Exit in duopoly under uncertainty

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Abstract

This paper examines a declining duopoly, where the firms must choose when to exit from the market. The uncertainty is modeled by letting the revenue stream follow a geometric Brownian motion. We consider the Markov-perfect equilibrium in firms' exit strategies. With a low degree of uncertainty there is a unique equilibrium, where one of the firms always exits before the other. However, when uncertainty is increased, another equilibrium with the reversed order of exit may appear ruining the uniqueness. Whether this happens or not depends on the degree of asymmetry in the firm specific parameters.

1 Introduction

While entry in growing markets has received more attention in the literature, declining markets and abandonment of existing businesses are also important phenomena in many industries. When considering a firm in isolation from competitors, the decision to exit is conceptually similar to entry. Just as the decision to enter, or more generally to trigger an investment project, the decision to exit is typically characterized by (partial) irreversibility, flexibility with respect to the timing, and continuously unfolding uncertainty over the profitability. Optimal timing of exit in such a setting is most naturally approached using the theory of irreversible investment under uncertainty, or in other words the theory of real options, as done by Dixit (1989) and Alvarez (1998, 1999).

However, in many cases industries are characterized by competitive interaction between the firms. Such strategic considerations induce a profound difference between the decisions of entry and exit. Assuming that the firms affect negatively each other's profitability, the strategic interaction in entry takes the form of preemption, while in exit it resembles war of attrition.

In this paper, we extend the methodology of irreversible investment under uncertainty to consider exit in an oligopolistic market structure. More particularly, we examine a declining duopoly with the following characteristics. The profitability evolves according to a geometric Brownian motion at such a low drift rate that it eventually becomes optimal for the firms to exit. The decision to exit is irreversible. The firms affect negatively each other's profitability, and thus both firms would like to see their competitor exit as early as possible. The questions to be analyzed are which of the firms exits first and when this happens.

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The main methodological feature of the model is that it incorporates in the real options framework the perfect Nash equilibrium concept similar to many deterministic timing games. We show that in equilibrium the firms always exit sequentially. With a given (arbitrarily small) asymmetry in the firm specific parameters, there always exists a unique equilibrium if the degree of uncertainty is sufficiently low. However, as the degree of uncertainty is increased, the uniqueness may break down. More specifically, in addition to the "normal" equilibrium, an equilibrium where the exit strategy of one of the firms is a disconnected stopping set in the state space may appear. Whether this happens or not depends on the degree of asymmetry between the firms.

The model builds on several existing streams of literature. In the following we review the most relevant literature, and discuss our contribution in relation to those papers.

The theory of irreversible investment under uncertainty considers problems, where a firm should choose the optimal timing of investment when the decision can not be reversed and the value of the project evolves stochastically. Important contributions to the theory include, e.g., McDonald and Siegel (1986) and Pindyck (1988), while a thorough review is given in Dixit and Pindyck (1994). A main conclusion is that due to an option value inherent in an investment opportunity, it is optimal to postpone the investment compared to the traditional investment theory.[1] Due to the similarity of the techniques and concepts to financial options pricing settings, this new view of investment is often referred to as the real options approach. The approach is also applicable in valuing other options associated with managing real assets, such as the abandonment of an existing business (option to exit), which has been studied by Dixit (1989) and Alvarez (1998, 1999). If the decision to exit is irreversible, then the theory suggests that the firm should stay in a declining market longer than if the exit decision is reversible.

Because in reality competitive aspects characterize many investment situations, one of the main shortcomings in basic real options models is that they do not account for the strategic interaction between the firms.[2] There is, however, a new stream of literature that incorporates game theoretic concepts in the real options framework to fix this deficit. Examples of models in discrete time are Smit and Ankum (1993), and Kulatilaka and Perotti (1998). Papers that model uncertainty using diffusion processes include Grenadier (1996), Joaquin and Butler (1999), Lambrecht (1999, 2001), Mason and Weeds (2001), Moretto (1996), Weeds (2002), and Williams (1993). Typical for these papers is that concepts from the industrial organization literature, particularly preemption models (e.g. Fudenberg et al., 1983, Fudenberg and Tirole, 1985), are used in the real options framework.

Our paper belongs naturally to this literature stream. We model uncertainty using the geometric Brownian motion, which is standard in the continuous time real options models. However, as we consider exit, the strategic interaction is opposite to most of the above mentioned references.[3] A crucial difference concerns the assignment of roles to the firms. In preemption models one of the firms typically acts as a leader, but the threat of preemption implies that in equilibrium firms are indifferent

[1] The traditional investment theory is based on the rule that an investment project should be undertaken whenever its net present value is positive. This reasoning, however, neglects the fact that the correct decision involves comparing the value of investing today with the value of being able to undertake the project at any possible time in the future.

[2] A related literature considers the rational expectations competitive equilibrium in the case with divisible investments (Leahy, 1993; Baldursson and Karatzas, 1996). There is a close relationship between these models and the basic real option models due to the result that the timing of equilibrium capacity investments is the same as the optimal timing of a myopic investor that ignores other firms.

[3] An exception is Lambrecht (2001), who considers the impact of debt financing on entry and exit.

with respect to the role assignment. In our model, however, the follower is always better off. This means that asymmetries in the firm specific parameters get special importance, because even a slight difference between the firms may determine which of the firms is the follower in equilibrium.

Whereas many papers mentioned above are influenced by deterministic models of preemption, we adopt concepts from the deterministic models of exit, e.g., Ghemawat and Nalebuff (1985, 1990), and Fudenberg and Tirole (1986). Ghemawat and Nalebuff (1985) is particularly related to our work. They model the exit in a duopoly, where the firms differ from each other with respect to the scale of operations. In that model the equilibrium is derived by backward induction. As the decline of the market is deterministic, the game is truncated at some finite time when it is evident that both of the firms must have exited irrespective of each others' strategies. The model has a unique equilibrium, where the larger firm exits before the smaller firm. In our stochastic framework we can not work backward from a fixed time moment, thus the formulation of the strategies is crucial. We adopt state-dependent Markov strategies, which are expressed as stopping sets in the state space such that the firm under consideration exits when the state variable hits the corresponding stopping set for the first time. The equilibrium resulting with such strategies is Markov-perfect.[4]

The contribution of our paper can thus be seen from two angles. First, the paper extends the real options literature by studying the strategic interaction associated with abandonment options in oligopoly. Second, the paper gives new insight on how adding uncertainty modifies some results in the deterministic literature on exit. This is important, because in reality industry conditions are always more or less uncertain. In particular, it is shown that the main properties of Ghemawat and Nalebuff (1985) remain unchanged if uncertainty is small. However, if the level of uncertainty is increased, the uniqueness of the equilibrium may break down at some point, depending on the degree of asymmetry in the firm specific parameters. Then it is no longer possible to predict which of the two firms should exit first.

Another recent paper that considers strategic interaction in exit from the real options perspective is Lambrecht (2001). While offering a very interesting analysis on how the capital structure and possibilities of debt renegotiation affect the exit patterns of the firms, the paper simplifies the strategic interaction by assuming the stopping sets to be connected in order to ensure a unique subgame perfect equilibrium. Our paper concerns the relaxation of this assumption: when uncertainty is increased, an equilibrium with a disconnected strategy may appear ruining the uniqueness.

Methodologically the model is closely related to some literature on stopping time games. The formulation of strategies is similar to Dutta and Rustichini (1995). Other related papers are Huang and Li (1990), who prove the existence of equilibrium for a class of continuous time stopping games under certain monotonicity conditions, and Dutta and Rustichini (1993), who characterize the pure strategy equilibria in another class of stopping games. Both of these papers are mainly concerned with symmetric settings, whereas in our model the most interesting results are obtained with asymmetric firms. Fine and Li (1989) present a model of exit where the market declines according to a discrete time stochastic process. They show that if the discrete jumps in the process are sufficiently large, there is no unique equilibrium. The present paper, however, shows that there need not be a unique equilibrium even when the underlying stochastic process is continuous. This is also in contrast to the presumption stated in Ghemawat and Nalebuff (1990)

[4] In many preemption models the strategy of a firm is defined simply as a threshold level, i.e., a level to which the shock variable has to rise in order to trigger the investment. If we would use similar definition in our model, there would be two equilibria in every relevant case. Our definition allows us to use the sub-game perfection criterium to rule out some outcomes.

according to which there is a unique equilibrium with stochastic demand as long as it changes continuously.

The paper is organized as follows. In Section 2 we introduce the model and notation. We derive the equilibrium in three steps. First, in Section 3, we derive the optimal strategy of a firm that is alone in the market. In Section 4 we derive the best response strategy of a firm as a reaction to any given strategy of its competitor. Finally, in Section 5 we analyze the equilibria of the model. In Section 6 we illustrate the main results with an example. Section 7 concludes.

2 Model

We consider an industry that is initially a duopoly. If one of the firms exits, the remaining firm enjoys monopoly profits until it exits as well. The model is in continuous time with an infinite horizon. The two firms are labelled 1 and 2. By index i we refer to an arbitrary firm and by j to the 'other firm'. Both firms discount their cash flows with a fixed discount rate ρ.

The profitability of an active firm depends on two factors. First, there is an exogenous shock process X that characterizes the general profitability in the industry. Second, the presence of a competitor lowers the revenues. To model this, we define constants M_i and D_i such that $0 < D_i < M_i$ $(i = 1, 2)$, and assume that the shock variable X affects revenues multiplicatively: the revenue flow is $\Pi_i^M(X) = XM_i$ when alone in the market, and $\Pi_i^D(X) = XD_i$ in the presence of a competitor (M for monopoly, D for duopoly). To be active in the market costs C_i units of money in a time unit. Thus, the profit flows of i in monopoly and duopoly are respectively:

$$\begin{aligned}\pi_i^M(X) &= \Pi_i^M(X) - C_i = XM_i - C_i,\\ \pi_i^D(X) &= \Pi_i^D(X) - C_i = XD_i - C_i.\end{aligned}$$

The exogenous shock variable follows a geometric Brownian motion:

$$\frac{dX}{X} = \alpha dt + \sigma dz, \tag{1}$$

where α and σ are constants and dz is the standard Brownian motion increment. We require that $\alpha < \rho$ to ensure finite firm values, and $\alpha < \sigma^2/2$ to ensure that the firms want to exit in a finite time. By X we refer generally to a solution process of (1), by X_t to the value of the process at time t, and in particular by $\{X_t^x\}$ we refer to a solution process of (1) that starts at x.[5]

The only decision the firms have to make is to choose when to exit. Firm i is free to exit permanently at any moment by paying a fixed exit cost U_i. To ensure that it can ever be optimal for a firm to exit, we assume that $C_i > \rho U_i$ $\forall$ $i = 1, 2$ (otherwise, a firm would rather pay the cost stream C_i forever than to pay the exit cost U_i). Once one of the firms has exited, the remaining firm enjoys monopoly profits until it finds it optimal to exit as well. However, for the sake of clarity, we transform the game so that it ends already at the moment when one of the firms exits. The remaining firm simply gets as the termination payoff the value function of its optimization problem as a monopoly firm. We derive this termination payoff in the next section, where we consider the optimal exit problem with no strategic interaction.

[5] $\{X_t^x\}$ is a sequence of random variables indexed by $t > 0$ defined on a complete probability space $(\Delta, \mathcal{F}, P)$ and adapted to the nondecreasing sequence $\{\mathcal{F}_t\}_{t\geq 0}$ of sub-σ-fields of $\mathcal{F}$. $\mathcal{F}_t$ contains the information generated by X on the interval $[0, t]$.

In order to utilize the subgame perfection criterion, the strategies must be defined so that they specify the actions of the firms at all possible courses of events. We restrict our attention to pure strategies. As X is a Markov process, it is also natural to restrict ourselves to Markov strategies.[6] Given the current shock value X_t, the strategy of i should tell whether to continue operating or to exit. Thus, it can be expressed as a *stopping set*, that is, a set of values of R_+ such that i exits if and only if X_t is within this set.[7] We call such a set an *exit region.* Firm i exits when X hits the corresponding exit region for the first time. We restrict the exit regions to be closed in order to present the subsequent equilibrium analysis in the simplest possible form (cf. *closed Markov strategies* in Dutta and Rustichini, 1995). This assumption is not restrictive, because an optimal exit region of a firm regardless of the strategy of the other firm can always be expressed as a closed stopping set. The formal definition of the strategies is given below:

Definition 1 *The strategy of i is a closed stopping set $S_i \subset R_+$, which defines i's actions as long as j is still in the market: if $X_t \in S_i$, then i exits immediately, otherwise i stays in the market. We denote by $S = \{S_1, S_2\}$ the strategy profile containing the strategies of both firms.*

To formalize the objective functions, we define $\tau(x, A)$ as the time when X hits some region $A \subset R_+$ for the first time when starting at some x. This *first-passage time* is a Markov time:

$$\tau(x, A) = \inf\{t > 0 : X_t^x \in A\}.$$

We denote by τ_i the time when the state variable hits the exit region of firm i for the first time, that is, $\tau_i = \tau(x, S_i)$. The game ends at time $\widetilde{\tau} = \tau(x, S_1 \cup S_2) = \min\{\tau_k\}_{k=1,2}$. If $\tau_i < \tau_j$, firm i exits and j gets a monopoly position. Given the strategies of both firms, the value of i at state value $X = x$ is:[8]

$$\begin{aligned} V_i^D(x, S) &= E_x\,[\int_0^{\widetilde{\tau}} \pi_i^D(X_t^x)\, e^{-\rho t} dt \\ &\quad + e^{-\rho\widetilde{\tau}} \left(\chi_{\{\tau_i > \tau_j\}} \cdot V_i^M(X_{\widetilde{\tau}}^x) - \chi_{\{\tau_i \le \tau_j\}} \cdot U_i\right)], \end{aligned} \tag{2}$$

where E_x denotes the expectation given the current state value x, and $\chi_{\{\cdot\}}$ is an indicator function, i.e. $\chi_{\{true\}} = 1$ and $\chi_{\{false\}} = 0$. The termination payoff function $V_i^M(\cdot)$ is the monopoly value of the firm i, which will be derived in the next section. The problem of firm i is to choose its strategy S_i so that (2) is maximized at all state values x.

[6] In general, strategies in a dynamic game are decision rules that associate an action at a given time for each history of the game. By Markov strategies one refers to a class of strategies that depend only on the current state. By restricting on such strategies, we assume that agents condition their actions only on variables that directly influence their payoffs. An equilibrium in Markov strategies remains an equilibrium even when history-dependent strategies are allowed.

[7] See Dutta and Rustichini (1995) for such a definition of strategies in a related class of stochastic games. They have also another component in the strategy in order to allow the firms to react instantaneously to each other's actions. This is not needed in our model, because it is never optimal for a firm to react instantly to the other firm's exit decision.

[8] The firms are assumed to be risk-neutral. This assumption is not crucial for our results. Alternatively we could incorporate risk-aversity by assuming that fluctuations in X are spanned by traded assets and interpret the objectives of the firms using equivalent risk-neutral valuation.

3 Optimal exit with no strategic interaction

In this section we consider the optimal exit strategy in the absence of strategic interaction. The firm under consideration either does not have any competitors, or has a passive competitor that never exits. The purpose is to derive the monopoly value function $V_i^M(x)$ and some other useful results for the subsequent analysis. To emphasize that we have no strategic interaction, we leave for a moment the subscript i out: the profit flow is $\pi = \Pi - C$ and the exit cost is U. By Π we represent all possible revenue terms introduced in the previous section ($\Pi_i^M(X)$, $\Pi_i^D(X)$, $i = 1, 2$). In any case, it is a multiplicative of X and therefore follows the same geometric Brownian motion:

$$\frac{d\Pi}{\Pi} = \alpha dt + \sigma dz.$$

The active firm is an asset that earns the revenue flow Π in return to the cost flow C. In addition, the firm owns the option to exit permanently from the market by giving up the fixed cost U. The firm should choose the optimal time to exit in order to maximize the expected present value of the future cash flows, in other words, consider when the momentary losses are so severe that it is no longer worth to stick to the market in the hope of future profits. This is a standard optimal stopping problem, and we will merely state the main results. For a more detailed solving of the problem see Dixit (1989), where also re-entry is allowed.

It is clear that the lower the revenue flow, the more tempting it becomes to exit. Therefore, the optimal exit region can be expressed as a threshold level Π^* such that it is optimal to exit whenever the revenue flow has a value in $(0, \Pi^*]$. Since Π is a Markov process, the value of the firm can be expressed as a function $V(\Pi)$. It is a standard result that as long as it is not optimal to stop, this value function must satisfy the following differential equation, which results from the Bellman's principle of optimality and the application of Ito's lemma on V:

$$\frac{1}{2}\sigma^2\Pi^2 V''(\Pi) + \alpha\Pi V'(\Pi) - \rho V(\Pi) + \Pi - C = 0, \tag{3}$$

where the primes denote derivatives with respect to Π. The general solution to this is:

$$V(\Pi) = a\Pi^{\beta_1} + b\Pi^{\beta_2} + \frac{\Pi}{\delta} - \frac{C}{\rho}, \tag{4}$$

where

$$\begin{aligned} \beta_1 &= \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left[\frac{\alpha}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2\rho}{\sigma^2}} > 1 \text{ and} \\ \beta_2 &= \frac{1}{2} - \frac{\alpha}{\sigma^2} - \sqrt{\left[\frac{\alpha}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2\rho}{\sigma^2}} < 0 \end{aligned}$$

are the roots of the characteristic equation $\frac{1}{2}\sigma^2\beta(\beta - 1) + \alpha\beta - \rho = 0$. Coefficients a and b are free parameters, which are solved by applying appropriate boundary conditions. In this case the conditions are derived from the value-matching and smooth-pasting conditions at Π^*, and the condition that eliminates the speculative bubbles (see Dixit, 1989; Dixit and Pindyck, 1994). Applying these in (4) results in an exact solution for Π^* and $V(\Pi)$:

$$\Pi^* = \frac{\beta_2(\rho-\alpha)}{\beta_2-1}\left(\frac{C}{\rho}-U\right), \tag{5}$$

$$V(\Pi) = \begin{cases} \frac{\Pi}{\rho-\alpha} - \frac{C}{\rho} + \left(-U - \frac{\Pi}{\rho-\alpha} + \frac{C}{\rho}\right)\left(\frac{\Pi}{\Pi^*}\right)^{\beta_2} & \text{, when } \Pi > \Pi^* \\ -U & \text{, when } \Pi \le \Pi^*. \end{cases} \tag{6}$$

To conclude this section, we apply the result to the original duopoly setting. Assume that firm j has exited, but i is still in the market. Then the revenue flow of i is $\Pi = XM_i$, and the optimal exit threshold in terms of the shock variable X must be $X_i^M = \Pi^*/M_i$. Simple substitution gives X_i^M and the corresponding monopoly value $V_i^M(\cdot)$ as a function of the current state value x:

$$X_i^M = \frac{\beta_2(\rho-\alpha)}{M_i(\beta_2-1)}\left(\frac{C_i}{\rho}-U_i\right), \tag{7}$$

$$V_i^M(x) = \begin{cases} \frac{xM_i}{\rho-\alpha} - \frac{C_i}{\rho} + \left(-U_i - \frac{xM_i}{\rho-\alpha} + \frac{C_i}{\rho}\right)\left(\frac{x}{X_i^M}\right)^{\beta_2} & \text{, when } x > X_i^M \\ -U_i & \text{, when } x \le X_i^M. \end{cases} \tag{8}$$

The value function (8) gives the termination payoff used in (2). In other words, it is the value of firm i when j leaves the market at the state value x. The function $V_i^M(\cdot)$ is increasing, continuous, and smooth everywhere.

For later use, we consider also firm i in duopoly under the pessimistic assumption (from i's point of view) that j is passive, i.e., will never exit. Then the only required modification is to replace M_i with D_i. We denote by X_i^D the optimal exit threshold for i under that assumption:

$$X_i^D = \frac{\beta_2(\rho-\alpha)}{D_i(\beta_2-1)}\left(\frac{C_i}{\rho}-U_i\right). \tag{9}$$

Clearly, $X_i^D > X_i^M$. Since X_i^D is the optimal exit threshold for i at the most pessimistic scenario where there is no hope that j exits before i, it can never be optimal for i to exit at values above X_i^D, irrespective of the strategy of j. On the other hand, since it is optimal for i to exit at X_i^M even when j has already exited, it can never be optimal for i to stay at values below X_i^M, irrespective of the strategy of j. We can thus already say that when S_i is optimally chosen, $\left(0, X_i^M\right] \subset S_i$ and $S_i \cap \left(X_i^D, \infty\right) = \varnothing$, irrespective of S_j.

4 Best response to a given strategy

In this section we consider the optimal strategy of i given that j adopts an arbitrary strategy S_j. In order to apply the subgame perfection criterion, the optimal actions must be specified at all state values. The main insight to be established is that such a best response strategy may be a disconnected set. We start with the following definition:

Definition 2 *The best response of i is a strategy S_i that maximizes (2) at all $x > 0$, given that j's strategy S_j is fixed. We denote the best response to S_j by $R_i(S_j)$.*

The problem considered is to find a closed set $R_i(S_j)$ that maximizes (2) given that $\tau_i = \tau(x, R_i(S_j))$ and $\tau_j = \tau(x, S_j)$. This is a time-homogenous, yet somewhat more complicated optimal stopping problem than the one considered in Section 3. To keep the key ideas as accessible as possible, we maintain a relatively low level of technical formality. For mathematical treatments of optimal stopping problems see, e.g., Friedman (1976), Karatzas and Shreve (1998), or Øksendal (2000).

In order to facilitate the communication of the main points, we adopt the following definitions:

Definition 3 *Given a point $x^0 > X_i^M$, the upper reaction of i to x^0 is the lowest value above it where it is optimal for i to exit given that j would exit as soon as X falls below x^0. We denote the upper reaction to x^0 by $r_i^+(x^0)$. Formally, $r_i^+(x^0) > x^0$ is the upper reaction to x^0 if it maximizes the expression:*

$$E_x \left[\int_0^{\widetilde{\tau}} \pi_i^D(X_t^x) e^{-\rho t} dt + e^{-\rho \widetilde{\tau}} \left(\chi_{\{\tau_i > \tau_j\}} \cdot V_i^M(X_{\widetilde{\tau}}^x) - \chi_{\{\tau_i \leq \tau_j\}} \cdot U_i \right) \right], \tag{10}$$

when $\tau_i = \inf\{t > 0 : X_t^x \geq r_i^+(x^0)\}$, $\tau_j = \inf\{t > 0 : X_t^x \leq x^0\}$, $\widetilde{\tau} = \min\{\tau_i, \tau_i\}$, and $x \in (x^0, r_i^+(x^0))$. If it is not optimal for i to exit above x^0, then we say that the upper reaction to x^0 does not exist.

Similarly, the lower reaction of i to some $x^0 > X_i^M$ is the highest value below x^0 where it is optimal for i to exit given that j would exit as soon as X rises above x^0. We denote the lower reaction to x^0 by $r_i^-(x^0)$. Formally, $r_i^-(x^0) < x^0$ is the lower reaction to x^0 if it maximizes (10) when $\tau_i = \inf\{t > 0 : X_t^x \leq r_i^-(x^0)\}$, $\tau_j = \inf\{t > 0 : X_t^x \geq x^0\}$, $\widetilde{\tau} = \min\{\tau_i, \tau_i\}$, and $x \in (r_i^-(x^0), x^0)$. Since it must eventually be optimal to exit if X falls low enough, the lower reaction exists for all $x^0 > X_i^M$.

We consider next the actual derivation of the upper and lower reactions to some arbitrary $x^0 > X_i^M$. Whenever it is optimal for i to wait, the value function must satisfy (3). Thus, the value function in the continuation region must be of the form:

$$V_i^C(x) = a(xD_i)^{\beta_1} + b(xD_i)^{\beta_2} + \frac{(xD_i)}{\rho - \alpha} - \frac{C_i}{\rho}, \tag{11}$$

where a and b are free parameters. It is now easy to check whether an upper reaction to a given x^0 exists (the lower reaction always exists for any $x^0 > X_i^M$). This is done by adding an artificial constraint that i *can not* exit above x^0. The corresponding value function, valid for $x \in (x^0, \infty)$ and denoted $\widetilde{V}_i(x; x^0, \infty)$, is given by (11), where ruling out 'speculative bubbles' implies that $a = 0$ (see Dixit and Pindyck, 1994), and the other free parameter b is fixed by requiring that the value function gives the correct termination payoff at x^0, that is $\widetilde{V}_i(x^0; x^0, \infty) = V_i^M(x^0)$. If $\widetilde{V}_i(x; x^0, \infty) > -U_i$, $\forall x > x^0$, then it can not be optimal for i to exit above x^0, because continuing until j exits gives a higher payoff. In that case there is no upper reaction to x^0. If $\widetilde{V}_i(x; x^0, \infty) \leq -U_i$ for some $x > x^0$, $r_i^+(x^0)$ exists.

The upper and lower reactions can be solved by applying appropriate value-matching and smooth-pasting conditions to (11). Let $\widetilde{x} \neq x$ be the optimal exit point (the upper or lower reaction to x^0). The following conditions must then hold:

$$V_i^C(x^0) = V_i^M(x^0), \tag{12}$$

$$V_i^C(\widetilde{x}) = -U_i, \tag{13}$$

$$\frac{\partial V_i^C(x)}{\partial x}\Big|_{x=\widetilde{x}} = 0. \tag{14}$$

Equations (12) and (13) are the value-matching conditions that actually result from the fact that the value function (2) is continuous in x everywhere for any strategy profile S. Thus, the value function in the continuation region must match the termination payoffs at the points where the game ends. Equation (14) is the smooth-pasting condition, which ensures the optimality of the exit point $\widetilde{x}$ (see Dixit and Pindyck, 1994, for a heuristic proof, or Friedman, 1976, for mathematical verification applicable to the present case). Since x^0 is given (or chosen by j), there is no optimality requirement from the point of view of i, and thus no smooth-pasting condition at x^0.

The optimal upper (lower) reaction can be solved by finding a, b, and $\widetilde{x} > x^0$ ($\widetilde{x} < x^0$) such that (12) - (14) are satisfied. Since these equations are linear in a and b, one can use two of the equations to determine a and b as functions of $\widetilde{x}$, and substitute these in the third, which results in a non-linear equation for $\widetilde{x}$. The lower reaction can be solved by finding $\widetilde{x} < x^0$ that satisfies this equation. Correspondingly, if an upper reaction exists, it can be solved by finding $\widetilde{x} > x^0$ that satisfies this equation.

Next, we move on to constructing the entire best-response exit region. We do that under the assumption that the strategy set S_j consists of a countable number of closed intervals (cf. *interval-based* Markov strategy in Dutta and Rustichini, 1995). This is sufficient for our purpose, since any closed set on R_+ may be approximated to an arbitrary precision by such a set.

Denote the value function of i associated with the best response by $V_i^R(x, S_j)$. This must be solved together with the optimal exit region. First, from (2) it is easy to conclude that in the region where j exits, the value is defined by the termination payoff $V_i^M(x)$:

$$V_i^R(x, S_j) = V_i^M(x) \text{ , when } x \in S_j. \tag{15}$$

On the other hand, when it is optimal for i to exit, the value is defined by the termination payoff $-U_i$:

$$V_i^R(x, S_j) = -U_i \text{ , when } x \in R_i(S_j). \tag{16}$$

Finally, in the region where j does not exit, and it is optimal for i to wait (in the continuation region $[S_j \cup R_i(S_j)]^C$), the value function must be of the form (11).

As shown in Section 3, it is always optimal for i to exit below X_i^M, so $\left(0, X_i^M\right] \subset R_i(S_j)$ for all S_j. Similarly, it can never be optimal for i to exit above X_i^D, so $\left(X_i^D, \infty\right) \cap R_i(S_j) = \varnothing$. The problem is thus only to split the region $\left(X_i^M, X_i^D\right]$ between continuation and stopping regions. We denote this interesting region by $\Omega_i = \left(X_i^M, X_i^D\right]$. Since $V_i^M(x) > -U_i$ for all $x > X_i^M$, conditions (15) and (16) imply that in Ω_i it is never optimal for the firms to exit together (that is, $S_j \cap R_i(S_j) \cap \Omega_i = \varnothing$). Thus, in Ω_i the state space is divided into three mutually exclusive sets, which we denote for a moment simply as:

$$\begin{aligned}
\Delta_j &= S_j \cap \Omega_i, \text{ where } j \text{ exits,} \\
\Delta_i &= R_i(S_j) \cap \Omega_i, \text{ where } i \text{ exits, and} \\
\Delta_c &= \Omega_i \setminus (\Delta_i \cup \Delta_j), \text{ where none of the firms exits.}
\end{aligned}$$

The best-response value function in these sets is given by (15), (16), and (11) respectively. The continuity of $V_i^R(x, S_j)$ together with (15) and (16) imply that the value matching condition could not be satisfied at a boundary between Δ_i and Δ_j, which means that these sets must be isolated from each other.

Since Δ_j is given, it remains to work out how the set $\Omega_i \setminus \Delta_j$ is split between stopping and continuation regions Δ_i and Δ_c. Because S_j is composed of closed intervals, $\Omega_i \setminus \Delta_j$ is composed of a number of intervals separated by the exit regions of j, and thus the optimal stopping region can be solved by considering each such interval separately. Consider an arbitrary interval $\left(x^L, x^R\right) \subset \Omega_i \setminus \Delta_j$ such that j exits at the end points x^L and x^R. Given that the current state value is within this interval, the first question is whether it is always optimal for i to wait until X hits one of the end points where j exits, or is it optimal to exit somewhere in the middle. The answer is easy to obtain by once again assuming that i *can not* exit within the interval. The 'artificial' value function under this condition, $\widetilde{V}_i\left(x; x^L, x^R\right)$, is found by taking a function of the form (11), and setting the free parameters to match the boundary conditions $\widetilde{V}_i\left(x^L; x^L, x^R\right) = V_i^M\left(x^L\right)$ and $\widetilde{V}_i\left(x^R; x^L, x^R\right) = V_i^M\left(x^R\right)$. This is a linear system for parameters a and b to be solved. Since $\widetilde{V}_i\left(x; x^L, x^R\right)$ is the value in the case where it is not possible to exit, it is straightforward to conclude that if $\widetilde{V}_i\left(x; x^L, x^R\right) > -U_i$ within the whole interval, it would not be optimal to exit in any case, and thus $\left(x^L, x^R\right) \subset \Delta_c$. On the contrary, if $\widetilde{V}_i\left(x; x^L, x^R\right) \leq -U_i$ for some $x \in \left(x^L, x^R\right)$, there must be an exit region in the middle. Since $r_i^+\left(x^L\right)$ is the lowest point where it is optimal to exit given that j exits at x^L, and $r_i^-\left(x^R\right)$ is the highest point where it is optimal to exit given that j exits at x^R, the optimal exit region must be $\left[r_i^+\left(x^L\right), r_i^-\left(x^R\right)\right]$. Thus, $\left(x^L, r_i^+\left(x^L\right)\right) \cup \left(r_i^-\left(x^R\right), x^R\right) \subset \Delta_c$ and $\left[r_i^+\left(x^L\right), r_i^-\left(x^R\right)\right] \subset \Delta_i$.

At the left and right ends of Ω_i there may be intervals in $\Omega_i \setminus \Delta_j$ where j exits at one endpoint only. At the left-hand side, this interval is $\left(X_i^M, x^R\right)$, where j exits only at x^R. In that case, $\left(X_i^M, r_i^-\left(x^R\right)\right] \subset \Delta_i$ and $\left(r_i^-\left(x^R\right), x^R\right) \subset \Delta_c$. At the right-hand side, the interval is $\left(x^L, X_i^D\right]$, where j exits only at x^L. In that case, if $r_i^+\left(x^L\right)$ exists, then $\left(x^L, r_i^+\left(x^L\right)\right) \subset \Delta_c$ and $\left[r_i^+\left(x^L\right), X_i^D\right] \subset \Delta_i$. If $r_i^+\left(x^L\right)$ does not exist, then $\left(x^L, X_i^D\right] \subset \Delta_c$.

The best response is constructed by processing all intervals of $\Omega_i \setminus \Delta_j$ in the way described above. This is illustrated in figure 1, where S_j is composed of three closed intervals, $[x_1, x_2]$, $[x_3, x_4]$, and $[x_5, x_6]$.[9] As can be seen in the figure, the best response value function is $V_i^R(x, S_j) = V_i^M(x)$ within all these intervals, as determined by (15). Since all these intervals are within Ω_i, Δ_j is the union of them, and $\Omega_i \setminus \Delta_j$ consists of four intervals: $\left(X_1^M, x_1\right)$, (x_2, x_3), (x_4, x_5), and $\left(x_6, X_i^D\right]$. For the first interval, $\left(X_i^M, r_i^-(x_1)\right] \subset \Delta_i$ and $\left(r_i^-(x_1), x_1\right) \subset \Delta_c$. The best response value function $V_i^R(x, S_j)$ is given by (16) in the former, and (11) in the latter. For the second interval, the 'artificial' value function $\widetilde{V}_i(x; x_2, x_3) \leq -U_i$ for some values, which means that there must be an exit region within the interval, which is determined by solving $r_i^+(x_2)$ and $r_i^-(x_3)$ using (12) - (14): the exit region is $\left[r_i^+(x_2), r_i^-(x_3)\right] \subset \Delta_i$. The value function $V_i^R(x, S_j)$ is again given by (16) in the exit region, and (11) in continuation regions. For the third interval, $\widetilde{V}_i(x; x_4, x_5) > -U_i \; \forall x \in (x_4, x_5)$. Thus it is not optimal to exit and $V_i^R(x, S_j) = \widetilde{V}_i(x; x_4, x_5)$. Finally, since $\widetilde{V}_i(x; x_6, \infty) > -U_i \; \forall x \in (x_6, \infty)$, it is not optimal to exit above x_6 and $V_i^R(x, S_j) = \widetilde{V}_i(x; x_6, \infty)$ everywhere above x_6. Summing up, the optimal exit region of i within Ω_i is $\Delta_i = \left(X_i^M, r_i^-(x_1)\right] \cup \left[r_i^+(x_2), r_i^-(x_3)\right]$ and the best-response strategy is $R_i(S_j) = \left(0, X_i^M\right] \cup \Delta_i = \left(0, r_i^-(x_1)\right] \cup \left[r_i^+(x_2), r_i^-(x_3)\right]$. Note that $V_i^R(x, S_j)$ is continuous everywhere (the value-matching conditions hold) and smooth at the boundaries between $R_i(S_j)$ and the continuation regions (the smooth-pasting conditions hold).

The main thing to note is that the best response strategy is disconnected. However, the strategies given in the figure do not form an equilibrium, because even if $R_i(S_j)$ is the best response to S_j, the converse is not true. Nevertheless, in the next section we will show that such disconnected strategies may appear in equilibrium.

[9] The parameter values used in the figure are $\rho = .05$, $\alpha = -.05$, $\sigma = .1$, $D_i = 1$, $M_i = 2$, $C_i = 2$, and $U_i = 5$, which result $X_i^M = .8$ and $X_i^D = 1.6$. The parameters that determine S_j are $x_1 = .91$, $x_2 = .93$, $x_3 = 1.22$, $x_4 = 1.24$, $x_5 = 1.4$, and $x_6 = 1.42$.

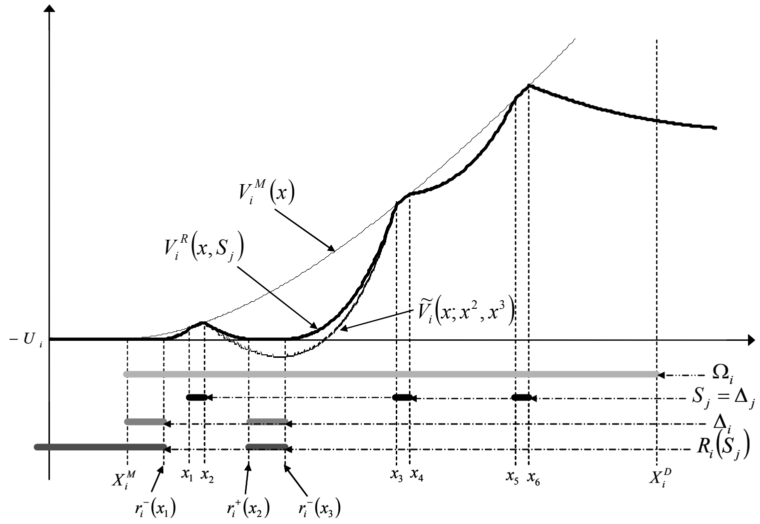

Figure 1: Construction of the best response.

5 Equilibrium

5.1 Definitions and preliminary results

Both firms wish to maximize their values at every state. The equilibrium is defined as follows:

Definition 4 *S is an equilibrium if and only if the strategies of both firms are the best responses to each other, that is, $S_1 = R_1(S_2)$ and $S_2 = R_2(S_1)$.*

In other words, the equilibrium is such a strategy pair S that it is not possible for any of the firms to unilaterally improve (2) at any $x > 0$. To characterize potential equilibria, we define two special classes of strategy profiles, one for each firm:

Definition 5 *We denote by Θ_i, $i \in \{1, 2\}$ the collection of all such strategy profiles $S = \{S_1, S_2\}$ where*

$$\sup S_i = X_i^D > \sup S_j.$$

The definition refers to classes within which all the strategies are equivalent in the sense that if the initial value is high enough, the game ends when firm i's exit is triggered by X falling below X_i^D. We use the term *outcome* to refer to the actual development of the game, that is, the order and the timing of exit. It should be noted that there is an infinite number of strategies that result in one outcome. The reason for our seemingly too rich strategy structure is that it enables the distinction between strategies that would imply different actions at a state that has been reached by disequilibrium actions in the past or by a low initial value. This is essential for applying the sub-game perfection criterion to sort out adequate equilibria.

The next proposition says that there are only two possible outcomes in equilibrium: either firm 1 exits first at threshold level X_1^D or firm 2 exits first at X_2^D:

Proposition 1 *Let S be an equilibrium strategy profile. Then, $S \in \Theta_1 \cup \Theta_2$.*

Proof. See the Appendix. ■

The proposition means that the potential equilibrium profiles can be divided into two classes in both of which the firms exit sequentially: Θ_1 where firm 1 exits first, and Θ_2 where firm 2 exits first. Stating who exits first actually identifies completely the outcome in equilibrium. We are interested in whether one or both of these exit orders are possible in equilibrium. Ghemawat and Nalebuff (1985) show in the deterministic framework that the application of the sub-game perfection criterion rules out one of the two exit orders. This result is extended to a stochastic market in Lambrecht (2001), who defines the strategies to be connected sets. We consider such connected strategies in the next subsection. However, after that we will show that when disconnected strategies are allowed, the sub-game perfection criterion may lose its power if the degree of uncertainty is high enough.

5.2 Equilibria with connected strategy sets

In this subsection we restrict ourselves to the simplest kind of strategies, namely those where S_i, $i = 1, 2$, are connected sets (i.e. single closed intervals on R_+). Consider first the symmetric game. Then, as can be expected, there is no unique equilibrium:

Proposition 2 *In a symmetric game, where $D_1 = D_2$, $M_1 = M_2$, $C_1 = C_2$, $U_1 = U_2$, the following two equilibria exist:*

$$\begin{array}{lll} \textit{1)} & \left\{S_1 = \left(0, X^D\right], S_2 = \left(0, X^M\right]\right\} & \in \Theta_1, \\ \textit{2)} & \left\{S_1 = \left(0, X^M\right], S_2 = \left(0, X^D\right]\right\} & \in \Theta_2, \end{array}$$

where $X^D = X_1^D = X_2^D$ and $X^M = X_1^M = X_2^M$.

Proof. See the Appendix. ■

However, in reality no two firms are exactly alike. Therefore, it is interesting to look at what happens when there is some difference between the firms. The asymmetries in the firm specific parameters generally imply that $X_1^D \neq X_2^D$ and $X_1^M \neq X_2^M$. We assume in the following that at least $X_1^M \neq X_2^M$.[10]

It will turn out that the most important factor for the nature of the equilibria is how the monopoly threshold levels X_1^M and X_2^M are located. In particular, it is important which of the firms has the lower X_i^M, because that firm would 'last' longer in the monopoly position. We say that the firm with the lower X_i^M is *stronger*.[11] Without loss of generality, we assume that it is the firm 1 that is stronger (Assumption 1 below). However, to avoid confusion with excessive special conditions, we do not want too much asymmetry. Therefore, we rule out the special case where one of the firms is so much less profitable in duopoly that it would be optimal to exit at the 'duopoly threshold' X_i^D even under the most optimistic scenario where the other firm exits at the corresponding level X_j^D (Assumption 2 below). If this assumption would not hold for one of the firms, there would be no interesting strategic interaction since it would be obvious that this firm exits first.

Summing up, we assume through the rest of the paper that the parameters are such that the following hold:

Assumption 1: Firm 1 is stronger than firm 2, i.e., $X_1^M < X_2^M$.

Assumption 2: $X_i^D \notin R_i^D\left(\left(0, X_j^D\right]\right) \; \forall i \in \{1, 2\}, \, j \neq i$.

The following proposition shows that even a slightest asymmetry destroys one of the two equilibria of the symmetric model:

[10] In section 6 we give an example where $X_1^M \neq X_2^M$, but $X_1^D = X_2^D$.

[11] Fine and Li (1989) use the term 'stronger' in a slightly different meaning than we do. They say that a firm is stronger than its competitor if its profit flow at any shock variable value is greater.

Proposition 3 *Under assumptions 1 and 2, there is an equilibrium where firm* 2 *exits first:*

$$\left\{S_1 = \left(0, X_1^M\right], \, S_2 = \left(0, X_2^D\right]\right\} \in \Theta_2. \tag{17}$$

However, the corresponding profile in Θ_1*,*

$$\left\{S_1 = \left(0, X_1^D\right], S_2 = \left(0, X_2^M\right]\right\} \in \Theta_1, \tag{18}$$

is not an equilibrium.

Proof. See the Appendix. ■

Proposition 3 together with Proposition 1 means that the equilibrium given in (17) is the only possible equilibrium with connected strategies. It is actually the subgame perfection criterion that rules out one of the two potential equilibria. The result is similar as in Ghemawat and Nalebuff (1985) and Lambrecht (2001).

As we have now shown that there is at least one equilibrium both in the symmetric and asymmetric cases, we have in practice shown that there always exists an equilibrium in the model.

5.3 Equilibria with disconnected strategy sets

To complete the discussion of equilibria, the possibility of equilibria where the exit regions are not connected sets must also be analyzed. In particular, we are interested in whether there can be an equilibrium where the strong firm exits first, that is, an equilibrium in Θ_1, because that would mean that there is no unique equilibrium outcome.

It was shown in Section 4 that the best response exit region of i is isolated from the exit region of j above X_i^M. On the other hand, it is easy to see that since firm 2 should always exit within the interval $\left(X_1^M, X_2^M\right]$, there can not be an equilibrium where firm 1 exits within that region. Therefore, if there is some equilibrium where firm 1 exits first, then there must be some 'empty space' between the exit regions of the firms such that there is an exit region of firm 1 above and of firm 2 below the space. In other words, there must be an interval $(\underline{x}, \overline{x})$ such that none of the firms exits within it, but firm 1 exits at $\overline{x}$ and firm 2 exits at $\underline{x}$. To make it possible that such an interval exists in equilibrium, $\overline{x}$ must be the upper reaction of firm 1 to $\underline{x}$, and $\underline{x}$ must be the lower reaction of firm 2 to $\overline{x}$. This is formalized in the following proposition:

Proposition 4 *Under assumptions 1 and 2, there is an equilibrium in* Θ_1 *if and only if there is a point* $\underline{x} > X_2^M$ *such that* $r_1^+(\underline{x})$ *exists and* $\underline{x} = r_2^-\left(r_1^+(\underline{x})\right)$*. This equilibrium is of the form:*

$$\left\{S_1 = \left(0, X_1^M\right] \cup \left[\overline{x}, X_1^D\right], \, S_2 = (0, \underline{x}]\right\} \in \Theta_1, \tag{19}$$

where $\overline{x} = r_1^+(\underline{x})$ *and* $\underline{x} = r_2^-\left(r_1^+(\underline{x})\right) = r_2^-(\overline{x})$*.*

Proof. See the Appendix. ■

We call this kind of an equilibrium where there is an empty gap between the exit regions of the firms a *gap equilibrium.* An interesting property of such an equilibrium is that if $X_t \in (\underline{x}, \overline{x})$ and none of the firms has exited, then it may be that the exit is triggered by an increase in the shock process. In other words, an improvement in the profitability of the industry may trigger firm 1 to exit. The reason is that firm 1 would like firm 2 to exit first, but as X rises, it becomes less likely that 2 is going to exit in the near future, so it becomes optimal for 1 to exit. It should be emphasized, however, that the state where neither of the firms has exited when $X_t \in (\underline{x}, \overline{x})$ can only be reached by a 'mistake' or by a low initial value, because normally firm 1 would already have exited when X crossed the exit region $\left[\overline{x}, X_1^D\right] \subset S_1$.

The next proposition says that a gap equilibrium may exist only if the degree of uncertainty is sufficiently high.

Proposition 5 *Let $\alpha < 0$ and all other parameters except σ are fixed at such values that assumptions 1 and 2 hold. Then, there is some non-empty interval $(0, \overline{\sigma}]$ such that when $\sigma \in (0, \overline{\sigma}]$, the equilibrium (17) given in Proposition 3 is unique.*

Proof. See the Appendix. ■

Intuitively, the result can be explained by noting that as uncertainty is decreased towards zero, the lower reaction to any $x > X_i^M$ moves towards x, but on the contrary, the upper reaction (if it exists) remains strictly above x no matter how low the uncertainty. Thus, as uncertainty is decreased, it becomes impossible to have such a pair $\underline{x}$ and $\overline{x}$ that the upper and lower reactions of the firms 'match each other'. Then a gap equilibrium can not exist.

The proposition provides the main conclusion of this paper: when there is little uncertainty, the game has a unique equilibrium where the stronger firm can force the weaker firm to exit first. However, it is equally important to acknowledge that when uncertainty is increased beyond some point, a gap-equilibrium with the reversed order of exit may appear ruining the uniqueness. We will demonstrate this in the next section through an example.

6 Example

In this section we illustrate the model by specifying it as in Ghemawat and Nalebuff (1985), except for the uncertainty.[12] Consider a market for a homogenous good, where the demand fluctuates stochastically. We assume that the two firms differ from each other with respect to their production capacities. We denote the capacities of the firms K_1 and K_2.

We assume that the exogenous shock variable affects demand multiplicatively. The price of the product is thus given by an inverse demand function of the form:

$$P = XD(q_1 + q_2),$$

where q_i is the output of firm i. We assume that function D satisfies $\partial D(q)/\partial q < 0$ and $\partial (D(q)q)/\partial q > 0$. The second assumption implies that marginal revenue is always positive, and is made in order to ensure that the firms want to fully utilize their capacities. It is satisfied by, e.g., an isoelastic demand function with a high enough elasticity. We thus specify further that D is of the form $D(q) = Aq^{-\frac{1}{\varepsilon}}$, where A is a positive parameter and price elasticity satisfies: $\varepsilon > 1$.

We also adopt the same assumption on the cost structure as Ghemawat and Nalebuff (1985): the cost of keeping the firm operating imposes a cost flow proportional to the firm's capacity. We denote by c the cost flow per unit of capacity. Since marginal revenue is always positive and there are no operating costs, firms always utilize fully their capacities as long as they stay in the market. Thus, in the following we associate the outputs of the firms with their capacities, i.e. $q_i = K_i$, $i = 1, 2$. The cost flow of operating is thus $C_i = cK_i$ and can only be avoided by exiting permanently. We assume that also the exit cost is proportional to the capacity of the firm, i.e., $U_i = uK_i$, where u is a parameter.[13]

[12] This is done in order to link the results directly to the deterministic industrial organization literature. The example is perfectly consistent with the original model of section 2 and captures all of its essential properties.

[13] Ghemawat and Nalebuff (1985) assume that exit is free. Of course, we may also assume that by setting $u = 0$.

The revenue flow of i when alone in the market and when j coexists are respectively:

$$\begin{aligned}\Pi_i^M(X) &= K_i X D(K_i),\\ \Pi_i^D(X) &= K_i X D(K_1+K_2).\end{aligned}$$

Thus, using the notation of Section 2, $M_i = K_i D(K_i)$, $D_i = K_i D(K_1+K_2)$.

The exit threshold levels X_i^M and X_i^D are now obtained from (7) and (9):

$$\begin{aligned}X_i^M &= \frac{\beta_2(\rho-\alpha)}{K_i D(K_i)(\beta_2-1)}\left(\frac{cK_i}{\rho}-uK_i\right) = \frac{\beta_2(\rho-\alpha)}{D(K_i)(\beta_2-1)}\left(\frac{c}{\rho}-u\right),\\ X_i^D &= \frac{\beta_2(\rho-\alpha)}{D(K_1+K_2)(\beta_2-1)}\left(\frac{c}{\rho}-u\right).\end{aligned}$$

The difference between the firms is in the production capacity: we assume that firm 2 produces at a larger scale than firm 1, i.e. $K_2 > K_1$. This implies that firm 1 is stronger: $X_1^M < X_2^M$. However, $X_1^D = X_2^D \equiv X^D$.

According to Proposition 3, there is an equilibrium where firm 2 exits first: $S_1 = \left(0, X_1^M\right]$, $S_2 = \left(0, X^D\right]$. Figure 2 shows the values $V_i^D(x,S)$ in that equilibrium as functions of x with two values of σ: .1 (dashed lines) and .3 (solid lines). The capacities of the firms are $K_1 = 1$, $K_2 = 1.12$. The other parameter values are: $\rho = .05$, $\alpha = -.05$, $c = 2$, $u = 5$, $A = 2$, $\varepsilon = 1.08$. It can be seen that the value of firm 1 is considerably larger than that of firm 2.[14] The kinks in the values of firm 1 are due to the fact that firm 2 exits at those points. It can be seen that increasing uncertainty increases the value of firm 2. This is a standard result in the real options literature. In the present case, however, there is a region where firm 1 is more valuable at a lower uncertainty. This is due to the strategic interaction: at the higher uncertainty, firm 2 is willing to stay in the market at values where it would exit at the lower uncertainty. This makes firm 1 worse off at those demand shock values. On the other hand, one can see that the increased uncertainty reduces the payoff difference arising from the strategic interaction: the difference in the firms' values is much lower in the case $\sigma = .3$ than $\sigma = .1$.

To examine whether there are equilibria where firm 1 exits first, we utilize Proposition 4: to have such an equilibrium, there must be a point $\underline{x} > X_2^M$ such that $r_1^+(\underline{x})$ exists and $\underline{x} = r_2^-\left(r_1^+(\underline{x})\right)$. For a given x, we can calculate $r_1^+(x)$ by setting $x^0 = x$ in (12) and finding a, b, and $\widetilde{x} > x$ such that (12)-(14) are satisfied. If the solution exits, then $r_1^+(x) = \widetilde{x}$. Similarly, $r_2^-(x)$ can be calculated by finding a, b, and $\widetilde{x} < x$ such that the equations are satisfied, and setting $r_2^-(x) = \underline{x}$.

Figure 3 shows the curves $y = r_2^-(x)$ and $x = r_1^+(y)$ with the same parameter values as used in figure 2. It can be seen that when $\sigma = .1$, the two curves do not intersect. However, as σ is increased the curves move closer to each other eventually crossing each other: when $\sigma = .3$, the two curves intersect at a point that we denote $x = \overline{x}$, $y = \underline{x}$. At this point $\underline{x} = y = r_2^-(x) = r_2^-\left(r_1^+(y)\right) = r_2^-\left(r_1^+(\underline{x})\right)$. Then, according to Proposition 4, there is a gap equilibrium where firm 1 exits first:[15] $S_1 = \left(0, X_1^M\right] \cup \left[\overline{x}, X^D\right)$, $S_2 = (0, \underline{x}]$. This is a demonstration of Proposition 5: with low uncertainty there is a unique equilibrium where the weak firm exits first. However, with higher uncertainty the uniqueness breaks down and there are suddenly two different equilibria: the "normal" equilibrium where the weak firm

[14] Except outside the figure area at very high values of X, where the high industry profitability makes firm 2 more valuable due to its larger capacity even if it is strategically weaker than firm 1.

[15] The curves could intersect at more than one point. This would mean that there are many gap equilibria. We are only interested in whether a gap equilibrium exists, not in the number of such equilibria.

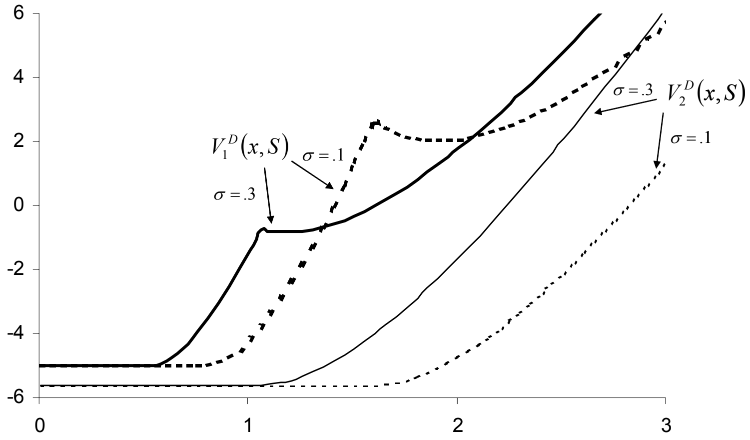

Figure 2: Values of the firms in equilibrium.

exits first, and the gap equilibrium where the strong firm exits first. Figure 4 shows the values of the firms in the gap equilibrium. It should be noted that contrary to the normal equilibrium, firm 2 is initially more valuable, because firm 1 exits first.

However, when asymmetry between the firms is large enough, the gap equilibrium does not exist anymore even when uncertainty is high. This is easy to demonstrate by increasing the difference in the firms' capacities while keeping their sum fixed. At some point, the curves $y = r_2^-(x)$ and $x = r_1^+(y)$ will not intersect, no matter how much uncertainty is increased.

This example demonstrates the main conclusion of this paper. The more there is uncertainty, the more likely it is that there is no unique equilibrium. On the other hand, the more there is asymmetry, the more likely it is that there is a unique equilibrium.

7 Conclusions

We have studied the devolution of a stochastically declining duopoly, where the firms choose when to exit permanently from the market. We have shown that there always exists at least one equilibrium, and in every equilibrium the firms exit sequentially.

We have demonstrated the effect of strategic interaction on the shapes of the curves that represent the firms values as functions of the shock variable value. In equilibrium, the value function of the firm that stays in the market longer has a kink that corresponds to the exit of its competitor. The strategic interaction also changes the effect of uncertainty on the value of this firm. Without competition, increasing uncertainty would increase the value of the firm with an abandonment option. In duopoly, however, the increased uncertainty changes the behavior of the competitor, and the total effect may be that the value of the firm is decreased at some demand shock values. It was also demonstrated that the increased uncertainty reduces the effect of strategic interaction on the payoff difference between the firms.

The main result of the paper is that a unique equilibrium exists only when uncertainty is sufficiently low or asymmetry between the firms is sufficiently high. In that case, one of the firms is doomed to exit before the other, because the other firm can credibly commit to staying in the market longer in case it is left alone.

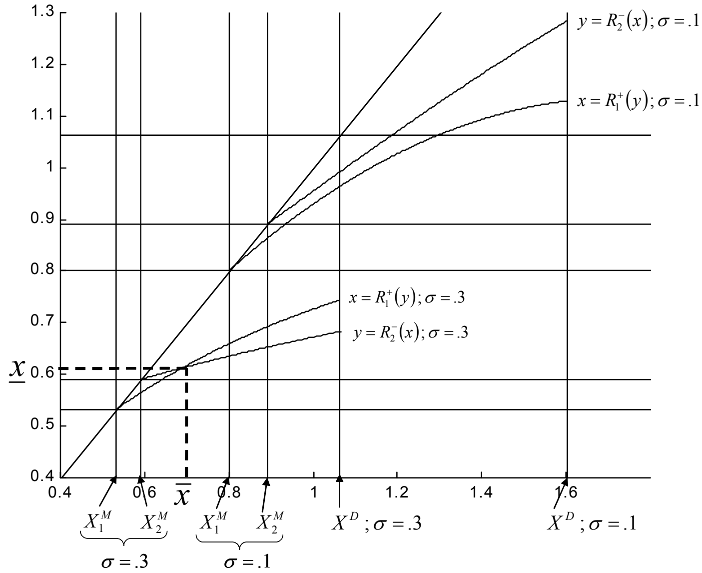

Figure 3: Reaction curves.

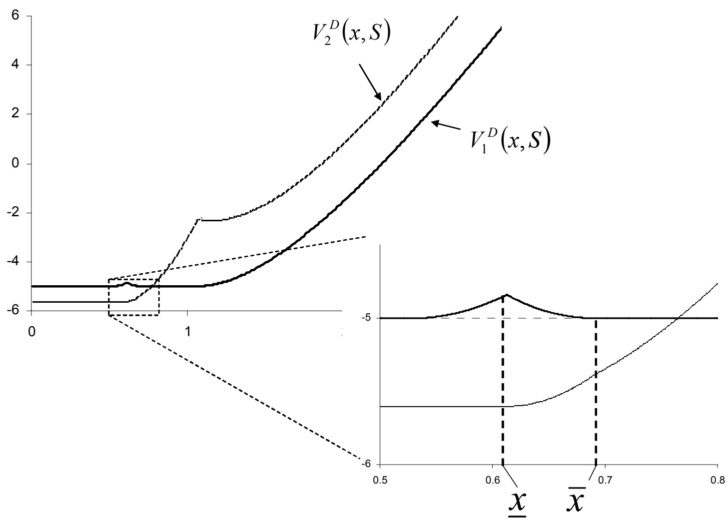

Figure 4: Values of the firms in gap equilibrium.

However, when the degree of uncertainty is high enough and asymmetry low enough, then a 'gap equilibrium', where the exit order of the firms is reversed, may appear. Then there is no unique equilibrium outcome, and it is no longer possible to predict which of the firms would exit first.

It is interesting to compare this main result with the paper by Ghemawat and Nalebuff (1985). Their model is deterministic and they specify that the difference between the firms is in the size. In their model, the larger firm exits first (unless there is a major cost advantage in favor of the large firm). The basic version of their model is in fact a special case of our model with demand shock volatility set to zero.[16] The present paper shows that if uncertainty is increasingly added in that model, then at some point the uniqueness of the equilibrium may suddenly break down. In that kind of a situation it is hard to predict the outcome of the game, particularly the order in which the firms exit. This is an important addition to the theory, because in reality uncertainty is a main property of many industries.

It is also interesting to compare our results with the paper by Fine and Li (1989). They also extend deterministic exit models to a stochastic environment, but work in discrete time. They have a result similar to ours: the uniqueness of the equilibrium, as in Ghemawat and Nalebuff (1985) and Fudenberg and Tirole (1986), does not hold. However, they assert that jumps in the demand process drive the multiple equilibria result, independent of whether the demand process is stochastic or deterministic. Our model shows that the multiple equilibria result may also be driven by uncertainty, even if the demand decline path is continuous.

[16] Except that they have a more general deterministic time trajectory for the evolution of the profitability.

A Appendix

Proofs of Propositions 1-5 follow.

Proof of Proposition 1. It is given that S is an equilibrium profile. Let i denote a firm for which it holds that $\sup S_i \geq \sup S_j$. Then, given a high enough initial value for X, firm i knows that j will not exit before i. Since S_i is the best response to S_j, it must be that i behaves exactly as it would do in the case where j never exits, that is, i exits at the first moment when X falls below X_i^D. Therefore, it must be that $\sup S_i = X_i^D$. If $\sup S_j = \sup S_i$, then i's strategy would intersect with S_j above X_i^M. However, it was shown in Section 4 that the best-response strategy of i can not intersect with S_j above X_i^M. Thus, it must be that $\sup S_j < \sup S_i$. Summing up, $\sup S_i = X_i^D > \sup S_j$, which means that $S \in \Theta_i$. Since $i = 1$ or 2, $S \in \Theta_1 \cup \Theta_2$. *Q.E.D.* ■

Proof of Proposition 2. Assume that firm j has adopted a strategy $S_j = \left(0, X^M\right]$, and consider i's best response $R_i\left(S_j\right)$. As shown before, $\left(0, X_i^M\right]$ must be contained in any best response strategy. Since $X_i^M = X_j^M = X^M$, j will not exit before i, and thus it must be optimal for i to exit everywhere between X^M and X^D. It must then be that $R_i\left(\left(0, X^M\right]\right) = \left(0, X^D\right]$. On the other hand, given that i has adopted $S_i = \left(0, X^D\right]$, j will get the termination payoff $V_j^M(x)$ anywhere within the interval $\left(X^M, X^D\right]$ by waiting. Since this is greater than the termination payoff in the case where j exits, namely $-U_j$, it can not be optimal for j to exit within $\left(X^M, X^D\right]$. Since it is never optimal for j to exit above X_j^D, it must be that $R_j\left(\left(0, X^D\right]\right) = \left(0, X^M\right]$. Being best responses to each other, strategies $\left\{S_i = \left(0, X^D\right], S_j = \left(0, X^M\right]\right\}$ form an equilibrium profile. Since i was arbitrary, both profiles in Proposition 2 are equilibria. *Q.E.D.* ■

Proof of Proposition 3. Using the same argumentation as in the proof of Proposition 2, it is straight-forward to show that $R_1\left(\left(0, X_2^D\right]\right) = \left(0, X_1^M\right]$ and $R_2\left(\left(0, X_1^M\right]\right) = \left(0, X_2^D\right]$, which means that (17) is an equilibrium. However, the same argumentation does not work for strategies (18), because if firm 2 adopts $S_2 = \left(0, X_2^M\right]$, then there is the interval $\left(X_1^M, X_2^M\right]$ where firm 1 would prefer to stay. Therefore, $S_1 = \left(0, X_1^D\right]$ can not be the optimal response to $S_2 = \left(0, X_2^M\right]$. *Q.E.D.* ■

Proof of Proposition 4. Because $\left(0, X_2^M\right] \in R_2\left(S_1\right)$ for any S_1, and $R_1\left(S_2\right)$ can not intersect with S_2 above X_1^M, the interval $\left(X_1^M, X_2^M\right]$ can not be contained in S_1 in equilibrium. By definition, for a profile S to be in Θ_1 means that $\sup S_1 = X_1^D$, so S_1 can not be connected. The "only if" part of the proposition is then clear from the way the best response strategies are constructed. To confirm the "if" part, it suffices to show that (19) is an equilibrium profile. Assuming that $\underline{x} = r_2^-\left(r_1^+\left(\underline{x}\right)\right)$, and denoting $\overline{x} = r_1^+\left(\underline{x}\right)$, it is straight-forward to check that $R_1\left(\left(0, \underline{x}\right]\right) = \left(0, X_1^M\right] \cup \left[\overline{x}, X_1^D\right]$ and $R_2\left(\left(0, X_1^M\right] \cup \left[\overline{x}, X_1^D\right]\right) = \left(0, \underline{x}\right]$, which confirms that the profile is an equilibrium. *Q.E.D.* ■

Proof of Proposition 5. Observe that the upper and lower reactions $r_i^+(x)$ and $r_i^-(x)$ depend on the parameter σ through the conditions (12) - (14). To account for that, we write them here as $r_i^+(x, \sigma)$ and $r_i^-(x, \sigma)$. It is easy to conclude that $r_i^+(x, \sigma)$ and $r_i^-(x, \sigma)$ are continuous in σ (when $r_i^+(x, \sigma)$ exists).

We utilize the following observations:

$$\lim_{\sigma \to 0} r_i^-(x, \sigma) = x, \ \forall X_i^M < x < X_i^D, \tag{A1}$$

$$\lim_{\sigma \to 0} r_i^+(x, \sigma) > x, \ \forall x > X_i^M \text{ s.t.} \lim_{\sigma \to 0} r_i^+(x, \sigma) \text{ exists.} \tag{A2}$$

Proving these rigorously would be lengthy, but they can be justified by considering the deterministic case where $\sigma = 0$. Consider first (A1). If j will not exit below x, then i should exit immediately below x, if there is no chance that X will rise again to x. Naturally, the limit of $r_i^-(x,\sigma)$ as σ goes to zero must then be x. On the other hand, since $V_i^M(x) > -U_i$ for all $x > X_i^M$, i is not willing to exit if j is going to exit within a very short interval for certainty. Thus, for any $x > X_i^M$, there must be a nonnegative interval above x where i rather waits for X to fall to x to get $V_i^M(x)$, than exits immediately. This justifies (A2).

From (A1) and (A2) we can directly state that there is a $\overline{\sigma} > 0$ such that:

$$x' - r_2^-(x',\sigma) < r_1^+(x'',\sigma) - x'' \tag{A3}$$

for all $\sigma < \overline{\sigma}$, $x' > X_i^M$, and $x'' > X_2^M$ for which $r_1^+(x'',\sigma)$ exists.

According to Proposition 4, if there is an equilibrium for $\sigma < \overline{\sigma}$ where firm 1 exits first, there must be a $x > X_2^M$ for which $r_1^+(x,\sigma)$ exists and $x = r_2^-\left(r_1^+(x,\sigma),\sigma\right)$. Assume that this is the case. However, if we take this particular x and write $x'' = x$, $x' = r_1^+(x,\sigma)$, then (A3) says that $r_1^+(x,\sigma) - r_2^-\left(r_1^+(x,\sigma),\sigma\right) < r_1^+(x,\sigma) - x$, i.e. $x < r_2^-\left(r_1^+(x,\sigma),\sigma\right)$. This is a contradiction. Thus, if $\sigma < \overline{\sigma}$, there can not be an equilibrium where firm 1 exits first.

It has now been shown that there can not be an 'empty' interval such that there is an exit region of firm 1 above, and of firm 2 below the interval. Since $\left(0, X_2^M\right] \subset R_2(S_1)$ for any S_1, the interval $\left(X_1^M, X_2^M\right]$ can not be contained in any S_1 that is a best response to $R_2(S_1)$. Thus, the only connected S_1 that can be the best response to a best response of firm 2 is $S_1 = \left(0, X_1^M\right]$. Since $R_2\left(\left(0, X_1^M\right]\right) = \left(0, X_2^D\right]$, the equilibrium (17) is unique. *Q.E.D.* ■

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