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Expansion of a high-frequency time-harmonic wavefield given on an initial surface into Gaussian beams

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Summary. A high-frequency asymptotic expansion of a time-harmonic wavefield given on a curved initial surface into Gaussian beams is determined. The time-harmonic wavefield is assumed to be specified on the initial surface in terms of a complex-valued amplitude and a phase. The asymptotic expansion has the form of a two-parametric integral superposition of Gaussian beams. The expansion corresponds to the relevant ray approximation in all regions, where the ray solution is sufficiently regular (smooth) in effective regions of the beams under consideration.

1 Introduction

Gaussian beams are the high-frequency asymptotic time-harmonic solutions of the elastodynamic equations concentrated close to rays. The solutions concentrated close to rays were first investigated by Babich (1968) and applied to elastodynamic equations by Kirpichnikova (1971). For details refer to Červený (1981) or Červený & Pšenčík (1983). A large number of references is given in Červený (1983).

In the Gaussian beam method, the high-frequency initial conditions for solving elastodynamic equations are decomposed into the initial conditions for Gaussian beams, the Gaussian beams are evaluated and the high-frequency asymptotic solution is obtained as a superposition of these beams. The method of Gaussian beams is shown to be a powerful generalization of the ray method. Its results correspond to the Maslov theory applied in a general 3-D subspace of a 6-D complex phase space. The Gaussian beam method has no caustics, only its limiting cases like the ray method or asymptotic solutions in a real mixed subspace of a phase space (Chapman & Drummond 1982) may have caustics. It is capable of providing a uniform asymptotic solution even without any blending of various asymptotic solutions with weighting functions.

Since we intend to study the choice of the initial conditions for Gaussian beams in this paper in the first place, we shall consider perfectly elastic media with smooth elastic parameters: density ρ, velocity v_P of P-waves and velocity v_S of S-waves. Dynamic ray tracing across curved interfaces is described in Červený & Hron (1980) and the transformation of Gaussian beams at curved interfaces is described in Červený (1984).

The initial high-frequency conditions for solving elastodynamic equations given on a curved initial surface are assumed to be decomposed into initial conditions for P- and S-waves. Each of these initial conditions with the corresponding wave is considered independently and is assumed to be described along the initial surface in terms of the complex-valued amplitude and the phase, see (28).

The paper is divided into four sections. Some quantities used frequently throughout the paper are introduced in Section 2. The reader is required to pay attention to the notation of vectors and matrices used. The notation is introduced at the beginning of Section 2. Section 3 contains the expressions for one Gaussian beam.

The high-frequency expansion of the wavefield given on an initial surface into Gaussian beams is performed in Section 4. The wavefield on the initial surface is assumed to be expressed in terms of the complex-valued amplitude and the phase. For comparison see also Kravtsov & Orlov (1980), Popov (1982) and Chapman & Drummond (1982). The obtained results involve the Maslov method of Chapman & Drummond (1982) as a special limiting case for infinitely broad Gaussian beams. The application of Gaussian beams with a finite width has very much the same sense as applying interpolation between individual paraxial approximations in the ray method and/or as applying a Gaussian integration window in the Chapman–Maslov method.

The parameters specifying the shape of the Gaussian beams are supposed to be given so that we can use the paraxial ray approximation in the effective regions of the beams along the screen. Some optimization of the parameters will be described elsewhere.

The expansion formulae derived in this paper may also be obtained by applying the Maslov asymptotic theory to a general 3-D subspace of the 6-D complex phase space. The rederivation using the Maslov theory is shown in Klimeš (1984).

For simple numerical examples see Červený & Klimeš (1984).

2 The specification of some used quantities

The capital-letter indices will take the values 1 and 2, lower-case indices will take the values 1, 2, 3. The indices will have the form of right-hand suffices. For instance $f(x_i) = f(x_1, x_2, x_3)$, $f(x_A) = f(x_1, x_2)$ and, for any function $f(x_i)$, $f|_{x_j=0} = f(0, 0, 0)$, $f|_{x_B=0} = f(0, 0, x_3)$ may be used. Pairs of identical indices will indicate summing. This means that we shall use the Einstein summation convention for the suffices instead of writing the summation symbol.

2×2 matrices with components A_{AB} will be parallely denoted by the symbols **A** or A_{AB}. 3×3 matrices will always be described by means of their components. The symbol A^{-1}_{AB} will indicate the components of the matrix inverse to A_{AB}, i.e.

$$A^{-1}_{AB} A_{BC} = \delta_{AC}, \tag{1}$$

where δ_{AC} is the Kronecker delta symbol. In other words, δ_{AB} denotes the components of the unit matrix. $A^{T}_{AB} = A_{BA}$ denote the components of the matrix transposed to A_{AB}.

We shall use three important coordinate systems throughout this paper:

(1) Cartesian coordinates x_i.

(2) Ray coordinates $\gamma_i = (\gamma_1, \gamma_2, \gamma_3 = \sigma)$, where $\gamma_A = (\gamma_1, \gamma_2)$ are the parameters of the ray (e.g. the take-off angles at a point source specifying the initial direction of rays or the coordinates along an initial surface), and σ is the coordinate along the ray connected with the travel time τ or with the arclength s by the relations

$$\sigma = \sigma_0 + \int_{\tau_0}^{\tau} v^2 d\tau = \sigma_0 + \int_{s_0}^{s} v ds. \tag{2}$$

Here v is the velocity of propagation of the corresponding wave.

(3) Orthogonal ray-centred coordinate system along a chosen ray. This coordinate system

and its computation is described in Popov & Pšenčík (1978a, b) and also in Červený & Hron (1980). We shall denote the ray-centred coordinates by $q_i = (q_1, q_2, q_3 = s)$ where s is the arclength along the ray and $q_A = (q_1, q_2)$ are the Cartesian coordinates in the plane perpendicular to the ray at points $s = q_3$ with the origin on the ray.

The travel time corresponding to the relevant ray approximation will be denoted by τ. We define the Cartesian components of the slowness vector

$$p_i = \frac{\partial \tau}{\partial x_i} . \tag{3}$$

The velocity of propagation of the appropriate elementary wave is denoted by v. The velocity on the central ray will be denoted by

$$V(s) = v \,|_{q_A = 0} . \tag{4}$$

We also denote the components of the velocity gradient in the ray-centred coordinate system on the central ray by

$$V_i(s) = \frac{\partial v}{\partial q_i}\bigg|_{q_A=0} = \frac{\partial x_k}{\partial q_i}\frac{\partial v}{\partial x_k}\bigg|_{q_A=0} . \tag{5}$$

Similarly we define the second derivatives of the velocity in the local Cartesian base of the ray-centred coordinate system taken on the central ray

$$V_{ij}(s) = \frac{\partial x_k}{\partial q_i}\frac{\partial x_l}{\partial q_j}\frac{\partial^2 v}{\partial x_k \partial x_l}\bigg|_{q_A=0} . \tag{6}$$

Note that the matrix

$$H_{ij} = \frac{\partial x_i}{\partial q_j} = \frac{\partial q_j}{\partial x_i} \tag{7}$$

is the unitary matrix, the columns of which constitute the local vector basis of the ray centred coordinate system expressed in general Cartesian coordinates x_i.

We define the matrix of transformation from the ray coordinates to the ray-centred coordinate system as

$$Q^R_{AB} = \frac{\partial q_A}{\partial \gamma_B}\bigg|_{q_C=0} . \tag{8}$$

We shall call it the matrix of geometrical spreading.

The components of the matrix

$$M^R_{AB} = \frac{\partial^2 \tau}{\partial q_A \partial q_B}\bigg|_{q_C=0} \tag{9}$$

are the second derivatives of the travel-time field in the ray-centred coordinate system.

We also define the matrix of the transformation from the ray coordinates to the components of the slowness vector in the ray-centred coordinate system

$$P^R_{AB} = \frac{\partial^2 \tau}{\partial \gamma_B \partial q_A}\bigg|_{q_C=0} = M^R_{AC}\, Q^R_{CB} . \tag{10}$$

Matrix $\mathbf{P}^R$ is related to matrix $\mathbf{Q}^R$ by means of ordinary differential equations called the dynamic ray tracing system:

$$\frac{d}{d\sigma} Q^R_{AB} = P^R_{AB}$$

$$\frac{d}{d\sigma} P^R_{AB} = -\frac{V_{AC}}{V^3} Q^R_{CB}. \tag{11}$$

Equations (11) follow from definitions (8) and (10) of matrices $\mathbf{P}^R$, $\mathbf{Q}^R$ and from the ray tracing system

$$\frac{d}{d\sigma} x_i = p_i$$

$$\frac{d}{d\sigma} p_i = -\frac{1}{v^3} \frac{\partial v}{\partial x_i}. \tag{12}$$

Equations (11) and (12) can also be directly used for numerical computation of the rays and of matrices $\mathbf{P}^R$, $\mathbf{Q}^R$ along the rays. The initial conditions for the numerical integration of the ordinary differential equations (11) must be chosen in accordance with (8) and (10) in this case. Considering (10), we can evaluate matrix $\mathbf{M}^R$ using the relation

$$\mathbf{M}^R = \mathbf{P}^R(\mathbf{Q}^R)^{-1}. \tag{13}$$

For the fixed point $s = s_1$ at the ray, we can transfer from the ray-centred coordinate system to the corresponding local Cartesian coordinates

$$r_i = H_{ji}(s_1)[x_j - \tilde{x}_j(q_A = 0, q_3 = s_1)], \tag{14}$$

where $\tilde{x}_j(q_k)$ are the general Cartesian coordinates x_j corresponding to point q_k in the ray-centred coordinate system. In coordinate system (14), $r_A = q_A$ for $r_3 = 0$ and the third coordinate r_3 is measured along the tangent to the ray at point $s = s_1$.

Now we evaluate the second derivatives of the travel-time field in the local Cartesian coordinates r_i. Since $r_A = q_A$ for $r_3 = 0$, we obtain

$$\left.\frac{\partial^2 \tau}{\partial r_A\, \partial r_B}\right|_{r_i=0} = \left.\frac{\partial^2 \tau}{\partial q_A\, \partial q_B}\right|_{r_i=0} = \left. M^R_{AB}\right|_{s=s_1}. \tag{15}$$

As r_i are Cartesian coordinates,

$$\frac{\partial \tau}{\partial r_i} = p_i, \tag{16}$$

where, for a while, p_i are the components of a slowness vector transformed to the co-ordinates r_i. Using the ray tracing system (12), we obtain

$$\left.\frac{\partial^2 \tau}{\partial r_3 \partial r_i}\right|_{r_j=0} = \left.\frac{\partial p_i}{\partial s}\right|_{r_j=0} = -\left.\left(\frac{1}{v^2} \frac{\partial v}{\partial r_i}\right)\right|_{r_j=0} = -\left.\frac{V_i}{V^2}\right|_{s=s_1}. \tag{17}$$

For V_i and V see (4) and (5). We rewrite (15) and (17) in the compact form

$$\left.\frac{\partial^2 \tau}{\partial r_i \partial r_j}\right|_{q_A=0} = \begin{pmatrix} M_{11}^{\mathrm{R}}, & M_{12}^{\mathrm{R}}, & -V_1/V^2 \\ M_{12}^{\mathrm{R}}, & M_{22}^{\mathrm{R}}, & -V_2/V^2 \\ -V_1/V^2, & -V_2/V^2, & -V_3/V^2 \end{pmatrix} . \tag{18}$$

We can easily transfer from the local Cartesian coordinates r_i to the general Cartesian coordinates x_i by means of the relation

$$\frac{\partial^2 \tau}{\partial x_i \partial x_j} = H_{ik} H_{jl} \frac{\partial^2 \tau}{\partial r_k \partial r_l} , \tag{19}$$

where the transformation matrix H_{ik} from the local coordinates r_k to the general Cartesian coordinates x_i is given by (7).

3 Gaussian beams

Gaussian beams are the high-frequency asymptotic time-harmonic solutions of elastodynamic equations concentrated close to their central rays.

The most natural expressions for the Gaussian beams can be written in the orthogonal ray-centred coordinate system, because the polarization of the beams does not change in this coordinate system. The principal component of the complex-valued vector amplitude of the Gaussian beam has the form

$$g = \frac{C}{\sqrt{v\rho \det \mathbf{Q}}} \exp[i\omega(\tau + \tfrac{1}{2} q_A M_{AB} q_B - t)], \tag{20}$$

where C is the normalization factor, constant along the ray, which may depend on the ray parameters γ_A. The quantity v denotes the velocity of propagation of the corresponding elementary wave, ρ is the density, ω is the angular frequency (positive), τ is the travel time along the central ray, t is the time and M_{AB} is the complex-valued matrix given by

$$\mathbf{M} = \mathbf{P}\mathbf{Q}^{-1}. \tag{21}$$

The complex-valued matrices **P** and **Q** are the functions of the parameter σ along the central ray and must satisfy the dynamic ray tracing system (11) which reads

$$\frac{d\mathbf{Q}}{d\sigma} = \mathbf{P}, \quad \frac{d\mathbf{P}}{d\sigma} = -\frac{\mathbf{V}}{V^3}\mathbf{Q}. \tag{22}$$

The initial conditions at $\sigma = \sigma_0$ for solving (22) have to be chosen in such a way that matrix

$$\mathbf{M}(\sigma_0) = \mathbf{P}(\sigma_0)\mathbf{Q}^{-1}(\sigma_0) \tag{23}$$

is symmetric in both its real and imaginary parts, and its imaginary part $\mathrm{Im}\,\mathbf{M}(\sigma_0)$ is positive definite. The matrix **M** will then be symmetric along the whole ray and its imaginary part will be positive definite everywhere, see Červený & Pšenčík (1983).

In (20) we must take the same branch of the complex-valued square root along the whole ray.

We denote by u_i the complex-valued components of the displacement vector in the ray-centred coordinate system.

The compressional Gaussian beam has only one principal component

$$u_3 = \frac{C}{\sqrt{v\rho \det \mathbf{Q}}} \exp[i\omega(\tau + \tfrac{1}{2}q_A M_{AB} q_B - t)] \tag{24}$$

and two additional components

$$u_A = vM_{AB} q_B u_3. \tag{25}$$

The quantity v in (24) and (25) denotes the velocity of P-waves.

The shear Gaussian beam has two principal components

$$u_B = \frac{C_B}{\sqrt{v\rho \det \mathbf{Q}}} \exp[i\omega(\tau + \tfrac{1}{2}q_A M_{AB} q_B - t)] \tag{26}$$

and one additional component

$$u_3 = -vq_A M_{AB} u_B. \tag{27}$$

The quantity v in (26) and (27) denotes the velocity of S-waves.

The detailed derivation and description of the properties of the beams are given in Červený & Pšenčík (1983).

4 Expansion of a high-frequency time-harmonic wavefield given on an initial surface into Gaussian beams

4.1 WAVEFIELD GIVEN ON AN INITIAL SURFACE – THE CENTRAL RAYS OF BEAMS

Assume a curved initial surface parameterized by curvilinear coordinates ξ_1, ξ_2. The elastodynamic wavefield propagating from the initial surface into the model is specified on the initial surface. Assume the wavefield to be time-harmonic and to be decomposed into the P-wavefield and into two independent linearly polarized S-waves. We shall choose one of these elementary waves and expand it into Gaussian beams. The expansion will be found asymptotically for high frequencies ω.

Assume that the complex-valued amplitude of the selected elementary wave is specified on the initial surface in the following form:

$$\phi(\xi_A) = A(\xi_A) \exp\{i\omega[T(\xi_B) - t]\}. \tag{28}$$

We require functions $A(\xi_A)$, $T(\xi_B)$ to be at least so smooth, to be able to replace the function $A(\xi_A)$ by the linear expansion and the function $T(\xi_A)$ by the Taylor expansion up to the second order in the effective regions of the Gaussian beams used below. In the same way suppose that the initial surface can be locally described by the Taylor expansion up to the second order in the effective regions of the beams. The mentioned assumptions can be satisfied especially for high frequencies ω, for which the Gaussian beams can be chosen sufficiently narrow.

Let us consider the local Cartesian coordinate system for every point of the initial surface with its origin at this point. The coordinates x_1, x_2 are measured tangentially to the initial surface, and x_3 is measured perpendicularly to the initial surface and is oriented into the model. The central ray of the Gaussian beam is specified at this point by the slowness vector

$$p_A = \left.\frac{\partial T}{\partial x_A}\right|_{x_j=0} = T_A,$$

$$p_3 = \sqrt{\frac{1}{v^2} - p_A p_A}. \tag{29}$$

The initial travel time τ on the initial surface will be equal to the phase T

$$\tau|_{\mathrm{IS}} = T. \tag{30}$$

Note that the local coordinates x_i may be introduced, e.g. in such a way that the coordinate x_2 is measured perpendicularly to the plane of 'incidence' of the beam, i.e.

$$p_2 = T_2 = 0. \tag{31}$$

In (29) we have used T_A to denote the first derivatives of $T(\xi_A)$ in coordinates x_A

$$T_A = \frac{\partial \xi_B}{\partial x_A}\bigg|_{x_i=0} \frac{\partial T}{\partial \xi_B}\bigg|_{x_N=0}. \tag{32}$$

In the same way we use

$$T_{AB} = \frac{\partial \xi_C}{\partial x_A}\bigg|_{x_i=0} \frac{\partial \xi_D}{\partial x_B}\bigg|_{x_j=0} \frac{\partial^2 T}{\partial \xi_C\, \partial \xi_D}\bigg|_{x_N=0} + \frac{\partial^2 \xi_C}{\partial x_A\, \partial x_B}\bigg|_{x_i=0} \frac{\partial T}{\partial \xi_C}\bigg|_{x_N=0} \tag{33}$$

and

$$A_A = \frac{\partial \xi_B}{\partial x_A}\bigg|_{x_i=0} \frac{\partial A}{\partial \xi_B}\bigg|_{x_M=0}. \tag{34}$$

The coordinates x_A can be locally used as the coordinates along the initial surface, because their projections upon the initial surface in the direction of the axis x_3 form local orthogonal coordinates on the initial surface. In these coordinates the time field $T(x_B)$ along the initial surface can be locally approximated by the Taylor expansion

$$T(x_E) = T + T_A x_A + \tfrac{1}{2} x_A T_{AB} x_B. \tag{35}$$

Note that $x_E = 0$ is understood everywhere, where the local coordinates are used but the independent variables x_E of the used quantities are not specified. Similarly we can write the expansion

$$A(x_E) = A + A_A x_A. \tag{36}$$

4.2 THE TRACE OF A GAUSSIAN BEAM ON THE INITIAL SURFACE

The complex-valued travel time of the Gaussian beam (20) will be denoted by

$$\vartheta(q_i) = \tau(q_3) + \tfrac{1}{2} q_A M_{AB}(q_3) q_B. \tag{37}$$

Assume the product of frequency ω and matrix $\mathbf{M}(q_3)$ (especially its imaginary part) to be sufficiently large, so that the Taylor expansions used below are valid in the effective regions of the Gaussian beams. Let us transform the time field $\vartheta(q_i)$ of the Gaussian beam into local Cartesian coordinates x_i. In these coordinates we use the Taylor expansion of the time field $\vartheta(x_i)$:

$$\vartheta = \tau|_{x_j=0} + p_i x_i + \tfrac{1}{2} x_i H_{ik} x_j H_{jl} \frac{\partial^2 \vartheta}{\partial r_k\, \partial r_l}\bigg|_{x_n=0}. \tag{38}$$

The local Cartesian coordinates r_i, corresponding to the ray-centred coordinates, were introduced in Section 2. The transformation matrix H_{ik} was defined by (7).

Assume that the initial surface may be described in the effective regions of the Gaussian beams by a Taylor expansion up to the second order,

$$x_3 = \tfrac{1}{2} x_A D_{AB} x_B, \tag{39}$$

where D_{AB} is the matrix of curvature of the initial surface.

The trace of the time field ϑ on the initial surface is then given by the relation

$$\vartheta|_{\mathrm{IS}} = \tau + p_A x_A + \tfrac{1}{2} p_3 x_A D_{AB} x_B + \tfrac{1}{2} x_A H_{Ak} x_B H_{Bl} \left.\frac{\partial^2 \vartheta}{\partial r_k \, \partial r_l}\right|_{x_n = 0} \tag{40}$$

in terms up to the second order.

The second derivatives of the complex-valued time field ϑ of the Gaussian beam can be written in the form of (18) similarly as the derivatives of the travel time τ. We may then write

$$\left.\frac{\partial^2 \vartheta}{\partial r_i \partial r_j}\right|_{r_n = 0} = \begin{pmatrix} M_{11}(\sigma_0), & M_{12}(\sigma_0), & -V_1 V^{-2} \\ M_{12}(\sigma_0), & M_{22}(\sigma_0), & -V_2 V^{-2} \\ -V_1 V^{-2}, & -V_2 V^{-2}, & -V_3 V^{-2} \end{pmatrix}, \tag{41}$$

where $M_{AB}(\sigma_0)$ is the initial value of the complex-valued matrix M_{AB} at point $r_n = 0$.

Substituting (41) into (40), we obtain the expansion

$$\vartheta|_{\mathrm{IS}} = \tau + p_A x_A + \tfrac{1}{2} x_A x_B [p_3 D_{AB} + H_{AC} H_{BD} M_{CD}(\sigma_0) - H_{A3} H_{BC} V_C V^{-2} - H_{B3} H_{AC} V_C V^{-2} - H_{A3} H_{B3} V_3 V^{-2}]. \tag{42}$$

Introducing matrix

$$E_{AB} = -H_{A3} H_{BC} V_C V^{-2} - H_{B3} H_{AC} V_C V^{-2} - H_{A3} H_{B3} V_3 V^{-2}$$

$$= -H_{A3} V^{-2} \left.\frac{\partial v}{\partial x_B}\right|_{x_m = 0} - H_{B3} V^{-2} \left.\frac{\partial v}{\partial x_A}\right|_{x_m = 0} + H_{A3} H_{B3} V_3 V^{-2} \tag{43}$$

we can rewrite (42) in the form

$$\vartheta|_{\mathrm{IS}} = \tau + p_A x_A + \tfrac{1}{2} x_A x_B [p_3 D_{AB} + H_{AC} H_{BD} M_{CD}(\sigma_0) + E_{AB}]. \tag{44}$$

Using initial conditions (29) and (30), we obtain

$$\vartheta|_{\mathrm{IS}} = T + T_A x_A + \tfrac{1}{2} x_A x_B [p_3 D_{AB} + H_{AC} H_{BD} M_{CD}(\sigma_0) + E_{AB}]. \tag{45}$$

The expression (45) describes the time field of the Gaussian beam propagating from point $x_n = 0$. The time field is computed at point x_A on the initial surface.

Similarly we may consider the beam with the central ray starting from point x_A and evaluate its time field at the origin $x_n = 0$

$$\vartheta_{xE} = T(x_E) - x_A \frac{\partial T(x_E)}{\partial x_A} + \tfrac{1}{2} x_A x_B [p_3 D_{AB} + H_{AC} H_{BD} M_{CD}(\sigma_0) + E_{AB}]. \tag{46}$$

Substituting (35) into (46), we obtain the expansion of the time field of the Gaussian beam with the central ray starting from point x_E. The value of the time field is taken at the origin of the local coordinates:

$$\vartheta_{xE} = T + \tfrac{1}{2} x_A x_B [p_3 D_{AB} - T_{AB} + H_{AC} H_{BD} M_{CD}(\sigma_0) + E_{AB}]. \quad (47)$$

The field is considered along the initial surface. Introducing matrix

$$R_{AB} = p_3 D_{AB} - T_{AB} + H_{AC} H_{BD} M_{CD}(\sigma_0) + E_{AB}, \quad (48)$$

we can rewrite (47) in a brief form:

$$\vartheta_{xE} = T + \tfrac{1}{2} x_A R_{AB} x_B. \quad (49)$$

We denote by G_{AB} the inverse matrix to the 2×2 matrix H_{AB}

$$G_{AB} H_{BC} = \delta_{AC}. \quad (50)$$

The matrix

$$M^R_{AB}(\sigma_0) = G_{AC} G_{BD}(T_{CD} - p_3 D_{CD} - E_{CD}) \quad (51)$$

is the initial value of matrix (9) of the second derivatives of the travel-time field corresponding to the ray approximation of the wave field generated from the initial surface. Using the initial value of the ray matrix $M^R_{AB}(\sigma_0)$ we can rewrite definition (48) to read

$$R_{AB} = H_{AC} H_{BD} [M_{CD}(\sigma_0) - M^R_{CD}(\sigma_0)]. \quad (52)$$

4.3 TWO-PARAMETRIC INTEGRAL SUPERPOSITION OF GAUSSIAN BEAMS

The integral superposition of the principal components of Gaussian beams has the form

$$u = \iint d\gamma_1 \wedge d\gamma_2 \, g(\gamma_1, \gamma_2), \quad (53)$$

where $g(\gamma_1, \gamma_2)$ is the wavefield of the Gaussian beam (20) concentrated close to the ray with the ray parameters γ_1, γ_2. The additional components are not considered in this expansion. The integral superposition of the additional components is assumed to be negligible in this paper.

Using expansion (49) we can evaluate integral superposition (53) locally as

$$u = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx_1 dx_2 \det \left(\frac{\partial \gamma_C}{\partial x_D}\right) \frac{C(x_E)}{\sqrt{v\rho \det \mathbf{Q}(x_F, \sigma_0)}} \exp[i\omega(T + \tfrac{1}{2} x_A R_{AB} x_B - t)]. \quad (54)$$

Using definition (8) of the matrix of the ray geometrical spreading, we have

$$\frac{\partial \gamma_A}{\partial x_B} = \frac{\partial q_k}{\partial x_B} \frac{\partial \gamma_A}{\partial q_k} = H_{BC} (Q^R)^{-1}_{AC}. \quad (55)$$

Denoting

$$K(x_C) = \frac{|\det H_{AB}| \, C(x_C)}{|\det Q^R_{DE}(\sigma_0)| \sqrt{\det Q_{FG}(\sigma_0)}}, \quad (56)$$

we can rewrite (54) as

$$u = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dx_1 dx_2}{\sqrt{\rho v}} K(x_C) \exp[i\omega(T + \tfrac{1}{2} x_A R_{AB} x_B - t)]. \quad (57)$$

Assume that the normalization factor $K(x_C)$ can be replaced by the linear Taylor expansion

$$K(x_C) = K + K_A x_A \tag{58}$$

in the effective region of the Gaussian beam. The quantity $(\rho v)^{-1/2}$ is taken at the 'receiver'. It is then independent of x_A. The odd part of the integrand, corresponding to $K_A x_A$, vanishes after integration and we have

$$u = \frac{1}{\sqrt{\rho v}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx_1\, dx_2\, K \exp[i\omega(T + \tfrac{1}{2} x_A R_{AB} x_B - t)]. \tag{59}$$

We can evaluate this integral (Červený 1982)

$$u = \frac{K}{\sqrt{\rho v}} \frac{2\pi}{\omega} \frac{1}{\sqrt{-\det(R_{AB})}} \exp[i\omega(T-t)], \tag{60}$$

where the square root $\sqrt{-\det R_{AB}}$ must be taken with a positive real part. We wish (60) to be equal to the given wavefield (28) on the initial surface. We then have to choose

$$K = \frac{\omega}{2\pi} \sqrt{\rho v} \sqrt{-\det R_{AB}}\, A \tag{61}$$

at each point of the initial surface. In this way we obtain the expression for the normalization factors C of the beams (see 56),

$$C = \frac{\omega}{2\pi} \sqrt{\rho v}\, A \sqrt{-\det R_{AB}}\, |\det H_{CD}|^{-1}\, |\det Q^R_{EF}(\sigma_0)| \sqrt{\det Q_{GH}(\sigma_0)}. \tag{62}$$

Now we substitute for $\sqrt{-\det R_{AB}}$ from (52)

$$C = \frac{\omega}{2\pi} \sqrt{\rho(\sigma_0)\, v(\sigma_0)}\, A(\sigma_0) \sqrt{-\det[\mathbf{M}(\sigma_0) - \mathbf{M}^R(\sigma_0)]}\, |\det \mathbf{Q}^R(\sigma_0)| \sqrt{\det \mathbf{Q}(\sigma_0)}. \tag{63}$$

The square root $\sqrt{-\det[\mathbf{M}(\sigma_0) - \mathbf{M}^R(\sigma_0)]}$ must be taken with a positive real part. The same branch of the square root as in (20) must be taken for $\sqrt{\det \mathbf{Q}(\sigma_0)}$. Using (10) and (21), we can rearrange (63) to read

$$\begin{aligned} C &= \frac{\omega}{2\pi} \sqrt{\rho(\sigma_0)\, v(\sigma_0)}\, A(\sigma_0)\, |\sqrt{\det \mathbf{Q}^R(\sigma_0)}| \\ &\quad \times \sqrt{-\det\{\mathbf{Q}^T(\sigma_0)[\mathbf{M}^T(\sigma_0) - \mathbf{M}^R(\sigma_0)]\, \mathbf{Q}^R(\sigma_0)\}} \\ &= \frac{\omega}{2\pi} \sqrt{\rho(\sigma_0)\, v(\sigma_0)}\, A(\sigma_0)\, |\sqrt{\det \mathbf{Q}^R(\sigma_0)}| \\ &\quad \times \sqrt{-\det[\mathbf{P}^T(\sigma_0)\, \mathbf{Q}^R(\sigma_0) - \mathbf{Q}^T(\sigma_0)\, \mathbf{P}^R(\sigma_0)]}. \end{aligned} \tag{64}$$

The argument of the square root $\sqrt{-\det[\mathbf{P}^T(\sigma_0)\, \mathbf{Q}^R(\sigma_0) - \mathbf{Q}^T(\sigma_0)\, \mathbf{P}^R(\sigma_0)]}$ must not differ by more than $\pi/2$ from the argument of the square root $\sqrt{\det \mathbf{Q}(\sigma_0)}$ in (20) to preserve the correct argument of C from (62) and (63).

The amplitude $A(\sigma_0)$ specified on the initial surface can be understood here as the complex-valued ray amplitude given on the initial surface.

4.4 THE NORMALIZATION FACTORS OF BEAMS

The normalization factor (64) can be expressed as the product of three factors, constant along the ray:

$$C(\gamma_A) = C_1\, C_2(\gamma_B)\, C_3(\gamma_C) \tag{65}$$

$$C_1 = \frac{\omega}{2\pi} \tag{66}$$

$$C_2 = A(\sigma_0)\sqrt{\rho(\sigma_0)\, v(\sigma_0)}\, |\sqrt{\det \mathbf{Q}^R(\sigma_0)}\,| \tag{67}$$

$$C_3 = \sqrt{-\det[\mathbf{P}^T(\sigma_0)\,\mathbf{Q}^R(\sigma_0) - \mathbf{Q}^T(\sigma_0)\,\mathbf{P}^R(\sigma_0)]}. \tag{68}$$

The argument of C_3 must not differ by more than $\pi/2$ from the argument of $\sqrt{\det \mathbf{Q}(\sigma_0)}$ in (20).

The constant C_2 may be evaluated at any point σ of the ray using the complex-valued ray amplitude $A(\sigma)$, density $\rho(\sigma)$, velocity $v(\sigma)$, geometrical spreading $|\sqrt{\det \mathbf{Q}^R(\sigma)}\,|$ and the phase shift $\phi(\sigma, \sigma_0) = -\pi/2$ KMAH between points σ_0 and σ due to caustics (see the KMAH index in Chapman & Drummond 1982)

$$C_2 = A(\sigma)\sqrt{\rho(\sigma)\, v(\sigma)}\, |\sqrt{\det \mathbf{Q}^R(\sigma)}\,| \exp[-i\phi(\sigma, \sigma_0)]. \tag{69}$$

The constant C_3 may be expressed at any point of the ray as

$$C_3 = \sqrt{-\det[\mathbf{P}^T(\sigma)\,\mathbf{Q}^R(\sigma) - \mathbf{Q}^T(\sigma)\,\mathbf{P}^R(\sigma)]}, \tag{70}$$

because the matrix

$$\mathbf{P}^T\mathbf{Q}^R - \mathbf{Q}^T\mathbf{P}^R, \tag{71}$$

where matrices $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{P}^R$, $\mathbf{Q}^R$ satisfy equations (22) (or 11), is constant along the ray. This can be proved by differentiating (71) with respect to σ and using equations (22) (see Červený & Pšenčík 1983). However, there is one problem in (70). The argument of the square root (70) must be taken in accordance with the argument of $\sqrt{\det \mathbf{Q}(\sigma_0)}$, not in accordance with the argument of $\sqrt{\det \mathbf{Q}(\sigma)}$. This choice of a constant argument of C_3 along the ray is connected with the phase shift in (69). To avoid the above-mentioned problem with the phase shift in (70), we shall rewrite (70) in the form

$$C_3 = \sqrt{\det \mathbf{Q}(\sigma)}\, |\sqrt{\det \mathbf{Q}^R(\sigma)}\,|\sqrt{-\det[\mathbf{M}(\sigma) - \mathbf{M}^R(\sigma)]}\, \exp[i\phi(\sigma, \sigma_0)], \tag{72}$$

where the same branch of the square root is taken for $\sqrt{\det \mathbf{Q}(\sigma)}$ as in expression (20) for the Gaussian beam, and $\sqrt{-\det[\mathbf{M}(\sigma) - \mathbf{M}^R(\sigma)]}$ is taken with a positive real part. Now we prove that the expression (72) is equivalent to (68) or (70). Argument of the quantity $\sqrt{\det \mathbf{Q}(\sigma)}$ is continuous along the whole ray. The phase shift $\phi(\sigma, \sigma_0)$ and the argument of $\sqrt{-\det[\mathbf{M}(\sigma) - \mathbf{M}^R(\sigma)]}$ are continuous along the elements of the ray between two consequent caustics. The phase shift $\phi(\sigma, \sigma_0)$ increases by $\pi/2$ or π at a caustic and the argument of $\sqrt{-\det[\mathbf{M}(\sigma) - \mathbf{M}^R(\sigma)]}$ taken with a positive real part decreases at the caustic by the same quantity $\pi/2$ or π. Therefore, the argument of (72) is continuous along the whole ray. Since $(C_3)^2$, given by (72), is constant along the ray, C_3 must be constant too.

Using (65), (66), (69), (72) we can evaluate the normalization factor $C(\gamma_A)$ of the beams (20) in superposition (53) at any point σ of the central ray as

$$C = \frac{\omega}{2\pi} A(\sigma)\sqrt{\rho(\sigma)\, v(\sigma)}\, |\det \mathbf{Q}^R(\sigma)|\sqrt{-\det[\mathbf{M}(\sigma) - \mathbf{M}^R(\sigma)]}\,\sqrt{\det \mathbf{Q}(\sigma)}, \tag{73}$$

where $A(\sigma)$ is the complex-valued ray amplitude including the corresponding phase shift, the square root $\sqrt{-\det[\mathbf{M}(\sigma)-\mathbf{M}^R(\sigma)]}$ is taken with a positive real part and the same branch of the square root is taken for $\sqrt{\det \mathbf{Q}(\sigma)}$ as in expression (20) for the Gaussian beam. The normalization factor (73) is constant along the whole ray, but may depend on the parameters γ_A of the central ray.

4.5 DISCUSSION

We have expanded the wavefield (28), given on a curved initial surface, into the integral superposition (53) of the Gaussian beams (20). The normalization factor C of the beams (20) in the superposition is given by (63), which is the special case of its generalization (73). Expression (73) of the normalization factor of the beams in the superposition offers more computational possibilities and theoretical conclusions than expression (63).

Only the quantities describing the Gaussian beam and the quantities corresponding to the relevant ray approximation of the wavefield generated from the initial surface appear in the evaluation (73) of the normalization factor of the beam. Therefore, the expansion of the wavefield into Gaussian beams need not be specified directly on the initial surface. The wavefield given on the initial surface may be extended on to another surface by means of the ray approximation and the expansion can be specified on the latter surface. Even if the ray wavefield is not regular on the latter surface, the expansion is valid, but it corresponds to the wavefield given on the initial surface or on any other surface on which the corresponding ray field is sufficiently regular. In this way our asymptotic expansion of the wavefield into Gaussian beams corresponds to the far-field approximation of the source.

On the other hand, in every region where the ray solution is regular in such a way that the ray diagram and the ray solution can be replaced by paraxial ray approximations in the effective regions of the used Gaussian beams, we obtain the ray solution as the integral superposition (53) of Gaussian beams. In particular, a frequency sufficiently high to obtain the ray solution as the integral superposition of Gaussian beams can be chosen for almost every point of the medium.

For practical numerical computations of Gaussian beams we determine the normalization factor (73) at a point of the ray at which we know simultaneously the ray quantities A, $\det \mathbf{Q}^R$, $\mathbf{M}^R$ (see 51) and the quantities $\det \mathbf{Q}$, $\mathbf{M}$ appropriate to the beam. For instance, at a point source $[Q_{AB}(\sigma_0)=0]$ we obtain the limiting value of (73) for $\sigma \to \sigma_0+$ as

$$C=\frac{i\omega}{2\pi}\sqrt{\rho(\sigma_0)\,v(\sigma_0)}\,v(\sigma_0)\,|\det \mathbf{P}^R(\sigma_0)|\sqrt{\det \mathbf{Q}(\sigma_0)}\,F, \tag{74}$$

where

$$F=\frac{1}{v(\sigma_0)}\lim_{\epsilon\to 0+}[\epsilon A(\sigma_0+\epsilon)] \tag{75}$$

is the radiation pattern of the point source. From the point at which we have determined the normalization factor (73), we need to compute numerically beams (20) to the vicinity of the receiver. For more details refer to Červený & Pšenčík (1983).

If we know the quantities A, $\det \mathbf{Q}^R$, $\mathbf{M}^R$ corresponding to the ray solution in the vicinity of the receiver, we can choose the quantities $\det \mathbf{Q}$, $\mathbf{M}$ describing the beams arbitrarily and determine the normalization factor (73) directly in the vicinity of the receiver. Superposition (53) of the beams (20) than takes the form

$$u=\frac{\omega}{2\pi}\iint d\gamma_1\wedge d\gamma_2 A\,|\det \mathbf{Q}^R|\sqrt{-\det(\mathbf{M}-\mathbf{M}^R)}\exp\{i\omega[\tau+\tfrac{1}{2}q_A M_{AB} q_B - t]\}\,, \tag{76}$$

where all the quantities A, det $\mathbf{Q}^R$, $\mathbf{M}$, $\mathbf{M}^R$, τ are taken at the point which is the orthogonal projection of the receiver upon the ray specified by the ray parameters γ_A. The quantity A is the complex-valued ray amplitude including phase shift due to the caustics; $|\det \mathbf{Q}^R|$ is the square of the ray geometrical spreading in ray coordinates γ_1, γ_2; τ is the travel time; $\mathbf{M}^R$ is the matrix of the second derivatives of the travel time; matrix $\mathbf{M}$ describes the shape of the beam; q_A are the relevant ray-centred coordinates of the receiver.

4.6 THE EXPANSION IN CARTESIAN COORDINATES

It is useful to rewrite the expansion (76) in the general Cartesian coordinates x_i. Applying (18) and (19) for the complex-valued time field of the Gaussian beams, we can modify (76) as

$$u(x_l) = \frac{\omega}{2\pi} \iint d\gamma_1 \wedge d\gamma_2 A\, |\det \mathbf{Q}^R|\, \sqrt{-\det(\mathbf{M}-\mathbf{M}^R)}$$
$$\times \exp\{i\omega[\tau + (x_j-\tilde{x}_j)p_j + \tfrac{1}{2}(x_j-\tilde{x}_j)(x_k-\tilde{x}_k)H_{jm}H_{kn}M_{mn} - t]\}, \qquad (77)$$

where the square root $\sqrt{-\det(\mathbf{M}-\mathbf{M}^R)}$ is taken with a positive real part. In (77) we have introduced the 3×3 matrix,

$$M_{mn} = \begin{pmatrix} M_{11}, & M_{12}, & -V_1/V^2 \\ M_{12}, & M_{22}, & -V_2/V^2 \\ -V_1/V^2, & -V_2/V^2, & -V_3/V^2 \end{pmatrix}, \qquad (78)$$

where M_{AB} is an arbitrary complex-valued symmetric 2×2 matrix with a positive definite imaginary part. All the quantities A, τ, p_k, det $\mathbf{Q}^R$, $\mathbf{M}^R$, $\mathbf{M}$, H_{ij}, V_i, V are taken at the termination point $\tilde{x}_i$ of the ray specified by the ray parameters γ_A. The quantities H_{ij}, V_i, V are defined by (4), (5) and (7).

For example, if all the third coordinates $\tilde{x}_3$ of the termination points of the rays are equal to the third coordinate x_3 of the receiver, expression (78) simplifies to

$$u(x_k) = \frac{\omega}{2\pi} \iint d\gamma_1 \wedge d\gamma_2 A\, |\det \mathbf{Q}^R|\, \sqrt{-\det(\mathbf{M}-\mathbf{M}^R)}$$
$$\times \exp\{i\omega[\tau + p_A(x_A-\tilde{x}_A) + \tfrac{1}{2}(x_A-\tilde{x}_A)(x_B-\tilde{x}_B)(H_{AC}H_{BD}M_{CD} + E_{AB}) - t]\}, \qquad (79)$$

where the matrix E_{AB} is given by (43). Note that (79) gives the generalization of the Maslov method of the second order (Chapman & Drummond 1982) and can also be derived using the Maslov asymptotic theory in a general 3-D subspace of 6-D complex phase space (Klimeš 1984).

The optimal choice of the matrix $\mathbf{M}$ in (76), (77) or (79) will be proposed elsewhere.

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