Expansion and lack thereof in randomly perturbed graphs

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1 Introduction

Developing models of complex networks has been a major industry in the fields of physics, mathematics, and computer science during the last decade. Empirical study of numerous large networks harvested from the real world has revealed that, unlike the classical models of random graphs developed by Erdős and Rényi for applications to probabilistic combinatorics, many of the complex networks which surround us today have high correlation coefficients and/or power-law degree sequences. This observation has driven the development of numerous alternative distributions for random graphs, which often are described by some generative procedure.

Unfortunately, it is much easier to propose a generative procedure than to refute one. However, the numerous models for real-world graphs in the literature today may not withstand the test of time any better than the Erdős-Rényi distribution. This motivates the approach pursued in the present paper. Instead of studying a particular model for generating graphs with the hopes of finding it "more realistic" than previously proposed models, this paper considers an approach for incorporating randomness into network modeling that is less model-specific.

In this paper, a complex network is viewed as composed of a *base graph* and a *random perturbation*. The general goal in this framework is to show that some property is likely to hold for a wide variety of base graphs and under a very gentle random perturbation. For example, [25] shows that if a network is generated from any connected base graph on n vertices, perturbed by adding ϵn random edges, then **whp** the network will have diameter $\mathcal{O}(\epsilon^{-1} \log n)$.

This can be viewed as work in the line of "How many random edges make a dense graph Hamiltonian?", and subsequent studies of the effects of adding a few random edges to dense graphs [12, 11, 31]. It is also similar in spirit to the smoothed analysis of algorithms of Spielman and Teng [37] which has been used to explain why algorithms perform better in practice than worst-case bounds predict (see also [38, 5, 6, 7, 19]). Also similar are the hybrid graphs studied in [15, 4] which explicitly model long and short edges.

In addition to the perturbation models previously considered on sparse random instances in [25], this paper will consider non-uniform perturbations, in the spirit of Jon Kleinberg's small-world model [30] and long-range percolation in finite graphs studied in [8, 18, 9], and also the graph which both these models build upon, the small-world model of Watts and Strogatz [39].

Perturbation	Expander **whp**?
1-out	Yes, for any connected $\bar{G}$
$\mathbb{G}_{n,\epsilon/n}$	Not if $\bar{G}$ has a bad partition
Watts-Strogatz Small World	Not if $\bar{G}$ has a bad partition
Kleinberg Small World	Yes, for any conn. $\bar{G}$, if $r < r_{\max}(\bar{G})$

Table 1: Conditions for expansion under several perturbations

1.1 Results and applications

The main technical development in this paper is a technique for understanding when randomly perturbed graphs exhibit expansion properties. This is motivated by the success of expansion bounds on more traditional random instances. For sufficiently dense Erdős-Rényi graphs, the First Moment Method provides a simple way to obtain a **whp** lower-bound on expansion. This paper provides a new method of accounting that permits a similar First-Moment-Method approach to be employed on randomly perturbed graphs.

For clear presentation, this new application of the First Moment Method is presented in the proof of an expansion property for a random graph G formed by perturbing any connected graph $\bar{G}$ by adding a random 1-out (which is the graph formed by adding an edge out of every vertex to another vertex chosen uniformly at random, and ignoring the directions of the edges).

Theorem 1 *For any sufficiently small $\delta > 0$, for any n-vertex connected graph $\bar{G}$, and for $R \sim \mathbb{G}_{n,\text{1-out}}$, the perturbed graph $G = \bar{G} + R$ has the following property* **whp***: for all $S \subset V$ with $|S| \leq \frac{3}{4}n$, at least $\delta|S|$ edges go between S and $\bar{S}$.*

This technology is also applied to similar random graphs, to yield results summarized in Table 1.

With the additional assumption that $\bar{G}$ does not contain any set $S \subset V$ with $|S| > (1-\epsilon)n$ and $2e_{\bar{G}}(S) + e_{\bar{G}}(S, \bar{S}) \leq |E(\bar{G})|$, the same technique can show that the perturbed graphs which are expanding are also rapidly mixing.

1.2 History of expansion in random graphs

A close connection between edge expansion, vertex expansion, spectral gap, and mixing time has emerged over the last 40 years [13, 3, 1, 36, 32, 40, 14]. Through this link, many different results on random graphs can be related to expansion properties. In regular and nearly-regular graphs, bounds on the second-largest eigenvalue of the adjacency matrix give bounds on expansion, [28, 29, 27, 2, 26, 22]. In a graph with a power-law degree distribution or other far-from-regular graphs, the eigenvalues of the adjacency matrix are not necessarily related to eigenvalues of the Laplacian and expansion. Both have been investigated theoretically and experimentally in recent years [21, 35, 8, 33, 17, 16, 34, 23].

In the empirical study of complex networks occurring in the real world, examining Laplacian eigenvalues has revealed that some real networks are expanders and others are not [10, 20]. This has led to the development of web graph models which specifically avoid being good expanders [24].

Algorithmically, there are many benefits to knowing that a graph is an expander (for example, rapid mixing, disjoint paths and routing, and robustness to attacks) and there are many other benefits to knowing that a graph is not an expander (for example, high-quality cuts, divide-and-conquer algorithms, and compressing data). Expansion may be less universal to real-world graphs than the some other properties observed empirically like local clustering and power-law degree distributions.

1.3 Notation

Undirected edges are sets of 2 vertices, but edge $\{u, v\}$ will be abbreviated as uv when it is not confusing to do so. For any graph H, let $E(H)$ denote the edge set of H, let $V(H)$ denote the vertex set of H, and for sets $S, T \subseteq V(H)$, let $e_H(S, T)$ denote the number of edges between S and T in H, and let $e_H(s)$ denote the number of edges in $E(H[S])$. Let $\deg_H(v)$ denote the degree of v in H. The subscripts for $e(S, T)$, $e(S)$, and $\deg(v)$ will be omitted when referring the graph G if it is not too confusing to do so.

1.4 Distributions for random graphs

Perturbed graph 1 ($\mathcal{P}_1$): The randomly perturbed graph that appears in Theorem 1 is a random graph generated by starting with base graph $\bar{G}$ and adding a random 1-out ($\mathbb{G}_{n,\text{1-out}}$ is the distribution of random graphs where every vertex chooses a neighbor uniformly at random and adds a edge to it.) The random graph $G = \bar{G} + R$ where $R \sim \mathbb{G}_{n,\text{1-out}}$ is studied primarily to illustrate the central technique of this paper, although it is a reasonably small perturbation. On average it changes the degree of every vertex by 2.

Perturbed graph 2 ($\mathcal{P}_2$): In the context of studying the effects of randomness in complex networks without making drastic assumptions about the distribution of randomness, it would be better to use a perturbation that does not change the base graph as much as a 1-out does. This can be accomplished by starting with base graph $\bar{G}$ and adding a sparse Erdős-Rényi random graph ($\mathbb{G}_{n,\epsilon/n}$ is the distribution of random graphs where each of the $\binom{n}{2}$ candidate edges appears independently with probability ϵ/n.) The random graph $G = \bar{G} + R$ where $R \sim \mathbb{G}_{n,\epsilon/n}$ is studied in [25], which shows that **whp** $\operatorname{diam}(G) = \mathcal{O}(\epsilon^{-1} \log n)$. Since, on average, this perturbation changes the degree of every vertex by only ϵ, the local effects of the perturbation are quite minimal.

Small-world graph 1 ($\mathcal{SW}_1$): The small-world model of Watts and Strogatz is generated by starting with a base graph $\bar{G}$ and an ordering of the edges $E(\bar{G})$ (in [39] $\bar{G}$ is a ring of n vertices with each vertex connected to its k nearest neighbors with $k \gg \ln n$, and the edges are ordered in a particular way that is implicit in the description of the perturbation). The base graph is perturbed in the following fashion: proceed through the edges according to the ordering, and for each edge, with probability p, randomly rewire this edge to a vertex chosen uniformly at random, with duplicate edges forbidden; otherwise leave the edge in place.

Small-world graph 2 ($\mathcal{SW}_2$): Kleinberg's small-world graph is a random digraph generated by starting with a base graph $\bar{G}$ and a distance function $d(\cdot,\cdot)$ on the vertices of $V(\bar{G})$ (in [30] $\bar{G}$ is primarily taken to be an $n \times n$ grid, where $V = [n]^2$, and uv is an edge if $d_1(u,v) \leq p$; the distance function is taken to be the ℓ_1 norm). The base graph is perturbed by adding q random edges out of every vertex independently at random, where the i-th edge out of vertex v is denoted by $e_{v,i}$ and is chosen according to the distribution $\Pr[e_{v,i} = vw] = d(v,w)^{-r} / \left(\sum_{u \neq v} d(v,u)^{-r}\right)$ for all $w \neq v$.

Comparison of $\mathcal{SW}_1$ and $\mathcal{SW}_2$: $\mathcal{SW}_2$ is often viewed as a generalization of $\mathcal{SW}_1$. The big difference is that while $\mathcal{SW}_1$ rewires edges uniformly at random, $\mathcal{SW}_2$ includes the parameter r, which controls the degree to which the underlying network is willing to try new things.

There is also a subtle difference between these two models. While $\mathcal{SW}_1$ randomly rewires each edge of the underlying graph with probability p (which, for an r-regular graph, results in rp random edges expected out of each vertex), $\mathcal{SW}_2$ *adds* q random edges out of each vertex. This sounds very similar for $q = rp$, and it is similar, but it is also different in a very important way. Graphs from the $\mathcal{SW}_2$ distribution are expanders **whp**, while graphs from the $\mathcal{SW}_1$ distribution are not necessarily so.

1.5 Outline of what follows

Section 2 proceeds with the proof of Theorem 1, which uses the new method of First-Moment-Method accounting to show that $G = \bar{G} + R$ has $e(S, \bar{S}) \geq \delta|S|$ for all S **whp** when R is a 1-out ($\mathcal{P}_1$).

Section 3 considers the more gentle perturbation, where R is distributed as $\mathbb{G}_{n,\epsilon/n}$ instead of as a 1-out. In this case, G is not necessarily an expander, and a criteria for $\bar{G}$ of having a "bad partition" is shown to prevent G from satisfying the expansion property **whp**. The same results are also shown to hold for Watts-Strogatz random graphs ($\mathcal{SW}_1$). In particular, when $\bar{G}$ is a cycle with edges connecting each vertex to its k nearest neighbors, or when $\bar{G}$ is a d-dimensional grid, it contains a bad partition and hence the perturbed graph is not an expander **whp**.

Section 4 considers the $\mathcal{SW}_2$ perturbation, where $\bar{G}$ is perturbed by a non-uniform q-out, in which each random edge out of v chooses a vertex w with probability related to distance from v to w under some distance function $d(\cdot,\cdot)$ according to $\Pr[e_{v,i} = vu] = d(v,u)^{-r} / \sum_{w \neq v} d(v,w)^{-r}$. For $q = 1$ and $r = 0$, this reduces to the base-graph-plus-1-out considered in Section 2, and the techniques from that section are shown to extend for $r > 0$ for grid-like graphs. These techniques show that when $\bar{G}$ is a d-dimensional grid, G is an expander for any $r < d$, and furthermore, there is a threshold at $r = d$, at which point G is no longer an expander.

This shows that the transition from expanding to non-expanding occurs precisely at the point where a local algorithm can find polylogarithmic length paths in the network.

2 Perturbing any connected $\bar{G}$ with a 1-out yields expander

The proof of Theorem 1 is an application of the First Moment Method, and relies on a moderately precise calculation of the expected number of sets S which violate the bound $e(S, \bar{S}) \leq \delta|S|$. This is achieved by considering separately the sets with $|S| \leq \gamma n$ and $|S| > \gamma n$ for an appropriately chosen constant γ.

Proof of Theorem 1: A straightforward way to obtain an upper bound on the probability that there exists a set $S \subseteq V$ with $|S| \leq \frac{3}{4}n$ and $e(S, \bar{S}) \leq \delta|S|$ is the following: let Z_S be an indicator random variable for the event that a particular set S satisfies these conditions, and calculate an upper bound on the expected value of the sum $Z = \sum_{S \subseteq V} Z_S$. Showing that the expected value tends to 0 with n yields a bound which proves the theorem, because for any non-negative random variable, $\Pr[Z \geq 1] \leq \mathrm{E}[Z]$ (this deceptively simple inequality is honored with the title "The First Moment Method").

Unlike the simple application of the First Moment Method which is sufficient to show that $G \sim \mathbb{G}_{n,k\text{-out}}$ is likely to be an expander when k is a sufficiently large constant, making this calculation precise enough to yield results about a perturbed graph requires an examination of the structure of the set S in the base graph $\bar{G}$.

The key trick is to use a special tour of $\bar{G}$ to describe each set S; let T be a spanning tree of $\bar{G}$, and let $W = (e_1, e_2, \dots, e_{2m})$ be an Euler tour of the multigraph formed by including every edge of T twice. That is to say that W is a sequence of edges which gives a circuit in G that traverses each edge of T exactly 2 times. Such a tour exists because doubling every edge of T makes the degree of every vertex even. For any set S, let $\mathbf{I}_S \in \{0,1\}^{2(n-1)}$ be the incidence vector with $\mathbf{I}_S(i) = 1$ iff $e_i \in E(T[S])$. Let $e(\mathbf{I}_S) = |\{i : \mathbf{I}_S(i) \neq \mathbf{I}_S(i+1)\}|$ denote the number of times the Euler tour crosses the boundary of S. There is a direct relationship between $e_T(S, \bar{S})$ and $e(\mathbf{I}_S)$. Since each edge of T appears twice in W,

$$e_{\bar{G}}(S, \bar{S}) \geq e_T(S, \bar{S}) \geq e(\mathbf{I}_S)/2. \tag{1}$$

To obtain a bound on the expected value of the sum $\sum_{S:|S|=s} Z_S$, let

$$\mathcal{S}_{s,k} = \{S : |S| = s, e(\mathbf{I}_S) = k\}$$

denote the collection of sets S of size s for which T crosses the boundary of S exactly k times. Since every S maps to a unique $\mathbf{I}_S$, it follows that

$$|\mathcal{S}_{s,k}| \leq 2\binom{2n}{k},$$

because an incidence vector with k changes in value can be described by giving the k "change positions" and specifying if the first bit is a 0 or a 1.

For $S \in \mathcal{S}_{s,k}$, equation (1) shows that $e_{\bar{G}}(S, \bar{S}) \geq k/2$, so in order to have $e(S, \bar{S}) \leq \delta s$, it is necessary that $e_R(S, \bar{S}) \leq \delta s - k/2$. This is impossible when $k > 2\delta s$, which leads to the following:

$$\begin{aligned}
\sum_{S:\,|S|=s} \mathrm{E}\left[Z_S\right] &\le \sum_{k=1}^{2\delta s} \sum_{S \in \mathcal{S}_{s,k}} \Pr\left[e_R(S,\bar{S}) \le \delta s\right] \\
&\le \sum_{k=1}^{2\delta s} 2\binom{2n}{k} \Pr\left[e_R(S,\bar{S}) \le \delta s\right] \\
&\le (4\delta s)\binom{2n}{2\delta s} \Pr\left[e_R(S,\bar{S}) \le \delta s\right] \\
&\le n\left(\frac{ne}{\delta s}\right)^{2\delta s} \Pr\left[e_R(S,\bar{S}) \le \delta s\right] \qquad \text{for } \delta \le 1/4.
\end{aligned}$$

Finishing the calculation requires an upper-bound on $\Pr\left[e_R(S,\bar{S}) \le \delta s\right]$, for which it is necessary to consider separately the large and small sets S.

Large sets expand: When $s = |S| \ge \gamma n$, $\mathrm{E}[e_R(S,\bar{S})] = s\left(1 - \frac{s}{n}\right)$ and Chernoff's bound gives

$$\Pr\left[e_R(S,\bar{S}) \le \delta s\right] \le \exp\left\{-s\left(1-\frac{s}{n}\right)\left(1-\frac{\delta}{1-s/n}\right)^2 \Big/ 2\right\}.$$

So, for $\gamma n \le s \le \frac{3}{4}n$,

$$\sum_{S:|S|=s} \mathrm{E}[Z_S] \le n\left[\left(\frac{e}{\delta\gamma}\right)^{2\delta} \exp\left\{-\frac{(1-4\delta)^2}{8}\right\}\right]^s.$$

For any constant γ, if δ is a sufficiently small constant then this upper-bound is exponentially small in n.

Small sets expand: When $s = |S| \le \gamma n$, a tighter bound on the probability can be obtained directly by

$$\Pr\left[e_R(S,\bar{S}) \le \delta s\right] \le \binom{s}{\delta s}\left(\frac{s}{n}\right)^{s-\delta s} \le \left[\left(\frac{e}{\delta}\right)^{\delta}\left(\frac{s}{n}\right)^{1-\delta}\right]^s.$$

So

$$\sum_{S:|S|=s} \mathrm{E}[Z_S] \le n\left[\left(\frac{ne}{\delta s}\right)^{2\delta}\left(\frac{e}{\delta}\right)^{\delta}\left(\frac{s}{n}\right)^{1-\delta}\right]^s = n\left[\left(\frac{e}{\delta}\right)^{3\delta}\left(\frac{s}{n}\right)^{1-3\delta}\right]^s.$$

For $\frac{3}{1-3\delta} \le s \le \gamma n$ and δ sufficiently small, this upper-bound is $o(1/n)$.

Tiny sets expand: For $\delta \le \frac{1}{6}$, the tiny sets S, of size $s \le \frac{3}{1-3\delta}$, will satisfy $e(S,\bar{S}) \ge \delta s$ because the base graph $\bar{G}$ is connected and so $e(S,\bar{S}) \ge 1 \ge \delta\frac{3}{1-3\delta}$.

Putting this all together shows that there exists $\delta > 0$ such that

$$\Pr\left[\text{exists } S : |S| \le \frac{3}{4}n \text{ and } e(S,\bar{S}) \le \delta|S|\right] \le \sum_{s=1}^{\frac{3}{4}n} \sum_{S:|S|=s} \mathrm{E}[Z_S] = o(1).$$

□

3 Gentler perturbation does not necessarily yield expander

Adding a 1-out to a graph increases the average degree of a vertex by 2. This is not much, but it is something. This section investigates the effects of perturbing $\bar{G}$ by adding a random instance of $\mathbb{G}_{n,\epsilon/n}$ (which is the Erdős-Rényi graph where every candidate edge is included independently with probability ϵ/n). The intention of the parameterization ϵ/n is to indicate that ϵ should be thought of as a small constant, although the results of this section apply to any constant ϵ.

An attempt to show that if $R \sim \mathbb{G}_{n,\epsilon/n}$ then the perturbed graph $G = \bar{G} + R$ is an expander can begin by following in the footsteps of the proof of Theorem 1. And such a proof attempt will succeed in showing that **whp** the large sets in G expand.

Theorem 2 *For any $\epsilon > 0$, for any sufficiently small $\delta > 0$, for any n-vertex connected graph $\bar{G}$, and for $R \sim \mathbb{G}_{n,\epsilon/n}$, the perturbed graph $G = \bar{G}+R$ has the following property* **whp***: for all $S \subseteq V$ with $e^{-\epsilon/64\delta}n \le |S| \le \frac{3}{4}n$, at least $\delta|S|$ edges go between S and $\bar{S}$.*

Proof Follow the proof of Theorem 1. The only new calculation this proof requires is a fresh application of Chernoff's bound. For this perturbation, $\mathrm{E}[e_R(S,\bar{S})] = \epsilon s\left(1 - \frac{s}{n}\right)$, and so

$$\Pr[e_R(S,\bar{S}) \le \delta s] \le \exp\left\{-\epsilon s\left(1-\frac{s}{n}\right)\left(1-\frac{\delta}{\epsilon(1-s/n)}\right)^2 \Big/ 2\right\}.$$

□

However, following the proof of Theorem 1 does not succeed in showing that small sets expand. And indeed, it should not show this, because it is not necessarily true. If $\bar{G}$ has a *bad partition* (defined below) then **whp** G is not an expander.

Definition 1 *$\bar{G}$ has a δ-*bad partition* iff $V(\bar{G})$ can be partitioned into $(S_1, \ldots, S_k, \bar{S})$ for which the following inequalities hold:*

$$
\begin{aligned}
|S_i| &\le \frac{1}{2}\epsilon^{-1}\ln n, && \text{for } i = 1, \ldots, k; \\
e_{\bar{G}}(S_i, \bar{S}_i) &< \delta |S_i|, && \text{for } i = 1, \ldots, k; \\
k &= \omega(n^{1/2}).
\end{aligned}
$$

Theorem 3 *For any $\epsilon > 0$, any $\bar{G}$, and $R \sim \mathbb{G}_{n,\epsilon/n}$, if $\bar{G}$ has a δ-bad partition, then* **whp** *there exists $i \in \{1, \ldots, k\}$ such that $e_R(S_i, \bar{S}_i) = 0$, and hence $G = \bar{G} + R$ has $e(S_i, \bar{S}_i) < \delta|S_i|$.*

The proof is a direct application of the Second Moment Method and is included in Appendix A.

This theorem applies to show that, for example, when $\bar{G}$ is the d-dimensional grid graph, the perturbed graph is not an expander **whp**.

Corollary 1 *Let $\bar{G}$ be a d-dimensional grid on $N = n^d$ vertices and let $R \sim \mathbb{G}_{n,\epsilon/n}$ for any $\epsilon > 0$. Then, for any $\delta > 0$,* **whp** *the graph $G = \bar{G} + R$ has some $S \subseteq V$ with $|S| = \frac{1}{2}\epsilon \ln N$ and $e(S, \bar{S}) < \delta|S|$.*

Proof Partition $V(\bar{G})$ into subcubes each containing $\ln n$ vertices. Each subcube S_i has sides of length $(\ln n)^{1/d}$, and, for any constant δ and n sufficiently large, $e_{\bar{G}}(S_i, \bar{S}_i) = \mathcal{O}((\ln n)^{(d-1)/d} < \delta \ln n$. □

On the other hand, if $\bar{G}$ is a graph such that all small partitions satisfying the expansion condition, then Theorem 2 is sufficient to show that G is an expander **whp**. For example, if $\bar{G}$ consists of 2 expander graphs, each on $n/2$ disjoint vertices, that are joined by a single edge, then G will be an expander **whp**.

The proof of Theorem 3 goes through without modification to show that the model studied by Watts and Strogatz ($\mathcal{SW}_1$ with $\bar{G}$ a k-connected cycle) is not an expander for any δ if k is any constant. On the other hand, when $k \gg \ln n$ (as in the original Watts-Strogatz specification), every vertex has at least 1 edge randomly rewired, so it follows from Theorem 1 that the resulting graph is an expander.

4 Conditions for expansion in Kleinberg's small-world graph

In $\mathcal{SW}_2$, when $r = 0$ and $q = 1$, this is exactly the case treated in Theorem 1. Making q larger only increases the number of edges across any cut, so any connected base graph leads to an expander when $r = 0$.

For $r > 0$, the proof of expansion can proceed as in the proof of Theorem 1 provided there is a upper-bound on $\Pr[e_{v,i} \in S]$ with any constants $C > 0$ and $\epsilon > 0$ of the form:

$$\text{for any } v \in V \text{ and } S \subseteq V \text{ with } |S| = s,\ \Pr[e_{v,i} \in S] \leq C \left(\frac{s}{n}\right)^{\epsilon}.$$

When the metric is the ℓ_1 norm on the lattice $[n]^d$, such a bound exists for any $r < d$:

Theorem 4 *Let $V = [n]^d$, and let $d(u, v) = \sum_{i=1}^{d} |u_i - v_i|$. Then, for any $0 \leq r < d$, and for any $v \in V$ and $S \subseteq V$ with $|S| = s$, $\Pr[e_{v,i} \in S] \leq C \left(\frac{s}{n}\right)^{d-r-1}$.*

The proof is a direct calculation and appears in Appendix B.

The upper-bound on r given in this bound is tight, and when $r = d$, the resulting graph is not an expander.

Theorem 5 *For $\bar{G}$ an $n \times n$ grid, $d(\cdot, \cdot) = d_1(\cdot, \cdot)$, and $r = 2$, Kleinberg's small-world graph is not rapidly mixing* **whp**.

Proof To verify the theorem, consider the set $S = \{(x, y) : x + y \leq k\}$, where $k = n/\ln n$, and calculate an upper-bound on the expected number of random edges between S and $\bar{S}$. This calculation can be simplified by considering sets $S_\ell = \{(x, y) \in V(\bar{G}) : x + y = \ell\}$. For any i and j with $i \leq k \leq j$,

$$\begin{aligned}
\sum_{v \in S_i} \sum_{w \in S_j} d_1(v, w)^{-2} &\leq |S_i| \left((j - i) \frac{1}{(j - i)^2} + 2 \sum_{\ell=1}^{|S_j| - (j-i)} \frac{1}{(j - i + 2\ell)^2} \right) \\
&= i \left(\frac{1}{j - i} + 2 \sum_{\ell=1}^{i} \frac{1}{(j - i + 2\ell)^2} \right) \\
&\leq 2i \left(\frac{1}{j - i} \right).
\end{aligned}$$

Also, for any $v \in V(\bar{G})$,

$$\sum_{w \neq v} d_1(v,w)^{-2} = \Theta(\ln n).$$

Thus, an upper-bound on the expected number of random edges between S and $\bar{S}$ is given by the following

$$\begin{aligned}
\mathrm{E}[e_R(S,\bar{S})] &= \sum_{i=1}^{k} \sum_{j=k+1}^{n} \sum_{v \in S_i} \sum_{w \in S_j} q \left(\frac{d_1(v,w)^{-2}}{\sum_{u \neq v} d_1(v,u)^{-2}} + \frac{d_1(w,v)^{-2}}{\sum_{u \neq w} d_1(w,u)^{-2}} \right) \\
&\leq (2q)\Theta\left((\ln n)^{-1}\right) \sum_{i=1}^{k} \sum_{j=k+1}^{n} 2i \left(\frac{1}{j-i} \right) \\
&\leq (4qk)\Theta\left((\ln n)^{-1}\right) \sum_{i=1}^{k} (H_n - H_i) \\
&= (4qk)\Theta\left((\ln n)^{-1}\right) (k + k(H_n - H_k)) \\
&= \Theta\left(\frac{k^2 \ln \ln n}{\ln n} \right).
\end{aligned}$$

Since $e_{\bar{G}}(S,\bar{S}) = \mathcal{O}(k)$, Markov's inequality shows that for any constant δ with $\delta > 0$, **whp** $e(S,\bar{S}) \leq \delta |S|$. □

5 Conclusion

It is necessary to conclude that the expansion of a randomly perturbed graph depends on the base graph and the perturbation, and even seemingly similar perturbations can produce vastly different results. Though adding a random 1-out makes any connected graph an expander **whp**, such a simple statement is impossible for the more gentle perturbation of adding a random $\mathbb{G}_{n,\epsilon/n}$. This is not a bad thing, however, as empirical observations show that among complex networks in the real world, some are expanders and others are not.

In the case of the small-world models of Watts and Strogatz and of Kleinberg, the difference in the distributions is quite subtle. Generally Kleinberg's model is viewed as a strict generalization of Watts and Strogatz', but in the context of expansion, the models are actually just different.

It is a pleasant surprise that Kleinberg's model stops being an expander exactly at the point where it becomes possible to find short paths with a decentralized algorithm. Perhaps the expansion threshold and existence of decentralized algorithms are fundamentally related in some way. But more likely not.

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A Proof of Theorem 3

The proof is an application of the Second Moment Method. Let Z_i be an indicator random variable for the event $e_R(S_i, \bar{S}_i) = 0$, and let $Z = \sum_{i=1}^k Z_i$. Then the probability that there exists $i \in \{1, \ldots, k\}$ such that S_i fails to expand is $\Pr[Z = 0] \geq \mathrm{E}[Z]^2 / \mathrm{E}[Z^2]$.

Let $s_i = |S_i|$.

$$\mathrm{E}[Z] = \sum_{i=1}^k \Pr[e_R(S_i, \bar{S}_i) = 0] = \sum_{i=1}^k (1 - \epsilon/n)^{s_i(n-s_i)}.$$

and

$$\begin{aligned}
\mathrm{E}[Z^2] &= \sum_{i=1}^k \sum_{j=1}^k \Pr\left[e_R(S_i, \bar{S}_i) = 0\right] \Pr\left[e_R(S_j, \bar{S}_j) = 0 \,\middle|\, e_R(S_i, \bar{S}_i) = 0\right] \\
&= \sum_{i=1}^k (1 - \epsilon/n)^{s_i(n-s_i)} \left(1 + \sum_{j \neq i} (1 - \epsilon/n)^{s_j(n-s_i-s_j)}\right) \\
&\leq \sum_{i=1}^k (1 - \epsilon/n)^{s_i(n-s_i)} \left(1 + (1 + o(1)) \sum_{j=1}^k (1 - \epsilon/n)^{s_j(n-s_j)}\right) \\
&\leq (1 + o(1)) \left(1 + \sum_{i=1}^k (1 - \epsilon/n)^{s_i(n-s_i)}\right)^2 \\
&= (1 + o(1))(1 + \mathrm{E}[Z])^2.
\end{aligned}$$

Since $\mathrm{E}[Z] \geq (1 - o(1)) k e^{-\ln n/2} \to \infty$,

$$\Pr[Z = 0] \geq (1 - o(1)) \frac{\mathrm{E}[Z]^2}{(1 + \mathrm{E}[Z])^2} = 1 - o(1).$$

□

B Proof of Theorem 4

For any $v \in V$, there are $\Theta(\ell^d)$ vertices with distance from v of at most ℓ, and there are $\Theta(\ell^{d-1})$ vertices with distance from v exactly ℓ. For any

$v \in V$ and $S \subseteq V$, let $s = |S|$, let s_ℓ be the number of vertices of S that are at distance ℓ from v and let n_ℓ be the number of vertices of V that are at distance ℓ from v:

$$s_\ell = |\{w \in S : d_1(v,w) = \ell\}|, \qquad n_\ell = |\{w \in V : d_1(v,w) = \ell\}|.$$

Then

$$\begin{aligned}
\Pr[e_{v,i} \in S] &= \left(\sum_{w \in S} d_1(v,w)^{-r}\right) \Bigg/ \left(\sum_{u \neq v} d_1(v,u)^{-r}\right) \\
&= \left(\sum_{\ell=1}^{dn} s_\ell \ell^{-r}\right) \Bigg/ \left(\sum_{\ell=1}^{dn} n_\ell \ell^{-r}\right) \\
&\leq \left(\sum_{\ell=1}^{\Theta(s^{1/d})} \Theta\left(\ell^{d-1}\right) \ell^{-r}\right) \Bigg/ \left(\sum_{\ell=1}^{\Theta(|V|^{1/d})} \Theta\left(\ell^{d-1}\right) \ell^{-r}\right) \\
&= \Theta\left(\frac{s^{(d-r)/d}}{|V|^{(d-r)/d}}\right).
\end{aligned}$$

□