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EXPANDING GRAVITATIONAL SYSTEMS

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Abstract. In this paper we obtain a classification of motion for Newtonian gravitational systems as time approaches infinity. The basic assumption is that the motion survives long enough to be studied, i.e., the solution exists in the interval $(0, \infty)$. From this classification it is possible to obtain a sketch of the evolving Newtonian universe.

The mathematical study of Newtonian gravitational systems has a long history and has inspired a considerable amount of modern mathematics such as, among other topics, ergodic theory, algebraic topology, qualitative theory of differential equations and some functional analysis. Yet very little seems to be known about gravitational systems beyond the two-body problem. In 1922, J. Chazy [1] was able to classify the motion of the three-body problem as time, t, approaches infinity. In 1967, H. Pollard [7] obtained the first general n-body results as $t \to \infty$. He obtained results concerning the maximum and minimum spacing between particles as $t \to \infty$. His work suggests that the behavior of systems with nonnegative energy is in some sense a generalization of the two- and three-body problems.

It is the purpose of this paper to sharpen these results and to provide the first classification of motion of the n-body problem, as $t \to \infty$ *independent of the sign of the energy*. With this classification of motion a sketch of the evolution of Newton's universe as $t \to \infty$ is possible. Also several remaining problems on the growth of systems are partially answered.

It is interesting to note that previous classifications of motion have been attempted in terms of the sign of the total energy of the system. It turns out that this approach is far too restrictive and that the classification should be made according to the rate of separation of the particles, as is done here.

It will be shown that in the absence of motion that we will call *oscillatory* and *pulsating*, the n-body problem is quite well behaved. It separates into *clusters* where the mutual distances between particles are bounded as $t \to \infty$. The clusters form subsystems characterized by the separation of clusters like $t^{2/3}$. The centers of mass of the subsystems separate asymptotically from each other as Ct.

Most of the results depend quite heavily on Tauberian theorems of the type of Landau [20, p. 194] (we follow the customary usage of the o—O symbols):

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$f \in C^2(0, \infty)$, $f(t)=o(t^\alpha)$ (or $f(t)=O(t^\alpha)$) and $f''(t) \geqq -At^{\alpha-2} \Rightarrow f'(t)=o(t^{\alpha-1})$ (or $f(t)=O(t^{\alpha-1})$) where A and α are constants.

Probably A. Wintner [21, p. 429] was the first to recognize the power of Tauberian arguments in celestial mechanics. He pointed out that some of Sundman's considerations in his discussion on binary collisions in the three-body problem [19] were of a Tauberian nature. (R. P. Boas, Jr., [2] extracted from Sundman's work a simplified Tauberian argument and J. Karamata [3] extended the result. Along this same line, papers of Pollard [8], Saari [15], and Pollard and Saari [12] are of interest.) This approach of using Tauberian theorems was subsequently exploited by H. Pollard in his paper on gravitational systems and by Pollard and Saari [9], [10] in their discussions of singularities and collisions in the n-body problem.

In addition to the above mentioned references, C. Siegel's paper [16] on collisions in the three-body problem is part of the literature leading to this work.

While we borrow freely some of the ideas from the literature, and some of these ideas have become almost standard arguments, most need to be strongly modified. This is so, because in our setting the major problem is the existence of error terms and the lack of integrals of motion. However, as a by-product, some of the altered arguments turn out to be an improvement in their original setting where more information is available.

The primary assumptions will be that the motion exists for all t in the interval $(0, \infty)$ and that the center of mass of the system is located at the origin of some inertial coordinate system.

The notation will be introduced as needed, but the following is basic. For positive continuous functions $f, g, f \approx g$ will imply that after some time there exist positive constants A and B such that $Ag(t) \leqq f(t) \leqq Bg(t)$.

The symbols m_k, $\boldsymbol{r}_k$, $\boldsymbol{v}_k$ denote respectively the mass, position and velocity of the kth particle. The same letter will be used to indicate the magnitude of a vector. For example, $r_k=|\boldsymbol{r}_k|$ and $r_{kj}=|\boldsymbol{r}_k-\boldsymbol{r}_j|$. We define further

$$T = \frac{1}{2}\sum m_k v_k^2,\ U = \sum_{1 \leqq j<k \leqq n} \frac{m_j m_k}{r_{jk}} \quad \text{and} \quad I = \frac{1}{2}\sum m_k r_k^2.$$

If we assume the gravitational constant to be unity, then the law of conservation of energy becomes $T=U+h$ where h is the constant total energy. The conservation of angular momentum is $\sum m_k \boldsymbol{r}_k \times \boldsymbol{v}_k = \boldsymbol{c}$ where $\boldsymbol{c}$ is a constant vector. The Lagrange-Jacobi formula [6, p. 41] is simply $d^2I/dt^2 = U+2h$ and the Sundman inequality is $c^2+(dI/dt)^2 \leqq 4IT$.

2. **The Sundman inequality.** Here we establish the validity of the Sundman inequality. There are several proofs available [2], [7, p. 605] but this one has the advantage that it is not only simple but it also includes the error term. (The proof is a consequence of correspondence between H. Pollard and the author, leading to [11].)

By definition of the quantities involved,

$$\frac{1}{2}\sum m_k\left(\frac{d}{dt}\frac{r_k}{I^{1/2}}\right)^2 = \frac{Q}{I}-\frac{1}{4}\left(\frac{dI}{dt}\right)^2 I^{-2}$$

where $Q=\frac{1}{2}\sum m_k\,(dr_k/dt)^2$. Hence

(2.1) $(dI/dt)^2 \leqq 4IQ$ and equality is achieved if, and only if, $r_k(t)=D_k(I(t))^{1/2}$ for $k=1, 2, \ldots, n$. D_k is a positive constant.

By definition of $\boldsymbol{c}$,

$$\boldsymbol{c} = \sum (m_k)^{1/2} r_k\left((m_k)^{1/2}\frac{\boldsymbol{r}_k\times\boldsymbol{v}_k}{r_k}\right).$$

Taking the absolute value of both sides and using the Cauchy inequality yields

$$c^2 \leqq \sum m_k r_k^2 \sum m_k\left(\frac{\boldsymbol{r}_k\times\boldsymbol{v}_k}{r_k}\right)^2 = 2I\sum m_k\left(\frac{\boldsymbol{r}_k\times\boldsymbol{v}_k}{r_k}\right)^2.$$

The sum on the right-hand side is equal to $2(T-Q)$. This follows from the definition of T and the relationship

$$r^2v^2 = (\boldsymbol{r}\cdot\boldsymbol{v})^2+(\boldsymbol{r}\times\boldsymbol{v})^2 = r^2\,(dr/dt)^2+(\boldsymbol{r}\times\boldsymbol{v})^2.$$

Hence

$$c^2 \leqq 4I(T-Q) \tag{2.2}$$

and

$$c^2+(dI/dt)^2 \leqq 4IT. \tag{2.3}$$

3. **Classification of motion.** In the three-body problem, there is the possibility that some initial conditions lead to motion with the property that, as $t\to\infty$, $\liminf(r_{12}/r_{23})=0$ and $\limsup(r_{12}/r_{23})>0$ where $\limsup r_{23}=\infty$ [1], [13]. We generalize this definition to the n-body problem as follows:

DEFINITION. Masses m_k, m_j, and m_i are *oscillatory* if, as $t\to\infty$, $\limsup r_{ij}=\infty$, $\limsup(r_{ki}/r_{ij})>0$ and $\liminf(r_{ki}/r_{ij})=0$. We say that r_l *participates* in oscillatory motion if l can be chosen as one of the above indices. The existence of such motion in the three-body problem has been shown by Sitnikov [17]. Bounds on its behavior in the three-body problem have been found by Saari [13].

In this section we lead to a precise statement of the classification of motion. But first we state and prove the following lemma. This will show what type of motion to expect and will motivate the development of the machinery which follows.

LEMMA. *Consider the p-body problem. Suppose* $\limsup r_{12}=\infty$.

$$0 < \liminf(r_{sj}/r_{12}) \leqq \limsup(r_{sj}/r_{12}) < \infty, \qquad s \neq j.$$

Then either

$$r_s = C_s t + O(\ln t)$$

or

$$r_{sj} \approx t^{2/3}, \qquad s \neq j.$$

Proof. We first show that the total energy must be nonnegative so assume the contrary, i.e., $h<0$. As $T \geqq 0$, it follows from the conservation of energy integral that $U+h \geqq 0$ or $U \geqq |h|$. By the second hypothesis $I^{1/2} \approx U^{-1}$ hence there exists positive A such that after some time $AI^{-1/2} \geqq |h|$. But this states that $I=O(1)$, which contradicts the first hypothesis.

If $h>0$, then by definition of U, it follows that $d^2I/dt^2 = U+2h \geqq 2h$, or $I \geqq ht^2$. Again from the second hypothesis $I^{1/2} \approx r_{sj}$, $s \neq j$, hence $r_{sj}^{-1} = O(t^{-1})$. That is $d^2\mathbf{r}_s/dt^2 = O(t^{-2})$. Integration yields $d\mathbf{r}_s(t_2)/dt - d\mathbf{r}_s(t_1)/dt = O(t_1^{-1} - t_2^{-1})$. As $t_1, t_2 \to \infty$, the right-hand side goes to zero, forcing the left-hand side to zero also. Hence for all s, $d\mathbf{r}_s/dt = \mathbf{C}_s + O(t^{-1})$, or $\mathbf{r}_s = \mathbf{C}_s t + O(\ln t)$. If all $\mathbf{C}_s = \mathbf{0}$, then $r_{sj} = O(\ln t)$ and $I^{1/2} = O(\ln t)$, contrary to $I \geqq ht^2$. Thus some $\mathbf{C}_s \neq \mathbf{0}$, say $s=1$. From the fact that the center of mass is fixed at $\mathbf{0}$, $m_1\mathbf{r}_1 t^{-1} = -\sum_2^p m_i \mathbf{r}_i t^{-1} \to m_1 \mathbf{C}_1$. Hence for some other index, $\mathbf{C}_s \neq \mathbf{0}$. It follows directly from the second hypothesis that $r_{sj} \approx t$, i.e., at most one $\mathbf{C}_s = \mathbf{0}$.

The last case is $h=0$. Again from the second hypothesis $I^{1/2} \approx U^{-1}$, hence the Lagrange-Jacobi relationship becomes $d^2I/dt^2 \approx I^{-1/2}$. Note that $d^2I/dt^2 > 0$, hence dI/dt is an increasing function. It must eventually become positive, otherwise I is bounded which is contrary to hypothesis. Once dI/dt becomes positive, it remains positive. This implies I is eventually monotonically increasing, and as it is unbounded, $I \to \infty$. Using these facts, after some time

$$(dI/dt)\, d^2I/dt^2 \approx (dI/dt) I^{-1/2} \quad \text{or} \quad (dI/dt)^2 \approx I^{1/2}.$$

(The constant of integration is absorbed by the fact $I \to \infty$.) Again by the fact that dI/dt eventually becomes positive,

$$(dI/dt)/I^{1/4} \approx 1 \quad \text{or} \quad I^{3/4} \approx t.$$

But as $I^{1/2} \approx r_{sj}$, $s \neq j$, this implies the conclusion and completes the proof of the lemma.

What we do next is to try to implement the intuitive idea that in the general n-body problem, at large enough distances "slower motions" can be viewed as point masses. Hence the motion should be in some sense similar to that given in the lemma.

Choose indices k, j such that $\limsup r_{kj} = \infty$ and $\mathbf{r}_k, \mathbf{r}_j$ do not participate in the same oscillatory motion. (If no such indices exist, the motion is either oscillatory and/or bounded.) r_{kj} divides, in a natural fashion, the n masses into clusters. We collect those indices i for which $r_{ki}/r_{kj} \to 0$ as $t \to \infty$ into set G_k, with a similar definition for set G_j. As the particles m_k and m_j do not participate in the same oscillatory motion, it follows from the triangle inequality that this can be done. (Note, $\boldsymbol{r}_k$ *may* participate in oscillatory motion, but the point is that $\boldsymbol{r}_k$ and $\boldsymbol{r}_j$ do not define the same oscillatory motion. That is, the growth properties of the oscillatory motion is either much slower or faster than r_{kj}.)

After r_{kj} has been chosen, we are simply interested in its asymptotic properties, so let $f(t) \in C^2(0, \infty)$ be such that $f(t) \approx r_{kj}$.

For all l such that l is not in G_k or G_j and $\liminf (r_{kl}/f(t)) < \infty$, we can define G_l. That is, we collect all indices into G_l which adhere to $\boldsymbol{r}_l$ in the sense defined above. Again by precluding participation in oscillatory motion, the triangle inequality yields $\limsup (r_{kl}/f(t)) < \infty$. We (relabel and) enumerate the sets G_s, $s = 1, 2, \ldots, p$, and define $M_s \boldsymbol{\rho}_s = \sum_{i \in G_s} m_i \boldsymbol{r}_i$ where $M_s = \sum_{i \in G_s} m_i$.

The equations of motion for $\boldsymbol{\rho}_s$ are

$$\begin{aligned} M_s\, d^2\boldsymbol{\rho}_s/dt^2 &= \sum_{i \in G_s} m_i\, d^2\boldsymbol{r}_i/dt^2 \\ &= \sum_{k,i \in G_s;\, k \neq i} \frac{m_k m_i(\boldsymbol{r}_k - \boldsymbol{r}_i)}{r_{ki}^3} + \sum_{j=1;\, j \neq s}^{p} \sum_{i \in G_s;\, k \in G_j} \frac{m_k m_i(\boldsymbol{r}_k - \boldsymbol{r}_i)}{r_{ki}^3} \\ &\quad + \sum_{i \in G_s;\, k \notin G_j} \frac{m_k m_i(\boldsymbol{r}_k - \boldsymbol{r}_i)}{r_{ki}^3}, \qquad j = 1, 2, \ldots, p. \end{aligned}$$

The first double sum on the right-hand side vanishes by the antisymmetry of the term $\boldsymbol{r}_k - \boldsymbol{r}_i$. Each term of the last double sum is of the magnitude $O(r_{ki}^{-2})$. For these values of k, $f(t)/r_{ki} \to 0$ as $t \to \infty$, hence the sum is $o(f(t)^{-2})$ and

$$\begin{aligned} M_s \frac{d^2\boldsymbol{\rho}_s}{dt^2} &= \sum_{j=1;\, j \neq p}^{p} \sum_{i \in G_s;\, k \in G_j} \frac{m_i m_k(\boldsymbol{r}_k - \boldsymbol{r}_i)}{r_{ki}^3} + o(f(t)^{-2}) \\ &= \sum_{j=1;\, j \neq s}^{p} \frac{M_j M_s(\boldsymbol{\rho}_j - \boldsymbol{\rho}_s)}{\rho_{js}^3} + \sum_{j=1;\, j \neq s}^{p} \sum_{i \in G_s;\, k \in G_j} m_i m_k(\boldsymbol{r}_k - \boldsymbol{r}_i)\left(\frac{1}{r_{ik}^3} - \frac{1}{\rho_{sj}^3}\right) + o(f(t)^{-2}). \end{aligned}$$

Each term of the triple sum is of the order

$$r_{ik}\left(\frac{1}{r_{ik}^3} - \frac{1}{\rho_{sj}^3}\right) = \frac{1}{r_{ik}^2}\left(1 - \frac{r_{ik}^3}{\rho_{sj}^3}\right).$$

By the triangle inequality and the definition of the sets G_s, $r_{ik} = \rho_{sj} + o(\rho_{sj})$. Hence each term of the triple sum is $o(f(t)^{-2})$.

This leads to the final form of the equations of motion:

$$M_s \frac{d^2\boldsymbol{\rho}_s}{dt^2} = \sum_{j=1;\, j \neq s}^{p} \frac{M_s M_j(\boldsymbol{\rho}_j - \boldsymbol{\rho}_s)}{\rho_{js}^3} + o(f(t)^{-2}), \qquad s = 1, 2, \ldots, p. \tag{3.1}$$

We first show that equation (3.1) retains the same form if $\boldsymbol{\rho}_s$ is expressed relative to the common center of mass of the vectors $\boldsymbol{\rho}_s$ rather than the origin of the inertial coordinate system. Let $M = \sum_{s=1}^{p} M_s$ and $M\boldsymbol{P} = \sum M_s \boldsymbol{\rho}_s$.

By the antisymmetry of $\boldsymbol{\rho}_j - \boldsymbol{\rho}_s$ and (3.1), $M\, d^2\boldsymbol{P}/dt^2 = o(f(t)^{-2})$. Hence, if we replace $\boldsymbol{\rho}_k$ by $\boldsymbol{\rho}_k - \boldsymbol{P}$ in (3.1), the above estimate for $d^2\boldsymbol{P}/dt^2$ and the fact $(\boldsymbol{\rho}_j - \boldsymbol{P})$

$-(\boldsymbol{\rho}_s-\boldsymbol{P})=(\boldsymbol{\rho}_j-\boldsymbol{\rho}_s)$ yield differential equations which are again of the form (3.1). For the remainder of this paper we assume equations (3.1) are expressed relative to their common center of mass.

The restrictions on oscillatory motion for the n-body problem implies the construction of the sets G_s. These sets define a system which is a perturbation of the p-body problem defined in the lemma at the beginning of this section. One wishes to prove a perturbation theorem, but needs an additional assumption: no "pulsating motion".

Motivated by the definition of U and T, define

$$V = \sum{}^{*} \frac{M_s M_j}{\rho_{sj}} \quad \text{and} \tag{3.2}$$
$$E = \frac{1}{2}\sum_{s=1}^{p} M_s \frac{d\boldsymbol{\rho}_s^2}{dt} = \frac{1}{2M}\sum{}^{*} M_s M_j\left(\frac{d\boldsymbol{\rho}_s}{dt}-\frac{d\boldsymbol{\rho}_j}{dt}\right)^2$$

where (*) denotes the double summation $1 \leqq s < j \leqq p$. The two sums in the definition of E are equal as a consequence of $\boldsymbol{\rho}_s$ being expressed relative to the center of mass $\boldsymbol{P}$, i.e., $\sum M_s\, d\boldsymbol{\rho}_s/dt=\mathbf{0}$.

With the definition of V, (3.1) can now be stated as

$$M_s\frac{d^2\boldsymbol{\rho}_s}{dt} = \frac{\partial V}{\partial \boldsymbol{\rho}_s}+o(f(t)^{-2}). \tag{3.3}$$

If the error term were not present, then a conservation of energy integral would follow: $E=V+H$ where H is a constant. As $E\geqq 0$, $V\approx f(t)^{-1}$ and $f(t)$ is not bounded, it follows that $H\geqq 0$. Hence $\liminf E/V\geqq 1$.

When the error term is present, a possible interpretation for $\liminf E/V\geqq 1$ is that in some sense (3.3) does satisfy a "conservation of energy" relationship. This will be made more explicit in Corollary 1.2.

We can now state the main theorem leading to the classification of motion.

THEOREM 1. *If* $\liminf E/V>\frac{1}{2}$ *as* $t\to\infty$ *then either*

$$\boldsymbol{\rho}_s = \boldsymbol{C}_s t+\boldsymbol{D}_s \ln t+o(\ln t), \qquad s = 1,2,\dots,p,$$

or

$$\rho_{sj} \approx t^{2/3}, \qquad s\neq j.$$

$\boldsymbol{C}_s$ *and* $\boldsymbol{D}_s$ *are constant vectors.* $|\boldsymbol{C}_s-\boldsymbol{C}_k|\neq 0$ *for* $s\neq k$.

COROLLARY 1.1. *For all i, k, at least one of the following occurs as* $t\to\infty$:

1. $\boldsymbol{r}_i$ *and* $\boldsymbol{r}_k$ *participate in the same oscillatory motion.*
2. $r_{ik}=O(1)$.
3. $r_{ik}\approx t^{2/3}$.
4. $r_{ik}\sim C_{ik}t$, *where* C_{ik} *is some positive constant.*
5. r_{ik} *defines a subsystem with the property* $\liminf E/V\leqq\frac{1}{2}$ *as* $t\to\infty$.

Proof of Corollary 1.1. If $\mathbf{r}_i$ and $\mathbf{r}_j$ do not participate in the same oscillatory motion and $r_{ij} \neq O(1)$, then r_{ij} can be used to define (3.1). Let $i \in G_s$ and $k \in G_j$, $s \neq j$. As $\mathbf{r}_i = \boldsymbol{\rho}_s + o(r_{ik})$ and $\mathbf{r}_k = \boldsymbol{\rho}_j + o(r_{ik})$, the conclusion of Corollary 1.1 follows directly from that of Theorem 1.

The condition "$\liminf E/V > \frac{1}{2}$" will be discussed in greater detail in §6. But here we state the following which shows that it does imply a conservation of energy relationship:

COROLLARY 1.2. *$\liminf E/V > \frac{1}{2}$ as $t \to \infty$ if and only if $E = V + H + o(V)$ as $t \to \infty$. H is a nonnegative constant.*

For purposes of identification we call the case $\liminf E/V \leqq \frac{1}{2}$ *pulsating motion.*

4. **Proof of Theorem 1 and Corollary 1.2.** First the proof of Theorem 1. Motivated by the definition of and properties of I, we define

$$J = \frac{1}{2} \sum M_s \boldsymbol{\rho}_s^2 = \frac{1}{2M} \sum{}^* M_s M_j (\boldsymbol{\rho}_s - \boldsymbol{\rho}_j)^2.$$

From the definition of V and Euler's theorem, $\sum \boldsymbol{\rho}_s \cdot \partial V / \partial \boldsymbol{\rho}_s = -V$. Hence, from (3.2), (3.3) and the fact $V \approx f(t)^{-1}$,

$$\frac{d^2 J}{dt^2} = \sum M_s \frac{d\boldsymbol{\rho}_s^2}{dt} + \sum M_s \boldsymbol{\rho}_s \cdot \frac{d^2 \boldsymbol{\rho}_s}{dt^2} = 2E - V + o(f(t)^{-1}). \tag{4.1}$$

As $\liminf E/V > \frac{1}{2}$, after some time there exists a positive constant B such that $2E - V \geqq BV$, or

$$d^2 J/dt^2 \geqq BV + o(V).$$

By the definition of J, $J \approx f(t)^2$, or

$$d^2 J/dt^2 \geqq BJ^{-1/2} + o(J^{-1/2}). \tag{4.2}$$

(B may have a different value with each usage.) But this implies d^2J/dt^2 is positive after some time, or that dJ/dt is monotonically increasing. As J is unbounded and positive, there is some t_1 for which $dJ(t_1)/dt \geqq 0$; hence, it follows that dJ/dt is positive for $t > t_1$. Using this fact and (4.2) we have

$$\frac{dJ}{dt} \frac{d^2 J}{dt^2} \geqq B \frac{dJ}{dt} J^{-1/2} + o\left(\frac{dJ}{dt} J^{-1/2}\right).$$

Integrating from a to t, we have

$$(dJ/dt)^2 \geqq BJ^{1/2} + o(J^{1/2}) + C$$

where C is a constant of integration.

As dJ/dt is positive and J is unbounded, it follows that $J \to \infty$; hence C can be incorporated into the error term.

Again by the fact that dJ/dt is positive, we have

$$(dJ/dt)/J^{1/4} \geqq B+o(1).$$

Integration yields

$$J^{3/4} \geqq Bt+o(t) \quad \text{or} \quad J \geqq Bt^{4/3}+o(t^{4/3}).$$

Now, as $J \approx f(t)^2$, we have $f(t) \geqq Bt^{2/3}$.

From (3.1), this implies

$$\frac{d^2\boldsymbol{\rho}_s}{dt^2} = O(1/t^{4/3}), \qquad s = 1, \dots, p.$$

Integrating from t_1 to t_2, $t_1 < t_2$,

$$d\boldsymbol{\rho}_s(t_2)/dt - d\boldsymbol{\rho}_s(t_1)/dt = O(t_2^{-1/3} - t_1^{-1/3}). \tag{4.3}$$

As $t_1, t_2 \to \infty$, the right-hand side approaches zero, carrying the left-hand side to zero with it. By the Cauchy criterion for the existence of a limit, this implies

$$d\boldsymbol{\rho}_s/dt \to \boldsymbol{C}_s, \qquad s = 1, \dots, p. \tag{4.4}$$

Integration again yields

$$\boldsymbol{\rho}_s \sim \boldsymbol{C}_s t, \tag{4.5}$$

where for any choice of s such that $\boldsymbol{C}_s = \boldsymbol{0}$, $\boldsymbol{\rho}_s = o(t)$.

We assume first that at least one $\boldsymbol{C}_s \neq \boldsymbol{0}$, say $\boldsymbol{C}_1$. Then as the $\boldsymbol{\rho}_s$ are expressed relative to their common center of mass, $\sum M_s \boldsymbol{\rho}_s = \boldsymbol{0}$ or

$$-M_1\boldsymbol{C}_1 = \lim_{t\to\infty} \sum_{s=2}^{p} M_s \boldsymbol{\rho}_s t^{-1} = \sum_{s=2}^{p} M_s \boldsymbol{C}_s. \tag{4.6}$$

Hence, for at least one other choice of subscript, say $s=2$, (4.4) has a nonzero limit. As the masses are positive and there is a (-1) term on the left-hand side of (4.6), $\boldsymbol{C}_2$ can be chosen so that $|\boldsymbol{C}_1 - \boldsymbol{C}_2| \neq 0$. This means that $|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2| \sim |\boldsymbol{C}_1 - \boldsymbol{C}_2| t$, or that $f(t)$ can be chosen as t.

But as $\rho_{sk} \approx t$, we have from (4.5) $\rho_{sk} = |\boldsymbol{\rho}_s - \boldsymbol{\rho}_k| \sim |\boldsymbol{C}_s - \boldsymbol{C}_k| t \approx t$, or $|\boldsymbol{C}_s - \boldsymbol{C}_k| \neq 0$ for $s \neq k$. This means that *at most one* $\boldsymbol{C}_s = \boldsymbol{0}$.

Substituting (4.5) back into (3.1) yields $d^2\boldsymbol{\rho}_s/dt^2 \sim \boldsymbol{D}_s t^{-2}$ where

$$\boldsymbol{D}_s = \sum \frac{M_j(\boldsymbol{C}_j - \boldsymbol{C}_k)}{|\boldsymbol{C}_j - \boldsymbol{C}_k|^3}.$$

Integrating twice gives us the desired result

$$\boldsymbol{\rho}_s \sim \boldsymbol{C}_s t - \boldsymbol{D}_s \ln t.$$

We return to (4.4) and assume now that all $\boldsymbol{C}_s = \boldsymbol{0}$. Letting $t_2 \to \infty$ in (4.3), this implies $d\boldsymbol{\rho}_s(t)/dt = O(t^{-1/3})$, or $\boldsymbol{\rho}_s = O(t^{2/3})$.

That is,

$$f(t) \approx \rho_{sj} = |\boldsymbol{\rho}_s - \boldsymbol{\rho}_j| = O(t^{2/3}).$$

As it has already been shown that $f(t) \geqq Bt^{2/3}$, it follows that $\rho_{sj} \approx t^{2/3}$, and the theorem is proved.

We now prove Corollary 1.2. One direction is obvious as $E/V = 1 + HV^{-1} + o(1) \geqq 1$. To show the other direction assume first that $f(t) \approx t^{2/3}$. Then (3.1) becomes

$$M_s \frac{d^2\boldsymbol{\rho}_s}{dt^2} = \frac{\partial V}{\partial \boldsymbol{\rho}_s} + o(1/t^{4/3})$$

or

$$M_s \frac{d\boldsymbol{\rho}_s}{dt} \cdot \frac{d^2\boldsymbol{\rho}_s}{dt^2} = \frac{\partial V}{\partial \boldsymbol{\rho}_s} \cdot \frac{d\boldsymbol{\rho}_s}{dt} + o\left(\left| \frac{d\boldsymbol{\rho}_s}{dt} \right| t^{-4/3} \right).$$

Now from the above proof of Theorem 1 and (4.3) it follows that $|d\boldsymbol{\rho}_s/dt| = O(t^{-1/3})$, hence the error term is $o(t^{-5/3})$. Thus summing over $s = 1, \ldots, p$,

$$\frac{dE}{dt} = \sum M_s \frac{d\boldsymbol{\rho}_s}{dt} \cdot \frac{d^2\boldsymbol{\rho}_s}{dt^2} = \frac{\partial V}{\partial \boldsymbol{\rho}_s} \cdot \frac{d\boldsymbol{\rho}_s}{dt} + o(t^{-5/3}) = \frac{dV}{dt} + o(t^{-5/3}).$$

Integrating from t_1 to t_2,

$$E(t_2) - E(t_1) = V(t_2) - V(t_1) + o(t_1^{-2/3} - t_2^{-2/3}). \tag{4.7}$$

As $V \approx t^{-2/3}$, we have, as $t_2, t_1 \to \infty$, that $E(t_2) - E(t_1) \to 0$, or by the Cauchy criterion for the existence of a limit $E \to H$. By definition of E, $H \geqq 0$.

Allowing $t_2 \to \infty$ in (4.7) and recalling that $V \approx t^{-2/3}$ it follows that

$$E(t) = V(t) + H + o(V). \tag{4.8}$$

From the definition of E and the fact $d\boldsymbol{\rho}_i/dt = O(t^{-1/3})$, it follows that $H = 0$.

Mimicking the above argument with $f(t) = t$ and the fact $d\boldsymbol{\rho}_s/dt \sim \boldsymbol{C}_s$, (4.8) follows. In this case, the definition of E and the fact that $d\boldsymbol{\rho}_s/dt \sim \boldsymbol{C}_s$, where not all $\boldsymbol{C}_s = \boldsymbol{0}$, implies that $H = \frac{1}{2} \sum M_s C_s^2 > 0$.

5. **Susbystems.** To summarize Theorem 1: In the absence of oscillatory motion and pulsating motion ($\liminf E/V \leqq \frac{1}{2}$), the n-body problem is quite well behaved. It separates into what we call *clusters*, where the mutual distances between p particles are bounded as $t \to \infty$. The clusters form *subsystems* characterized by the separation of clusters like $t^{2/3}$. The centers of mass of the subsystems separate asymptotically from each other as $\boldsymbol{C}t$.

By imposing additional conditions and restrictions upon possible oscillatory and pulsating motion, improvements of Theorem 1 can be made and some partial answers to questions about the n-body problem can be obtained. In this section we concentrate on the relationship between clusters, oscillatory motion and subsystems. Hence, for the remainder of this section assume $J \approx t^{4/3}$. We will need the following:

Lemma 1. *If* $J \approx t^{4/3}$ *then* $|dE/dt|$, $|dV/dt| = O(t^{-5/3})$.

Proof. As $J \approx t^{4/3}$ we have $f(t) \approx t^{2/3}$ and $\rho_{sj} \approx t^{2/3}$, $s \neq j$. Substituting this value of $f(t)$ into (3.1) and recalling that the $\boldsymbol{\rho}_s$, $s = 1, \ldots, p$, are expressed relative to their common center of mass, $d\boldsymbol{\rho}_s/dt = O(t^{-1/3})$. Hence $|d\rho_{sj}/dt| \leqq |d\boldsymbol{\rho}_s/dt - d\boldsymbol{\rho}_j/dt| = O(t^{-1/3})$. By definition

$$|dV/dt| \leqq \sum \frac{M_s M_j |d\rho_{sj}/dt|}{\rho_{sj}^2} = O(t^{-5/3}).$$

The conclusion for dE/dt follows from the expression prior to (4.7) and the above estimates for $d\boldsymbol{\rho}_s/dt$ and $f(t)$.

We rewrite (4.1) as

$$d^2J/dt^2 = 2E - V + e_1(t) \tag{5.1}$$

where $e_1(t)$ is the error term. Likewise we define $e_2(t)$ to be the error term of (4.8). Note that e_1, $e_2 = o(t^{-2/3})$.

We seek a more precise statement in terms of J and V concerning the behavior of the subsystem. The following theorem gives us this information by stating that under certain conditions we have asymptotic behavior for J and V. It is motivated by a theorem of Pollard [7, p. 607]. Using the present terminology Pollard finds a similar conclusion for *one particle* clusters where it turns out that no other motion besides $t^{2/3}$ separation is permitted. Also he requires the total energy of the system to be zero. We make no restrictions on the total energy or the number of particles in a cluster. The only restrictions are on possible oscillatory and pulsating motion.

Theorem 2. *If* $\int_{t_1}^{t} (e_1(u) + e_2(u)) J^{-1/4}\, du$ *converges as* $t \to \infty$, *then* $J \sim At^{4/3}$, $dJ/dt \sim (4/3)At^{1/3}$ *and* $V \sim (4/9)At^{-2/3}$ *as* $t \to \infty$. *A is some positive constant.*

It is interesting to note that the conclusion of this theorem is similar to the type of results one obtains in the problem of collision [9], [10], [21, pp. 255–257]. Of course in the present setting $t \to \infty$, whereas in collisions $t \to 0$. But, in collisions, J and V exhibit the same type of behavior.

Theorem 2 will be extended by Corollaries 2.4 and 2.5. However the statement and proof of these corollaries will be deferred until the end of this section to permit the development of the necessary machinery. The proof of the theorem and corollaries are complicated by the existence of the error term resulting from possible bounded, pulsating or oscillatory motion. We employ Tauberian arguments and the Sundman inequality to overcome this difficulty.

This is our first assumption on bounds for possible oscillatory and pulsating motion. It is quite liberal as the following corollaries show.

Corollary 2.1. *If oscillatory and pulsating motion do not occur, then* $J \sim At^{4/3}$ *and* $V \sim (4/9)At^{-2/3}$.

COROLLARY 2.2. *If oscillatory and/or pulsating motion occur only for those particles having indices in* G_s *and if* $i, j \in G_s$, $s=1, 2, \ldots, p$; $\mathbf{r}_i$, $\mathbf{r}_j$ *participating in this motion implies* $r_{ij}=O(t^{2/3-\varepsilon})$, $\varepsilon>0$, *then* $J \sim At^{4/3}$ *and* $V \sim (4/9)At^{-2/3}$.

Proof of the corollaries. We prove only the second corollary. The first will follow directly. As $\boldsymbol{\rho}_s$ is the common center of mass for $\mathbf{r}_i$, $i \in G_s$,

$$\sum_{i\in G_s} m_i(\mathbf{r}_i-\boldsymbol{\rho}_s)^2 = \frac{1}{2M_s}\sum_{i,j\in G_s; i\neq j} m_i m_j[(\mathbf{r}_i-\boldsymbol{\rho}_s)-(\mathbf{r}_j-\boldsymbol{\rho}_s)]^2$$
$$= \frac{1}{2M_s}\sum m_i m_j r_{ij}^2.$$

By hypothesis and Corollary 1.1, the right-hand side is $O(t^{4/3-2\varepsilon})$. As the terms in the sum on the left-hand side are all nonnegative, $\mathbf{r}_i=\boldsymbol{\rho}_s+O(t^{2/3-\varepsilon})$. Hence for arbitrary $i\in G_s$ and $\alpha \in G_k$, $r_{i\alpha}=\rho_{sk}+O(t^{2/3-\varepsilon})$ and $r_{i\alpha}/\rho_{sk}=1+O(t^{-\varepsilon})$. The error term in (3.1) is now found to be $O(t^{-4/3-\varepsilon})$. The error term due to particles with indices not in G_s, $s=1,\ldots,p$, is $O(t^{-2})$ (from Theorem 1) hence $e_1(t), e_2(t)$ $=O(t^{-2/3-\varepsilon})$ and these values clearly satisfy the conditions of Theorem 2.

Proof of the theorem. We first show that $(dJ/dt)/J^{1/4}\sim l>0$, as $t\to\infty$. As $f(t)=t^{2/3}$, $J\approx t^{4/3}$ and $V\approx t^{-2/3}$. From the proof of Theorem 1, $H=0$, hence (4.1) and (4.8) imply $d^2J/dt^2\approx t^{-2/3}$, i.e. $dJ/dt\approx t^{1/3}$. Hence $(dJ/dt)/J^{1/4}\approx 1$. Thus if $(dJ/dt)/J^{1/4}$ has a limit as $t\to\infty$, it is positive.

By definition of the terms involved,

$$4\frac{d}{dt}\left(\frac{dJ/dt}{J^{1/4}}\right) = \frac{4J\,d^2J/dt^2-(dJ/dt)^2}{J^{5/4}} = \frac{4JE-(dJ/dt)^2}{J^{5/4}}+\frac{4(E-V+e_1)}{J^{1/4}}$$
$$= \frac{4JE-(dJ/dt)^2}{J^{5/4}}+\frac{4(e_2+e_1)}{J^{1/4}}.$$

Integrating from t_1 to t,

$$4\left.\frac{dJ/dt}{J^{1/4}}\right|_{t_1}^{t} = \int_{t_1}^{t}\frac{4JE-dJ^2/dt}{J^{5/4}}+\int_{t_1}^{t}\frac{4(e_2(u)+e_1(u))}{J^{1/4}}\,du.$$

As $J\approx t^{4/3}$, $J^{1/4}\approx t^{1/3}$ and by hypothesis, the second integral on the right-hand side is bounded as $t\to\infty$. As $dJ/dt\approx t^{1/3}$ and $J^{1/4}\approx t^{1/3}$, the left-hand side is bounded as $t\to\infty$. Hence

$$\int_{t_1}^{t}\frac{4JE-(dJ/dt)^2}{J^{5/4}}\,du = O(1) \quad \text{as } t\to\infty. \tag{5.2}$$

By an argument similar to that in §2, $dJ^2/dt\leqq 4JE$, hence the integrand is positive and the integral converges. That is, there exists a nonnegative constant B such that

$$\int_{t_1}^{t}\frac{4JE-dJ^2/dt}{J^{5/4}}-B = o(1).$$

By hypothesis the integral containing the error term converges as $t \to \infty$. As the right-hand side converges, so does the left-hand side. That is,

$$(5.3)\qquad \begin{aligned} &(dJ/dt)/J^{1/4} \sim l > 0, \quad \text{or} \\ &J^{3/4} \sim lt \Rightarrow J \sim At^{4/3} \quad \text{for some positive constant } A. \end{aligned}$$

From (5.3) we have $dJ/dt \sim Bt^{1/3}$ where B is some positive constant. Integrating both sides and comparing with $J \sim At^{4/3}$ shows $dJ/dt \sim (4/3)At^{1/3}$. This implies (from (5.1) and (4.8)) $d^2J/dt^2 = V + o(t^{-2/3})$ that

$$\int_{t_1}^{t} V(u)\,du \sim \frac{4}{3}At^{1/3}.$$

We would like to differentiate both sides of this relationship to obtain the desired result that $V(t) \sim (4/9)At^{-2/3}$. By a well-known Tauberian theorem, this can be done if $dV/dt = O(t^{-5/3})$. But as this is the case, (by Lemma 1) the theorem is proved.

COROLLARY 2.3. *Under the conditions of Theorem* 2,

$$\frac{d}{dt}\left(\frac{\boldsymbol{\rho}_s}{t^{2/3}}\right) = o(t^{-1}).$$

Proof. By definition of the terms

$$(5.4)\qquad \frac{1}{2}\sum M_s\left(\frac{d}{dt}\frac{\boldsymbol{\rho}_s}{t^{2/3}}\right)^2 = \frac{E}{t^{4/3}} - \frac{2}{3}\frac{dJ/dt}{t^{7/3}} + \frac{4}{9}\frac{J}{t^{10/3}}.$$

As $E = V + o(V)$, the conclusion of Theorem 2 implies that the right-hand side is $o(t^{-2})$. As the left-hand side is the sum of positive quantities, the proof is completed.

In the case of subsystems, we would like to state $\boldsymbol{\rho}_s \sim C_s t^{2/3}$. While we are unable to prove this, we can obtain a result of equal interest, namely $\boldsymbol{\rho}_s/t^{2/3}$, $s = 1, 2, \ldots, p$, asymptotically approach the vertices of central configurations.

DEFINITION [21, p. 273]. We say that $\boldsymbol{r}_i$, $i = 1, \ldots, n$, forms a central configuration at time t_1 if, for all i, $\lambda \boldsymbol{r}_i(t_1) = d^2\boldsymbol{r}_i(t_1)/dt^2$ where λ is a constant independent of i. That is, $\lambda m_i \boldsymbol{r}_i = \partial U/\partial \boldsymbol{r}_i$ at time t_1.

For example, if $n = 3$, there are 3 collinear solutions (depending on the arrangement of the masses) and one noncollinear central configuration—an equilateral triangle [21, pp. 274–277].

THEOREM 3. *Under the hypothesis of Theorem* 2 *as* $t \to \infty$, $\boldsymbol{\rho}_s t^{-2/3}$ *asymptotically approaches the vertices of some central configuration.*

(Again a result similar to this exists for a complete collapse of the n-body problem [21, pp. 280–282]. With the new additional information available on collisions [10], [11] Wintner's proof carries over with only minor modifications to show that for any collision in the n-body problem the participating particles must tend asymptotically to some central configuration. However, again because of possible bounded and oscillatory motion, Wintner's proof cannot be directly generalized to prove Theorem 3.)

Proof. What we show is, for $\boldsymbol{R}_s = \boldsymbol{\rho}_s t^{-2/3}$,

$$\frac{2}{9}\boldsymbol{R}_s - \frac{1}{M_s}\frac{\partial V}{\partial \boldsymbol{R}_s} \to 0 \quad \text{as } t \to \infty, s = 1, 2, \ldots, p.$$

From (4.1),

$$\begin{aligned}\frac{d^2\boldsymbol{\rho}_s}{dt^2} &= \frac{dt}{dt^2}(\boldsymbol{R}_s t^{2/3}) = \frac{d^2\boldsymbol{R}_s}{dt^2}t^{2/3} + \frac{4}{3}\frac{d\boldsymbol{R}_s}{dt}t^{-1/3} - \frac{2}{9}\boldsymbol{R}_s t^{-4/3} \\ &= \frac{1}{M_s}\left(\frac{\partial V}{\partial \boldsymbol{R}_s}\right)t^{-4/3} + o(t^{-4/3})\end{aligned}$$

where $\partial V/\partial \boldsymbol{R}_s$ is defined by

$$\sum \frac{M_s M_k(\boldsymbol{R}_k - \boldsymbol{R}_s)}{R_{ks}^3}.$$

Note $\partial V/\partial \boldsymbol{R}_s \approx 1$, i.e., R_{sk} is bounded away from zero as $t \to \infty$ for all s, k, $s \neq k$.

By Corollary 2.3, the above can be expressed as

$$\frac{d^2\boldsymbol{R}_s}{dt^2}t^2 - \frac{2}{9}\boldsymbol{R}_s = \frac{1}{M_s}\frac{\partial V}{\partial \boldsymbol{R}_s} + o(1).$$

Integrating from t_1 to t,

$$\int_{t_1}^{t}\frac{d^2\boldsymbol{R}_s}{dt^2}u^2\,du = \int_{t_1}^{t}\left(\frac{2}{9}\boldsymbol{R}_s + \frac{1}{M_s}\frac{\partial V}{\partial \boldsymbol{R}_s}\right)du + o(t). \tag{5.5}$$

Integrating the left-hand side by parts,

$$\int_{t_1}^{t}\frac{d^2\boldsymbol{R}_s}{dt^2}u^2\,du = u^2\frac{d\boldsymbol{R}}{dt}(u)\Big|_{t_1}^{t} - 2\int_{t_1}^{t}\frac{d\boldsymbol{R}_s}{dt}u\,du.$$

Again by Corollary 2.3, the right-hand side of the above is $o(t)$. Hence (5.5) can be expressed as

$$\int_{t_1}^{t}\left(\frac{2}{9}\boldsymbol{R}_s + \frac{1}{M_s}\frac{\partial V}{\partial \boldsymbol{R}_s}\right)du = o(t).$$

We would like to differentiate both sides of the above expression to obtain the conclusion

$$\frac{2}{9}\boldsymbol{R}_s + \frac{1}{M_s}\frac{\partial V}{\partial \boldsymbol{R}_s} \to \boldsymbol{0} \quad \text{as } t \to \infty.$$

But again from the Tauberian theorem $g(t) = o(t)$ and $d^2g(t)/dt^2 = O(t^{-1}) \Rightarrow dg(t)/dt = o(1)$, this can be done if

$$\frac{2}{9}\frac{d\boldsymbol{R}_s}{dt} + \frac{1}{M_s}\frac{d}{dt}\left(\frac{\partial V}{\partial \boldsymbol{R}_s}\right) = O(t^{-1}).$$

That this is so follows from Corollary 2.3, and the fact $R_{ij} \approx 1$. This completes the proof.

As stated earlier, we would like to have $\boldsymbol{\rho}_s \sim C_s t^{2/3}$. Clearly a sufficient condition that this is the case would be that the square root of the right-hand side of (5.4) is integrable. However this refined information on the behavior of E, J, and dJ/dt is missing both for this problem and the problem of collision [10], [11].

What we can show is that in certain cases $\rho_s \sim C_s t^{2/3}$ where C_s is a nonnegative constant.

THEOREM 4. *We assume the hypothesis of Theorem 2. For those values of p and M_s, $s=1,\ldots,p$, for which only a finite number of central configurations exist, $\rho_s \sim C_s t^{2/3}$ where C_s is some nonnegative constant. At most one $C_s=0$.*

This proof and the observation of a finite number of central configurations is essentially that of Wintner [21, p. 282] as applied to complete collapse of the system. As the proof carries over directly, we simply outline the details.

Proof. We first show that at most one $C_s=0$. Assume $\rho_s \sim C_s t^{2/3}$. As $|\boldsymbol{\rho}_s - \boldsymbol{\rho}_k| \leqq \rho_s + \rho_k = (C_s + C_k)t^{2/3} + o(t^{2/3})$, if C_s and C_k are both zero for some s and k, $s \neq k$; then $\rho_{sk} = o(t^{2/3})$. But this is a contradiction to the fact $V \approx t^{-2/3}$.

As we have only a finite number of central configurations, the $\boldsymbol{R}_s$ must converge to *one* central configuration as $t \to \infty$. Hence $|\boldsymbol{R}_i - \boldsymbol{R}_j| = \lambda(t)C_{ij} + O(1)$, where C_{ij} is the distance between vertices i and j. We first show that $\lambda(t)$ can be chosen as a constant. As we are assuming that the vectors $\boldsymbol{\rho}_s$, $s=1,\ldots,p$, are expressed relative to the common center of mass,

$$J = \frac{1}{2M} \sum_{1 \leqq i < j \leqq p} M_i M_j (\boldsymbol{\rho}_i - \boldsymbol{\rho}_j)^2 + o(t^{4/3})$$

where M is the total mass. As $J \sim At^{4/3}$ we have

$$\begin{aligned} Jt^{-4/3} &= \frac{1}{2M} \sum M_i M_j (\boldsymbol{R}_i - \boldsymbol{R}_j)^2 + o(1) \\ &= \frac{\lambda(t)^2}{2M} \sum M_i M_j C_{ij}^2 + o(1) \sim A. \end{aligned}$$

Hence $\lambda(t)^2$ is a constant plus terms $o(1)$. As $\lambda(t)$ is continuous and nonzero, $(V \approx t^{-2/3})\lambda(t)$ can clearly be chosen as a constant.

Hence the vectors $\boldsymbol{R}_i$ are asymptotic to the vertices of a rigid body which may be rotating about its center of mass. But as the center of mass of this body is fixed, it is a simple matter to show that $|\boldsymbol{R}_i(t)| \to C_i$ as $t \to \infty$. The C_i are nonnegative constants.

COROLLARY 2.4. *If $\int_{t_1}^{t} (e_1(u) + e_2(u)) J^{-1/4}\, du = O(1)$ as $t \to \infty$, then the conclusions of Corollary 2.3 and Theorem 3 follow.*

Proof. The problem is to find new estimates for J, dJ/dt and d^2J/dt^2 to substitute into (5.4). With this error term and following the proof of Theorem 2, we still obtain the statement

$$\int_{t_1}^{t} \frac{4JE-(dJ/dt)^2}{J^{5/4}} = B+o(1).$$

We would like to show

$$\frac{4JE-(dJ/dt)^2}{J^{5/4}} = o(t^{-1}).$$

The above estimates for J, dJ/dt, E, V, d^2J/dt^2 and dE/dt (from Lemma 1) imply

$$\begin{aligned}\frac{d}{dt}\left(\frac{4JE-(dJ/dt)^2}{J^{5/4}}\right) &= \frac{4(dJ/dt)E+4J\,dE/dt-2J\,d^2J/dt^2}{J^{5/4}}-\frac{5}{4}\frac{4JE-(dJ/dt)^2}{J^{9/4}}\\ &= O(t^{-2}).\end{aligned}$$

To summarize we have a function $g \in C^2(0,\infty)$ such that $g=o(1)$ and $d^2g/dt^2 = O(t^{-2})$. But by the Tauberian theorem this yields $dg/dt=o(t^{-1})$. Hence

$$\frac{4JE-(dJ/dt)^2}{J^{5/4}} = o(t^{-1}) \quad \text{or} \quad 4JE-(dJ/dt)^2 = o(t^{2/3}).$$

By (5.1) and (4.4) $d^2J/dt^2=E+o(t^{-2/3})$, so

$$4J\,d^2J/dt^2 = (dJ/dt)^2+o(t^{2/3}).$$

As $J\,d^2J/dt^2$ and $(dJ/dt)^2 \approx t^{2/3}$,

$$4J\,d^2J/dt^2 \sim (dJ/dt)^2.$$

As $dJ/dt \approx t^{1/3}$, it is positive and

(5.6) $$4(d^2J/dt^2)/(dJ/dt) \sim (dJ/dt)/J.$$

Now

$$\begin{aligned}d((dJ/dt)/J)/dt &= (4J d^2J/dt^2-(dJ/dt)^2)J^{-2}-3(d^2J/dt^2)J^{-1}\\ &= -3(d^2J/dt^2)J^{-1}+o(t^{-2}).\end{aligned}$$

From (5.6) $4(d^2J/dt^2)J^{-1} \sim ((dJ/dt)J^{-1})^2$, or by defining $e(t)=(dJ/dt)J^{-1}$, $de/dt=-\frac{3}{4}e^2+o(t^{-2})$ and $e^2 \approx t^{-2}$. This implies that $(de/dt)e^{-2}=-\frac{3}{4}+o(1)$ or $-e^{-1}|_{t_0}^{t}=-\frac{3}{4}(t-t_0)+o(t)$.
That is,

(5.7) $$J/(dJ/dt) \sim \tfrac{3}{4}t.$$

By (5.6), $(d^2J/dt^2)/(dJ/dt) \sim (3t)^{-1}$.

From (5.7) we have that $\ln J = \ln t^{4/3}+o(\ln t)$ or that $J=h(t)t^{4/3}$. As $J \approx t^{4/3}$, $h(t) \approx 1$. By (5.7) $dJ/dt \sim (4/3)t^{1/3}h(t)$. In the same fashion,

$$d^2J/dt^2 \sim (4/9)t^{-2/3}h(t).$$

With this information the proof of Corollary 2.3 follows. (These estimates are substituted into (5.4).) The proof of Theorem 3 depends upon the validity of Corollary 2.3, and so the conclusion of Theorem 3 still holds.

Note that from (5.7) $t(dh/dt)h^{-1} \to 0$. From this it follows that $h(t)$ is a slowly varying function in the sense of Karamata, i.e., $h(\beta t)/h(t) \to 1$ for constant $\beta > 0$. This does not imply that $h(t) \to A$ as the example $2+\cos(\ln \ln t)$ shows. So a remaining question is under what additional conditions can one say that $h(t) \to A$, i.e., that the conclusion of Theorem 2 holds?

COROLLARY 2.5. *Under the hypothesis of Corollary* 2.4, *if the* $\boldsymbol{R}_s$ *approach the vertices of one central configuration, then* $J \sim At^{4/3}$ *and* $V \sim (4/9)At^{-2/3}$.

Proof. As the $\boldsymbol{R}_s$ converge to *one* central configuration, $|\boldsymbol{R}_i - \boldsymbol{R}_j| = \lambda(t)C_{ij} + o(1)$, where $\lambda(t) \approx 1$. Hence $Jt^{-4/3} = (1/2M) \sum^* M_i M_j C_{ij}^2 \lambda(t)^2 + o(1) = h(t)$ or $\lambda(t)^2 B \sim h(t)$ where B is a positive constant. We showed in the proof of Corollary 2.4 that $d^2J/dt^2 \sim (4/9)t^{-2/3}h(t)$, or that $V \sim (4/9)t^{-2/3}h(t)$. That is

$$t^{2/3}V = \sum{}^* \frac{M_i M_j}{\lambda(t)C_{ij} + o(1)} \sim \frac{4}{9} h(t),$$

or $\lambda(t)^{-1} D \sim h(t)$. This implies that $\lambda^3 \sim D/B$ or that $\lambda(t)$ is asymptotic to a constant. This in turn implies that $h(t) \sim A$ where A is some positive constant and the proof is completed.

6. **Pulsating motion.** It seems to be questionable whether "pulsating motion" exists. The nonexistence of an energy relationship may simply be a technical difficulty which has not been surmounted by the present technique. However, as this is an open question, this section will consider some results which give some flavor to the notion. No attempt will be made to provide an exhaustive study nor to obtain the sharpest possible results.

An investigation of (3.1) as to what may cause the nonexistence of a conservation of energy relationship leads to the tentative conclusion that there must exist either a strong rotational and/or a pulsating action (in the sense of continual contractions and expansions). This is partially confirmed by the next theorem which states, as a special case, that if for *some* mutual distance, say ρ_{12}, that $d\rho_{12}/dt$ is eventually nonnegative and eventually the magnitudes of the velocities are of the same order as $d\rho_{12}/dt$ ($|d\boldsymbol{\rho}_s/dt| = O(d\rho_{12}/dt)$, $s = 1, \ldots, p$) then we *do not* have pulsating motion.

Let $K(\alpha) = \sum^* a_{ij}\rho_{ij}^{\alpha}$ where α is a nonzero constant and the a_{ij} are nonnegative constants, not all zero. The summation is $1 \leqq i < j \leqq p$. Note that $K(\alpha)^{1/\alpha} \approx f(t) \approx V^{-1}$.

THEOREM 5. *If there exists some* $K(\alpha)$ *such that* $\alpha\, dK/dt$ *is eventually nonnegative and*

$$\left|\frac{d\boldsymbol{\rho}_s}{dt}\right| = O(K^{(1-\alpha)/\alpha}\, dK/dt) \quad \textit{for } s = 1, \ldots, p,$$

then $E = V + H + o(V)$ *where* H *is a nonnegative constant.*

Notice that a natural choice for $K(2)$ would be J and for $K(-1)$ would be V. A choice for $K(1)$ would be ρ_{12}.

Proof. From (3.1)

$$M_s \frac{d^2\boldsymbol{\rho}_s}{dt^2} = \frac{\partial V}{\partial \boldsymbol{\rho}_s} + o(K^{-2/\alpha})$$

and

$$\begin{aligned}\frac{dE}{dt} &= \sum M_s \frac{d\boldsymbol{\rho}_s}{dt}\cdot\frac{d^2\boldsymbol{\rho}_s}{dt^2} = \sum \frac{\partial V}{\partial \boldsymbol{\rho}_s}\cdot\frac{d\boldsymbol{\rho}_s}{dt} + o\left(\left|\frac{d\boldsymbol{\rho}_s}{dt}\right| K^{-2/\alpha}\right)\\ &= \frac{dV}{dt} + o\left(\left|\frac{dK}{dt}\right| K^{(-1-\alpha)/\alpha}\right).\end{aligned}$$

Integrating from t_1 to t where t_1 is large enough so that dK/dt is of one sign for $t \geqq t_1$,

$$E(t) = V(t) + A + o(K^{-1/\alpha}) = V(t) + A + o(V(t))$$

where A is a constant of integration.

If A is negative, then as $E \geqq 0$,

$$|A| \leqq V(t) + o(V(t)) \approx f(t)^{-1},$$

which implies that $f(t) = O(1)$. As this is a contradiction, A is nonnegative. This implies

$$E/V = 1 + AV^{-1} + o(1).$$

As the left-hand side is eventually greater than $\frac{1}{2}$, the conclusion of the theorem follows from Corollary 1.2.

By using the fact $E = (1/2M)\sum^* M_i M_j (d\boldsymbol{\rho}_i/dt - d\boldsymbol{\rho}_j/dt)^2$, the above proof can be modified to replace the condition $|d\boldsymbol{\rho}_s/dt| = O(K^{(1-\alpha)/\alpha}\, dK/dt)$ with $|d\boldsymbol{\rho}_i/dt - d\boldsymbol{\rho}_j/dt| = O(K^{(1-\alpha)/\alpha}\, dK/dt)$.

A rough estimate on the growth properties of such motion can be readily found by making a slight assumption, which is motivated by lim inf $E/V \leqq \frac{1}{2}$.

Theorem 6. *If $E = O(V)$ then $f(t) = O(t^{2/3})$.*

Proof. From $E = (1/2M)\sum^* M_s M_j (d\boldsymbol{\rho}_s/dt - d\boldsymbol{\rho}_j/dt)^2$ and $\rho_{sj} \approx f(t) \approx V^{-1}$, it follows that

$$\rho_{sj} E = O(1) \quad \text{or} \quad \rho_{sj} |d\boldsymbol{\rho}_s/dt - d\boldsymbol{\rho}_j/dt|^2 = O(1).$$

This implies $|\rho_{sj}^{1/2}\, d\rho_{sj}/dt| = O(1)$ or that $\rho_{sj}^{3/2} = O(t)$. That is, $\rho_{sj} = O(t^{2/3})$ which implies the conclusion of the theorem.

This upper bound is given some credence by the following:

Theorem 7. *In pulsating motion, $\int_a^\infty f(s)^{-3/2}\, ds = \infty$.*

Proof. As $V^{-1} \approx J^{1/2} \approx f(t)$, this is equivalent to showing that $\int_a^\infty V/J^{1/4}\, dt = \infty$.

From the proof of Theorem 2, (§5)

$$4\frac{d}{dt}\left(\frac{dJ/dt}{J^{1/4}}\right)+\frac{V+e_1}{J^{1/4}} = \frac{4JE-(dJ/dt)^2}{4J^{5/4}}+\frac{E}{J^{1/4}} \geqq 0.$$

Hence

$$g(t) = \frac{dJ/dt}{J^{1/4}}+\int_a^t \frac{V+e_1}{J^{1/4}}\,dt$$

is a monotonically nondecreasing function, that is $g(t)\to l$ as $t\to\infty$ where l is finite or $l=\infty$. First we show that $l=\infty$.

Assume that $|l|<\infty$. As the integrand is eventually positive ($e_1=o(V)$), $\int_a^t (V+e_1)/J^{1/4}\,dt\to l_2$. If l_2 is finite, then $(dJ/dt)J^{-1/4}\to l-l_2$, or $J^{3/4}\sim(l-l_2)t$. Note that $l-l_2\geqq 0$. As l, l_2 are finite, this implies $f(t)=O(t^{2/3})$, i.e., after some time there exists some positive constant C such that $C/t^{2/3}\leqq f(t)^{-1}$, or

$$C^{3/2}\frac{\ln t}{a} \leqq \int_a^t f(t)^{-3/2} \approx \int_a^t \frac{V+e_1}{J^{1/4}}\,dt.$$

But this is a contradiction to the convergence of the integral on the right. Hence $\int_a^t (V+e_1)/J^{1/4}\,dt\to\infty$. This implies that $(dJ/dt)J^{-1/4}\to-\infty$, or that dJ/dt is eventually negative. However this contradicts the fact J is unbounded and positive, hence we have that $l=\infty$.

For $l=\infty$, either $\int_a^t V/J^{1/4}\,dt\to\infty$ or it does not. If this integral converges, then $(dJ/dt)J^{-1/4}\to\infty$ as $t\to\infty$. This means that eventually $(dJ/dt)J^{-1/4}>1$, or $J^{3/4}>\frac{3}{4}t$, i.e. there exists a positive constant B such that eventually $f(t)>Bt^{2/3}$. But from the proof of Theorem 1, and Corollary 1.2, this implies that an energy relationship exists, contrary to the assumption of "pulsating" motion. Hence the theorem is proved.

7. **Separation of subsystems and oscillatory motion.** A refinement for the separation of subsystems is possible as the following theorem shows.

THEOREM 8. *If* $\boldsymbol{\rho}_s=\boldsymbol{C}_st+\boldsymbol{D}_s\ln t+o(\ln t)$, $s=1,\dots,p$, *and either oscillatory and pulsating motion do not occur or the mutual distances between particles participating in this motion with indices in some* G_s *is* $O(t^{2/3})$, *then* $\boldsymbol{\rho}_s=\boldsymbol{C}_st-\boldsymbol{D}_s\ln t+\boldsymbol{E}_s+O(t^{-1/3})$.

Proof. The first half of this proof is the same as the proof of Theorem 1. The hypothesis implies, by means of analysis similar to that employed in the proof of Corollary 2.2, that the error term in (3.1) is $O(t^{-7/3})$. Substituting $\boldsymbol{\rho}_s=\boldsymbol{C}_st+O(\ln t)$ into (3.1) yields

$$d^2\boldsymbol{\rho}_s/dt^2 = \boldsymbol{D}_st^{-2}+O((\ln t)/t^3)+O(t^{-7/3}) = \boldsymbol{D}_st^{-2}+O(t^{-7/3})$$

or

$$d\boldsymbol{\rho}_s/dt = \boldsymbol{C}_s-\boldsymbol{D}_st^{-1}+O(t^{-4/3}).$$

Integrating from t_1 to t_2, $t_1<t_2$,

$$(\boldsymbol{\rho}_s(t_2)-\boldsymbol{C}_st_2+\boldsymbol{D}_s\ln t_2)-(\boldsymbol{\rho}_s(t_1)-\boldsymbol{C}_st_1+\boldsymbol{D}_s\ln t_1) = O(t_2^{-1/3}-t_1^{-1/3}).$$

As t_1, $t_2 \to \infty$, the right-hand side approaches zero, carrying the left-hand side to zero with it. Hence, by the Cauchy criterion for the existence of a limit, the conclusion of the theorem follows.

Actually, in the absence of oscillatory and pulsating motion, the evolution of the n-body problem as $t \to \infty$ is quite well behaved. The clusters form subsystems and (under certain hypotheses) asymptotically separate like $t^{2/3}$, possibly in pinwheel fashion, to the vertices of expanding central configurations. The center of mass of these subsystems separate as described in Theorem 8. With additional hypothesis, oscillatory and pulsating motion can be included in this sketch in a straightforward fashion. Various combinations of the classifications give all possible motion for the n-body problem as $t \to \infty$.

It follows from the above that many of the outstanding questions about the n-body problem are reduced to a study of oscillatory and/or pulsating motion. For example, is it true that for all initial conditions $I = O(t^2)$ as $t \to \infty$, i.e., $\boldsymbol{r}_i = O(t)$ for all i [7, p. 604]?

It has recently been shown by Saari [13] that for the three-body problem, this is true. Clearly for the n-body problem this condition can be violated only by oscillatory or pulsating motion.

THEOREM 9. *$I = O(t^2)$ as $t \to \infty$, if, and only if, the mutual distance between any particles participating in oscillatory and pulsating motion is $O(t)$.*

Proof. This follows directly from $I = (1/2M) \sum_{1 \le i < j \le n} m_i m_j (\boldsymbol{r}_i - \boldsymbol{r}_j)^2$ and Theorem 1. M is the total mass of the system.

Another open question is whether or not $r(t)$, the minimum distance between particles (i.e., $r(t) = \min r_{ij}(t)$), can approach zero as $t \to \infty$ [7, p. 603].

COROLLARY 8.1. *If $r(t) \to 0$ as $t \to \infty$, then there exists oscillatory motion and particles participating in the same oscillatory motion with indices i and j such that* $\limsup r_{ij}(t) t^{-1} = \infty$ *as $t \to \infty$.*

Proof. Pollard has shown [7, p. 605] that this condition implies $I/t^2 \to \infty$ as $t \to \infty$. If $I/t^2 \to \infty$ could be explained solely in terms of pulsating motion, then $\rho t^{-1} \to \infty$ or $1/f(t) = o(t^{-1})$. This contradicts Theorem 7. Hence the conclusion follows.

In the absence of oscillatory and pulsating motion, the behavior of I can be approximated by use of the classification of motion.

THEOREM 10. *In the absence of oscillatory and pulsating motion, I has one of the following forms.*

(a) $I = At^2 + Bt^{4/3} + o(t^{4/3})$.

(b) $I = At^2 + Bt \ln t + O(t)$.

(c) $I = At^{4/3} + o(t^{4/3})$.

(d) $I = O(1)$.

A and B denote positive constants, not necessarily the same with each usage.

Proof. By definition $I=\frac{1}{2}\sum m_i r_i^2$. Each r_i belongs to some subsystem, the center of mass given by P_α, and to some cluster. The center of mass of this cluster relative to P_α is $\rho_{\alpha\mu}$ and $r_i=P_\alpha+\rho_{\alpha\mu}+O(1)$. That is

$$\begin{aligned} I &= \tfrac{1}{2}\sum m_i(P_\alpha^2+2P_\alpha\cdot\rho_{\alpha\mu}+\rho_{\alpha\mu}^2)+O(P_\alpha) \\ &= \tfrac{1}{2}\sum_\alpha\sum_{\mu\in G_\alpha} M_{\alpha\mu}(P_\alpha^2+2P_\alpha\cdot\rho_{\alpha\mu}+\rho_{\alpha\mu}^2)+O(P_\alpha) \end{aligned}$$

where the first sum is over the indices of the clusters in a given subsystem and the second is over all the subsystems. $M_{\alpha\mu}$ is the mass of the cluster given by $\rho_{\alpha\mu}$. Note that $\sum\sum M_{\alpha\mu}P_\alpha\cdot\rho_{\alpha\mu}=\sum M_\alpha P_\alpha^2$ where M_α is the total mass of the subsystem corresponding to P_α. Hence

$$I = \tfrac{1}{2}\sum M_\alpha P_\alpha^2+\sum\sum M_{\alpha\mu}\rho_\mu^2+O(P_\alpha).$$

If there are at least two subsystems and at least two clusters in some subsystem, then (a) follows from Theorems 1 and 3. If there are at least two subsystems and only one cluster per subsystem, then $\rho_{\alpha\mu}=\mathbf{0}$ and (b) follows. To complete (b) it remains to show that B is a positive constant. This follows from (4.1) and (4.8) where $H>0$, i.e., $d^2J/dt^2=V+2H+o(V)$. Note in this case $V\sim B/t$ where B is some positive constant. Hence $J=Ht^2+B\ln t+o(\ln t)$.

If there is only one subsystem and at least two clusters then $P_\alpha=0$ and (c) follows. If there is only one subsystem and one cluster, then (d) follows.

Of course by refining the assumptions on the type of existing motion both Theorems 8 and 10 can be extended. But as long as bounded motion is permitted the results are marginal and omitted here.

An example of how Theorem 10 can be used is given here.

COROLLARY 10.1. *If U is quasi-periodic then either I is quasi-periodic or $I=Ct^2+Dt+O(1)$ and oscillatory motion exists. C is a positive constant.*

Proof. A singularity at $t=t_1$ can occur if and only if $U\to\infty$ as $t\to t_1$ [21, p. 326]. As U is quasi-periodic, $U=O(1)$ for all t, hence the solution exists for all time. As U is quasi-periodic, $d^2I/dt^2=U+2h$ is quasi-periodic and $dI/dt=2Ct+$quasi-periodic motion. Hence, $I=Ct^2+Dt+$quasi-periodic motion. Note $C\geqq 0$. If $C\neq 0$ then from Theorem 10 oscillatory and/or pulsating motion exist. If only pulsating motion existed, then $f(t)\approx t$, contradicting Theorem 7. If $C=0$, then $D=0$, otherwise $I\to-\infty$ as t approaches either $+\infty$ or $-\infty$ (depending on the sign of D). This would contradict the existence of the solution and the definition of I ($\geqq 0$).

The question of escape from systems with nonnegative energy [11] is another problem reduced to the study of oscillatory motion.

COROLLARY 10.2. *For $h>0$, if oscillatory and pulsating motion do not occur and $U\to 0$ as $t\to\infty$, then $U\sim A/t^{2/3}$ or $U\sim B/t$. A and B are some positive constants. If, in addition, in all the subsystems all the clusters approach the vertices of a central* *configuration, then at least $n-1$ particles escape, i.e., for at least $n-1$ values of the indices, $\mathbf{r}_i \to \infty$ as $t \to \infty$.*

Proof. As $U \to 0$, all $r_{ij} \to \infty$ and each cluster has only one particle. In an obvious fashion $U = \sum V_\mu + o(V_\mu)$ where the sum is over all the subsystems and V_μ is the generalized self-potential of the subsystem. If any of the subsystems has at least two clusters, then $U \sim A/t^{2/3}$ (from Theorem 2). If all subsystems have only one cluster, then $U \sim A/t$. The proof of the escape statement follows from Theorems 2 and 3.

It is a conjecture of the author that for some $\mathbf{r}_i$ and $\mathbf{r}_j$ participating in oscillatory motion, lim inf $r_{ij} < \infty$ as $t \to \infty$. If this is true, then the condition $U \to 0$ automatically excludes oscillatory motion.

8. **Inverse q law.** In the classification of motion the exponents on t are 2/3 and 1. The mechanism that leads to these values is the force law. To see this we consider the force law r^{-q}, where $1 < q < 3$. The value $q = 2$ is the Newtonian force law. The inverse q central force law leads to a similar classification of motions except that $t^{2/3}$ is replaced with $t^{2/q+1}$. The central configuration results hold also.

The only real differences in the proof are that $V = \sum (M_i M_j / \rho_{ij}^{q-1})$ and (4.1) becomes

$$d^2J/dt^2 = 2E + (1-q)V + o(f(t)^{1-q}) = (3-q)V + 2H + o(f(t)^{1-q}).$$

Pulsating motion is now defined to be the case where lim inf $E/V \leqq (q-1)/2$. However, with the above restrictions on q, all of the proofs are essentially the same. With q outside of this range modifications are necessary and will not be discussed here.

It also turns out that the problem of collision can be generalized from the inverse square law. If a collision occurs as $t \to 0+$ then the colliding particles approach each other like $t^{2/3}$. In the inverse q law, the colliding particles approach each other like $t^{2/(q+1)}$. Again the details of the proof for the case $q = 2$ [9] can be generalized with only minor modifications.

9. **A remark and extensions.** Care was taken in the proofs of the theorems to allow the greatest latitude for the error term. It essentially turns out that the error term can be "almost r^{-2}". Hence these results are directly applicable to a wider class of problems than simply mass particles subjected to the inverse square law. The results are equally valid in central force systems where the inverse square term is the dominant term for large values of r_{ij}. That is, it applies to models which include oblateness effects, nongravitational forces for close encounters, or even a crude approximation to the theory of relativity where the force law is assumed to be $ur^{-2} + Br^{-4}$ [6, p. 88]. B is some constant.

It is interesting to note that vectors describing the separation of clusters in a subsystem and vectors describing the separation of subsystems have the property

$$|d\rho_{ij}/dt| / \rho_{ij} \approx t^{-1}.$$

This is the Newtonian version of "Hubble's constant".

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