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EXPANDING RICCI SOLITONS WITH PINCHED RICCI CURVATURE

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Abstract

In this paper, we prove that there is no expanding gradient Ricci soliton with (positively) pinched Ricci curvature in dimension three. The same result is true for higher dimensions with the extra decay condition about the full curvature.

1. Introduction

In this paper we consider Problem 9.62 in the famous book [2], which is also the unanswered question in [6]. Namely, when $n \geq 3$, do there exist expanding gradient Ricci solitons with (positively) pinched Ricci curvature? In dimension three, we settle the problem completely. Here we recall that the (positively) pinched Ricci curvature for the Riemannian manifold (M^n, g) is in the sense that

$$Rc \geq \varepsilon R g \geq 0, \tag{1}$$

where R and Rc are the scalar and Ricci curvatures of the metric g, $\varepsilon > 0$ is a small constant. This concept plays an important role in the seminal work of R. Hamilton [3]. We remark that the compact expanding gradient Ricci solitons are Einstein. This result is known in G. Perelman [7]. Then we may assume that the expanding gradient Ricci soliton $(M^n, g(t), \phi)$ under consideration is complete, non-compact, and in canonical form that

$$D^2\phi = Rc + \frac{1}{2t} g, \quad t > 0.$$

We denote by $d_g(x, o)$ the distance between the points x and o in (M, g).

We show that there are only trivial ones.

Theorem 1. *There is no expanding gradient Ricci soliton $(M, g(t))$, $t > 0$, with (positively) pinched Ricci curvature and curvature decay at the order $d_{g(t)}(x, o)^{-2-\delta}$ for any $\delta > 0$.*

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We remark that in dimension three, the curvature decay condition is automatically true [6]. This result has been used in [5]. Since we are studying the Ricci flow, we should have the curvature decay order as that of the Ricci curvature, at least with some curvature pinching condition.

This paper is organized as follows. In section 2 we recall some famous results, which will be in use in section 3. Theorem 1 is proved in section 3.

We shall use r denote various uniform positive constants.

2. Preliminary

Before we prove our main result Theorem 1, we cite the following results, which will be in use in next section. The first is

Proposition 2 (Hamilton, Proposition 9.46 in [2]). *If* $(M, g(t))$, $t > 0$, *is a complete non-compact expanding gradient Ricci soliton with* $Rc > 0$, *then*

$$AVR(g(t)) := \lim_{r \to \infty} \frac{Vol(B_{g(t)}(o, r))}{r^n} > 0,$$

where the definition of $AVR(g(t))$ *is independent of the base point* $o \in M$.

The second is

Proposition 3 ([6]). *If* $(M, g(t))$, $t > 0$, *is a complete non-compact expanding gradient Ricci soliton with* (1), *then the scalar curvature is quadratic exponential decay.*

The third one is Theorem 1.1 in [1]. Roughly speaking, the result says that for the complete and non-compact (M, g), if $AVR(g) > 0$ and the curvature decay suitably, then it is an asymptotic manifold. We invite the readers to papers [1] and [4] for the definitions of asymptotic flat manifold M_τ and coordinates $(z) = (z^i)$ at infinity (also called the asymptotic coordinates).

The last one is Proposition 10.2 in [4]. Namely,

Proposition 4 ([4]). *If* (M, g) *is asymptotic flat with* $g \in \mathbf{M}_\tau$, *for some* $\tau > \dfrac{n-2}{2}$, *and the Ricci curvature is non-negative. Then the mass*

$$m(g) := \lim_{r \to \infty} \int_{S_r} \mu \lrcorner dz$$

is non-negative, with $m(g) = 0$ *if and only if* (M, g) *is isometric to* R^n *with its Euclidean metric. Here* $S_r : \{x \in M; d_g(o, x) = 1\}$,

$$\mu = (\partial_i g_{ij} - \partial_j g_{ii})\partial_j, \quad \partial_j = \frac{\partial}{\partial z^j},$$

and (z^j) *are the asymptotic coordinates.*

3. Proof of Theorem 1

We argue by contradiction. So we assume that $(M, g(t))$ is not flat.

Using the strong maximum principle [8] to the Ricci soliton, we may assume that the scalar curvature is positive, i.e., $R > 0$. Hence we know that $Rc > 0$. According to the arguments in [2] and [6], we know that ϕ is a proper strict convex function, which implies by using the Morse theory that M^n is diffeomorphic to R^n. Using Proposition 2, Proposition 3 (in dimension three case), and Theorem 1.1 in [1] we know that $(M, g(t))$ is an asymptotic flat manifold. We also know that

$$\phi(x) \approx d_g(x,o)^2, \quad |\nabla\phi(x)| \approx d_g(x,o).$$

Recall that in coordinates (z^j) at infinity, we have the Ricci soliton equation

$$\phi_{ij} = R_{ij} + \frac{1}{2t} g_{ij}.$$

For notation simple, we let $t = 1/2$. Then we have $g_{ij} = \phi_{ij} - R_{ij}$. By Ricci pinching condition we know that R_{ij} decay exponentially and

$$\Delta\phi = \phi_{ii} = R + n.$$

In the computation below, we shall use the Ricci formula

$$\phi_{iji} = \phi_{iij} + R_{ji}\phi_i$$

and the soliton relation

$$\nabla_j R = -2R_{ij}\phi_j.$$

Note that ϕ grows quadratically in distance and the growth of volume of the metric ball is of polynomial type (see the proof of Proposition 9.46 in [2]). We now compute the mass. We have

$$\begin{aligned}
m(g) &= \lim_{r\to\infty} \int_{S_r} (\partial_i g_{ij} - \partial_j g_{ii})\partial_j \lrcorner dz \\
&= \lim_{r\to\infty} \int_{S_r} (\partial_i \phi_{ij} - \partial_j \phi_{ii})\partial_j \lrcorner dz \\
&= \lim_{r\to\infty} \int_{S_r} \phi_{iji}\partial_j \lrcorner dz \\
&= \lim_{r\to\infty} \int_{S_r} (\phi_{iij} + R_{ij}\phi_i)\partial_j \lrcorner dz \\
&= \lim_{r\to\infty} \int_{S_r} R_j \partial_j \lrcorner dz \\
&= 0.
\end{aligned}$$

We remark that in dropping the term $\partial_j\phi_{ii}$ (equivalently, $\nabla_j R$) in the computations above, we have used the exponential decay of the scalar curvature, the

Ricci pinching condition, and the linear growth of $|\nabla\phi|$ in distance. Using Proposition 4 we know that $(M, g(1/2))$ is R^n with the Euclidean metric. This is a contradiction. This completes the proof of Theorem 1. q.e.d.

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