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Expansion of the global error for numerical schemes solving stochastic differential equations — Source link

Denis Talay, Luciano Tubaro

Institutions: French Institute for Research in Computer Science and Automation, University of Trento

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Denis Talay, Luciano Tubaro

UNITÉ DE RECHERCHE
INRIA-SOPHIA ANTIPOLIS

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P.105
78153 Le Chesnay Cedex
France
Tél. (1) 39 63 55 11

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EXPANSION OF THE GLOBAL ERROR FOR NUMERICAL SCHEMES SOLVING STOCHASTIC DIFFERENTIAL EQUATIONS

Denis TALAY
Luciano TUBARO

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Expansion of the global error for numerical schemes solving Stochastic Differential Equations

Développement de l'erreur globale pour des schémas de discrétisation d'Equations Différentielles Stochastiques

Denis Talay
INRIA
Sophia-Antipolis
06560 VALBONNE
FRANCE

Luciano Tubaro
Università di Trento
Dipartimento di Matematica
38050 POVO (TN)
ITALY

Abstract

Let (X_t) the solution of a Stochastic Differential System. We consider 2 situations : computation of $\mathbf{E}f(X_t)$ by a Monte-Carlo method (for example, computation of moments of the solution) ; integration of f w.r.t the invariant probability law of (X_t) (when this process is ergodic) by simulating a single trajectory.

In each case, we expand the global error of approximation for a class of discretisation schemes and of functions $f(\cdot)$, in powers of the discretisation step size ; the first result extends a result of Gragg for ordinary differential equations. This expansion permits to justify an extrapolation between approximate values due to first order schemes ; this algorithm provides a second order accuracy, is much simpler to implement than second order schemes, and seems to have interesting numerical properties (short computation time, numerical stability). This is illustrated by results of numerical simulations.

It permits also to have a better understanding of the numerical behaviour of discretization schemes, and to explain a priori strange results obtained for two particular examples.

Résumé

Soit (X_t) la solution d'un système différentiel stochastique. Nous considérons 2 situations : le calcul de $\mathbf{E}f(X_t)$ par une méthode de Monte-Carlo (par exemple, calcul des moments de la solution) ; l'intégration de f par rapport à la loi invariante de (X_t) (quand ce processus est ergodique) en simulant une unique trajectoire.

Dans chaque cas, nous effectuons un développement asymptotique de l'erreur d'approximation pour une classe de schémas de discrétisation et de fonctions f, selon les puissances du pas de discrétisation. Le premier résultat étend un résultat de Gragg concernant les équations différentielles ordinaires. Le développement permet d'effectuer des extrapolations à partir des résultats fournis par des schémas du premier ordre, et d'obtenir ainsi une précision d'ordre 2 ; l'algorithme est beaucoup plus simple à implémenter que les schémas d'ordre 2, et semble avoir des propriétés numériques intéressantes (temps de calcul réduit, stabilité numérique). Nous donnons les résultats d'essais numériques.

Par ailleurs, le développement permet de mieux comprendre le comportement numérique des schémas de discrétisation, et d'expliquer les résultats a priori étranges observés dans deux situations particulières.

1 Introduction

Let us consider the following Itô stochastic differential equation :

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \tag{1}$$

where (X_t) is a stochastic process in $\mathbb{R}^d$, (W_t) is a Wiener process in $\mathbb{R}^\ell$, $b(t,x)$ is a d-vector and $\sigma(t,x)$ is a $d \times \ell$-matrix.

We will denote the solution of (1) with the deterministic initial condition x at time s (i.e $X_s = x$), by $(X_t^{s,x})$. (We will suppose below that the coefficients b and σ are smooth, so that the existence and uniqueness of this solution are ensured).

Let Y be a random variable independent of any increment $W_t - W_s$ of the Wiener process, and having moments of any order. The solution of (1) with initial condition $X_0 = Y$ at time 0 will be denoted by (X_t).

Let us first consider the numerical evaluation of the quantity

$$\mathbf{E}f(X_T)$$

where T is some fixed time, and f a given smooth function from $\mathbb{R}^d \to \mathbb{R}$, by a Monte-Carlo method based upon the simulation of a piecewise constant approximating process $(\overline{X}_p^h, p \in \mathbb{N})$.

We divide the interval $[0,T]$ in n subintervals with the same length h : in other words

$$h = T/n$$

According to our notation, $\bar{X}_n^h$ is $\bar{X}_T^h$; we want to study the global error

$$Err(T,h) = \mathbf{E}f(X_T) - \mathbf{E}f(\bar{X}_T^h) \tag{2}$$

It is already known ([5] or [4]) that the Euler and Milshtein discretization schemes (defined below) are of first order, i.e the global error satisfies : there exists a positive constant $C(T)$, independent of h, such that :

$$|Err(T,h)| \leq C(T)h \tag{3}$$

Here we want to study $Err(T,h)$ as a function of h ; more precisely, after preliminary results given in Section 2.1, in Section 2.2, we will give an expansion of $Err(T,h)$ of the form :

$$Err(T,h) = e_1(T)h + e_2(T)h^2 + ... + e_m(T)h^m + \mathcal{O}(h^{m+1})$$

This expansion justifies an extrapolation technique, described in Subsection 2.3 and tested in Subsections 2.4 and 2.5, which permits to obtain results of the same theoretical accuracy as those provided by second order schemes. The algorithm is very simple, and our numerical experiments seem to show also that this procedure has an interesting numerical behaviour, and permits to save CPU time.

Another motivation of this work was a priori strange numerical results obtained at least on the situation described in Subsection 2.5 : we consider a (not so much) particular system (1), whose coefficients depend on a parameter ν ; for $\nu \leq 1$, we observe that the numerical results provided by Euler and Milshtein schemes have the same accuracy, as predicted by the estimation (3). For $\nu > 1$, we observe that Milshtein scheme (but not Euler scheme) provides very inaccurate results, except for very small h. We will completely explain this difference of behaviour of the two schemes, thanks to an explicit computation of the corresponding functions $e_1(t)$.

In Section 3, an analogous work will be achieved for the numerical integration of f w.r.t. the invariant probability law of the process (X_t) (in the ergodic case).

2 The case of approximations of $\mathbf{E}f(X_T)$

2.1 Preliminary results

First, we introduce the following numerical scheme (Euler scheme) :

$$\begin{aligned} \bar{X}_0^h &= X_0 \\ \bar{X}_{p+1}^h &= \bar{X}_p^h + \sigma(ph, \bar{X}_p^h)\Delta_{p+1}^h W + b(ph, \bar{X}_p^h)h \end{aligned} \tag{4}$$

where we have set $\Delta_{p+1}^h W = W_{(p+1)h} - W_{ph}$.

Now, let us define the Milshtein scheme.

Let σ_j denote the jth column of σ, and $\partial\sigma_j$ denote the matrix whose element of the ith row and k^{th} column is $\partial_k \sigma_j^i$.

We begin by defining the family $(\tilde{U}_{p+1}^{kj})$ of i.i.d. random variables, their common law being defined by :

$$P(\tilde{U}_p^{kj} = \frac{1}{2}) = P(\tilde{U}_p^{kj} = -\frac{1}{2}) = \frac{1}{2}$$

and such that the $(\tilde{U}^{kj}_{q+1}, \Delta^h_{p+1}W^l)_{p,q,k,j,l}$'s are mutually independent.

Then we define :

$$\begin{aligned} Z^{kj}_{p+1} &= \frac{1}{2}\Delta^h_{p+1}W^k\Delta^h_{p+1}W^j + \tilde{U}^{kj}_{p+1}h \quad , \quad k<j \\ Z^{kj}_{p+1} &= \frac{1}{2}\Delta^h_{p+1}W^k\Delta^h_{p+1}W^j - \tilde{U}^{jk}_{p+1}h \quad , \quad k>j \\ Z^{jj}_{p+1} &= \frac{1}{2}\left((\Delta^h_{p+1}W^j)^2 - h\right) \end{aligned}$$

and finally the Milshtein scheme by :

$$\begin{aligned} \overline{X}^h_{p+1} &= \overline{X}^h_p + \sum_{j=1}^{r}\sigma_j(ph,\overline{X}^h_p)\Delta^h_{p+1}W^j + b(ph,\overline{X}^h_p)h \\ &\quad + \sum_{j,k=1}^{r}\partial\sigma_j(ph,\overline{X}^h_p)\sigma_k(ph,\overline{X}^h_p)Z^{kj}_{p+1} \end{aligned}$$

The proofs of the following two lemmas are easy, the arguments can be found in Talay [5].

Lemma 1 *Suppose that the b_i's and the σ_{ij}'s are Lipschitz continuous functions. Then, for any integer k, there exists a strictly positive constant C_k such that :*

$$\mathbf{E}|\overline{X}^h_p|^k \leq e^{C_kT}$$

for any p between 0 and n.

Let us now consider the class $\mathcal{F}_T$ of functions $\phi : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ with the following properties : ϕ is of class $\mathcal{C}^\infty$, and, for some positive integer s and positive $C(T)$:

$$\forall\theta\in[0,T] \ , \ \forall x\in\mathbb{R}^d \ : \ |\ \phi(\theta,x)\ | \leq C(T)(1+\ |\ x\ |^s) \tag{5}$$

A function ϕ of $\mathcal{F}_T$ will be called **homogeneous** if it does not depend on the time variable : $\phi(\theta,x)=\phi(x)$.

We will denote by $\mathcal{L}$ the differential operator associated to (1) :

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{d} a^i_j(t,x)\partial_{ij} + \sum_{i=1}^{d} b^i(t,x)\partial_i \tag{6}$$

where $a(t,x) = \sigma(t,x)\sigma^*(t,x)$.

If $\phi \in \mathcal{F}_T$ then the function $u(\theta;t,x)$, defined by

$$u(\theta;t,x) = \mathbf{E}\phi(\theta, X_T^{t,x}) = E_{t,x}\phi(\theta, X_T)$$

verifies the following equation :

$$\left\{ \begin{array}{rcl} \frac{\partial u}{\partial t} + \mathcal{L}u & = & 0 \\ u(\theta;T,x) & = & \phi(\theta,x) \end{array} \right. \tag{7}$$

Moreover, we have the following result :

Lemma 2 *Let us suppose that the functions b and σ are C^∞ functions, whose derivatives of any order are bounded.*

For any multi-index α, there exist strictly positive constants $k_\alpha(T)$, $C_\alpha(T)$ such that :

$$\forall\theta \in [0,T] \quad : \mid \partial_\alpha u(\theta;t,x) \mid \leq C_\alpha(T)(1+ \mid x \mid^{k_\alpha(T)})$$

Here by ∂_α we mean the mixed partial derivative of order $|\alpha|$

$$\frac{\partial^{|\alpha|}}{\partial_1^{\alpha_1}\partial_2^{\alpha_2}...\partial_r^{\alpha_r}}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_r)$, and $|\alpha| = \alpha_1 + \ldots \alpha_r$.

2.2 Main result

In all the sequel, we suppose that the functions b and σ are C^∞ functions, whose derivatives of any order are bounded.

Consider a **homogeneous** function f of $\mathcal{F}_T$, and $u(t,x) = \mathbf{E}f(X_T^{t,x})$ which solves :

$$\left\{ \begin{array}{rcl} \frac{\partial u}{\partial t} + \mathcal{L}u & = & 0 \\ u(T,x) & = & f(x) \end{array} \right. \tag{8}$$

Then we have

$$Err(T,h) = \mathbf{E}u(T, \bar{X}_n^h) - \mathbf{E}u(0,Y)$$

Let us first consider the Euler scheme.

We compute $\mathbf{E}u(T,\bar{X}_n^h) - \mathbf{E}u((n-1)h,\bar{X}_{n-1}^h)$ by performing a Taylor expansion at the point $((n-1)h,\bar{X}_{n-1}^h)$, of the form :

$$\begin{aligned} u(t+\Delta t, x+\Delta x) &= u(t,x) + \Delta t\frac{\partial}{\partial t}u(t,x) + \frac{1}{2}(\Delta t)^2\frac{\partial^2}{\partial t^2}u(t,x) \\ &+ \Delta t\sum_{|\alpha|=1}\Delta x^\alpha\frac{\partial}{\partial t}u(t,x)\partial_\alpha u(t,x) \\ &+ \frac{1}{2}\Delta t\sum_{|\alpha|=2}\Delta x^\alpha\frac{\partial}{\partial t}u(t,x)\partial_\alpha u(t,x) \\ &+ \frac{1}{6}\Delta t\sum_{|\alpha|=3}\Delta x^\alpha\frac{\partial}{\partial t}u(t,x)\partial_\alpha u(t,x) \\ &+ \frac{1}{|\alpha|!}\sum_{|\alpha|=1}^{5}\Delta x^\alpha\partial_\alpha u(t,x) + \dots \end{aligned}$$

where $\alpha = (\alpha_1,\dots,\alpha_r)$ and Δx^α means :

$$\Delta x^\alpha = (\Delta x_1)^{\alpha_1}\dots(\Delta x_r)^{\alpha_r}$$

One gets after some easy but tedious computations (cf [5]) :

$$\mathbf{E}u(T,\bar{X}_n^h) = \mathbf{E}u((n-1)h,\bar{X}_{n-1}^h) + h^2\mathbf{E}\psi_e((n-1)h,\bar{X}_{n-1}^h) + h^3R_n^h$$

where

$$\begin{aligned} \psi_e(t,x) &= \frac{1}{2}\sum_{i,j=1}^{d} b^i(t,x)b^j(t,x)\partial_{ij}u(t,x) \\ &+\frac{1}{2}\sum_{i,j,k=1}^{d} b^i(t,x)a_k^j(t,x)\partial_{ijk}u(t,x) \\ &+\frac{1}{8}\sum_{i,j,k,l=1}^{d} a_j^i(t,x)a_l^k(t,x)\partial_{ijkl}u(t,x) + \frac{1}{2}\frac{\partial^2}{\partial t^2}u(t,x) \\ &+\sum_{i=1}^{d} b^i(t,x)\frac{\partial}{\partial t}\partial_i u(t,x) + \frac{1}{2}\sum_{i,j=1}^{d} a_j^i(t,x)\frac{\partial}{\partial t}\partial_{ij}u(t,x) \qquad (9) \end{aligned}$$

and there exists a constant $C(T)$ independent of h such that :

$$|R_n^h| \le C(T)$$

We can use the same expansion for $u((n-1)h, \bar{X}^h_{n-1})$ and, continuing in this way n times, we arrive to :

$$\mathbf{E}u(T, \bar{X}^h_n) = \mathbf{E}u(0, Y) + h^2 \sum_{j=0}^{n-1} \mathbf{E}\psi_e(jh, \bar{X}^h_j) + h^2 \mathcal{R}^h_n \tag{10}$$

with

$$|\mathcal{R}^h_n| \leq C(T)$$

Proposition 1 *There exists a real number $C(T)$, independent on h, such that :*

$$h \sum_{j=0}^{n-1} \mathbf{E}|\psi_e(jh, \bar{X}^h_j)| \leq C(T)$$

Proof. That follows from Lemma 1 and Lemma 2. □

Proposition 2 *For any function ϕ of $\mathcal{F}_T$, there exists a real number $C(T)$, independent on h, such that :*

$$\mathbf{E}\phi(\theta, \bar{X}^h_T) = \mathbf{E}\phi(\theta, X_T) + R_T(h)$$

with $|R_T(h)| \leq C(T)h$.

Proof. This easily follows from Proposition 1 and equality (10). □

Proposition 3

$$\left| h \sum_{j=1}^{n-1} \mathbf{E}\psi_e(jh, \bar{X}^h_j) - \int_0^T \mathbf{E}\psi_e(s, X_s) ds \right| = \mathcal{O}(h).$$

Proof.

$$\left| h \sum_{j=1}^{n-1} \mathbf{E}\psi_e(jh, \bar{X}^h_j) - \int_0^T \mathbf{E}\psi_e(s, X_s) ds \right| \leq$$

$$h \sum_{j=1}^{n-1} |\mathbf{E}\psi_e(jh, \bar{X}^h_j) - \mathbf{E}\psi_e(jh, X_{jh})| +$$

$$+\left|h\sum_{j=1}^{n-1}\mathbf{E}\psi_e(jh,X_{jh})-\int_0^T\mathbf{E}\psi_e(s,X_s)ds\right|.$$

Now we have for the first term:

$$\left|\sum_{j=1}^{n-1}\mathbf{E}\psi_e(jh,\bar{X}_j^h)-\sum_{j=1}^{n-1}\mathbf{E}\psi_e(jh,X_{jh})\right|=\mathcal{O}(1)$$

because one may easily check that the function $\psi_e(t,x)$ belongs to $\mathcal{F}_T$, and therefore Proposition 2 implies :

$$|\mathbf{E}\psi_e(jh,\bar{X}_j^h)-\mathbf{E}\psi_e(jh,X_{jh})|\leq C(T)h.$$

On the other hand

$$\left|h\sum_{j=1}^{n-1}\mathbf{E}\psi_e(jh,X_{jh})-\int_0^T\mathbf{E}\psi_e(s,X_s)ds\right|=\mathcal{O}(h),$$

since the function

$$s\longrightarrow\mathbf{E}\psi_e(s,X_s)$$

has a continuous first derivative. □

We can conclude, summarising (and extending in an obvious way) the preceding results, with the

Theorem 1 *(i) For the Euler scheme, the error is given by :*

$$Err_e(T,h)=-h\int_0^T\mathbf{E}\psi_e(s,X_s)ds+\mathcal{O}(h^2) \tag{11}$$

where $\psi_e(\cdot)$ is defined by (9).

(ii) The same result extends to the Milshtein scheme :

$$Err_m(T,h)=-h\int_0^T\mathbf{E}\psi_m(s,X_s)ds+\mathcal{O}(h^2)$$

where $\psi_m(\cdot)$ is defined by :

$$\psi_m(t,x)=\psi_e(t,x)+\frac{1}{4}\sum_{i_1,i_2,j,k,l}a_k^l(t,x)\partial_l\sigma_j^{i_1}(t,x)\partial_k\sigma_j^{i_2}(t,x)\partial_{i_1i_2}u(t,x)$$
$$+\frac{1}{2}\sum_{\substack{i_1,i_2,i_3\\ j_1,j_2,k}}\sigma_{j_1}^{i_1}(t,x)\sigma_{j_2}^{i_2}(t,x)\sigma_{j_1}^{k}(t,x)\partial_k\sigma_{j_2}^{i_3}(t,x)\partial_{i_1i_2i_3}u(t,x) \tag{12}$$

(iii) For the second order schemes of [5], the global error can be written

$$Err(T,h) = h^2 \int_0^T \mathbf{E}\gamma(s, X_s)ds + \mathcal{O}(h^3)$$

for some smooth function γ.

(iv) Moreover, for all these schemes, it is possible to obtain an expansion of the form :

$$Err(T,h) = e_r(T)h^r + e_{r+1}(T)h^{r+1} + ... + e_m(T)h^m + \mathcal{O}(h^{m+1}) \quad (13)$$

Remark

Gragg [2] has obtained an analogous result for the expansion of the global error of a very large class of discretization schemes of ordinary differential equations (which correspond to the particular case $\sigma(\cdot) = 0$). But the method of Gragg, consisting in searching O.D.E's whose solutions are the functions $e_i(t)$, here is hopeless. Of course, our method provides an alternative proof of the result of Gragg (for the schemes treated in the previous Theorem).

2.3 Some applications

Just as in the case of ordinary differential equations, it is possible to carry on to the stochastic case some usual applications of the expansion of the error.

First, we can control the discretization step size, choosing at each step a new h, on the base of an estimate of $e_1(T)$: for example, (cf [2] e.g.), we first perform an approximation with the step size h, then a second one with the step size $h/2$. By (13) we have

$$\mathbf{E}f(\bar{X}_n^h) - \mathbf{E}f(\bar{X}_n^{h/2}) = e_1(T)\frac{h}{2} + \mathcal{O}(h^2)$$

from which we can estimate $e_1(T)$: now again by (13), we can choose a new h in order to get the error less than a given tolerance ϵ.

The second application is the use of polynomial (or rational) extrapolation methods, (again see [1] e.g.), to improve the approximation to the true solution : it is essential for this purpose to have an expansion such as (13).

Let us consider a scheme such that :

$$Err(T,h) = e_1(T)h + \mathcal{O}(h^2)$$

and consider the following new approximation (the Romberg extrapolation)

$$Z_T^h = 2\mathbf{E}f(\bar{X}_T^{h/2}) - \mathbf{E}f(\bar{X}_T^h) \tag{14}$$

then it is easy to check that

$$\mathbf{E}f(X_T) - Z_T^h = \mathcal{O}(h^2)$$

That is, it is possible to get a result of precision of order h^2 from results given by a first-order scheme .

Actually, it is possible to extend the argument if (13) is verified. In such a case consider a sequence

$$h_0 > h_1 > h_2 > \ldots > h_m$$

and the corresponding values

$$Z_{i0} = \mathbf{E}f(\bar{X}_T^{h_i})$$

We can construct the following scheme

$$\begin{array}{ccccc}
Z_{00} & & & & \\
 & Z_{01} & & & \\
Z_{10} & & Z_{02} & & \\
 & Z_{11} & \vdots & \vdots & Z_{mm} \\
Z_{20} & \vdots & Z_{m2} & & \\
\vdots & Z_{m1} & & & \\
Z_{m0} & & & &
\end{array}$$

where

$$Z_{i,k} = Z_{i,k-1} + \frac{Z_{i,k-1} - Z_{i-1,k-1}}{\frac{h_{i-k}}{h_i} - 1}.$$

Then

$$\mathbf{E}f(X_T) - Z_{m,m}(h) = \mathcal{O}(h^{m+1}).$$

Let us underline that, in Talay [5], the Romberg extrapolation was used in a very different way, since the extrapolation was not performed on the

expectations corresponding to the Euler scheme, but on the process defined by the Milshtein scheme, in order to construct a second order scheme (the MCRK scheme). Like other second order schemes (and the Milshtein scheme), the MCRK scheme involves the first derivatives of σ (contrary to the Euler scheme), and also (when the dimension of the Wiener process is larger than 1) requires the simulation of random variables other than the increments of (W_t) : this scheme requires a larger number of computations than the total one of the Euler scheme with the step-sizes h and $\frac{h}{2}$, and therefore a larger CPU time.

In the next Sections, we will numerically compare the two strategies.

2.4 Numerical tests : first example

Let us consider the first of our two examples ; the second order schemes numerically behave as predicted by the theory, i.e obviously are better than the first order schemes (Euler e.g.).

The function b is defined by :

$$\begin{aligned} b^1(x^1, x^2) &= -x^2 \\ b^2(x^1, x^2) &= x^1 \end{aligned}$$

and the matrix σ is defined by :

$$\begin{aligned} \sigma_1^1(x^1, x^2) &= 0 \\ \sigma_2^1(x^1, x^2) &= \frac{\sin(x^1 + x^2)}{\sqrt{t+1}} \\ \sigma_1^2(x^1, x^2) &= \frac{\cos(x^1 + x^2)}{\sqrt{t+1}} \\ \sigma_2^2(x^1, x^2) &= 0 \end{aligned}$$

It is easily proved that :

$$\mathbf{E}|X_t|^2 = \mathbf{E}|X_0|^2 + \log(1+t)$$

Now, we compute $\mathbf{E}|X_t|^2$ by simulating 10,000 independent paths of the Markov chain defined by a scheme, with the initial condition $X_0 = (1, 1)$.

The two following figures permit to compare results given by Euler scheme, Milshtein scheme, and a Romberg extrapolation.

The first figure shows the evolution in time of the true value (thin line) and of the approximate value computed using the Milshtein scheme with a step size $h = 0.01$ (thick line), for which the bad behaviour of the Milshtein scheme is patent.

The second one shows the evolution in time of the error corresponding to a Romberg extrapolation of the values given by the Euler scheme with step sizes $h = 0.02$ and $h = 0.01$ (thick line) and of the error corresponding to the Euler scheme with $h = 0.01$ (thin line). The extrapolation has notably improved the accuracy. Moreover, obviously the numerical stability has also been improved.

These figures show that the Euler and Milshtein schemes lead to almost equal errors, quickly negative, and whose absolute values are almost strictly increasing functions of the time. This is completely explained by the results of Theorem 1. Indeed, from the above expression for $\mathbf{E}|X_t|^2$, it is easy to check :

$$\mathbf{E}|X_T^{t,x}|^2 = |x|^2 + \log(1+T) - \log(1+t)$$

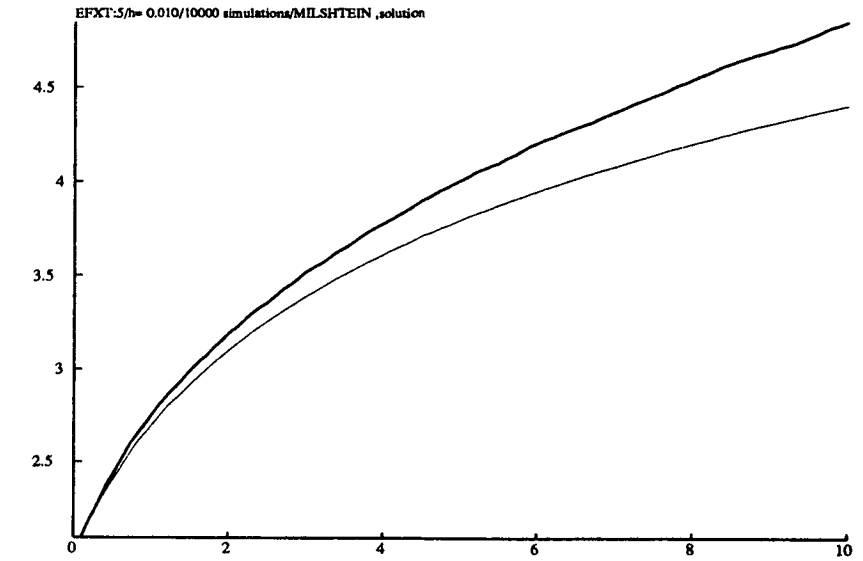
and therefore :

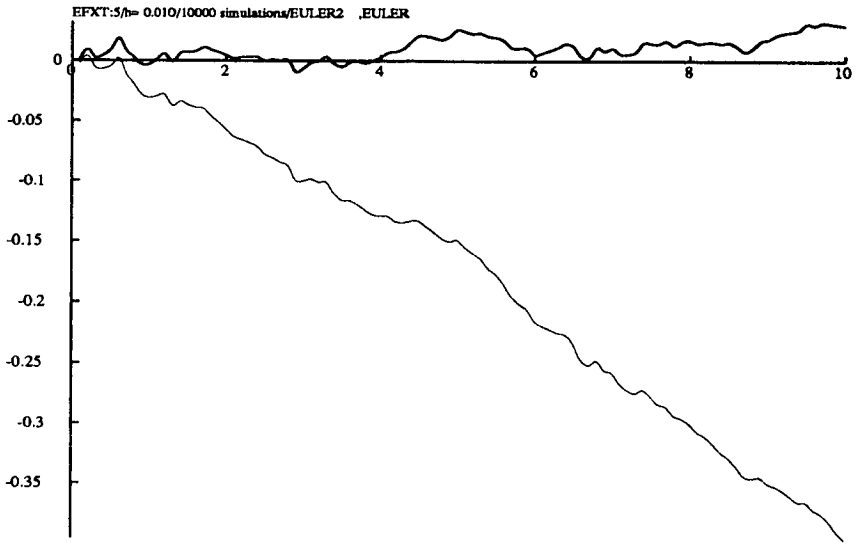
- for the Euler scheme, the function $e_1(T)$ is defined by :

$$-\int_0^T \mathbf{E}\psi_e(s, X_s)ds = (1-\mathbf{E}|X_0|^2)T - (1+T)\log(1+T) - \frac{1}{2} + \frac{1}{2(1+T)}$$

- and, for the Milshtein scheme, by :

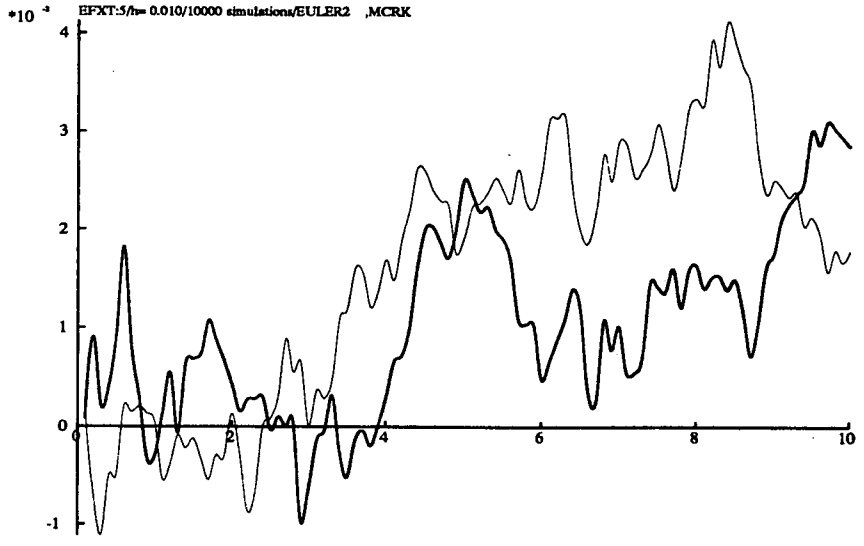
$$-\int_0^T \mathbf{E}\psi_m(s, X_s)ds = (1-\mathbf{E}|X_0|^2)T - (1+T)\log(1+T)$$



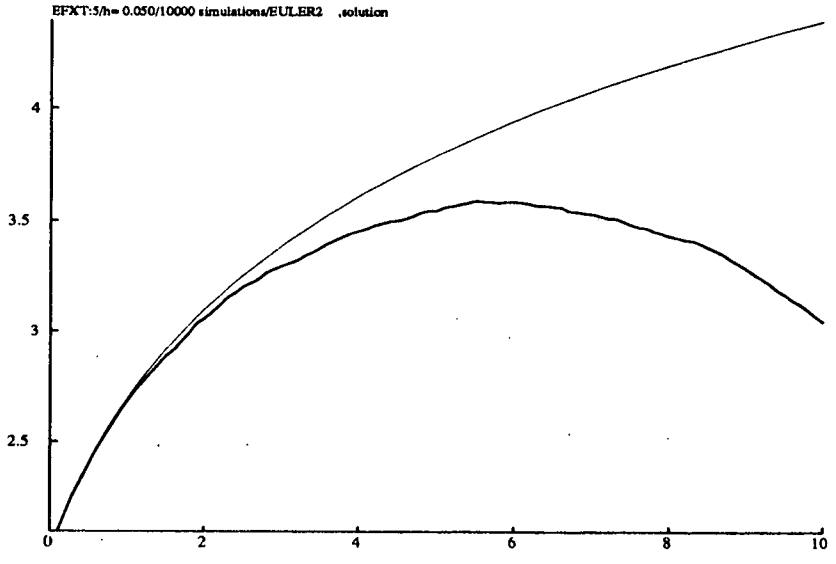


Now, we compare the errors due to the above Romberg extrapolation (thick line) and to a second order scheme (the MCRK scheme of Talay [5] : thin line), with a step size $h = 0.003$: they are equivalent. But the CPU time corresponding to the extrapolation algorithm is about 3 times less than the CPU time corresponding to any of the second order schemes we have

used, for the reasons indicated at the end of Subsection 2.3.



Of course, if the step sizes are too large, the results of the extrapolation may also be very bad : below we show the approximate solution due to an extrapolation of values due to Euler scheme with $h = 0.05$ and $h = 0.1$ (thick line), and the true solution (thin line).



2.5 Numerical tests : second example

In the second example, a strange fact occurs : Euler scheme gives results equivalent to those given by second order schemes, whereas Milshtein scheme, which also is of first order, may lead to extremely inaccurate results. We will explain why ; moreover, an interesting point is that the extrapolation even based on bad results permits to obtain a good accuracy !

We first define the functions $\alpha(t)$ and $\beta(t)$ by :

$$\begin{aligned}\alpha(t) &= \frac{3\sqrt{2} - 2\Omega\sin(\Omega t) - 12}{4(2+\cos(\Omega t))} \\ \beta(t) &= \frac{6\sqrt{2} - 6}{4(2+\cos(\Omega t))}\end{aligned}$$

The function b is defined by :

$$\begin{aligned}b^1 &= \alpha(t)x^1 + \beta(t)x^2 \\ b^2 &= \beta(t)x^1 + \alpha(t)x^2\end{aligned}$$

and the matrix σ is defined by :

$$\begin{aligned}\sigma_1^1(x^1,x^2) &= \sin\left(\nu\left(x^1+x^2\right)\right) \\ \sigma_2^1(x^1,x^2) &= \cos\left(\nu\left(x^1+x^2\right)\right) \\ \sigma_1^2(x^1,x^2) &= \sin\left(\nu\left(x^1+x^2\right)+\frac{\pi}{3}\right) \\ \sigma_2^2(x^1,x^2) &= \cos\left(\nu\left(x^1+x^2\right)+\frac{\pi}{3}\right)\end{aligned}$$

It is easy to check that, if the initial law is gaussian with zero mean and covariance matrix equal to

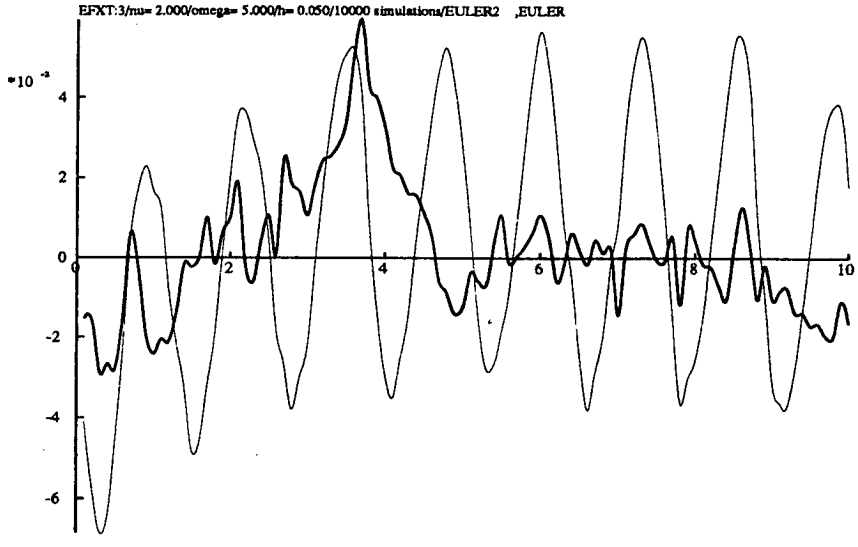
$$C = \begin{bmatrix} 1 & \sqrt{2}/2 \\ \sqrt{2}/2 & 1 \end{bmatrix} \tag{15}$$

then the law of $X(t)$ is also gaussian with zero mean and covariance matrix equal to $\frac{2+\cos(\Omega t)}{3}C$.

We compute $\mathbf{E}|X_t^1|^2 = \frac{2+\cos(\Omega t)}{3}$.

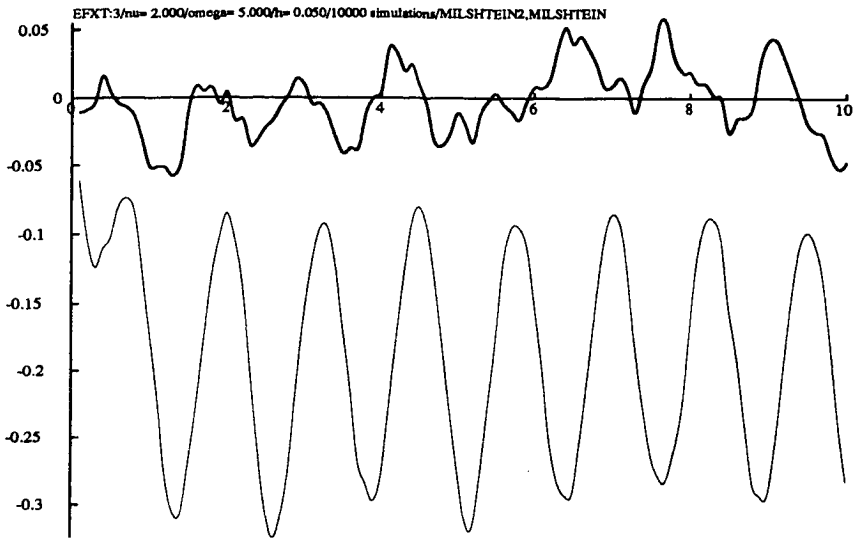
The two next figures compare the results given by the Euler scheme, the Milshtein scheme, and Romberg extrapolations.

The first figure shows the time evolution of the error due to the extrapolation based on the Euler scheme with the 2 step-sizes $h = 0.05$ and $h = 0.1$ (thick line), and of the error due to the Euler scheme itself with the step-size $h = 0.05$ (thin line) : the extrapolation has still improved the accuracy.



The second figure is devoted to the Milshtein scheme.

Numerical experiments show that, for $\nu \leq 1$, it behaves as the Euler scheme ; but, for $\nu > 1$, the error becomes very large, as illustrated by the next figure (corresponding to the situation $\nu = 2$), which compares the time evolution of the errors due to the extrapolation based on the Milshtein scheme with the 2 step-sizes $h = 0.05$ and $h = 0.1$ (thick line), and to the Milshtein scheme itself with the step-size $h = 0.05$ (thin line) : the extrapolation has extremely improved the accuracy.



The following Proposition explains why the Milshtein scheme may be so bad.

Proposition 4 *Let $\psi_e(t)$ and $\psi_m(t)$ be defined by (9) and (12).*
Let $K(t)$ be :

$$K(t) = \left[\frac{\Omega t + \pi}{2\pi}\right]$$

and $\gamma(t)$ be defined by :

$$\begin{aligned}\gamma(t) &= \int_0^t \frac{1}{2+\cos(\Omega s)} ds \\ &= \frac{2}{\sqrt{3}\Omega}\left(K(t)\pi + \arctan\left[\frac{1}{\sqrt{3}}\tan\left(\frac{\Omega t}{2} - K(t)\pi\right)\right]\right)\end{aligned}$$

Let A and B denote :

$$\begin{aligned}A &= \frac{18 - 9\sqrt{2}}{2} \\ B &= \frac{3\sqrt{2}+6}{2}\end{aligned}$$

Then

$$\begin{aligned}\psi_m(t,x) &- \psi_e(t,x) = \\ &= \frac{3\nu^2}{8}\frac{2+\cos(\Omega T)}{2+\cos(\Omega t)}\left[3e^{-A(\gamma(T)-\gamma(t))} + e^{-B(\gamma(T)-\gamma(t))}\right]\end{aligned}$$

and the difference between the errors due to the Milshtein and Euler schemes is given by :

$$Err_m(T,h) - Err_e(T,h) = \frac{3\nu^2}{8}(2+\cos(\Omega T))\left[\frac{3(1-e^{-A\gamma(T)})}{A} + \frac{1-e^{-A\gamma(T)}}{B}\right]$$

Before giving the proof, let us remark that the above result explains the increasing of the error due to the Milshtein scheme as ν increases.

It also explains why, in the case of the previous figure, the Milshtein scheme error is much larger than the Euler scheme one, and why the absolute value of the difference between these two errors oscillates (when t increases from 0 to 10) between 0.1 and 0.3 .

Proof.

Using Itô formula, we obtain easily that the function

$$\theta \longrightarrow \begin{pmatrix} \mathbf{E}|X^1(\theta)|^2 \\ \mathbf{E}(X^1(\theta)X^2(\theta)) \\ \mathbf{E}|X^2(\theta)|^2 \end{pmatrix}$$

solves the following differential system :

$$\frac{d}{d\theta}V(\theta) = \begin{bmatrix} 2\alpha(\theta) & 2\beta(\theta) & 0 \\ \beta(\theta) & 2\alpha(\theta) & \beta(\theta) \\ 0 & 2\beta(\theta) & 2\alpha(\theta) \end{bmatrix} V(\theta) + \begin{pmatrix} 1 \\ \frac{1}{2} \\ 1 \end{pmatrix} \tag{16}$$

It is possible to compute explicitly the evolution operator $V(\theta,t)$ of the homogeneous system : remarking

$$\begin{bmatrix} 2\alpha(\theta) & 2\beta(\theta) & 0 \\ \beta(\theta) & 2\alpha(\theta) & \beta(\theta) \\ 0 & 2\beta(\theta) & 2\alpha(\theta) \end{bmatrix} = 2\alpha(\theta)I + \beta(\theta)\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

one gets :

$$V(\theta,t) = \exp\left(2\int_t^\theta \alpha(\xi)d\xi\right)\exp\left(\int_t^\theta \beta(\xi)d\xi \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}\right)$$

and therefore easy computations lead to :

$$V(\theta,t) = \begin{bmatrix} V^1(\theta,t) & V^2(\theta,t) & V^3(\theta,t) \\ \frac{1}{2}V^2(\theta,t) & V^4(\theta,t) & \frac{1}{2}V^2(\theta,t) \\ V^3(\theta,t) & V^2(\theta,t) & V^1(\theta,t) \end{bmatrix}$$

with $V^1(\theta,t), V^2(\theta,t), V^3(\theta,t), V^4(\theta,t)$ defined by :

$$\begin{aligned} V^1(\theta,t) &= \frac{1}{4}(s(\theta,t)+d(\theta,t)) + \frac{1}{2}\exp\left(2\int_t^\theta \alpha(\xi)d\xi\right) \\ V^2(\theta,t) &= \frac{1}{2}(s(\theta,t)-d(\theta,t)) \\ V^3(\theta,t) &= \frac{1}{4}(s(\theta,t)+d(\theta,t)) - \frac{1}{2}\exp\left(2\int_t^\theta \alpha(\xi)d\xi\right) \\ V^4(\theta,t) &= \frac{1}{2}(s(\theta,t)+d(\theta,t)) \end{aligned}$$

where

$$\begin{aligned} s(\theta,t) &= \exp\left(2\int_t^\theta (\alpha(\xi)+\beta(\xi))d\xi\right) \\ d(\theta,t) &= \exp\left(2\int_t^\theta (\alpha(\xi)-\beta(\xi))d\xi\right) \end{aligned}$$

As already mentioned, if the initial law is the centered gaussian law of covariance matrix C defined by (15), then the law of (X_t) is gaussian, of covariance matrix $\frac{2+\cos(\Omega t)}{3}C$. This provides the solution of (16) corresponding to the initial condition $(1, \frac{\sqrt{2}}{2}, 1)$. The solution corresponding to the initial condition $(|x^1|^2, x^1x^2, |x^2|^2)$ at time t will be :

$$V(\theta,t)\begin{bmatrix} |x^1|^2 - \frac{2+\cos(\Omega t)}{3} \\ x^1x^2 - \frac{\sqrt{2}(2+\cos(\Omega t))}{6} \\ |x^2|^2 - \frac{2+\cos(\Omega t)}{3} \end{bmatrix} + \frac{2+\cos(\Omega\theta)}{3}\begin{bmatrix} 1 \\ \frac{\sqrt{2}}{2} \\ 1 \end{bmatrix}$$

Therefore, if we define the function $u(t,x)$ by :

$$u(t,x) = \mathbf{E}_{t,x}|X_T^1|^2$$

then :

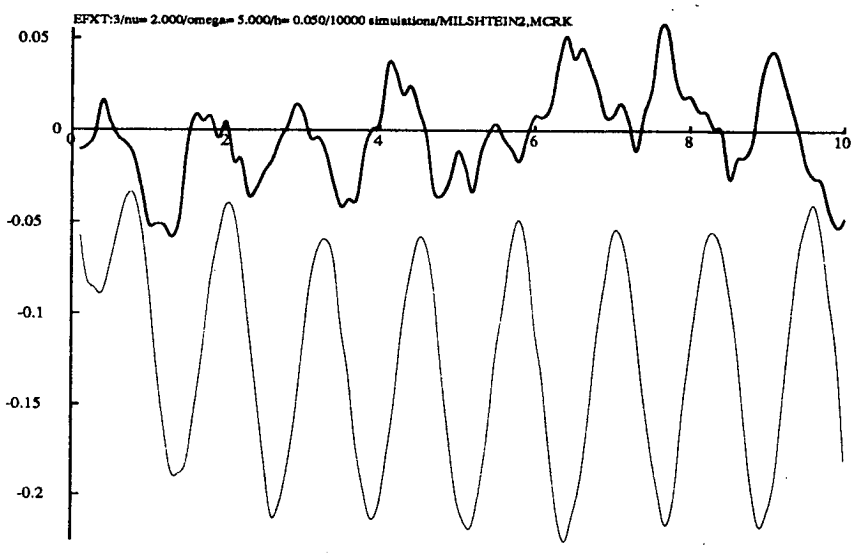
$$\begin{aligned} u(t,x) &= V^1(T,t)\left(|x^1|^2 - \frac{2+\cos(\Omega t)}{3}\right) + V^2(T,t)\left(x^1x^2 - \frac{\sqrt{2}(2+\cos(\Omega t))}{6}\right) \\ &\quad + V^3(T,t)\left(|x^2|^2 - \frac{2+\cos(\Omega t)}{3}\right) + \frac{2+\cos(\Omega T)}{3} \end{aligned}$$

To conclude, it just remains to remark :

$$\begin{aligned} \psi_m(t,x) - \psi_e(t,x) &= \frac{1}{4}\sum_{i_1,i_2,j,k,l} a_k^l(t,x)\partial_l\sigma_j^{i_1}(t,x)\partial_k\sigma_j^{i_2}(t,x)\partial_{i_1i_2}u(t,x) \\ &= \frac{3\nu^2}{8}(3s(T,t) + d(T,t)) \end{aligned}$$

and to use the above expressions for the functions $s(\theta,t)$ and $s(\theta,t)$. □

Finally, we compare the extrapolation method based on the Milshtein scheme (thick line) and the second order MCRK scheme (thin line). For $h = 0.05$, the MCRK scheme is also numerically unstable, therefore the results due to the extrapolation are much better.



3 The ergodic case

We consider the system (1) with time independent coefficients :

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

Let us suppose that the process (X_t) is ergodic : it has a unique invariant probability law μ, and for any deterministic condition $X_0 = x$, and any μ-integrable function f :

$$a.s. \lim_{T\to+\infty} \frac{1}{T}\int_0^T f(X_s)ds = \int f(y)\mu(dy)$$

With an analogous aim to that of the previous Section, now we want to prove the following expansion for the Euler and Milshtein schemes, with any deterministic initial condition $\overline{X}_0^h$:

$$a.s. \lim_{N\to+\infty} \frac{1}{N}\sum_{p=1}^{N} f(\overline{X}_p^h) = \int f(y)\mu(dy) + e_1 h + O(h^2)$$

where e_1 is independent of h.

3.1 Preliminary results

First, we state the Hypotheses.

We suppose :

(H1) the functions b, σ are of class C^∞ with bounded derivatives of any order ; the function σ is bounded ;

(H2) the operator $\mathcal{L}$ is uniformly elliptic : there exists a positive constant α such that :

$$\forall x, \xi \in \mathbb{R}^d \quad , \quad \sum_{i,j} a_i^j(\xi)x^i x^j \geq \alpha|x|^2$$

(H3) there exist a strictly positive constant β and a compact set K such that :

$$\forall x \in \mathbb{R}^d - K \quad , \quad x \cdot b(x) \leq -\beta|x|^2$$

It is well known that (H1) and (H3) is a (even too strong) sufficient condition for (X_t) to be ergodic (see Hasminskii [3] e.g.).

We introduce the class $\mathcal{F}$ of functions ϕ of class C^∞, such that, for some positive integer s and positive constants C, λ :

$$\forall \theta \in \mathbb{R}_+ \ , \ \forall x \in \mathbb{R}^d \ : \ | \ \phi(\theta, x) \ | \leq Ce^{-\lambda\theta}(1 + |x|^s) \tag{17}$$

We will use the results collected in the

Theorem 2 (Talay [6]) *(i) Under (H1) and (H3), the following holds :*

$$\forall n \in \mathbb{N} \ , \ \exists C_n > 0 \ , \ \exists \gamma_n > 0 \ :$$
$$\mathbf{E}|X_t^{0,x}|^n \leq C_n(1 + |x|^n e^{-\gamma_n t}) \ , \ \forall t \ , \ \forall x$$

The same kind of estimate holds for the Euler and Milshtein schemes : if $(\overline{X}_p^h(\xi)$ denotes the process starting at $\overline{X}_0^h = \xi$,

$$\forall n \in \mathbb{N} \ , \ \exists C_n > 0 \ , \ \exists \gamma_n > 0 \ , \ \exists H > 0 \ , \ \forall h \leq H \ ,$$
$$\mathbf{E}|\overline{X}_p^h(\xi)|^n \leq C_n(1 + |x|^n e^{-\gamma_n p h}) \ , \ \forall p \ , \ \forall x$$

(ii) The Markov chain defined by the Euler (resp. Milshtein) scheme is ergodic. Its unique invariant probability law has finite moments of any order, as well as the unique invariant probability measure of (X_t), μ.

(iii) For the Euler and Milshtein schemes, if f is a function of the class $\mathcal{F}$, there exist positive C, λ such that, for any deterministic initial condition ξ :

$$| \lim_{N \to \infty} \frac{1}{N} \sum_{p=1}^{N} f(\theta, \overline{X}_p^h(\xi)) - \int f(\theta, y)\mu(dy)| \leq Ce^{-\lambda\theta} h \ , \ a.s.$$

Remark

In Talay [6], the part *(iii)* is obtained only for homogeneous functions f (in that case, $\lambda = 0$ of course).

It is not difficult to check that the proof can be adapted to the non homogeneous situation.

Let us just give an indication concerning the crucial step of this proof, which is the following Lemma : if f is a homogeneous function, and $v(t, x)$

is defined by $v(t,x) = Ef(X_t^{0,x})$, then, under (H1), (H2), (H3), for any multi-index I indexing a time and space derivation, there exist an integer s_I and strictly positive constants Γ_I and γ_I such that the derivative $\partial_I v(t,x)$ satisfies :

$$|\partial_I v(t,x)| \leq \Gamma_I(1+|x|^{s_I})e^{-\gamma_I t} \tag{18}$$

In fact, it is not difficult to show that the constant Γ_I can be choosen proportional to $|f|_{L^2(\mu)}$. Therefore :

Theorem 3 *Suppose that the hypotheses (H1), (H2), (H3) hold.*

Let $\phi(\theta,x)$ be a function of the class $\mathcal{F}$, and $v(\theta;t,x) = Ef(\theta, X_t^{0,x})$.

Then, for any multi-index I indexing a time and space differentiation, there exist an integer s_I and strictly positive constants Γ_I, λ and γ_I such that the derivative $\partial_I v(\theta;t,x)$ satisfies :

$$|\partial_I v(\theta;t,x)| \leq \Gamma_I(1+|x|^{s_I})e^{-\gamma_I t-\lambda\theta}$$

3.2 Main result

As well as in the computation of $Ef(X_t)$ in finite time, the discretization error can be expanded in terms of h.

Theorem 4 *Suppose that the hypotheses (H1), (H2), (H3) hold, and let f be a function of the class $\mathcal{F}$. Let :*

$$\begin{aligned}
\phi_e(t,x) &= \frac{1}{2}\sum_{i,j=1}^{d} b^i(x)b^j(x)\partial_{ij}u(t,x) \\
&+\frac{1}{2}\sum_{i,j,k=1}^{d} b^i(x)a_k^j(x)\partial_{ijk}u(t,x) \\
&+\frac{1}{8}\sum_{i,j,k,l=1}^{d} a_j^i(x)a_l^k(x)\partial_{ijkl}u(t,x) + \frac{1}{2}\frac{\partial^2}{\partial t^2}u(t,x) \\
&-\sum_{i=1}^{d} b^i(x)\frac{\partial}{\partial t}\partial_i u(t,x) - \frac{1}{2}\sum_{i,j=1}^{d} a_j^i(x)\frac{\partial}{\partial t}\partial_{ij}u(t,x)
\end{aligned} \tag{19}$$

Let λ_e be defined by :

$$\lambda_e = \int_0^{+\infty} \int_{\mathbb{R}^d} \phi_e(t,y)\mu(dy)dt$$

Then the Euler scheme error satisfies : for any deterministic initial condition $\xi = \overline{X}_0^h$

$$\int f(y)\mu(dy) - a.s. \lim_{N\to+\infty} \frac{1}{N}\sum_{p=1}^{N} f(\overline{X}_p^h(\xi)) = -\lambda_e h + O(h^2) \qquad (20)$$

For the Milshtein scheme, an analogous result can be written, substituting

$$\lambda_m = \int_0^{+\infty} \int_{\mathbb{R}^d} \phi_m(t,y)\mu(dy)dt$$

to λ_e, where

$$\begin{aligned} \phi_m(t,x) &= \phi_e(t,x) + \frac{1}{4}\sum_{i_1,i_2,j,k,l} a_k^l(x)\partial_l\sigma_j^{i_1}(x)\partial_k\sigma_j^{i_2}(x)\partial_{i_1 i_2}u(t,x) \\ &\quad + \frac{1}{2}\sum_{\substack{i_1,i_2,i_3\\ j_1,j_2,k}} \sigma_{j_1}^{i_1}(x)\sigma_{j_2}^{i_2}(x)\sigma_{j_1}^{k}(x)\partial_k\sigma_{j_2}^{i_3}(x)\partial_{i_1 i_2 i_3}u(t,x) \qquad (21) \end{aligned}$$

Proof.

We will only consider the Euler scheme. Let $\xi = \overline{X}_0^h$ be the initial condition. We will write $\overline{X}_p^h$ instead of $\overline{X}_p^h(\xi)$.

Let $v(t,x) = Ef(X_t^{0,x})$. It solves :

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) &= \mathcal{L}v(t,x) \\ v(0,x) &= f(x) \end{cases} \qquad (22)$$

Using (22), Theorem (2)(ii) and (18), one may write that the remainder term $\mathcal{R}_p^h$ of the expansion :

$$\mathbf{E}f(\overline{X}_p^h) = \mathbf{E}v(0,\overline{X}_p^h) = v(ph,\xi) + h^2\sum_{j=1}^{p}\mathbf{E}\psi_e(jh,\overline{X}_{p-j}^h) + h^3\mathcal{R}_p^h \qquad (23)$$

satisfies : there exists a constant C independent of h and p (but depending on ξ) such that

$$|\mathcal{R}_p^h| \leq \frac{C}{h} \tag{24}$$

Summing (23) from 1 to N, and reordering the sum, we get :

$$\begin{aligned}\frac{1}{N}\sum_{p=1}^{N}\mathbf{E}f(\overline{X}_p^h) &= \frac{1}{N}\sum_{p=1}^{N}v(ph,\xi) + h^2\frac{1}{N}\sum_{j=1}^{N}\sum_{p=0}^{N-j}\mathbf{E}\psi_e(jh,\overline{X}_p^h) \\ &\quad + h^3\frac{1}{N}\sum_{p=1}^{N}\mathcal{R}_p^h\end{aligned}$$

In order to prove (20), we are going to make N tend to infinity in the previous equality.

Let $\overline{\mu}^h$ denote the invariant law of $(\overline{X}_p^h)$: this law integrates the polynomial functions (Theorem (2)(*ii*)), and therefore the function f, so that, as N goes to infinity, almost surely $\frac{1}{N}\sum_{p=1}^{N} f(\overline{X}_p^h)$ tends to $\int f(y)\overline{\mu}^h(dy)$.

Besides, Theorem (2)(*iii*) shows the existence of a constant C such that :

$$\forall N \quad , \quad \mathbf{E}|\frac{1}{N}\sum_{p=1}^{N} f(\overline{X}_p^h)|^2 \leq C$$

Therefore, the sequence $(\frac{1}{N}\sum_{p=1}^{N}\mathbf{E}f(\overline{X}_p^h))$ converges when N goes to infinity, and :

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\sum_{p=1}^{N}\mathbf{E}f(\overline{X}_p^h) &= \int f(y)\overline{\mu}^h(dy) \\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{p=1}^{N} f(\overline{X}_p^h) \quad a.s.\end{aligned}$$

In the same way, we can show :

$$\lim_{N\to\infty}\frac{1}{N}\sum_{p=1}^{N} v(ph,\xi) = \int f(y)\mu(dy)$$

Now, we remark that (24) implies :

$$h^3\frac{1}{N}\sum_{p=1}^{N}\mathcal{R}_p^h = \mathcal{O}(h^2)$$

Thus, it just remains to prove :

$$\lim_{N\to\infty} h\frac{1}{N}\sum_{j=1}^{N}\sum_{p=0}^{N-j}\mathbf{E}\psi_e(jh,\overline{X}_p^h) = \int_0^{+\infty}\int_{\mathbb{R}^d}\phi_e(t,y)\mu(dy)dt$$

Let us define, for each j, the sequence $(\rho_N^h(j))$ by :

$$\rho_N^h(j) = \frac{1}{N}\sum_{p=0}^{N}\mathbf{E}\psi_e(jh,\overline{X}_p^h)$$

and the function $\rho^h(t)$ by :

$$\rho^h(t) = \int_{\mathbb{R}^d}\psi_e(t,y)\overline{\mu}^h(dy)$$

With similar arguments as those used above, using the inequality (18) and Theorem (2), we can show that, as N goes to infinity, $(\rho_N^h(j))$ tends to $\rho^h(jh)$.

Moreover, there exist strictly positive constants C_1, C_2 such that :

$$\frac{1}{N}\sum_{j=1}^{N}\sum_{p=N-j+1}^{N}|\mathbf{E}\psi_e(jh,\overline{X}_p^h)| \leq \frac{C_1}{N}\sum_{j=1}^{N} je^{-jC_2h}$$

and therefore the left side of the previous inequality tends to 0 as N goes to infinity.

We conclude :

$$\lim_{N\to+\infty}\frac{1}{N}\sum_{j=1}^{N}\sum_{p=0}^{N-j}\mathbf{E}\psi_e(jh,\overline{X}_p^h) = \rho^h(jh)$$

Let us define :

$$\rho(t) = \int_{\mathbb{R}^d}\psi_e(t,y)\mu(dy)$$

Theorem (2)(iii) implies that, for any t :

$$|\rho^h(t) - \rho(t)| \leq Ce^{-\lambda t}h$$

As there exist positive constants C and λ such that

$$|\rho(t)| + |\frac{d}{dt}\rho(t)| \leq Ce^{-\lambda t}$$

we may write (as a consequence of the Lebesgue Dominated Convergence Theorem) :

$$\sum_{j=1}^{+\infty} \int_{\mathbb{R}^d} \psi_e(jh, y)\mu(dy) = \frac{1}{h} \int_0^{+\infty} \int_{\mathbb{R}^d} \psi_e(t, y)\mu(dy) + \mathcal{O}(1)$$

That ends the proof. □

3.3 Numerical tests

We consider the example of Section (2.5), with $\Omega = 0$.

The invariant law μ is normal, of zero mean and covariance matrix (15).

Again, we choose $f(x) = |x^1|^2$, so that $\int f(y)\mu(dy) = 1$.

Again, we may observe that, in this example, the Euler scheme gives better results than the Milshtein scheme.

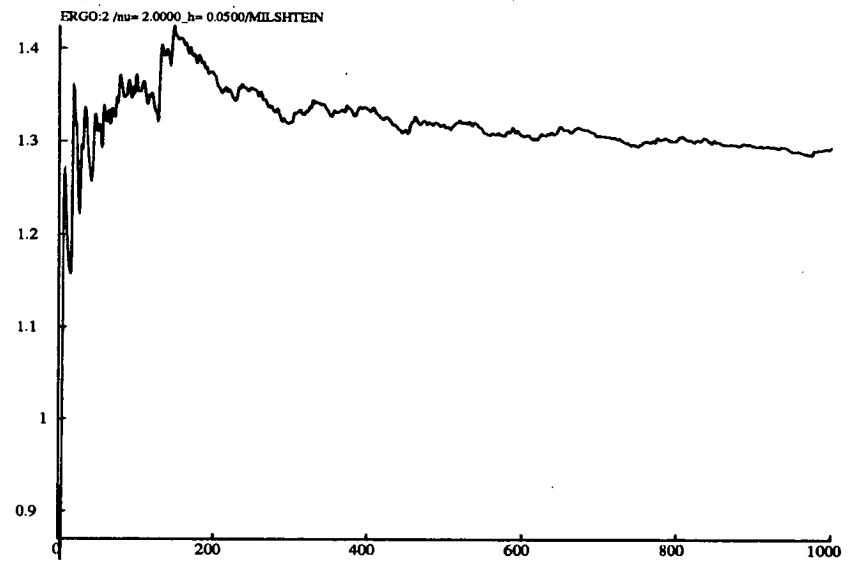
Indeed, explicit computations of the functions ϕ_e and ϕ_m show that, for the Euler scheme, the error is equal to

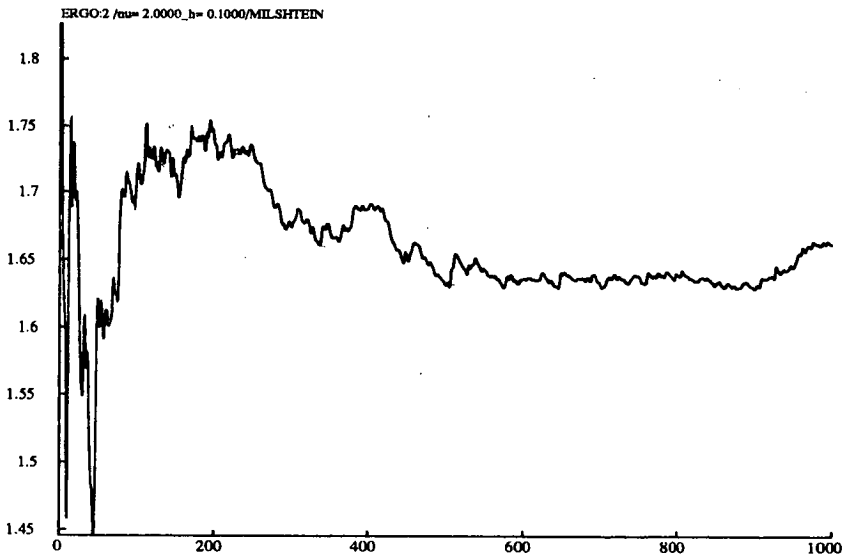
$$-\frac{1}{4}h + \mathcal{O}(h^2)$$

whereas, for the Milshtein scheme, this error is equal to

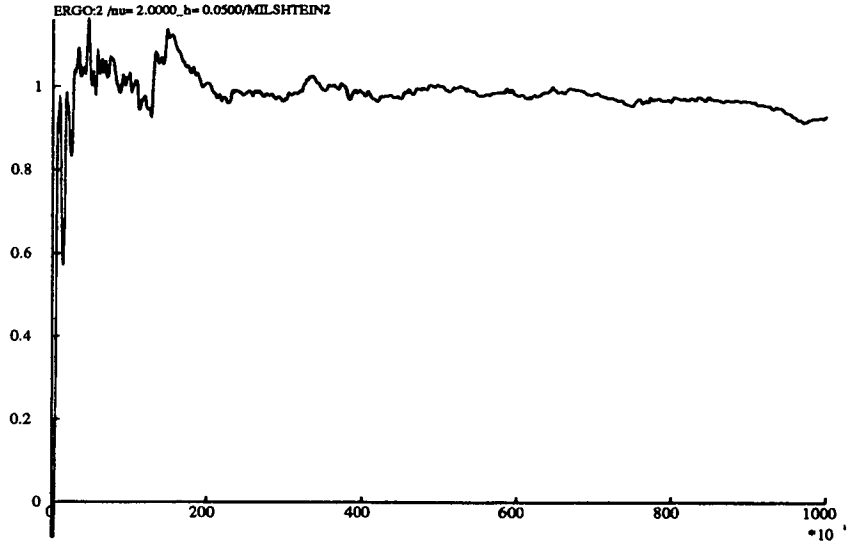
$$-(\frac{1}{4} + \frac{3}{2}\nu^2)h + \mathcal{O}(h^2)$$

This result also permits to understand the unaccuracy of the Milshtein scheme for $\nu = 2$, when the Milshtein scheme is run with $h = 0.1$ and $h = 0.05$. The errors are, respectively, about 0.3 and 0.65, as shown by the two following figures.


ERGO:2 /mu= 2.0000_h= 0.0500/MILSHTEIN
1.4
1.3
1.2
1.1
1
0.9
0
200
400
600
800
1000


ERGO:2 /mu= 2.0000_h= 0.1000/MILSHTEIN
1.8
1.75
1.7
1.65
1.6
1.55
1.5
1.45
0
200
400
600
800
1000

A Romberg interpolation based upon these catastrophic results gives a satisfying accuracy.



Let us also underline that, in any situation, the result of Theorem (4) helps in the numerical choice of N (which is a crucial difficulty) : at least, we must observe that, for several N large enough, the quantity :

$$\frac{1}{Nh}\left[\sum_{p=1}^{N} f(\overline{X}_p^h) - \frac{1}{2}\sum_{p=1}^{2N} f(\overline{X}_p^{h/2})\right]$$

is independent from N.

Finally, similar results can be obtained for the approximation of Lyapunov exponents of bilinear systems (cf Talay [7]) : the theoretical and numerical details are relevant from what has been done in that Section.

4 Conclusion

We have shown that the approximation error of $\mathbf{E}f(X_t)$, as well as $\int f(y)\mu(dy)$, can be expanded in terms of the discretization step h, in particular when the approximating process is defined by the Euler scheme. This expansion justifies a Romberg extrapolation of the computed values corresponding to two different choices of the step-size.

This procedure has an accuracy of order 2, as well as the second order schemes, for example the MCRK scheme (obtained by a different use of Romberg extrapolation, based on the process defined by the Milshtein scheme), but is much simpler to implement and requires less C.P.U. time.

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