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EXPANDING THE APPLICABILITY OF LAVRENTIEV REGULARIZATION METHODS FOR ILL-POSED EQUATIONS UNDER GENERAL SOURCE CONDITION

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Abstract. In this paper, we expand the applicability of Lavrentiev regularization method for ill-posed equations, recently presented in Mahale and Nair (2013). We use a modified center-type Lipschitz condition in our convergence analysis instead of Lipschitz-type condition used in earlier studies such as Mahale and Nair (2000), (2013) and Tautenhn (2002).

1. Introduction

In this paper, we consider the problem of approximately solving the nonlinear ill-posed operator equation of the form

$$F(x) = y, \tag{1.1}$$

where $F : D(F) \subset X \to X$ is a monotone operator and X is a real Hilbert space. We denote the inner product and the corresponding norm on a Hilbert space by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $U(x, r)$ stand for the open ball in X with center $x \in X$ and radius $r > 0$. Recall that F is said to be a monotone operator if it satisfies the relation

$$\langle F(x_1) - F(x_2), x_1 - x_2 \rangle \geq 0 \tag{1.2}$$

for all $x_1, x_2 \in D(F)$.

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We assume, throughout this paper, that $y^\delta \in Y$ are the available noisy data with

$$\|y - y^\delta\| \leq \delta \tag{1.3}$$

and (1.1) has a solution $\hat{x}$. Since (1.1) is ill-posed, the regularization methods are used ([9, 10, 11, 13, 14, 19, 21, 22]) for approximately solving (1.1). Lavrentiev regularization is used to obtain a stable approximate solution of (1.1). In the Lavrentiev regularization, the approximate solution is obtained as a solution of the equation

$$F(x) + \alpha(x - x_0) = y^\delta, \tag{1.4}$$

where $\alpha > 0$ is the regularization parameter and x_0 is an initial guess for the solution $\hat{x}$. For deriving the error estimates, we shall make use of the following equivalent form of (1.4),

$$x_\alpha^\delta = x_0 + (A_\alpha^\delta + \alpha I)^{-1}[y^\delta - F(x_\alpha^\delta) + A_\alpha^\delta(x_\alpha^\delta - x_0)], \tag{1.5}$$

where $A_\alpha^\delta = F'(x_\alpha^\delta)$.

In [15], Mahale and Nair, motivated by the work of Tautenhan [22], considered Lavrentieve regularization of (1.1) under a general source condition on $\hat{x} - x_0$ and obtained an order optimal error estimate.

In the present paper, we are motivated by [15]. In particular, we expand the applicability of the Lavrentieve regularization of (1.1) by weakening one of the major hypotheses in [15] (see below Assumption 2.1 (ii) in the next section).

In Section 2, we consider basic assumptions and some preliminaries required throughout the paper. The main order optimal result using the apriori and a posteriori parameter choice is provided in Section 3. Finally the paper ends with some numerical examples in Section 4.

2. Basic assumptions and some preliminary results

We use the following assumptions to prove the results in this paper.

Assumption 2.1. (1) There exists $r > 0$ such that $U(x_0, r) \subseteq D(F)$ and $F : U(x_0, r) \to X$ is Fréchet differentiable.

(2) There exists $K_0 > 0$ such that, for all $u_\theta = u + \theta(x_0 - u) \in U(x_0, r)$, $\theta \in [0, 1]$ and $v \in X$, there exists an element, say $\phi(x_0, u_\theta, v) \in X$, satisfying

$$[F'(x_0) - F'(u_\theta)]v = F'(u_\theta)\phi(x_0, u_\theta, v),$$
$$\|\phi(x_0, u_\theta, v)\| \leq K_0\|v\|\|x_0 - u_\theta\|$$

for all $u_\theta \in U(x_0, r)$ and $v \in X$.

(3) $\|F'(u) + \alpha I)^{-1} F'(u_\theta)\| \leq 1$ for all $u_\theta \in U(x_0, r)$.
(4) $\|(F'(u) + \alpha I)^{-1}\| \leq \frac{1}{\alpha}$.

The condition (2) in Assumption 2.1 weakens the popular hypotheses given in [15], [17] and [20].

Assumption 2.2. There exists a constant $K > 0$ such that, for all $x, y \in U(\hat{x}, r)$ and $v \in X$, there exists an element denoted by $P(x, u, v) \in X$ satisfying

$$[F'(x) - F'(u)]v = F'(u)P(x, u, v), \quad \|P(x, u, v)\| \leq K\|v\|\|x - u\|.$$

Clearly, Assumption 2.2 implies Assumption 2.1 (2) with $K_0 = K$, but not necessarily vice versa. Note that $K_0 \leq K$ holds in general and $\frac{K}{K_0}$ can be arbitrarily large [1]-[5]. Indeed, there are many classes of operators satisfying Assumption 2.1 (2), but not Assumption 2.2 (see the numerical examples at the end of this study). Moreover, if K_0 is sufficiently smaller than K which can happen since $\frac{K}{K_0}$ can be arbitrarily large, then the results obtained in this study provide a tighter error analysis than the one in [15].

Finally, note that the computation of constant K is more expensive than the computation of K_0.

Assumption 2.3. There exists a continuous and strictly monotonically increasing function $\varphi : (0, a] \to (0, \infty)$ with $a \geq \|F'(x_0)\|$ satisfying
(1) $\lim_{\lambda \to 0} \varphi(\lambda) = 0$;
(2) $\sup_{\lambda \geq 0} \frac{\alpha\varphi(\lambda)}{\lambda + \alpha} \leq \varphi(\alpha)$ for all $\alpha \in (0, a]$;
(3) there exists $v \in X$ with $\|v\| \leq 1$ such that

$$\hat{x} - x_0 = \varphi(F'(x_0))v. \tag{2.1}$$

Note that the source condition (2.1) is suitable for both mildly and severely ill-posed problems [16], [17]. Further note that the source condition (2.1) involves the known initial approximation x_0 whereas the source condition considered in [15] requires the knowledge of the unknown $\hat{x}$.

We need the auxiliary results based on Assumption 2.1.

Proposition 2.4. *For any $u \in U(x_0, r)$ and $\alpha > 0$,*

$$\begin{aligned} &\|(F'(u) + \alpha I)^{-1}[F(\hat{x}) - F(u) - F'(u)(\hat{x} - u)\| \\ &\leq \frac{5K_0}{2}\|\hat{x} - u\|^2 + 2K_0\|\hat{x} - x_0\|\|\hat{x} - u\|. \end{aligned}$$

Proof. Using the fundamental theorem of integration, for any $u \in U(x_0, r)$, we get

$$F(\hat{x}) - F(u) = \int_0^1 F'(u + t(\hat{x} - u))(\hat{x} - u)dt.$$

Hence, by Assumption 2.1,

$$\begin{aligned} & F(\hat{x})-F(u)-F'(u)(\hat{x}-u) \\ = & \int_0^1 [F'(u+t(\hat{x}-u))-F'(x_0)+F'(x_0)-F'(u)](\hat{x}-u)dt \\ = & \int_0^1 F'(x_0)[\phi(x_0,u+t(\hat{x}-u),u-\hat{x})-\phi(x_0,u,\hat{x}-u)]dt. \end{aligned}$$

Then, by (2), (3) in Assumptions 2.1 and the inequality $\|(F'(u)+\alpha I)^{-1}F'(u_\theta)\| \leq 1$, we obtain in turn

$$\begin{aligned} & \|(F'(u)+\alpha I)^{-1}[F(\hat{x})-F(u)-F'(u)(\hat{x}-u)\| \\ \leq & \int_0^1 \|\phi(x_0,u+t(\hat{x}-u),u-\hat{x})+\phi(x_0,u,\hat{x}-u)\|dt. \\ \leq & \left[\int_0^1 K_0(\|u-x_0\|+\|\hat{x}-u\|t)dt+K_0\|u-x_0\|\right]\|\hat{x}-u\| \\ \leq & \left[\int_0^1 K_0(\|u-\hat{x}\|+\|\hat{x}-x_0\|+\|\hat{x}-u\|t)dt\right. \\ & \left.+K_0(\|u-\hat{x}\|+\|\hat{x}-x_0\|)\right]\|\hat{x}-u\| \\ \leq & \frac{5K_0}{2}\|\hat{x}-u\|^2+2K_0\|\hat{x}-x_0\|\|\hat{x}-u\|. \end{aligned}$$

This completes the proof. □

Theorem 2.5. ([16, Theorem 2.2], also see [22]) *Let x_α be the solution of* (1.4) *with y in place of y^δ. Then*

(1) $\|x_\alpha^\delta - x_\alpha\| \leq \frac{\delta}{\alpha}$,
(2) $\|x_\alpha - \hat{x}\| \leq \|x_0 - \hat{x}\|$,
(3) $\|F(x_\alpha^\delta) - F(x_\alpha)\| \leq \delta$.

Remark 2.6. From Theorem 2.5 and triangle inequality we have

$$\|x_\alpha^\delta - \hat{x}\| \leq \frac{\delta}{\alpha} + \|x_0 - \hat{x}\|.$$

So (1.5) is meaningful if

$$r > \frac{\delta}{\alpha} + \|x_0 - \hat{x}\|.$$

3. Error Estimates

The error estimate in this section is obtained by finding error bounds for $\|x_\alpha - \hat{x}\|$.

Theorem 3.1. *Let Assumption* 2.1, 2.3 *hold, and let* $K_0\|x_0 - \hat{x}\| < \frac{2}{9}$. *Then*

$$\|x_\alpha - \hat{x}\| \leq C\varphi(\alpha),$$

where $C = \frac{1+2K_0\|x_0-\hat{x}\|}{1-9K_0\|x_0-\hat{x}\|/2}$.

Proof. Let $A_\alpha = F'(x_\alpha)$ and $A_0 = F'(x_0)$. Then

$$x_\alpha = x_0 + (A_\alpha + \alpha I)^{-1}[y - F(x_\alpha) + A_\alpha(x_\alpha - x_0)].$$

So

$$\begin{aligned} x_\alpha - \hat{x} &= x_0 - \hat{x} + (A_\alpha + \alpha I)^{-1}[y - F(x_\alpha) + A_\alpha(x_\alpha - x_0)] \\ &= x_0 - \hat{x} + (A_\alpha + \alpha I)^{-1}[F(\hat{x}) - F(x_\alpha) + A_\alpha(x_\alpha - \hat{x}) + A_\alpha(\hat{x} - x_0)] \\ &= (A_\alpha + \alpha I)^{-1}[\alpha(x_0 - \hat{x}) + F(\hat{x}) - F(x_\alpha) + A_\alpha(x_\alpha - \hat{x})] \\ &= (A_\alpha + \alpha I)^{-1}\alpha(x_0 - \hat{x}) \\ &\quad +(A_\alpha + \alpha I)^{-1}[F(\hat{x}) - F(x_\alpha) + A_\alpha(x_\alpha - \hat{x})] \\ &= (A_0 + \alpha I)^{-1}\alpha(x_0 - \hat{x}) + [(A_\alpha + \alpha I)^{-1} - (A_0 + \alpha I)^{-1}]\alpha(x_0 - \hat{x}) \\ &\quad +(A_\alpha + \alpha I)^{-1}[F(\hat{x}) - F(x_\alpha) + A_\alpha(x_\alpha - \hat{x})] \\ &= v_0 + (A_\alpha + \alpha I)^{-1}(A_0 - A_\alpha)v_0 + \Delta_1, \end{aligned} \tag{3.1}$$

where $v_0 = (A_0 + \alpha I)^{-1}\alpha(x_0 - \hat{x})$ and

$$\Delta_1 = (A_\alpha + \alpha I)^{-1}[F(\hat{x}) - F(x_\alpha) + A_\alpha(x_\alpha - \hat{x})].$$

By Assumption 2.3, we have

$$\|v_0\| \leq \varphi(\alpha), \tag{3.2}$$

by Assumption 2.1, 2.3 and Theorem 2.5, we have

$$\begin{aligned} (A_\alpha + \alpha I)^{-1}(A_0 - A_\alpha)v_0 &= (A_\alpha + \alpha I)^{-1}A_0\varphi(x_0, x_\alpha, v_0) \\ &\leq K_0\|x_0 - x_\alpha\|\|v_0\| \\ &\leq K_0(\|x_0 - \hat{x}\| + \|\hat{x} - x_\alpha\|)\|v_0\| \\ &\leq 2K_0\|x_0 - \hat{x}\|\varphi(\alpha) \end{aligned} \tag{3.3}$$

and by Proposition 2.4

$$\begin{aligned} \Delta_1 &\leq \frac{5K_0}{2}\|\hat{x} - x_\alpha\|^2 + 2K_0\|\hat{x} - x_0\|\|\hat{x} - x_\alpha\| \\ &\leq \frac{9K_0}{2}\|\hat{x} - x_0\|\|\hat{x} - x_\alpha\|. \end{aligned} \tag{3.4}$$

The result now follows from (3.1), (3.2), (3.3) and (3.4). □

The following Theorem is a consequence of Theorem 2.5 and Theorem 3.1.

Theorem 3.2. *Under the assumptions of Theorem* 3.1

$$\|x_\alpha^\delta - \hat{x}\| \le C\left(\frac{\delta}{\alpha} + \varphi(\alpha)\right),$$

where C *is as in Theorem* 3.1.

3.1. **Apriori parameter choice.** Let $\psi : (0, \varphi(a)] \to (0, a\varphi(a)]$ be defined as

$$\psi(\lambda) := \lambda\varphi^{-1}(\lambda). \tag{3.5}$$

Then $\frac{\delta}{\alpha} = \varphi(\alpha) \Leftrightarrow \delta = \psi(\varphi(\alpha))$.

Theorem 3.3. *Let the assumptions of Theorem* 3.1 *be satisfied. If the regularization parameter is chosen as* $\alpha = \varphi^{-1}(\psi^{-1}(\delta))$ *with* ψ *defined as in* (3.5), *then*

$$\|x_\alpha^\delta - \hat{x}\| \le 2C\psi^{-1}(\delta). \tag{3.6}$$

3.2. **Aposteriori parameter choice.** Note that the choice of α in the above Theorem depends on the unknown source function φ. In applications, it is desirable that α is chosen independent of the source function φ, but may depend on the data (δ, y^δ), and consequently on the regularized solution. For Lavrentiev regularization (1.5), Tautenhan (cf. [22]) considered the following discrepancy principal for chosing the regularization parameter α,

$$\|\alpha(A_\alpha^\delta + \alpha I)^{-1}[F(x_\alpha^\delta) - y^\delta]\| = c\delta, \tag{3.7}$$

where $c > 0$ is an appropriate constant. The error estimate in [22] is obtained under the assumption that the solution satisfies a Holder-type source condition. In [15], Mahale and Nair considered the discrepancy principle (3.7) and extended the analysis of Tautenhan [22] to include both Holder type and logarithmic type source conditions. In this paper we consider discrepancy principle (3.7) and derive order optimal error estimte under the Assumption 2.3.

We will be using the following proposition, proved in [22], which shows the existence of the regularization parameter α satisfying (3.7).

Proposition 3.4. (cf. [22, Proposition 4.1]) *Let* F *be monotone and* $\|F(x_0) - y^\delta\| \ge c\delta$ *with* $c > 2$. *Then there exists an* $\alpha \ge \beta_0 := \frac{(c-1)\delta}{\|x_0 - \hat{x}\|}$ *satisfying* (3.7).

Lemma 3.5. *Let Assumption* 2.1 *and assumptions in Proposition* 3.4 *hold and* $\alpha := \alpha(\delta)$ *is chosen according to* (3.7). *Then*

$$\|\alpha(A_0 + \alpha I)^{-1}(F(x_\alpha) - y)\| \ge \frac{(c-2)\delta}{1 + k_1}, \tag{3.8}$$

where $k_1 = \frac{K_0(2c-1)\|x_0-\hat{x}\|}{c-1}$.

Proof. By (3.7), we have

$$\begin{aligned}
& |c\delta - \alpha\|(A_\alpha^\delta + \alpha I)^{-1}(F(x_\alpha) - y)\|| \\
= \; & |\alpha(A_\alpha^\delta + \alpha I)^{-1}(F(x_\alpha^\delta) - y^\delta)\| - \alpha\|(A_\alpha^\delta + \alpha I)^{-1}(F(x_\alpha) - y)\|| \\
\leq \; & \|\alpha(A_\alpha^\delta + \alpha I)^{-1}[F(x_\alpha^\delta) - F(x_\alpha) - (y^\delta - y)]\| \\
\leq \; & \|F(x_\alpha^\delta) - F(x_\alpha)\| + \|y^\delta - y\| \\
\leq \; & 2\delta.
\end{aligned}$$

The last step follows from Theorem 2.5. So

$$(c-2)\delta \leq \alpha\|(A_\alpha^\delta + \alpha I)^{-1}(F(x_\alpha) - y)\| \leq (c+2)\delta.$$

Let

$$a_\alpha := \alpha(A_0 + \alpha I)^{-1}(F(x_\alpha) - y).$$

Then

$$\begin{aligned}
& \alpha\|(A_\alpha^\delta + \alpha I)^{-1}(F(x_\alpha) - y)\| \\
\leq \; & \|a_\alpha\| + \|\alpha[(A_\alpha^\delta + \alpha I)^{-1} - (A_0 + \alpha I)^{-1}](F(x_\alpha) - y)\| \\
\leq \; & \|a_\alpha\| + \|(A_\alpha^\delta + \alpha I)^{-1}(A_0 - A_\alpha^\delta)a_\alpha\| \\
\leq \; & \|a_\alpha\| + \|(A_\alpha^\delta + \alpha I)^{-1}A_\alpha^\delta \varphi(x_0, x_\alpha^\delta, a_\alpha)\| \\
\leq \; & \|a_\alpha\|(1 + K_0\|x_0 - x_\alpha^\delta\|) \\
\leq \; & \|a_\alpha\|(1 + K_0(\|x_0 - \hat{x}\| + \|\hat{x} - x_\alpha^\delta\|)) \\
\leq \; & \|a_\alpha\|\left(1 + K_0\left(\|x_0 - \hat{x}\| + \frac{\delta}{\alpha} + \|\hat{x} - x_0\|\right)\right) \\
\leq \; & \|a_\alpha\|\left(1 + K_0\left(2\|x_0 - \hat{x}\| + \frac{\delta}{\alpha}\right)\right) \\
\leq \; & \|a_\alpha\|\left(1 + K_0\left(2\|x_0 - \hat{x}\| + \frac{\|x_0 - \hat{x}\|}{c-1}\right)\right) \\
\leq \; & \|a_\alpha\|(1 + k_1),
\end{aligned}$$

which in turn implies $\|\alpha(A_0 + \alpha I)^{-1}(F(x_\alpha) - y)\| \geq \frac{(c-2)\delta}{1+k_1}$. □

Lemma 3.6. *Let the assumptions of Theorem* 3.1 *and Proposition* 3.4 *hold. Then*

(1) $\|\alpha(A_0 + \alpha I)^{-1}(F(x_\alpha) - y)\| \leq \mu\alpha\varphi(\alpha)$,
(2) $\alpha \geq \varphi^{-1}(\psi^{-1}(\xi\delta))$, *where* $\mu = C(1 + \frac{3K_0}{2}\|x_0 - \hat{x}\|)$ *and* $\xi = \frac{c-2}{1+k_1}$.

Proof. By Assumption 2.1, we know that for $x, z \in U(x_0, r)$ and $u \in X$,

$$F'(z)u = F'(x_0)u - F'(z)\varphi(x_0, z, u), \;\; \|\varphi(x_0, z, u)\| \leq K_0\|x_0 - z\|\|u\|.$$

So

$$\begin{aligned} F(x_\alpha) - F(\hat{x}) &= \int_0^1 F'(\hat{x} + t(x_\alpha - \hat{x}))(x_\alpha - \hat{x})dt \\ &= A_0(x_\alpha - \hat{x}) - \int_0^1 F'(\hat{x} + t(x_\alpha - \hat{x})) \\ &\quad \times \varphi(x_0, \hat{x} + t(x_\alpha - \hat{x}), x_\alpha - \hat{x})dt. \end{aligned}$$

Hence

$$\begin{aligned} & \|\alpha(A_0 + \alpha I)^{-1}(F(x_\alpha) - F(\hat{x}))\| \\ = & \Big\|\alpha(A_0 + \alpha I)^{-1}A_0(x_\alpha - \hat{x}) \\ & - \int_0^1 \alpha(A_0 + \alpha I)^{-1}F'(\hat{x} + t(x_\alpha - \hat{x}))\varphi(x_0, \hat{x} + t(x_\alpha - \hat{x}), x_\alpha - \hat{x})dt\Big\| \\ \leq & \alpha\left(\|x_\alpha - \hat{x}\| + K_0\left(\|x_0 - \hat{x}\| + \frac{\|x_\alpha - \hat{x}\|}{2}\right)\|x_\alpha - \hat{x}\|\right) \\ \leq & \alpha\left(1 + \frac{3K_0}{2}\|x_0 - \hat{x}\|\right)\|x_\alpha - \hat{x}\|. \end{aligned}$$

The last follows from Theorem 2.5. Now by using Theorem 3.1, we have

$$\|\alpha(A_0 + \alpha I)^{-1}(F(x_\alpha) - F(\hat{x}))\| \leq C\left(1 + \frac{3K_0}{2}\|x_0 - \hat{x}\|\right)\alpha\varphi(\alpha) = \mu\alpha\varphi(\alpha).$$

In view of (3.8), we get

$$\frac{(c-2)\delta}{1+k_1} \leq \mu\alpha\varphi(\alpha)$$

which implies, by definition of ψ

$$\psi(\varphi(\alpha)) = \alpha\varphi(\alpha) \geq \frac{(c-2)\delta}{1+k_1} := \xi\delta,$$

where $\xi = \frac{(c-2)}{1+k_1}$. Thus $\alpha \geq \varphi^{-1}(\psi^{-1}(\xi\delta))$. This completes the proof. □

Theorem 3.7. *Let Assumption* 2.1 *be satisfied and* $4K_0\|x_0 - \hat{x}\| < 1$. *Then for* $0 < \alpha_0 \leq \alpha$,

$$\|x_\alpha - x_{\alpha_0}\| \leq \frac{\|\alpha(A_0 + \alpha I)^{-1}(F(x_\alpha) - F(\hat{x}))\|}{(1 - 4\|x_0 - \hat{x}\|)\alpha_0}.$$

Proof. Since

$$F(x_\alpha) - y + \alpha(x_\alpha - x_0) = 0, \tag{3.9}$$

$$F(x_{\alpha_0}) - y + \alpha_0(x_{\alpha_0} - x_0) = 0, \tag{3.10}$$

and

$$\alpha_0(x_\alpha - x_{\alpha_0}) = (\alpha - \alpha_0)(x_0 - x_\alpha) + \alpha_0(x_0 - x_{\alpha_0}) - \alpha(x_0 - x_\alpha),$$

we have by (3.9) and (3.10),

$$\alpha_0(x_\alpha - x_{\alpha_0}) = \frac{\alpha - \alpha_0}{\alpha}(F(x_\alpha) - y) + F(x_{\alpha_0}) - F(x_\alpha),$$

and

$$\begin{aligned}&(A_\alpha + \alpha_0 I)(x_\alpha - x_{\alpha_0})\\ &= \frac{\alpha - \alpha_0}{\alpha}(F(x_\alpha) - y) + [F(x_{\alpha_0}) - F(x_\alpha) + A_\alpha(x_\alpha - x_{\alpha_0})].\end{aligned}$$

Hence

$$\begin{aligned}x_\alpha - x_{\alpha_0} &= \frac{\alpha - \alpha_0}{\alpha}(A_\alpha + \alpha_0 I)^{-1}(F(x_\alpha) - y)\\ &\quad + (A_\alpha + \alpha_0 I)^{-1}[F(x_{\alpha_0}) - F(x_\alpha) + A_\alpha(x_\alpha - x_{\alpha_0})].\end{aligned}$$

Thus by Proposition 3.4

$$\|x_\alpha - x_{\alpha_0}\| \le \|\frac{\alpha - \alpha_0}{\alpha}(A_\alpha + \alpha_0 I)^{-1}(F(x_\alpha) - y)\| + \Gamma_1, \tag{3.11}$$

where $\Gamma_1 = \|(A_\alpha + \alpha_0 I)^{-1}[F(x_{\alpha_0}) - F(x_\alpha) + A_\alpha(x_\alpha - x_{\alpha_0})]\|$. By Fundamental Theorem of Integration and Assumption 2.1, we have

$$\begin{aligned}&F(x_{\alpha_0}) - F(x_\alpha) + A_\alpha(x_\alpha - x_{\alpha_0})\\ =& \int_0^1 [F'(x_\alpha + t(x_{\alpha_0} - x_\alpha)) - A_\alpha](x_{\alpha_0} - x_\alpha)dt\\ =& \int_0^1 [F'(x_\alpha + t(x_{\alpha_0} - x_\alpha)) - F'(x_0) + F'(x_0) - A_\alpha](x_{\alpha_0} - x_\alpha)dt\\ =& \int_0^1 [-F'(u_\theta)\varphi(x_0, u_\theta, x_{\alpha_0} - x_\alpha) + A_\alpha\varphi(x_0, x_\alpha, x_{\alpha_0} - x_\alpha)]dt,\end{aligned}$$

where $u_\theta = x_0 + \theta(x_0 - (x_\alpha + t(x_{\alpha_0} - x_\alpha)))$. Therefore again by Assumption 2.1, we have

$$\begin{aligned}
\Gamma_1 &\le \left\| \int_0^1 \varphi(x_0, u_\theta, x_{\alpha_0} - x_\alpha) dt \right\| + \|\varphi(x_0, x_\alpha, x_{\alpha_0} - x_\alpha)\| \\
&\le K_0 \left[\int_0^1 \|x_0 - u_\theta\| dt + \|x_0 - x_\alpha\| \right] \|x_{\alpha_0} - x_\alpha\| \\
&\le K_0[\|x_0 - x_\theta\|/2 + \|x_0 - x_\alpha\|]\|x_{\alpha_0} - x_\alpha\| \\
&\le K_0[\|x_0 - x_\alpha\|/2 + \|x_0 - x_{\alpha_0}\|/2 + \|x_0 - x_\alpha\|]\|x_{\alpha_0} - x_\alpha\| \\
&\le K_0[3(\|x_0 - \hat{x}\| + \|\hat{x} - x_\alpha\|)/2 + (\|x_0 - \hat{x}\| + \|\hat{x} - x_{\alpha_0}\|)/2]\|x_{\alpha_0} - x_\alpha\| \\
&\le 4K_0\|x_0 - \hat{x}\|\|x_{\alpha_0} - x_\alpha\|. \qquad (3.12)
\end{aligned}$$

The last step follows from Theorem 2.5. Now since $\frac{\alpha - \alpha_0}{\alpha} < 1$, we have

$$\begin{aligned}
&\left\| \frac{\alpha - \alpha_0}{\alpha} (A_\alpha + \alpha_0 I)^{-1} (F(x_\alpha) - y) \right\| \\
\le\ & \|(A_\alpha + \alpha_0 I)^{-1}(F(x_\alpha) - y)\| \\
\le\ & \frac{1}{\alpha_0 \alpha} \|\alpha_0 (A_\alpha + \alpha_0 I)^{-1} (A_\alpha + \alpha_0 I) \alpha (A_\alpha + \alpha_0 I)^{-1} (F(x_\alpha) - y)\| \\
\le\ & \sup_{\lambda \ge 0} \left| \frac{\alpha_0(\lambda + \alpha)}{\alpha(\lambda + \alpha_0} \right| \frac{\|\alpha(A_\alpha + \alpha_0 I)^{-1}(F(x_\alpha) - y)\|}{\alpha_0} \\
\le\ & \frac{\|\alpha(A_\alpha + \alpha_0 I)^{-1}(F(x_\alpha) - y)\|}{\alpha_0}. \qquad (3.13)
\end{aligned}$$

So by (3.11), (3.12) and (3.13) we have

$$\|x_\alpha - x_{\alpha_0}\| \le \frac{\|\alpha(A_\alpha + \alpha_0 I)^{-1}(F(x_\alpha) - y)\|}{\alpha_0(1 - 4K_0\|x - \hat{x}\|)}.$$

This completes the proof. □

Lemma 3.8. *Let assumptions of Lemma* 3.6 *hold. Then*

$$\|\alpha(A_\alpha + \alpha_0 I)^{-1}(F(x_\alpha) - y)\| \le \beta\delta,$$

where $\beta = (c+2)(1 + \frac{K_0(4c-3)\|x_0 - \hat{x}\|}{c-1})$.

Proof. Observe that

$$\begin{aligned}
& \|\alpha(A_\alpha+\alpha_0 I)^{-1}(F(x_\alpha)-y)\| \\
\leq\ & \|\alpha(A_\alpha^\delta+\alpha_0 I)^{-1}(F(x_\alpha)-y)\| \\
& +\|\alpha[(A_\alpha+\alpha_0 I)^{-1}-(A_\alpha^\delta+\alpha_0 I)^{-1}](F(x_\alpha)-y)\| \\
\leq\ & \|\alpha(A_\alpha^\delta+\alpha_0 I)^{-1}(F(x_\alpha)-y)\| \\
& +\|\alpha(A_\alpha+\alpha_0 I)^{-1}(A_\alpha^\delta-A_\alpha)(A_\alpha^\delta+\alpha_0 I)^{-1}(F(x_\alpha)-y)\|.
\end{aligned} \tag{3.14}$$

Let $a_\delta=\alpha(A_\alpha^\delta+\alpha_0 I)^{-1}(F(x_\alpha)-y)$. Then

$$\begin{aligned}
& \|\alpha(A_\alpha+\alpha_0 I)^{-1}(F(x_\alpha)-y)\| \\
\leq\ & \|a_\delta\|+\|\alpha(A_\alpha+\alpha_0 I)^{-1}(A_\alpha^\delta-A_0+A_0-A_\alpha)(A_\alpha^\delta+\alpha_0 I)^{-1}(F(x_\alpha)-y)\| \\
\leq\ & \|a_\delta\|+\|-(A_\alpha+\alpha_0 I)^{-1}A_\alpha^\delta\varphi(x_0,x_\alpha^\delta,a_\delta)\| \\
& +\|(A_\alpha+\alpha_0 I)^{-1}A_\alpha\varphi(x_0,x_\alpha,a_\delta)\| \\
\leq\ & \|a_\delta\|(1+K_0(\|x_0-x_\alpha^\delta\|+\|x_0-x_\alpha\|)) \\
\leq\ & \|a_\delta\|(1+K_0(\|x_\alpha-x_\alpha^\delta\|+2\|x_0-x_\alpha\|)) \\
\leq\ & \|a_\delta\|\left(1+K_0\left(\frac{\delta}{\alpha}+4\|x_0-\hat{x}\|\right)\right) \\
\leq\ & \|a_\delta\|\left(1+K_0\left(\frac{1}{c-1}+4\right)\|x_0-\hat{x}\|\right) \\
\leq\ & \|a_\delta\|\left(1+K_0\frac{4c-3}{c-1}\|x_0-\hat{x}\|\right).
\end{aligned}$$

Now since $\|a_\delta\|\leq(c+2)\delta$, we have

$$\|\alpha(A_\alpha+\alpha_0 I)^{-1}(F(x_\alpha)-y)\|\leq\left(1+K_0\frac{4c-3}{c-1}\|x_0-\hat{x}\|\right)(c+2)\delta.$$

This completes the proof. □

The main results of the paper is the following.

Theorem 3.9. *Let assumptions of Lemma* 3.6 *hold. If, in addition,* $9K_0\|x_0-\hat{x}\|<2$, *then*

$$\|x_\alpha^\delta-\hat{x}\|\leq\wp\psi^{-1}(\eta\delta),$$

where $\wp=\frac{1}{\xi}+\frac{1}{1-4K_0\|x_0-\hat{x}\|}+C$ *and* $\eta=\max\{\xi,\beta\}$.

Proof. First consider the case $\alpha:=\alpha(\delta)\leq\alpha_0$. Then by Theorem 3.2 we have

$$\|x_\alpha^\delta-\hat{x}\|\leq C\left(\frac{\delta}{\alpha}+\varphi(\alpha_0)\right). \tag{3.15}$$

Now consider the case $\alpha := \alpha(\delta) > \alpha_0$. In this case by Theorem 3.7, we have

$$\begin{aligned}\|x_\alpha^\delta - \hat{x}\| &\leq \|x_\alpha^\delta - x_\alpha\| + \|x_\alpha - x_{\alpha_0}\| + \|x_{\alpha_0} - \hat{x}\| \\ &\leq \frac{\delta}{\alpha} + C\varphi(\alpha_0) + \frac{\|\alpha(A_\alpha + \alpha_0 I)^{-1}(F(x_\alpha) - y)\|}{\alpha_0(1 - 4K_0\|x - \hat{x}\|)}. \quad (3.16)\end{aligned}$$

From (3.15) and (3.16)

$$\|x_\alpha^\delta - \hat{x}\| \leq \frac{\delta}{\alpha} + C\varphi(\alpha_0) + \frac{\|\alpha(A_\alpha + \alpha_0 I)^{-1}(F(x_\alpha) - y)\|}{\alpha_0(1 - 4K_0\|x - \hat{x}\|)}$$

for all $\alpha \in (0, a]$. Let $\alpha_0 := \varphi^{-1}\psi^{-1}(\beta\delta)$ with $\beta = (1 + K_0(4c - 3)\|x_0 - \hat{x}\|(c + 2)/(c - 1)$. Then by Lemma 3.6 and 3.8, we get

$$\begin{aligned}\|x_\alpha^\delta - \hat{x}\| &\leq \frac{\delta}{\alpha} + C\varphi(\alpha_0) + \frac{\|\alpha(A_\alpha + \alpha_0 I)^{-1}(F(x_\alpha) - y)\|}{\alpha_0(1 - 4K_0\|x - \hat{x}\|)} \\ &\leq \frac{\delta}{\varphi^{-1}\psi^{-1}(\xi\delta)} + \frac{\beta\delta}{(1 - 4K_0\|x - \hat{x}\|)\varphi^{-1}\psi^{-1}(\beta\delta)} + C\psi^{-1}(\beta\delta).\end{aligned}$$

Now since $\varphi^{-1}\psi^{-1}(\lambda) = \frac{\lambda}{\psi^{-1}(\lambda)}$ we have

$$\begin{aligned}\|x_\alpha^\delta - \hat{x}\| &\leq \frac{\psi^{-1}(\xi\delta)}{\xi} + \frac{\psi^{-1}(\beta\delta)}{1 - 4K_0\|x - \hat{x}\|} + C\psi^{-1}(\beta\delta) \\ &\leq \left(\frac{1}{\xi} + \frac{1}{1 - 4K_0\|x_0 - \hat{x}\|} + C\right)\psi^{-1}(\eta\delta), \quad (3.17)\end{aligned}$$

where $\eta = \max\{\xi, \beta\}$. This completes the proof. □

4. Numerical Examples

We provide two numerical examples, where $K_0 < K$.

Example 4.1. Let $X = \mathbb{R}$, $D(F) = \overline{U(0,1)}$, $x_0 = 0$ and define a function F on $D(F)$ by

$$F(x) = e^x - 1. \quad (4.1)$$

Then, using (4.1) and Assumptions 2.1 (2) and 2.2, we get

$$K_0 = e - 1 < K = e.$$

Example 4.2. Let $X = C([0,1])$ (:the space of continuous functions defined on $[0,1]$ equipped with the max norm) and $D(F) = \overline{U(0,1)}$. Define an operator F on $D(F)$ by

$$F(h)(x) = h(x) - 5\int_0^1 x\theta h(\theta)^3 d\theta. \quad (4.2)$$

Then the Fréchet-derivative is given by

$$F'(h[u])(x) = u(x) - 15\int_0^1 x\theta h(\theta)^2 u(\theta)d\theta \tag{4.3}$$

for all $u \in D(F)$. Using (4.2), (4.3), Assumptions 2.1 (2), 2.2 for $x_0 = 0$, we get $K_0 = 7.5 < K = 15$.

Next, we provide an example where $\frac{K}{K_0}$ can be arbitrarily large.

Example 4.3. Let $X = D(F) = \mathbb{R}$, $x_0 = 0$ and define a function F on $D(F)$ by

$$F(x) = d_0 x - d_1 \sin 1 + d_1 \sin e^{d_2 x}, \tag{4.4}$$

where d_0, d_1 and d_2 are the given parameters. Note that $F(x_0) = F(0) = 0$. Then it can easily be seen that, for d_2 sufficiently large and d_1 sufficiently small, $\frac{K}{K_0}$ can be arbitrarily large.

We now present two examples where Assumption 2.2 is not satisfied, but Assumption 2.1 (2) is satisfied.

Example 4.4. Let $X = D(F) = \mathbb{R}$, $x_0 = 0$ and define a function F on D by

$$F(x) = \frac{x^{1+\frac{1}{i}}}{1+\frac{1}{i}} + c_1 x - c_1 - \frac{i}{i+1}, \tag{4.5}$$

where c_1 is a real parameter and $i > 2$ is an integer. Then $F'(x) = x^{1/i} + c_1$ is not Lipschitz on D. Hence Assumption 2.2 is not satisfied. However, the central Lipschitz condition in Assumption 2.2 (2) holds for $K_0 = 1$. We also have that $F(x_0) = 0$. Indeed, we have

$$\begin{aligned}\|F'(x) - F'(x_0)\| &= |x^{1/i} - x_0^{1/i}| \\ &= \frac{|x - x_0|}{\hat{x}^{\frac{i-1}{i}} + \cdots + x^{\frac{i-1}{i}}}\end{aligned}$$

and so

$$\|F'(x) - F'(x_0)\| \leq K_0|x - x_0|.$$

Example 4.5. We consider the integral equation

$$u(s) = f(s) + \lambda \int_a^b G(s,t)u(t)^{1+1/n}dt \tag{4.6}$$

for all $n \in \mathbb{N}$, where f is a given continuous function satisfying $f(s) > 0$ for all $s \in [a, b]$, λ is a real number and the kernel G is continuous and positive in $[a, b] \times [a, b]$.

For example, when $G(s,t)$ is the Green kernel, the corresponding integral equation is equivalent to the boundary value problem

$$u'' = \lambda u^{1+1/n},$$
$$u(a) = f(a),\ u(b) = f(b).$$

These type of the problems have been considered in [1]–[5]. The equation of the form (4.6) generalize the equation of the form

$$u(s) = \int_a^b G(s,t)u(t)^n dt, \tag{4.7}$$

which was studied in [1]-[5]. Instead of (4.6), we can try to solve the equation $F(u) = 0$, where

$$F : \Omega \subseteq C[a,b] \to C[a,b], \quad \Omega = \{u \in C[a,b] : u(s) \geq 0, s \in [a,b]\}$$

and

$$F(u)(s) = u(s) - f(s) - \lambda \int_a^b G(s,t)u(t)^{1+1/n} dt.$$

The norm we consider is the max-norm. The derivative F' is given by

$$F'(u)v(s) = v(s) - \lambda(1+\frac{1}{n}) \int_a^b G(s,t)u(t)^{1/n} v(t) dt$$

for all $v \in \Omega$. First of all, we notice that F' does not satisfy the Lipschitz-type condition in Ω. Let us consider, for instance, $[a,b] = [0,1]$, $G(s,t) = 1$ and $y(t) = 0$. Then we have $F'(y)v(s) = v(s)$ and

$$\|F'(x) - F'(y)\| = |\lambda|(1+\frac{1}{n}) \int_a^b x(t)^{1/n} dt.$$

If F' were the Lipschitz function, then we have

$$\|F'(x) - F'(y)\| \leq L_1 \|x - y\|$$

or, equivalently, the inequality

$$\int_0^1 x(t)^{1/n} dt \leq L_2 \max_{x \in [0,1]} x(s) \tag{4.8}$$

would hold for all $x \in \Omega$ and for a constant L_2. But this is not true. Consider, for example, the function

$$x_j(t) = \frac{t}{j}$$

for all $j \geq 1$ and $t \in [0,1]$. If these are substituted into (4.7), then we have

$$\frac{1}{j^{1/n}(1+1/n)} \leq \frac{L_2}{j} \quad \Longleftrightarrow \quad j^{1-1/n} \leq L_2(1+1/n)$$

for all $j \geq 1$. This inequality is not true when $j \to \infty$. Therefore, Assumption 2.2 is not satisfied in this case. However, Assumption 2.1 (2) holds. To show this, suppose that $x_0(t) = f(t)$ and $\gamma = \min_{s\in[a,b]} f(s)$. Then, for all $v \in \Omega$, we have

$$\begin{aligned} & \|[F'(x) - F'(x_0)]v\| \\ = \; & |\lambda|\Big(1 + \frac{1}{n}\Big) \max_{s\in[a,b]} \Big| \int_a^b G(s,t)(x(t)^{1/n} - f(t)^{1/n})v(t)dt \Big| \\ \leq \; & |\lambda|\Big(1 + \frac{1}{n}\Big) \max_{s\in[a,b]} G_n(s,t), \end{aligned}$$

where $G_n(s,t) = \frac{G(s,t)|x(t)-f(t)|}{x(t)^{(n-1)/n} + x(t)^{(n-2)/n} f(t)^{1/n} + \cdots + f(t)^{(n-1)/n}} \|v\|$. Hence it follows that

$$\begin{aligned} \|[F'(x) - F'(x_0)]v\| &= \frac{|\lambda|(1+1/n)}{\gamma^{(n-1)/n}} \max_{s\in[a,b]} \int_a^b G(s,t)dt \|x - x_0\| \\ &\leq K_0 \|x - x_0\|, \end{aligned}$$

where $K_0 = \frac{|\lambda|(1+1/n)}{\gamma^{(n-1)/n}} N$ and $N = \max_{s\in[a,b]} \int_a^b G(s,t)dt$. Then Assumption 2.1 (2) holds for sufficiently small λ.

Remark 4.6. The results obtained here can also be realized for the operators F satisfying an autonomous differential equation of the form

$$F'(x) = P(F(x)),$$

where $P : X \to X$ is a known continuous operator. Since $F'(x_0) = P(F(x_0)) = P(0)$, we can compute K_0 in Assumption 2.1 (2) without actually knowing x_0. Returning back to Example 4.1, we see that we can set $P(x) = x + 1$.

References

[1] I.K. Argyros, Convergence and Application of Newton-type Iterations, Springer, 2008.

[2] I.K. Argyros, *Approximating solutions of equations using Newton's method with a modified Newton's method iterate as a starting point*, Rev. Anal. Numer. Theor. Approx., **36** (2007), 123–138.

[3] I.K. Argyros, *A semilocal convergence for directional Newton methods*, Math. Comput. (AMS), **80** (2011), 327–343.

[4] I.K. Argyros and S. Hilout, *Weaker conditions for the convergence of Newton's method*, J. Complexity, **28** (2012), 364–387.

[5] I.K. Argyros, Y.J. Cho and S. Hilout, Numerical methods for equations and its applications, CRC Press, Taylor and Francis, New York, 2012.

[6] A. Bakushinskii and A. Seminova, *On application of generalized discrepancy principle to iterative methods for nonlinear ill-posed problems*, Numer. Funct. Anal. Optim., **26** (2005), 35–48.

[7] A. Bakushinskii and A. Seminova, *A posteriori stopping rule for regularized fixed point iterations*, Nonlinear Anal., **64** (2006), 1255–1261.

[8] A. Bakushinskii and A. Seminova, *Iterative regularization and generalized discrepancy principle for monotone operato equations*, Numer. Funct. Anal. Optim., **28** (2007), 13–25.

[9] A. Binder, H.W. Engl, C.W. Groetsch, A. Neubauer and O. Scherzer, *Weakly closed nonlinear operators and parameter identification in parabolic equations by Tikhonov regularization*, Appl. Anal., **55** (1994), 215–235.

[10] H.W. Engl, M. Hanke and A. Neubauer, Regularization of Inverse Problems, Dordrecht, Kluwer, 1993.

[11] H.W. Engl, K. Kunisch and A. Neubauer, *Convergence rates for Tikhonov regularization of nonlinear ill-posed problems*, Inverse Problems, **5** (1989), 523–540.

[12] Q. Jin, *On the iteratively regularized Gauss-Newton method for solving nonlinear ill-posed problems*, Math. Comp., **69** (2000), 1603–1623.

[13] Q. Jin and Z.Y. Hou, *On the choice of the regularization parameter for ordinary and iterated Tikhonov regularization of nonlinear ill-posed problems*, Inverse Problems, **13** (1997), 815–827.

[14] Q. Jin and Z.Y. Hou, *On an a posteriori parameter choice strategy for Tikhonov regularization of nonlinear ill-posed problems*, Numer. Math., **83** (1990), 139–159.

[15] P. Mahale and M.T. Nair, *Lavrentiev regularization of nonlinear ill-posed equations under general source condition*, JNAO(to appear).

[16] P. Mahale and M.T. Nair, *Iterated Lavrentiev regularization for nonlinear ill-posed problems*, ANZIAM, **51** (2009), 191–217.

[17] P. Mahale and M.T. Nair, *General source conditions for nonlinear ill-posed problems*, Numer. Funct. Anal. Optim., **28** (2007), 111–126.

[18] M.T. Nair and U. Tautanhahn, "Lavrentiev regularization for linear ill-posed problems under general source conditions", Z. Anal. Anwendungen, **23** (2004), 167–185. *Gauss-Newton method for nonlinear ill-posed problems*, Math. Comp., **78(265)** (2009), pp.171–184.

[19] O. Scherzer, H.W. Engl and K. Kunisch, *Optimal a posteriori parameter choice for Tikhonov regularization for solving nonlinear ill-posed problems*, SIAM J. Numer. Anal., **30** (1993), 1796–1838.

[20] E.V. Semenova, *Lavrentiev regularization and balancing principle for solving ill-posed problems with monotone operators*, Comput. Methods Appl. Math., **4** (2010), 444–454.

[21] U. Tautenhahn, *Lavrentiev regularization of nonlinear ill-posed problems*, Vietnam J. Math., **32** (2004), 29–41.

[22] U. Tautenhahn, *On the method of Lavrentiev regularization for nonlinear ill-posed problems*, Inverse Problems, **18** (2002), 191–207.

[23] U. Tautenhahn and Q. Jin, *Tikhonov regularization and a posteriori rule for solving nonlinear ill-posed problems*, Inverse Problems, **19** (2003), 1–21.