\theta(X(T_a^+)-a)}; T_a^+ < \infty\right) = \frac{\Phi_\lambda - \Phi}{\Phi_\lambda + \theta}e^{-\Phi(a-x)}, \quad (16)$$

$$\mathbb{E}_x\left(e^{-\theta(X(T_a^+)-a)}; T_a^+ < \tau_0^-\right) = \frac{\lambda}{\Phi_\lambda + \theta}\frac{W(x)}{Z(a,\Phi_\lambda)}, \quad (17)$$

$$\mathbb{E}_x\left(e^{\theta X(\tau_0^-)}; \tau_0^- < T_a^+\right) = Z(x,\theta) - \frac{W(x)}{\theta - \Phi_\lambda}\left(\psi(\theta) - \lambda\frac{Z(a,\theta)}{Z(a,\Phi_\lambda)}\right), \quad (18)$$

where ratios for $\theta = \Phi$ and $\theta = \Phi_\lambda$ should be interpreted in the limiting sense.

Remark 3.1. When $\mathbb{E}X(1) > 0$ and $q = \theta = 0$, we have $\Phi = 0$ and $Z(x,0) = 1$, so that (14) reduces to (11). Secondly, note the resemblance between (5) and (15), and that the first is retrieved from the second when $\lambda \to \infty$ cf. (13). Next, (16) is an extension of identity (1) (which is retained for $\theta = 0$ and $\lambda \to \infty$). This formula also implies that the overshoot $X(T_a^+) - a$ given $\{T_a^+ < \infty\}$ and $q = 0$ is exponentially distributed with rate Φ_λ for all $x \leq a$ (indeed it is not hard to establish the memoryless property of this overshoot). The identity (17) is a variation of (2) and (12), and identity (18) is the counterpart of (17) for the process $-X$ (reproducing (5) for $\lambda \to \infty$ and (7) for $\lambda \downarrow 0$).

Remark 3.2. There is a close link between some of our results and transforms of certain occupation times:

$$\mathbb{E}_x\left(e^{\theta X(\tau_0^-)}; \tau_0^- < T_a^+\right) = \mathbb{E}_x\left(e^{\theta X(\tau_0^-)}; \tau_0^- < \infty, N(A) = 0\right)$$
$$= \mathbb{E}_x\left(e^{\theta X(\tau_0^-) - \lambda\int_0^{\tau_0^-} 1_{\{X(t)>a\}}\,dt}; \tau_0^- < \infty\right),$$

where $A = \{t \in [0, \tau_0^-)\colon\ X(t) > a\}$ and N is an independent Poisson random measure with intensity $\lambda\,dt$. A similar identity also holds for $\mathbb{P}_x(\tau_a^+ < T_0^-)$. Hence, (12) and (18) for $\theta = 0$ can alternatively be obtained from [19], Corollaries 1 and 2, and taking appropriate limits followed by somewhat tedious simplifications.

The next result considers two-sided exit for exclusively Poissonian observation of the process.

Theorem 3.2. *For $a, \theta \geq 0$, $x \leq a$ and implicit killing rate $q \geq 0$, we have*

$$\mathbb{E}_x\left(e^{\theta X(T_0^-)}; T_0^- < T_a^+\right) = \frac{\lambda}{\lambda - \psi(\theta)}\left(Z(x,\theta) - Z(x,\Phi_\lambda)\frac{\tilde{Z}(a,\Phi_\lambda,\theta)}{\tilde{Z}(a,\Phi_\lambda,\Phi_\lambda)}\right), \quad (19)$$

$$\mathbb{E}_x\left(e^{-\theta(X(T_a^+)-a)}; T_a^+ < T_0^-\right) = \frac{\lambda}{\Phi_\lambda + \theta}\frac{Z(x,\Phi_\lambda)}{\tilde{Z}(a,\Phi_\lambda,\Phi_\lambda)}, \quad (20)$$

where we define a third scale function as

$$\tilde{Z}(x,\alpha,\beta) := \frac{\psi(\alpha)Z(x,\beta)-\psi(\beta)Z(x,\alpha)}{\alpha-\beta}, \qquad \alpha,\beta \geq 0. \tag{21}$$

Again there is a striking similarity between (15) and (19), as well as between (17) and (20). Note that for $\alpha = \beta$ the definition (21) results in

$$\tilde{Z}(x,\alpha,\alpha) = \psi'(\alpha)Z(x,\alpha) - \psi(\alpha)Z'(x,\alpha),$$

where the differentiation of Z is with respect to the second argument.

We now present results for reflected processes. Write $\mathbb{E}_x^0$ for the law of X reflected at 0 (from below) and $\mathbb{E}_x^a$ for the law of X reflected at a from above, and let R be the regulator at the corresponding barrier. That is $(X(t), R(t))$ under $\mathbb{E}_x^0$ and under $\mathbb{E}_x^a$ is given by $(X(t) + (-\underline{X}(t))^+, (-\underline{X}(t))^+)$ and $(X(t) - (\overline{X}(t) - a)^+, (\overline{X}(t) - a)^+)$ under $\mathbb{E}_x$, respectively, where

$$\underline{X}(t) := \inf\{X(s)\colon 0 \leq s \leq t\}, \qquad \overline{X}(t) := \sup\{X(s)\colon 0 \leq s \leq t\}.$$

Theorem 3.3. *For $a > 0, \theta, \vartheta \geq 0, x \leq a$ and implicit killing rate $q \geq 0$, we have the following identities for the reflected processes*:

$$\begin{aligned} &\mathbb{E}_x^0\big(\mathrm{e}^{-\vartheta R(T_a^+)-\theta(X(T_a^+)-a)}; T_a^+ < \infty\big) \\ &\quad = \frac{\lambda(\vartheta-\Phi_\lambda)Z(x,\vartheta)}{(\Phi_\lambda+\theta)(\psi(\vartheta)Z(a,\Phi_\lambda)-\lambda Z(a,\vartheta))} = \frac{\lambda}{\Phi_\lambda+\theta}\frac{Z(x,\vartheta)}{\tilde{Z}(a,\vartheta,\Phi_\lambda)}, \end{aligned} \tag{22}$$

$$\begin{aligned} &\mathbb{E}_x^a\big(\mathrm{e}^{-\vartheta R(T_0^-)+\theta X(T_0^-)}; T_0^- < \infty\big) \\ &\quad = \frac{\lambda}{\lambda-\psi(\theta)}\left(Z(x,\theta) + Z(x,\Phi_\lambda)\frac{W(a)\psi(\theta)-(\theta+\vartheta)Z(a,\theta)}{Z'(a,\Phi_\lambda)+\vartheta Z(a,\Phi_\lambda)}\right), \end{aligned} \tag{23}$$

where the derivative of Z is taken with respect to the first argument.

Note that T_a^+ and T_0^- can be infinite due to the implicit killing rate q. This result for $\lambda = \infty$ is to be compared with

$$\mathbb{E}_x^0\big(\mathrm{e}^{-\vartheta R(\tau_a^+)}; \tau_a^+ < \infty\big) = \frac{Z(x,\vartheta)}{Z(a,\vartheta)}, \tag{24}$$

$$\begin{aligned} &\mathbb{E}_x^a\big(\mathrm{e}^{-\vartheta R(\tau_0^-)+\theta X(\tau_0^-)}; \tau_0^- < \infty\big) \\ &\quad = Z(x,\theta) + W(x)\frac{W(a)\psi(\theta)-(\theta+\vartheta)Z(a,\theta)}{W'_+(a)+\vartheta W(a)}, \end{aligned} \tag{25}$$

where W'_+ denotes the right derivative of W (see, e.g., [17], Theorem 8.10, for $\vartheta = 0$ and [15] for the general case). Furthermore, letting $a \to \infty$ in (23) and using (6) we obtain (14), which provides a nice check (ϑ indeed cancels out). Similarly, (22) leads to (16) if we put $a = \Delta + x$ and let $x \to \infty$ (note that (16) depends only on the difference Δ).

Finally, we note that yet another exit identity for a reflected process with Poissonian observations can be found in [5], Corollary 6.1. In particular, letting $\rho_y := \inf\{t \geq 0\colon\ R(t) > y\}$ be the first passage time of R, it holds that

$$\mathbb{P}_x^a\big(\rho_y < T_0^-\big) = \frac{Z(x, \Phi_\lambda)}{Z(a, \Phi_\lambda)} \exp\biggl(-\frac{Z'(a, \Phi_\lambda)}{Z(a, \Phi_\lambda)} y\biggr), \tag{26}$$

where $q \geq 0$ is implicit and $x \in [0, a]$. If X is some surplus process, it is natural to interpret $R(t)$ as the dividend payments up to time t according to a horizontal dividend barrier strategy. Identity (26) then allows to obtain the expected discounted dividends until ruin:

$$\begin{aligned}
\mathbb{E}_x^a \int_0^\infty \mathrm{e}^{-qt} 1_{\{t<T_0^-\}}\,\mathrm{d}R(t) &= \mathbb{E}_x^a \int_0^\infty \mathrm{e}^{-q\rho_y} 1_{\{\rho_y<T_0^-\}}\,\mathrm{d}y \\
&= \int_0^\infty \mathbb{E}_x^a\big(\mathrm{e}^{-q\rho_y}; \rho_y < T_0^-\big)\,\mathrm{d}y \\
&= \frac{Z_q(x, \Phi_{\lambda+q})}{Z_q(a, \Phi_{\lambda+q})} \int_0^\infty \exp\biggl(-\frac{Z_q'(a, \Phi_{\lambda+q})}{Z_q(a, \Phi_{\lambda+q})} y\biggr)\,\mathrm{d}y \\
&= \frac{Z_q(x, \Phi_{\lambda+q})}{Z_q'(a, \Phi_{\lambda+q})},
\end{aligned} \tag{27}$$

where $x \in [0, a]$. In the case of continuous observations, that is, $\lambda = \infty$, this expression reduces to $W_q(x)/W_{q+}'(a)$, see, for example, [21], Proposition 2. Also, in the absence of discounting ($q = 0$), one obtains from (23) for $\theta = 0$ that

$$\mathbb{E}_a^a\big(\mathrm{e}^{-\vartheta R(T_0^-)}\big) = Z'(a, \Phi_\lambda)/\big(Z'(a, \Phi_\lambda) + \vartheta Z(a, \Phi_\lambda)\big),$$

that is, the distribution of total dividend payments until ruin (observed at Poissonian times) is exponentially distributed with parameter $Z'(a, \Phi_\lambda)/Z(a, \Phi_\lambda)$, if the initial surplus level is at the barrier. The exponential parameter reduces to $W_+'(a)/W(a)$ for $\lambda \to \infty$, cf. [21], Section 5.

Remark 3.3. We emphasize again that each of the formulas in Theorems 3.1–3.3 can also be written for $q > 0$ in an explicit form, see also (27). For instance, (15) can be read as

$$\mathbb{E}_x\big(\mathrm{e}^{-qT_0^- + \theta X(T_0^-)}; T_0^- < \tau_a^+\big) = \frac{\lambda}{\lambda - \psi_q(\theta)}\biggl(Z_q(x, \theta) - Z_q(x, \Phi_{\lambda+q}) \frac{Z_q(a, \theta)}{Z_q(a, \Phi_{\lambda+q})}\biggr).$$

4. Proofs

Some of the proofs below will rely on the following intriguing identity first observed in [19], Equation (6):

$$(p - q) \int_0^a W_p(a - x) W_q(x)\,\mathrm{d}x = W_p(a) - W_q(a), \tag{28}$$

which as a consequence yields

$$\int_0^a W_q(a-x)W_q(x)\,\mathrm{d}x = \frac{\partial W_q(a)}{\partial q}.$$

The following result generalizes the second part of [19], Equation (6):

Lemma 4.1. *For $\theta, \alpha, p, q \geq 0$, it holds that*

$$(p-q)\int_0^a W_p(a-x)Z_q(x,\theta)\,\mathrm{d}x = Z_p(a,\theta) - Z_q(a,\theta).$$

Proof. First, we show that

$$\begin{aligned}(p-q)&\int_0^a W_p(a-x)\int_0^x \mathrm{e}^{\theta(x-y)}W_q(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \mathrm{e}^{\theta a}\biggl(\int_0^a \mathrm{e}^{-\theta x}W_p(x)\,\mathrm{d}x - \int_0^a \mathrm{e}^{-\theta x}W_q(x)\,\mathrm{d}x\biggr)\end{aligned} \tag{29}$$

by taking transforms of both sides. The left-hand side gives, for sufficiently large s,

$$\int_0^\infty \mathrm{e}^{-sa}\biggl((p-q)\int_0^a W_p(a-x)\int_0^x \mathrm{e}^{\theta(x-y)}W_q(y)\,\mathrm{d}y\,\mathrm{d}x\biggr)\mathrm{d}a = \frac{p-q}{(s-\theta)\psi_p(s)\psi_q(s)},$$

and for the right-hand side we have

$$\int_0^\infty \mathrm{e}^{-sa}\biggl(\mathrm{e}^{\theta a}\int_0^a \mathrm{e}^{-\theta x}W_p(x)\,\mathrm{d}x\biggr)\mathrm{d}a = \frac{1}{(s-\theta)\psi_p(s)}$$

and similarly for the second term. Then (29) follows by noting that

$$\frac{1}{\psi_p(s)} - \frac{1}{\psi_q(s)} = \frac{p-q}{\psi_p(s)\psi_q(s)}.$$

Finally, using (29) we get

$$\begin{aligned}(p-q)&\int_0^a W_p(a-x)Z_q(x,\theta)\,\mathrm{d}x \\ &= (p-q)\int_0^a W_p(a-x)\mathrm{e}^{\theta x}\,\mathrm{d}x - \psi_q(\theta)\mathrm{e}^{\theta a}\biggl(\int_0^a \mathrm{e}^{-\theta x}W_p(x)\,\mathrm{d}x - \int_0^a \mathrm{e}^{-\theta x}W_q(x)\,\mathrm{d}x\biggr) \\ &= \mathrm{e}^{\theta a}\bigl(p-q-\psi_q(\theta)\bigr)\int_0^a \mathrm{e}^{-\theta x}W_p(x)\,\mathrm{d}x + \mathrm{e}^{\theta a}\psi_q(\theta)\int_0^a \mathrm{e}^{-\theta x}W_q(x)\,\mathrm{d}x \\ &= Z_p(a,\theta) - Z_q(a,\theta)\end{aligned}$$

finishing the proof. □

4.1. Proof of Theorem 3.1

We split the proof into several parts.

Proof of Equation (15). Denoting $f(x,\theta,a) := \mathbb{E}_x(\mathrm{e}^{\theta X(T_0^-)}; T_0^- < \tau_a^+)$ one can write, using the strong Markov property,

$$f(x,\theta,a) = \int_{-\infty}^{0} \mathbb{P}_x\big(X\big(\tau_0^-\big) \in \mathrm{d}z, \tau_0^- < \tau_a^+\big)\big(\mathbb{P}_z\big(\tau_0^+ < e_\lambda\big) f(0;\theta,a) + \mathbb{E}_z\big(\mathrm{e}^{\theta X(e_\lambda)}, e_\lambda < \tau_0^+\big)\big).$$

Recall that $\mathbb{P}_z(\tau_0^+ < e_\lambda) = \mathrm{e}^{\Phi_\lambda z}$, $z \le 0$ and also

$$\mathbb{E}_z\big(\mathrm{e}^{\theta X(e_\lambda)}, e_\lambda < \tau_0^+\big) = \mathbb{E}_z\big(\mathrm{e}^{\theta X(e_\lambda)}\big) - \mathrm{e}^{\Phi_\lambda z}\mathbb{E}\big(\mathrm{e}^{\theta X(e_\lambda)}\big) = \frac{\lambda}{\lambda - \psi(\theta)}\big(\mathrm{e}^{\theta z} - \mathrm{e}^{\Phi_\lambda z}\big) \tag{30}$$

for θ small enough such that $\psi(\theta) < \lambda$. The result can then be analytically continued to any $\theta \ge 0$. Thus, using (5) we arrive at

$$\begin{aligned} f(x,\theta,a) = {}& \left(Z(x,\Phi_\lambda) - Z(a,\Phi_\lambda)\frac{W(x)}{W(a)}\right)\left(f(0,\theta,a) - \frac{\lambda}{\lambda - \psi(\theta)}\right) \\ & + \left(Z(x,\theta) - Z(a,\theta)\frac{W(x)}{W(a)}\right)\frac{\lambda}{\lambda - \psi(\theta)}. \end{aligned} \tag{31}$$

Note that due to $Z(0,\theta) = 1$ we get for $x = 0$

$$Z(a,\Phi_\lambda)\frac{W(0)}{W(a)} f(0,\theta,a) = \frac{\lambda}{\lambda - \psi(\theta)}\left(Z(a,\Phi_\lambda)\frac{W(0)}{W(a)} - Z(a,\theta)\frac{W(0)}{W(a)}\right).$$

This equation is trivial when $W(0) = 0$, that is, when X has sample paths of unbounded variation, see, for example, [17], Equation (8.26), but otherwise we have

$$f(0,\theta,a) = \frac{\lambda}{\lambda - \psi(\theta)}\left(1 - \frac{Z(a,\theta)}{Z(a,\Phi_\lambda)}\right) \tag{32}$$

and the result follows combining (31) and (32).

It is only left to show that (32) holds also when X has sample paths of unbounded variation, that is, when $W(0) = 0$. For $x \in (0,a)$, we have

$$\begin{aligned} f(0,\theta,a) &= \mathbb{P}\big(\tau_x^+ < e_\lambda\big) f(x,\theta,a) + A(x) + B(x), \\ \text{where } A(x) &:= \mathbb{E}\big(\mathrm{e}^{\theta X(e_\lambda)}; e_\lambda < \tau_x^+, X(e_\lambda) < 0\big), \\ B(x) &:= \int_0^x \mathbb{P}\big(e_\lambda < \tau_x^+, X(e_\lambda) \in \mathrm{d}y\big) f(y,\theta,a). \end{aligned} \tag{33}$$

It is well known that for $\theta \ge 0$

$$\int_0^x \mathrm{e}^{-\theta y} W(y)\,\mathrm{d}y / W(x) \to 0 \qquad \text{as } x \downarrow 0, \tag{34}$$

which can be seen by interpreting the ratio of scale functions. In a similar way one can show, using (28), that $W_\lambda(x)/W(x) \to 1$ as $x \downarrow 0$. Using (9), observe that

$$\mathbb{P}\big(e_\lambda < \tau_x^+, X(e_\lambda) \geq 0\big) = \lambda \mathrm{e}^{-\Phi_\lambda x} \int_0^x W_\lambda(y)\,\mathrm{d}y = \mathrm{o}\big(W(x)\big)$$

as $x \downarrow 0$. Next, using (30) observe that $B(x) = \mathrm{o}(W(x))$ and

$$\begin{aligned} A(x) &:= \mathbb{E}\big(\mathrm{e}^{\theta X(e_\lambda)}; e_\lambda < \tau_x^+\big) - \mathbb{E}\big(\mathrm{e}^{\theta X(e_\lambda)}; e_\lambda < \tau_x^+, X(e_\lambda) \geq 0\big) \\ &= \frac{\lambda}{\lambda - \psi(\theta)}\big(1 - \mathrm{e}^{(\theta - \Phi_\lambda)x}\big) + \mathrm{o}\big(W(x)\big). \end{aligned}$$

Plugging (33) into (31) and rearranging it we obtain

$$\begin{aligned} &f(x,\theta,a)\left[1 - \left(Z(x,\Phi_\lambda) - Z(a,\Phi_\lambda)\frac{W(x)}{W(a)}\right)\mathrm{e}^{-\Phi_\lambda x}\right] \\ &\quad = -\frac{\lambda}{\lambda - \psi(\theta)} \\ &\qquad \times \left(Z(x,\Phi_\lambda)\mathrm{e}^{(\theta-\Phi_\lambda)x} - Z(x,\theta) + \frac{W(x)}{W(a)}\big(Z(a,\theta) - Z(a,\Phi_\lambda)\mathrm{e}^{(\theta-\Phi_\lambda)x}\big)\right) \\ &\qquad + \mathrm{o}\big(W(x)\big). \end{aligned} \tag{35}$$

Divide (35) by $W(x)$ and take the limit as $x \downarrow 0$, using the representation (3) and then also (34), to obtain

$$f(0,\theta,a)Z(a,\Phi_\lambda)\frac{1}{W(a)} = -\frac{\lambda}{\lambda - \psi(\theta)}\frac{1}{W(a)}\big(Z(a,\theta) - Z(a,\Phi_\lambda)\big),$$

which immediately yields (32). □

Proof of Equation (17). We only need to consider $x \in [0, a]$. Putting

$$f(x,\theta) := \mathbb{E}_x\big(\mathrm{e}^{-\theta(X(T_a^+)-a)}; T_a^+ < \tau_0^-\big)$$

we write

$$f(x,\theta) = \mathbb{P}_x\big(\tau_a^+ < \tau_0^-\big)f(a,\theta) = \frac{W(x)}{W(a)}f(a,\theta). \tag{36}$$

Using (10) and conditioning on the first Poisson observation time, we get

$$f(a,\theta) = \int_a^\infty \mathrm{e}^{-\theta(x-a)}\lambda W_\lambda(a)\mathrm{e}^{-\Phi_\lambda x}\,\mathrm{d}x + \int_0^a \lambda\big(W_\lambda(a)\mathrm{e}^{-\Phi_\lambda x} - W_\lambda(a-x)\big)f(x,\theta)\,\mathrm{d}x.$$

Using (36), we obtain

$$f(a,\theta)\left(W(a)-\lambda W_\lambda(a)\int_0^a W(x)\mathrm{e}^{-\Phi_\lambda x}\,\mathrm{d}x+\lambda\int_0^a W_\lambda(x)W(a-x)\,\mathrm{d}x\right)$$
$$=\frac{\lambda W_\lambda(a)W(a)}{\Phi_\lambda+\theta}\mathrm{e}^{-\Phi_\lambda a}.$$

With the help of (28), the expression in the brackets reduces to

$$W_\lambda(a)\left(1-\lambda\int_0^a W(x)\mathrm{e}^{-\Phi_\lambda x}\,\mathrm{d}x\right)=W_\lambda(a)\mathrm{e}^{-\Phi_\lambda a}Z(a,\Phi_\lambda),$$

which shows that $f(a,\theta)=\frac{\lambda}{\Phi_\lambda+\theta}\frac{W(a)}{Z(a,\Phi_\lambda)}$, completing the proof in view of (36). □

Proof of Equations (14) and (16). Identity (14) for $\theta>\Phi$ follows immediately from (15) and (6); by analytic continuation it is also true for any $\theta\geq 0$. Similarly, (16) follows from (17) by plugging in $x+u$ and $a+u$ instead of x and u, respectively, letting $u\to\infty$ and using (6) together with

$$\lim_{u\to\infty}W(x+u)/W(a+u)=\mathbb{P}_x\left(\tau_a^+<\infty\right)=\mathrm{e}^{-\Phi(a-x)}.$$ □

Proof of Equation (18). Consider $f(x):=\mathbb{E}_x(\mathrm{e}^{\theta X(\tau_0^-)};\tau_0^-<T_a^+)$, which can be written as

$$\begin{aligned}f(x)&=\mathbb{P}_x\left(\tau_a^+<\tau_0^-\right)f(a)+\mathbb{E}_x\left(\mathrm{e}^{\theta X(\tau_0^-)};\tau_0^-<\tau_a^+\right)\\&=\frac{W(x)}{W(a)}f(a)+Z(x,\theta)-W(x)\frac{Z(a,\theta)}{W(a)}\end{aligned}\tag{37}$$

and also

$$f(a)=\int_0^a\mathbb{P}_a\left(X(e_\lambda)\in\mathrm{d}x,e_\lambda<\tau_0^-\right)f(x)+\mathbb{E}_a\left(\mathrm{e}^{\theta X(\tau_0^-)};\tau_0^-<e_\lambda\right),$$

where the last term is $Z_\lambda(a,\theta)-W_\lambda(a)\frac{\psi(\theta)-\lambda}{\theta-\Phi_\lambda}$ according to (7). Hence, we can determine $f(a)$ by plugging in (37) and computing the integrals of $W(x)$ and $Z(x,\theta)$ with respect to (10). In particular, using (28) we find

$$\begin{aligned}&\int_0^a\mathbb{P}_a\left(X(e_\lambda)\in\mathrm{d}x,e_\lambda<\tau_0^-\right)W(x)\\&=\int_0^a\lambda\left(W_\lambda(a)\mathrm{e}^{-\Phi_\lambda x}-W_\lambda(a-x)\right)W(x)\,\mathrm{d}x\\&=\lambda W_\lambda(a)\int_0^a\mathrm{e}^{-\Phi_\lambda x}W(x)\,\mathrm{d}x-\left(W_\lambda(a)-W(a)\right)\\&=W(a)-Z(a,\Phi_\lambda)\mathrm{e}^{-\Phi_\lambda a}W_\lambda(a).\end{aligned}$$

Next we compute

$$\begin{aligned}
&\int_0^a \mathrm{e}^{-\Phi_\lambda x} Z(x,\theta)\,\mathrm{d}x \\
&\quad= \frac{1}{\theta-\Phi_\lambda}\big(\mathrm{e}^{(\theta-\Phi_\lambda)a}-1\big) - \frac{\psi(\theta)}{\theta-\Phi_\lambda}\left(\mathrm{e}^{(\theta-\Phi_\lambda)a}\int_0^a \mathrm{e}^{-\theta y}W(y)\,\mathrm{d}y - \int_0^a \mathrm{e}^{-\Phi_\lambda y}W(y)\,\mathrm{d}y\right) \\
&\quad= \frac{1}{\theta-\Phi_\lambda}\left(\mathrm{e}^{-\Phi_\lambda a}Z(a,\theta) - 1 + \psi(\theta)\int_0^a \mathrm{e}^{-\Phi_\lambda y}W(y)\,\mathrm{d}y\right),
\end{aligned}$$

which together with Lemma 4.1 implies that

$$\begin{aligned}
T &:= \int_0^a \mathbb{P}_a\big(X(e_\lambda)\in\mathrm{d}y, e_\lambda < \tau_0^-\big) Z(y,\theta) \\
&= \frac{\lambda W_\lambda(a)}{\theta-\Phi_\lambda}\left(\mathrm{e}^{-\Phi_\lambda a}Z(a,\theta) - 1 + \psi(\theta)\int_0^a \mathrm{e}^{-\Phi_\lambda y}W(y)\,\mathrm{d}y\right) - Z_\lambda(a,\theta) + Z(a,\theta).
\end{aligned}$$

This finally yields

$$\begin{aligned}
f(a) = {}&\left(1 - Z(a,\Phi_\lambda)\mathrm{e}^{-\Phi_\lambda a}\frac{W_\lambda(a)}{W(a)}\right) f(a) + T - \left(1 - Z(a,\Phi_\lambda)\mathrm{e}^{-\Phi_\lambda a}\frac{W_\lambda(a)}{W(a)}\right) Z(a,\theta) \\
&+ Z_\lambda(a,\theta) - W_\lambda(a)\frac{\psi(\theta)-\lambda}{\theta-\Phi_\lambda},
\end{aligned}$$

which reduces to

$$\begin{aligned}
&\left(Z(a,\Phi_\lambda)\mathrm{e}^{-\Phi_\lambda a}\frac{W_\lambda(a)}{W(a)}\right) f(a) \\
&\quad= \frac{W_\lambda(a)}{\theta-\Phi_\lambda}\big(\lambda \mathrm{e}^{-\Phi_\lambda a}Z(a,\theta) - \psi(\theta)\mathrm{e}^{-\Phi_\lambda a}Z(a,\Phi_\lambda)\big) + Z(a,\Phi_\lambda)\mathrm{e}^{-\Phi_\lambda a}\frac{W_\lambda(a)}{W(a)} Z(a,\theta),
\end{aligned}$$

and hence

$$f(a) = Z(a,\theta) - \frac{W(a)}{\theta-\Phi_\lambda}\left(\psi(\theta) - \lambda\frac{Z(a,\theta)}{Z(a,\Phi_\lambda)}\right).$$

Now the result follows from (37). □

4.2. Proof of Theorem 3.2

Using (3) and changing the order of integration, we can show that

$$(\alpha-\beta)\int_0^a \mathrm{e}^{-\alpha x}Z(x,\beta)\,\mathrm{d}x = 1 + \big(\mathrm{e}^{-\alpha a}Z(a,\alpha) - 1\big)\psi(\beta)/\psi(\alpha) - \mathrm{e}^{-\alpha a}Z(a,\beta),$$

and hence $\tilde{Z}$ has an alternative representation

$$\tilde{Z}(a,\alpha,\beta)=\mathrm{e}^{\alpha a}\frac{\psi(\alpha)-\psi(\beta)}{\alpha-\beta}-\psi(\alpha)\int_0^a \mathrm{e}^{\alpha(a-x)}Z(x,\beta)\,\mathrm{d}x.$$

Plugging in $\alpha=\Phi_\lambda$ and $\beta=\theta$, we obtain

$$\lambda\int_0^a \mathrm{e}^{-\Phi_\lambda x}Z(x,\theta)\,\mathrm{d}x=\frac{\psi(\theta)-\lambda}{\theta-\Phi_\lambda}-\mathrm{e}^{-\Phi_\lambda a}\tilde{Z}(a,\Phi_\lambda,\theta). \tag{38}$$

Proof of Equation (19). Defining $f(x,\theta):=\mathbb{E}_x(\mathrm{e}^{\theta X(T_0^-)};T_0^-<T_a^+)$ for $x\le a$, we write

$$f(x,\theta)=\mathbb{E}_x\big(\mathrm{e}^{\theta X(T_0^-)};T_0^-<\tau_a^+\big)+\mathbb{P}_x\big(\tau_a^+<T_0^-\big)f(a,\theta).$$

Plugging in the corresponding identities, we first get for $x=0$ that

$$f(0,\theta)=\frac{\lambda}{\lambda-\psi(\theta)}\left(1-\frac{Z(a,\theta)}{Z(a,\Phi_\lambda)}\right)+\frac{f(a,\theta)}{Z(a,\Phi_\lambda)},$$

and some simplifications yield

$$f(x,\theta)=\frac{\lambda}{\lambda-\psi(\theta)}\big(Z(x,\theta)-Z(x,\Phi_\lambda)\big)+f(0,\theta)Z(x,\Phi_\lambda). \tag{39}$$

Using (8) and conditioning on the first observation epoch, we get

$$f(0,\theta)=\frac{\lambda}{\psi'(\Phi_\lambda)}\int_0^a \mathrm{e}^{-\Phi_\lambda x}f(x,\theta)\,\mathrm{d}x+\int_{-\infty}^0 \mathrm{e}^{\theta x}\mathbb{P}\big(X(e_\lambda)\in\mathrm{d}x\big),$$

where the latter term evaluates to $\frac{\lambda}{\lambda-\psi(\theta)}+\frac{\lambda}{\psi'(\Phi_\lambda)(\theta-\Phi_\lambda)}$. Plugging in (39), we get

$$\begin{aligned}&f(0,\theta)\left(1-\frac{\lambda}{\psi'(\Phi_\lambda)}\int_0^a \mathrm{e}^{-\Phi_\lambda x}Z(x,\Phi_\lambda)\,\mathrm{d}x\right)\\&\quad=\frac{\lambda}{\psi'(\Phi_\lambda)}\frac{\lambda}{\lambda-\psi(\theta)}\int_0^a \mathrm{e}^{-\Phi_\lambda x}\big(Z(x,\theta)-Z(x,\Phi_\lambda)\big)\,\mathrm{d}x+\frac{\lambda}{\lambda-\psi(\theta)}+\frac{\lambda}{\psi'(\Phi_\lambda)(\theta-\Phi_\lambda)}.\end{aligned}$$

Using (38) this reduces to

$$\begin{aligned}&f(0,\theta)\mathrm{e}^{-\Phi_\lambda a}\tilde{Z}(a,\Phi_\lambda,\Phi_\lambda)/\psi'(\Phi_\lambda)\\&\quad=\mathrm{e}^{-\Phi_\lambda a}\frac{\lambda}{\psi'(\Phi_\lambda)(\lambda-\psi(\theta))}\big(\tilde{Z}(a,\Phi_\lambda,\Phi_\lambda)-\tilde{Z}(a,\Phi_\lambda,\theta)\big),\end{aligned}$$

which readily leads to

$$f(0,\theta)=\frac{\lambda}{\lambda-\psi(\theta)}\left(1-\frac{\tilde{Z}(a,\Phi_\lambda,\theta)}{\tilde{Z}(a,\Phi_\lambda,\Phi_\lambda)}\right)$$

and then the result follows from (39). □

Proof of Equation (20). We write for $x \le a$ that

$$\mathbb{E}_x\big(e^{-\theta(X(T_a^+)-a)}\big) = \mathbb{E}_x\big(e^{-\theta(X(T_a^+)-a)}; T_a^+ < T_0^-\big) + \int_{-\infty}^0 \mathbb{P}_x\big(X(T_0^-) \in dy, T_0^- < T_a^+\big)\mathbb{E}_y\big(e^{-\theta(X(T_a^+)-a)}\big).$$

Using (16), we obtain

$$\mathbb{E}_x\big(e^{-\theta(X(T_a^+)-a)}; T_a^+ < T_0^-\big) = \frac{\Phi_\lambda - \Phi}{\Phi_\lambda + \theta} e^{-\Phi a}\big(e^{\Phi x} - \mathbb{E}_x\big(e^{\Phi X(T_0^-)}; T_0^- < T_a^+\big)\big).$$

With (21), we see that $\tilde{Z}(a, \Phi_\lambda, \Phi) = \frac{\lambda}{\Phi_\lambda - \Phi} e^{\Phi a}$ and then it follows from (19) that

$$\mathbb{E}_x\big(e^{\Phi X(T_0^-)}; T_0^- < T_a^+\big) = e^{\Phi x} - Z(x, \Phi_\lambda)\frac{\lambda}{\Phi_\lambda - \Phi} e^{\Phi a}/\tilde{Z}(a, \Phi_\lambda, \Phi_\lambda),$$

which completes the proof. □

4.3. Proof of Theorem 3.3

Proof of Equation (22). Let $f(x) := \mathbb{E}_x^0(e^{-\vartheta R(T_a^+)-\theta(X(T_a^+)-a)}; T_a^+ < \infty)$, then

$$f(x) = \mathbb{E}_x^0\big(e^{-\vartheta R(\tau_a^+)}; \tau_a^+ < \infty\big) f(a) = \frac{Z(x, \vartheta)}{Z(a, \vartheta)} f(a)$$

according to (24), and also

$$f(a) = \mathbb{E}_a\big(e^{-\theta(X(T_a^+)-a)}; T_a^+ < \tau_0^-\big) + \mathbb{E}_a\big(e^{\vartheta X(\tau_0^-)}; \tau_0^- < T_a^+\big) f(0),$$

which is equal to $Z(a, \vartheta) f(0)$. Plugging in (17) and (18) we solve for $f(0)$:

$$f(0) = \frac{\lambda(\vartheta - \Phi_\lambda)}{(\Phi_\lambda + \theta)(\psi(\vartheta)Z(a, \Phi_\lambda) - \lambda Z(a, \vartheta))}$$

and the result follows. □

Proof of Equation (23). Let $f(x) = \mathbb{E}_x^a(e^{-\vartheta R(T_0^-)+\theta X(T_0^-)}; T_0^- < \infty)$, then

$$f(x) = \mathbb{E}_x\big(e^{\theta X(T_0^-)}; T_0^- < \tau_a^+\big) + \mathbb{P}_x\big(\tau_a^+ < T_0^-\big) f(a), \tag{40}$$

which using (15) and (12) yields

$$f(0) = \frac{\lambda}{\lambda - \psi(\theta)}\left(1 - \frac{Z(a, \theta)}{Z(a, \Phi_\lambda)}\right) + \frac{f(a)}{Z(a, \Phi_\lambda)}. \tag{41}$$

Also

$$f(a)=\int_{-\infty}^{0}\big(\mathbb{E}_a^a\big(\mathrm{e}^{-\vartheta R(\tau_0^-)};X\big(\tau_0^-\big)\in \mathrm{d}z,\tau_0^-<\infty\big)\big(\mathrm{e}^{\Phi_\lambda z}f(0)+\mathbb{E}_z\big(\mathrm{e}^{\theta X(e_\lambda)};e_\lambda<\tau_0^+\big)\big)\big).$$

Using (30), we arrive at

$$\begin{aligned} f(a)&=\mathbb{E}_a^a\big(\mathrm{e}^{-\vartheta R(\tau_0^-)+\Phi_\lambda X(\tau_0^-)}\big)f(0)\\ &\quad+\frac{\lambda}{\lambda-\psi(\theta)}\big(\mathbb{E}_a^a\big(\mathrm{e}^{-\vartheta R(\tau_0^-)+\theta X(\tau_0^-)}\big)-\mathbb{E}_a^a\big(\mathrm{e}^{-\vartheta R(\tau_0^-)+\Phi_\lambda X(\tau_0^-)}\big)\big). \end{aligned} \tag{42}$$

Substituting (25) and (41) into (42), we obtain after some simplifications

$$\begin{aligned} &f(a)\left((\Phi_\lambda+\vartheta)-\lambda\frac{W(a)}{Z(a,\Phi_\lambda)}\right)\\ &\quad=\frac{\lambda}{\lambda-\psi(\theta)}\left(W(a)\left(\psi(\theta)-\lambda\frac{Z(a,\theta)}{Z(a,\Phi_\lambda)}\right)+(\Phi_\lambda-\theta)Z(a,\theta)\right), \end{aligned}$$

which yields the result after plugging $f(a)$ into (40) and yet another round of simplifications. □

The above proofs mostly rely on the strong Markov property and various identities from fluctuation theory. We note that often there are several possibilities to approach a problem, but some of them may require significantly more effort to obtain a simple formula resembling the classical case. One could, for instance, consider using exit theory of random walks for a purely Poissonian observation. This approach builds upon some general formulas, see, for example, [12], Theorem 4, ignoring the crucial assumption of one-sided jumps. Consequently, one loses structure, making it hard to rewrite these formulas in terms of scale functions. Martingale techniques, as used, for example, in [9] to obtain some classical exit identities, also do not seem to be immediately appropriate for our setting. Moreover, one needs to guess the right martingale and for this one typically needs to know already the resulting expression. Finally, independent exponential inter-observation times may suggest using Wiener–Hopf factorization, exploited, for example, in [16] to design simulation algorithms. Indeed, this factorization is one way to prove (8)–(10), which are building blocks of our results.

5. Cramér–Lundberg risk model with exponential claims

As mentioned in Section 1, one application area for identities of the above type is the ruin analysis for an insurance portfolio with surplus value $X(t)=x+ct-S(t)$ at time t, where $x\ge 0$ is the initial capital and $c>0$ is a constant premium intensity. The classical Cramér–Lundberg risk model in this context assumes $S(t)$ to be a compound Poisson process, where independent and identically distributed claims arrive according to a homogeneous Poisson process with rate ν

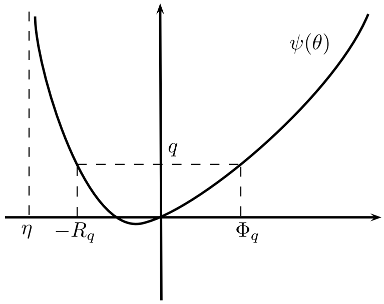

Figure 1. The function $\psi(\theta)$ and the inverses R_q and Φ_q.

(see, e.g., [7]). Assume now that claims are Exp(η) distributed. Then

$$\psi_q(\theta) = c\theta - \nu\bigl(1 - \mathbb{E}\bigl(\mathrm{e}^{-\theta e_\eta}\bigr)\bigr) - q = c\theta - \frac{\nu\theta}{\theta+\eta} - q.$$

If either $q > 0$ or $c - \nu/\eta \neq 0$ (the usual safety loading condition is $c - \nu/\eta > 0$), then the scale function has the form

$$W_q(x) = u_q \mathrm{e}^{\Phi_q x} - v_q \mathrm{e}^{-R_q x},$$

where $u_q, v_q > 0$ and $R_q, \Phi_q \geq 0$ (not simultaneously 0). Moreover, $-R_q$ and Φ_q are the two roots of $\psi_q(\theta) = 0$, see Figure 1 (note that R_0 is the classical Lundberg adjustment coefficient), and

$$\frac{u_q}{\theta - \Phi_q} - \frac{v_q}{\theta + R_q} = \frac{1}{\psi_q(\theta)}$$

yielding $u_q = 1/\psi_q'(\Phi_q) = \Phi_q'$, $v_q = -1/\psi_q'(-R_q) = R_q'$.

So we also obtain

$$\begin{aligned}
Z_q(x,\theta) &= \mathrm{e}^{\theta x}\biggl(1 - \psi_q(\theta)\biggl(\frac{u_q}{\Phi_q - \theta}\bigl(\mathrm{e}^{(\Phi_q-\theta)x} - 1\bigr) - \frac{v_q}{-R_q - \theta}\bigl(\mathrm{e}^{(-R_q-\theta)x} - 1\bigr)\biggr)\biggr) \\
&= \psi_q(\theta)\biggl(\frac{u_q \mathrm{e}^{\Phi_q x}}{\theta - \Phi_q} - \frac{v_q \mathrm{e}^{-R_q x}}{\theta + R_q}\biggr) \\
&= \frac{\psi_q(\theta)\Phi_q'}{\theta - \Phi_q}\bigl(\mathrm{e}^{\Phi_q x} - \mathrm{e}^{-R_q x}\bigr) + \mathrm{e}^{-R_q x}.
\end{aligned}$$

Consider (14), which for the present model immediately simplifies to

$$\begin{aligned}
\mathbb{E}_x\bigl(\mathrm{e}^{-qT_0^- + \theta X(T_0^-)}; T_0^- < \infty\bigr) &= \mathrm{e}^{-R_q x}\frac{\lambda - \psi_q(\theta)((\Phi_q - \Phi_{\lambda+q})/(\Phi_q - \theta))}{\lambda - \psi_q(\theta)} \\
&= \mathrm{e}^{-R_q x}\biggl(1 - \frac{\psi_q(\theta)}{\psi_{\lambda+q}(\theta)}\frac{\Phi_{\lambda+q} - \theta}{\Phi_q - \theta}\biggr).
\end{aligned}$$

Since $\psi_{\lambda+q}(\theta)\frac{\theta+\eta}{c}=(\theta+R_{\lambda+q})(\theta-\Phi_{\lambda+q})$, we get

$$\mathbb{E}_x\big(\mathrm{e}^{-qT_0^-+\theta X(T_0^-)};T_0^-<\infty\big)=\mathrm{e}^{-R_qx}\left(1-\frac{\theta+R_q}{\theta+R_{\lambda+q}}\right)=\mathrm{e}^{-R_qx}\frac{R_{\lambda+q}-R_q}{R_{\lambda+q}+\theta},$$

which agrees with the result of [2], Example 4.2 (take an exponential penalty function $w_2(y)=\mathrm{e}^{-\theta y}$ for the overshoot). Note also that $R_{\lambda+q}\to\eta$ as $\lambda\to\infty$, because $\theta=-\eta$ is the asymptote of $\psi_q(\theta)$. Hence, we also retain the classical formula for the Laplace transform of the (discounted) ruin deficit under continuous observation

$$\mathbb{E}_x\big(\mathrm{e}^{-q\tau_0^-+\theta X(\tau_0^-)};\tau_0^-<\infty\big)=\mathrm{e}^{-R_qx}\frac{\eta-R_q}{\eta+\theta},$$

cf. [13], Equation (5.42), which can alternatively be obtained using a direct argument (exchange the meaning of claims and interarrivals).

Next, identity (15) simplifies to

$$\begin{aligned}
&\mathbb{E}_x\big(\mathrm{e}^{-qT_0^-+\theta X(T_0^-)};T_0^-<\tau_a^+\big)\\
&\quad=\frac{-\lambda}{\psi_{\lambda+q}(\theta)}\frac{(\mathrm{e}^{-R_qa+\Phi_qx}-\mathrm{e}^{\Phi_qa-R_qx})(\psi_q(\theta)\Phi_q'/(\theta-\Phi_q)-\lambda\Phi_q'/(\Phi_{\lambda+q}-\Phi_q))}{(\lambda\Phi_q'/(\Phi_{\lambda+q}-\Phi_q))(\mathrm{e}^{\Phi_qa}-\mathrm{e}^{-R_qa})+\mathrm{e}^{-R_qa}}\\
&\quad=\frac{R_{\lambda+q}-R_q}{R_{\lambda+q}+\theta}\frac{\mathrm{e}^{\Phi_qa+R_q(a-x)}-\mathrm{e}^{\Phi_qx}}{\mathrm{e}^{\Phi_qa+R_qa}-1+(\Phi_{\lambda+q}-\Phi_q)/\lambda\Phi_q'}.
\end{aligned}$$

It is not hard to see that $(\Phi_{\lambda+q}-\Phi_q)/\lambda\to1/c$ as $\lambda\to\infty$, and hence we have

$$\begin{aligned}
\mathbb{E}_x\big(\mathrm{e}^{-q\tau_0^-+\theta X(\tau_0^-)};\tau_0^-<\tau_a^+\big)&=\frac{\eta-R_q}{\eta+\theta}\frac{\mathrm{e}^{\Phi_qa+R_q(a-x)}-\mathrm{e}^{\Phi_qx}}{\mathrm{e}^{\Phi_qa+R_qa}-1+\psi_q'(\Phi_q)/c}\\
&=\frac{\eta-R_q}{\eta+\theta}\frac{\mathrm{e}^{\Phi_qa+R_q(a-x)}-\mathrm{e}^{\Phi_qx}}{\mathrm{e}^{\Phi_qa+R_qa}+(R_q-\eta)/(\Phi_q+\eta)},
\end{aligned}$$

where the last equality follows from the observation that $\frac{\psi_q(\theta)}{\theta-\Phi_q}=c\frac{\theta+R_q}{\theta+\eta}$ and hence $\psi_q'(\Phi_q)=c\frac{\Phi_q+R_q}{\Phi_q+\eta}$.

Finally, (27) provides a formula for the expected (continuously) discounted dividends until ruin:

$$\begin{aligned}
\mathbb{E}_x^a\left(\int_0^\infty\mathrm{e}^{-qt}1_{\{t<T_0^-\}}\,\mathrm{d}R(t)\right)&=\frac{(\lambda\Phi_q'/(\Phi_{\lambda+q}-\Phi_q))(\mathrm{e}^{\Phi_qx}-\mathrm{e}^{-R_qx})+\mathrm{e}^{-R_qx}}{((\lambda\Phi_q'/(\Phi_{\lambda+q}-\Phi_q))(\Phi_q\mathrm{e}^{\Phi_qa}+R_q\mathrm{e}^{-R_qa})-R_q\mathrm{e}^{-R_qa}}\\
&=\frac{\mathrm{e}^{\Phi_qx}+\mathrm{e}^{-R_qx}((\Phi_{\lambda+q}-\Phi_q)/\lambda\Phi_q'-1)}{\Phi_q\mathrm{e}^{\Phi_qa}-R_q\mathrm{e}^{-R_qa}((\Phi_{\lambda+q}-\Phi_q)/\lambda\Phi_q'-1)}\\
&=\frac{(R_{\lambda+q}+\Phi_q)\mathrm{e}^{\Phi_qx}-(R_{\lambda+q}-R_q)\mathrm{e}^{-R_qx}}{(R_{\lambda+q}+\Phi_q)\Phi_q\mathrm{e}^{\Phi_qa}+R_q(R_{\lambda+q}-R_q)\mathrm{e}^{-R_qa}},
\end{aligned}\tag{43}$$

because $\frac{\Phi_{\lambda+q}-\Phi_q}{\lambda\Phi'_q} = \frac{\Phi_q+R_q}{\Phi_q+R_{\lambda+q}}$. This expression is similar to [1], Equation (24), but not identical, because there the dividends are paid at Poissonian times only (see also [8] for a spectrally positive model setup). Identity (43) is, however, the analogue of [3], Equation (20), where it was derived for a diffusion process $X(t)$.

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