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## Expansions For Gaussian Processes And Parseval Frames

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### Abstract

We derive a precise link between series expansions of Gaussian random vectors in a Banach space and Parseval frames in their reproducing kernel Hilbert space. The results are applied to pathwise continuous Gaussian processes and a new optimal expansion for fractional Ornstein-Uhlenbeck processes is derived.

**Key words:** Gaussian process, series expansion, Parseval frame, optimal expansion, fractional Ornstein-Uhlenbeck process.

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# 1 Introduction

Series expansions is a classical issue in the theory of Gaussian measures (see [2], [9], [17]). Our motivation for a new look on this issue finds its origin in recent new expansions for fractional Brownian motions (see [14], [1], [5], [6], [7]).

Let  $(E, \|\cdot\|)$  be a real separable Banach space and let  $X : (\Omega, \mathcal{A}, P) \rightarrow E$  be a centered Gaussian random vector with distribution  $P_X$ . In this article we are interested in series expansions of  $X$  of the following type. Let  $\xi_1, \xi_2, \dots$  be i.i.d.  $N(0, 1)$ -distributed real random variables. A sequence  $(f_j)_{j \geq 1} \in E^{\mathbb{N}}$  is called *admissible* for  $X$  if

$$\sum_{j=1}^{\infty} \xi_j f_j \text{ converges a.s. in } E \quad (1.1)$$

and

$$X \stackrel{d}{=} \sum_{j=1}^{\infty} \xi_j f_j. \quad (1.2)$$

By adding zeros finite sequences in  $E$  may be turned into infinite sequences and thus also serve as admissible sequences.

We observe a precise link to frames in Hilbert spaces. A sequence  $(f_j)_{j \geq 1}$  in a real separable Hilbert space  $(H, (\cdot, \cdot))$  is called *Parseval frame* for  $H$  if  $\sum_{j=1}^{\infty} (f_j, h) f_j$  converges in  $H$  and

$$\sum_{j=1}^{\infty} (f_j, h) f_j = h \quad (1.3)$$

for every  $h \in H$ . Again by adding zeros, finite sequences in  $H$  may also serve as frames. For the background on frames the reader is referred to [4]. (Parseval frames correspond to tight frames with frame bounds equal to 1 in [4].)

**Theorem 1.** *Let  $(f_j)_{j \geq 1} \in E^{\mathbb{N}}$ . Then  $(f_j)$  is admissible for  $X$  if and only if  $(f_j)$  is a Parseval frame for the reproducing kernel Hilbert space of  $X$ .*

We thus demonstrate that the right notion of a "basis" in connection with expansions of  $X$  is a Parseval frame and not an orthonormal basis for the reproducing kernel Hilbert space of  $X$ . The first notion provides the possibility of redundancy and is more flexible as can be seen *e.g.* from wavelet frames. It also reflects the fact that "sums" of two (or more) suitable scaled expansions of  $X$  yield an expansion of  $X$ .

The paper is organized as follows. In Section 2 we investigate the general Banach space setting in the light of frame theory and provide the proof of Theorem 1. Section 3 contains applications to pathwise continuous processes  $X = (X_t)_{t \in I}$  viewed as  $C(I)$ -valued random vectors where  $I$  is a compact metric space. Furthermore, we comment on optimal expansions. Fractional Brownian motions serve as illustration. Section 4 contains a new optimal expansion for fractional Ornstein-Uhlenbeck processes.

It is convenient to use the symbols  $\sim$  and  $\approx$  where  $a_n \sim b_n$  means  $a_n/b_n \rightarrow 1$  and  $a_n \approx b_n$  means  $0 < \liminf a_n/b_n \leq \limsup a_n/b_n < \infty$ .

## 2 The Banach space setting

Let  $(E, \|\cdot\|)$  be a real separable Banach space. For  $u \in E^*$  and  $x \in E$ , it is convenient to write

$$\langle u, x \rangle$$

in place of  $u(x)$ . Let  $X : (\Omega, \mathcal{A}, P) \rightarrow E$  be a centered Gaussian random vector with distribution  $P_X$ . The covariance operator  $C = C_X$  of  $X$  is defined by

$$C : E^* \rightarrow E, \quad Cu := E\langle u, X \rangle X. \quad (2.1)$$

This operator is linear and (norm-)continuous. Let  $H = H_X$  denote the reproducing kernel Hilbert space (Cameron Martin space) of the symmetric nonnegative definite kernel

$$E^* \times E^* \rightarrow \mathbb{R}, (u, v) \mapsto \langle u, Cv \rangle$$

(see [17], Propositions III.1.6. and III.1.7). Then  $H$  is a Hilbert subspace of  $E$ , that is  $H \subset E$  and the inclusion map is continuous. The reproducing property reads

$$(h, Cu)_H = \langle u, h \rangle, \quad u \in E^*, h \in H \quad (2.2)$$

where  $(\cdot, \cdot)_H$  denotes the scalar product on  $H$  and the corresponding norm is given by

$$\|h\|_H = \sup\{|\langle u, h \rangle| : u \in E^*, \langle u, Cu \rangle \leq 1\}. \quad (2.3)$$

In particular, for  $h \in H$ ,

$$\|h\| \leq \sup_{\|u\| \leq 1} \langle u, Cu \rangle^{1/2} \|h\|_H = \|C\|^{1/2} \|h\|_H. \quad (2.4)$$

The  $\|\cdot\|_H$ -closure of  $A \subset H$  is denoted by  $\overline{A}^{(H)}$ . Furthermore,  $H$  is separable,  $C(E^*)$  is dense in  $(H, \|\cdot\|_H)$ , the unit ball

$$U_H := \{h \in H : \|h\|_H \leq 1\}$$

of  $H$  is a compact subset of  $E$ ,

$$\text{supp}(P_X) = (\ker C)^\perp := \{x \in E : \langle u, x \rangle = 0 \text{ for every } u \in \ker C\} = \overline{H} \text{ in } E$$

and

$$H = \{x \in E : \|x\|_H < \infty\} \quad (2.5)$$

where  $\|x\|_H$  is formally defined by (2.3) for every  $x \in E$ . As for the latter fact, it is clear that  $\|h\|_H < \infty$  for  $h \in H$ . Conversely, let  $x \in E$  with  $\|x\|_H < \infty$ . Observe first that  $x \in \overline{H}$ . Otherwise, by the Hahn-Banach theorem, there exists  $u \in E^*$  such that  $u|_{\overline{H}} = 0$  and  $\langle u, x \rangle > 0$ . Since,  $\langle u, Cu \rangle = 0$  this yields

$$\|x\|_H \geq \sup_{a>0} a \langle u, x \rangle = \infty,$$

a contradiction. Now consider  $C(E^*)$  as a subspace of  $(H, \|\cdot\|_H)$  and define  $\varphi : C(E^*) \rightarrow \mathbb{R}$  by  $\varphi(Cu) := \langle u, x \rangle$ . If  $Cu_1 = Cu_2$ , then using  $(\ker C)^\perp = \overline{H}$ ,  $\langle u_1 - u_2, x \rangle = 0$ . Therefore,  $\varphi$  is well defined. The map  $\varphi$  is obviously linear and it is bounded since

$$\|\varphi\| = \sup\{|\varphi(Cu)| : u \in E^*, \|Cu\|_H \leq 1\} = \|x\|_H < \infty$$

by (2.2). By the Hahn-Banach theorem there exists a linear bounded extension  $\tilde{\varphi} : \overline{C(E^*)}^{(H)} \rightarrow R$  of  $\varphi$ . Then, since  $\overline{C(E^*)}^{(H)} = H$ , by the Riesz theorem there exists  $g \in H$  such that  $\tilde{\varphi}(h) = (h, g)_H$  for every  $h \in H$ . Consequently, using (2.2),

$$\langle u, x \rangle = \varphi(Cu) = (Cu, g)_H = \langle u, g \rangle$$

for every  $u \in E^*$  which gives  $x = g \in H$ .

The key is the following characterization of admissibility. It relies on the Ito-Nisio theorem. Condition (v) is an abstract version of Mercer's theorem (cf. [15]. p. 43). Recall that a subset  $G \subset E^*$  is said to be separating if for every  $x, y \in E, x \neq y$  there exists  $u \in G$  such that  $\langle u, x \rangle \neq \langle u, y \rangle$ .

**Lemma 1.** *Let  $(f_j)_{j \geq 1} \in E^M$ . The following assertions are equivalent.*

(i) *The sequence  $(f_j)_{j \geq 1}$  is admissible for  $X$ .*

(ii) *There is a separating linear subspace  $G$  of  $E^*$  such that for every  $u \in G$ ,*

$$(\langle u, f_j \rangle)_{j \geq 1} \text{ is admissible for } \langle u, X \rangle.$$

(iii) *There is a separating linear subspace  $G$  of  $E^*$  such that for every  $u \in G$ ,*

$$\sum_{j=1}^{\infty} \langle u, f_j \rangle^2 = \langle u, Cu \rangle.$$

(iv) *For every  $u \in E^*$ ,*

$$\sum_{j=1}^{\infty} \langle u, f_j \rangle f_j = Cu.$$

(v) *For every  $a > 0$ ,*

$$\sum_{j=1}^{\infty} \langle u, f_j \rangle \langle v, f_j \rangle = \langle u, Cv \rangle$$

*uniformly in  $u, v \in \{y \in E^* : \|y\| \leq a\}$ .*

**Proof.** Set  $X_n := \sum_{j=1}^n \xi_j f_j$ . (i)  $\Rightarrow$  (v).  $X_n$  converges a.s. in  $E$  to some  $E$ -valued random vector  $Y$ , say, with  $X \stackrel{d}{=} Y$ . It is well known that this implies  $X_n \rightarrow Y$  in  $L_E^2$ . Therefore,

$$\begin{aligned} & \left| \sum_{j=1}^n \langle u, f_j \rangle \langle v, f_j \rangle - \langle u, Cv \rangle \right| = \left| E \langle u, X_n \rangle \langle v, X_n \rangle - E \langle u, Y \rangle \langle v, Y \rangle \right| \\ & = \left| E \langle u, Y - X_n \rangle \langle v, Y - X_n \rangle \right| \leq a^2 E \|Y - X_n\|^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

uniformly in  $u, v \in \{y \in E^* : \|y\| \leq a\}$ . (v)  $\Rightarrow$  (iv)  $\Rightarrow$  (iii) is obvious. (iii)  $\Rightarrow$  (i). For every  $u \in G$ ,

$$E \exp(i \langle u, X_n \rangle) = \exp\left(-\sum_{j=1}^n \langle u, f_j \rangle^2 / 2\right) \rightarrow \exp(-\langle u, Cu \rangle / 2) = E \exp(i \langle u, X \rangle).$$

The assertion (i) follows from the Ito-Nisio theorem (cf. [17], p. 271). (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii) is obvious.  $\square$

Note that the preceding lemma shows in particular that  $(f_j)_{j \geq 1}$  is admissible for  $X$  if and only if  $(f_{\sigma(j)})_{j \geq 1}$  is admissible for  $X$  for (some) every permutation  $\sigma$  of  $N$  so that  $\sum \xi_j f_j$  converges unconditionally a.s. in  $E$  for such sequences and all the a.s. limits under permutations of  $N$  have distribution  $P_X$ .

It is also an immediate consequence of Lemma 1(v) that admissible sequences  $(f_j)$  satisfy  $\|f_j\| \rightarrow 0$  since by the Cauchy criterion,  $\lim_{j \rightarrow \infty} \sup_{\|u\| \leq 1} \langle u, f_j \rangle^2 = 0$ .

The corresponding lemma for Parseval frames reads as follows.

**Lemma 2.** *Let  $(f_j)_{j \geq 1}$  be a sequence in a real separable Hilbert space  $(K, (\cdot, \cdot)_K)$ . The following assertions are equivalent.*

(i) *The sequence  $(f_j)$  is a Parseval frame for  $K$ .*

(ii) *For every  $k \in K$ ,*

$$\lim_{n \rightarrow \infty} \left\| \sum_{j=1}^n (k, f_j)_K f_j \right\|_K = \|k\|_K.$$

(iii) *There is a dense subset  $G$  of  $K$  such that for every  $k \in G$ ,*

$$\sum_{j=1}^{\infty} (k, f_j)_K^2 = \|k\|_K^2.$$

(iv) *For every  $k \in K$ ,*

$$\sum_{j=1}^{\infty} (k, f_j)_K^2 = \|k\|_K^2.$$

**Proof.** (i)  $\Rightarrow$  (ii) is obvious. (ii)  $\Rightarrow$  (iv). For every  $k \in K, n \in N$ ,

$$\begin{aligned} 0 &\leq \left\| \sum_{j=1}^n (k, f_j)_K f_j - k \right\|_K^2 \\ &= \left\| \sum_{j=1}^n (k, f_j)_K f_j \right\|_K^2 - 2 \sum_{j=1}^n (k, f_j)_K^2 + \|k\|_K^2 \end{aligned}$$

so that

$$2 \sum_{j=1}^n (k, f_j)_K^2 \leq \left\| \sum_{j=1}^n (k, f_j)_K f_j \right\|_K^2 + \|k\|_K^2.$$

Hence

$$\sum_{j=1}^{\infty} (k, f_j)_K^2 \leq \|k\|_K^2.$$

Using this inequality we obtain conversely for  $k \in K, n \in N$

$$\begin{aligned} \left\| \sum_{j=1}^n (k, f_j)_K f_j \right\|_K^2 &= \sup_{\|g\|_K \leq 1} \left( g, \sum_{j=1}^n (k, f_j)_K f_j \right)_K^2 \\ &= \sup_{\|g\|_K \leq 1} \left( \sum_{j=1}^n (k, f_j)_K (g, f_j)_K \right)^2 \\ &\leq \sum_{j=1}^n (k, f_j)_K^2 \sup_{\|g\|_K \leq 1} \sum_{j=1}^n (g, f_j)_K^2 \\ &\leq \sum_{j=1}^n (k, f_j)_K^2. \end{aligned}$$

Hence

$$\|k\|_K^2 \leq \sum_{j=1}^{\infty} (k, f_j)_K^2.$$

(iv)  $\Rightarrow$  (iii) is obvious. (iii)  $\Rightarrow$  (i). Since  $G$  is dense in  $K$ , for  $k \in K$  there exist  $k_n \in G$  satisfying  $k_n \rightarrow k$  so that  $\lim_{n \rightarrow \infty} (k_n, f_j)_K^2 = (k, f_j)_K^2$  for every  $j$ . Fatou's lemma for the counting measure in  $N$  implies

$$\begin{aligned} \sum_{j=1}^{\infty} (k, f_j)_K^2 &\leq \liminf_{n \rightarrow \infty} \sum_{j=1}^{\infty} (k_n, f_j)_K^2 \\ &= \lim_{n \rightarrow \infty} \|k_n\|_K^2 = \|k\|_K^2. \end{aligned}$$

Therefore, one easily checks that  $\sum_{j=1}^{\infty} c_j f_j$  converges in  $K$  for every  $c = (c_j) \in l_2(N)$  and

$$T : l_2(N) \rightarrow K, T(c) := \sum_{j=1}^{\infty} c_j f_j$$

is linear and continuous (see [4], Theorem 3.2.3). Consequently, the frame operator

$$TT^* : K \rightarrow K, TT^*k = \sum_{j=1}^{\infty} (k, f_j)_K f_j$$

is linear and continuous. By (ii),

$$(TT^*k, k)_K = \sum_{j=1}^{\infty} (k, f_j)_K^2 = \|k\|_K^2$$

for every  $k \in G$  and thus  $(TT^*k, k)_K = \|k\|_K^2$  for every  $k \in K$ . This implies  $TT^*k = k$  for every  $k \in K$ .  $\square$

The preceding lemma shows that the series (1.3) converges unconditionally. Note further that a Parseval frame  $(f_j)$  for  $K$  satisfies  $\{f_j : j \geq 1\} \subset U_K$ , since

$$\|f_m\|_K^4 + \sum_{j \neq m} (f_m, f_j)_K^2 = \sum_{j=1}^{\infty} (f_m, f_j)_K^2 = \|f_m\|_K^2,$$

$\overline{\text{span}}\{f_j : j \geq 1\} = K$  and it is an orthonormal basis for  $K$  if and only if  $\|f_j\|_K = 1$  for every  $j$ .

**Proof of Theorem 1.** The "if" part is an immediate consequence of the reproducing property (2.2) and Lemmas 1 and 2 since for  $u \in E^*$ ,

$$\sum_{j=1}^{\infty} \langle u, f_j \rangle^2 = \sum_{j=1}^{\infty} (Cu, f_j)_H^2 = \|Cu\|_H^2 = \langle u, Cu \rangle.$$

The "only if" part. By Lemma 1,

$$\|f_j\|_H = \sup\{|\langle u, f_j \rangle| : \langle u, Cu \rangle \leq 1\} \leq 1$$

so that by (2.5),  $\{f_j : j \geq 1\} \subset H$ . Again the assertion follows immediately from (2.2) and Lemmas 1 and 2 since  $C(E^*)$  is dense in  $H$ .  $\square$

The covariance operator admits factorizations  $C = SS^*$ , where  $S : K \rightarrow E$  is a linear continuous operator and  $(K, (\cdot, \cdot)_K)$  a real separable Hilbert space, which provide a useful tool for expansions. It is convenient to allow that  $S$  is not injective. One gets

$$\begin{aligned} S(K) &= H, & (2.6) \\ * [.4em] (Sk_1, Sk_2)_H &= (k_1, k_2)_K, k_1 \in K, k_2 \in (\ker S)^\perp, \\ * [.4em] \|S\| &= \|S^*\| = \|C\|^{1/2}, \\ * [.4em] \overline{S^*(E^*)} &= (\ker S)^\perp \text{ in } K, \\ * [.4em] (\ker S^*)^\perp &:= \{x \in E : \langle u, x \rangle = 0 \forall u \in \ker S^*\} = \overline{H} \text{ in } E. \end{aligned}$$

Notice that factorizations of  $C$  correspond to linear continuous operators  $T : K \rightarrow H$  satisfying  $TT^* = I$  via  $S = JT$ , where  $J : H \rightarrow E$  denotes the inclusion map.

A sequence  $(e_j)$  in  $K$  is called *Parseval frame sequence* if it is a Parseval frame for  $\overline{\text{span}}\{e_j : j \geq 1\}$ .

**Proposition 1.** *Let  $C = SS^*$ ,  $S : K \rightarrow E$  be a factorization of  $C$  and let  $(e_j)$  be a Parseval frame sequence in  $K$  satisfying  $(\ker S)^\perp \subset \overline{\text{span}}\{e_j : j = 1, 2, \dots\}$ . Then  $(S(e_j))$  is admissible for  $X$ . Conversely, if  $(f_j)$  is admissible for  $X$  then there exists a Parseval frame sequence  $(e_j)$  in  $K$  satisfying  $(\ker S)^\perp = \overline{\text{span}}\{e_j : j = 1, 2, \dots\}$  such that  $S(e_j) = f_j$  for every  $j$ .*

**Proof.** Let  $K_0 := \overline{\text{span}}\{e_j : j = 1, 2, \dots\}$ . Since by (2.6)

$$S^*(E^*) \subset (\ker S)^\perp \subset K_0,$$

one obtains for every  $u \in E^*$ , by Lemma 2,

$$\sum_j \langle u, Se_j \rangle^2 = \sum_j (S^*u, e_j)_K^2 = \|S^*u\|_K^2 = \langle u, Cu \rangle.$$

The assertion follows from Lemma 1. Conversely, if  $(f_j)$  is admissible for  $X$  then  $(f_j)$  is a Parseval frame for  $H$  by Theorem 1. Set  $e_j := (S|_{(\ker S)^\perp})^{-1}(f_j) \in (\ker S)^\perp$ . Then by (2.6) and Lemma 2, for every  $k \in (\ker S)^\perp$ ,

$$\sum_j (k, e_j)_K^2 = \sum_j (Sk, f_j)_H^2 = \|Sk\|_H^2 = \|k\|_K^2$$

so that again by Lemma 2,  $(e_j)$  is a Parseval frame for  $(\ker S)^\perp$ . □

EXAMPLES • Let  $S : H \rightarrow E$  be the inclusion map. Then  $C = SS^*$ .

• Let  $K$  be the closure of  $E^*$  in  $L^2(P_X)$  and  $S : K \rightarrow E, Sk = Ek(X)X$ . Then  $S^* : E^* \rightarrow K$  is the natural embedding. Thus  $C = SS^*$  and  $S$  is injective (see (2.6)). ( $K$  is sometimes called the energy space of  $X$ .) One obtains

$$H = S(K) = \{Ek(X)X : k \in K\}$$

and

$$(Ek_1(X)X, Ek_2(X)X)_H = \int k_1 k_2 dP_X.$$

• Let  $E$  be a Hilbert space,  $K = E$  and  $S = C^{1/2}$ . Then  $C = SS^* = S^2$  and  $(\ker S)^\perp = \overline{H}$ . Consequently, if  $(e_j)$  is an orthonormal basis of the Hilbert subspace  $\overline{H}$  of  $E$  consisting of eigenvectors of  $C$  and  $(\lambda_j)$  the corresponding nonzero eigenvalues, then  $(\sqrt{\lambda_j}e_j)$  is admissible for  $X$  and an orthonormal basis of  $(H, (\cdot, \cdot)_H)$  (Karhunen-Loève basis).

Admissible sequences for  $X$  can be characterized as the sequences  $(Se_j)_{j \geq 1}$  where  $(e_j)$  is a fixed orthonormal basis of  $K$  and  $S$  provides a factorization of  $C$ . That every sequence  $(Se_j)$  of this type is admissible follows from Proposition 1.

**Theorem 2.** Assume that  $(f_j)_{j \geq 1}$  is admissible for  $X$ . Let  $K$  be an infinite dimensional real separable Hilbert space and  $(e_j)_{j \geq 1}$  an orthonormal basis of  $K$ . Then there is a factorization  $C = SS^*, S : K \rightarrow E$  such that  $S(e_j) = f_j$  for every  $j$ .

**Proof.** First, observe that  $\sum_{j=1}^{\infty} c_j f_j$  converges in  $E$  for every  $(c_j)_j \in l_2(N)$ . In fact, using Lemma 1,

$$\begin{aligned} \left\| \sum_{j=n}^{n+m} c_j f_j \right\|^2 &= \sup_{\|u\| \leq 1} \left\langle u, \sum_{j=n}^{n+m} c_j f_j \right\rangle^2 \\ &\leq \sum_{j=n}^{n+m} c_j^2 \sup_{\|u\| \leq 1} \sum_{j=1}^{\infty} \langle u, f_j \rangle^2 \\ &= \sum_{j=n}^{n+m} c_j^2 \sup_{\|u\| \leq 1} \langle u, Cu \rangle \\ &= \sum_{j=n}^{n+m} c_j^2 \|C\| \rightarrow 0, \quad n, m \rightarrow \infty \end{aligned}$$

and thus the sequence is Cauchy in  $E$ . Now define  $S : K \rightarrow E$  by

$$S(k) := \sum_{j=1}^{\infty} (k, e_j)_K f_j$$

where  $\sum (k, e_j)_K f_j$  converges in  $E$  since  $((k, e_j)_K)_j \in l_2(N)$ .  $S$  is obviously linear. Moreover, for  $k \in K$ , using again Lemma 1,

$$\begin{aligned} \|Sk\|^2 &= \sup_{\|u\| \leq 1} \langle u, Sk \rangle^2 \\ &= \sup_{\|u\| \leq 1} \left( \sum_{j=1}^{\infty} (k, e_j)_K \langle u, f_j \rangle \right)^2 \\ &\leq \|k\|_K^2 \|C\|. \end{aligned}$$



Consequently,  $S$  is continuous and  $S(e_j) = f_j$  for every  $j$ . (At this place one needs orthonormality of  $(e_j)$ .) Finally,  $S^*(u) = \sum_{j=1}^{\infty} \langle u, f_j \rangle e_j$  and hence

$$SS^*u = \sum_{j=1}^{\infty} \langle u, f_j \rangle f_j = Cu$$

for every  $u \in E^*$  by Lemma 1.  $\square$

It is an immediate consequence of the preceding theorem that an admissible sequence  $(f_j)$  for  $X$  is an orthonormal basis for  $H$  if and only if  $(f_j)$  is  $l_2$ -independent, that is  $\sum_{j=1}^{\infty} c_j f_j = 0$  for some  $(c_j) \in l_2(N)$  implies  $c_j = 0$  for every  $j$ . In fact,  $l_2$ -independence of  $(f_j)$  implies that the operator  $S$  in Theorem 2 is injective.

Let  $F$  be a further separable Banach space and  $V : E \rightarrow F$  a  $P_X$ -measurable linear transformation, that is,  $V$  is Borel measurable and linear on a Borel measurable subspace  $D_V$  of  $E$  with  $P_X(D_V) = 1$ . Then  $H_X \subset D_V$ , the operator  $VJ_X : H_X \rightarrow F$  is linear and continuous, where  $J_X : H_X \rightarrow E$  denotes the inclusion map and  $V(X)$  is centered Gaussian with covariance operator

$$C_{V(X)} = VJ_X(VJ_X)^* \quad (2.7)$$

(see [11], [2], Chapter 3.7). Consequently, by (2.6)

$$\begin{aligned} H_{V(X)} &= V(H_X), \\ *[\cdot 4em](Vh_1, Vh_2)_{H_{V(X)}} &= (h_1, h_2)_{H_X}, h_1 \in H_X, h_2 \in (\ker(V|_{H_X}))^{\perp}. \end{aligned} \quad (2.8)$$

Note that the space of  $P_X$ -measurable linear transformation  $E \rightarrow F$  is equal to the  $L_F^p(P_X)$ -closure of the space of linear continuous operators  $E \rightarrow F$ ,  $p \in [1, \infty)$  (see [11]).

>From Theorem 1 and Proposition 1 one may deduce the following proposition.

**Proposition 2.** *Assume that  $V : E \rightarrow F$  is a  $P_X$ -measurable linear transformation. If  $(f_j)_{j \geq 1}$  is admissible for  $X$  in  $E$ , then  $(V(f_j))_{j \geq 1}$  is admissible for  $V(X)$  in  $F$ . Conversely, if  $V|_{H_X}$  is injective and  $(g_j)_{j \geq 1}$  is an admissible sequence for  $V(X)$  in  $F$ , then there exists a sequence  $(f_j)_{j \geq 1}$  in  $E$  which is admissible for  $X$  such that  $V(f_j) = g_j$  for every  $j$ .*

EXAMPLE Let  $X$  and  $Y$  be jointly centered Gaussian random vectors in  $E$  and  $F$ , respectively. Then  $E(Y|X) = V(X)$  for some  $P_X$ -measurable linear transformation  $V : E \rightarrow F$ . The cross covariance operator  $C_{YX} : E^* \rightarrow F$ ,  $C_{YX}u = E\langle u, x \rangle Y$  can be factorized as  $C_{YX} = U_{YX}S_X^*$ , where  $C_X = S_X S_X^*$  is the energy factorization of  $C_X$  with  $K_X$  the closure of  $E^*$  in  $L^2(P_X)$  and  $U_{YX} : K_X \rightarrow F$ ,  $U_{YX}k = Ek(X)Y$ . Then

$$V = U_{YX}S_X^{-1} \text{ on } H_X$$

(see [11]). Consequently, if  $(f_j)_{j \geq 1}$  is admissible for  $X$  in  $E$  then  $(U_{YX}S_X^{-1}f_j)_{j \geq 1}$  is admissible for  $E(Y|X)$  in  $F$ .

### 3 Continuous Gaussian processes

Now let  $I$  be a compact metric space and  $X = (X_t)_{t \in I}$  be a real pathwise continuous centered Gaussian process. Let  $E := \mathcal{C}(I)$  be equipped with the sup-norm  $\|x\| = \sup_{t \in I} |x(t)|$  so that the norm dual  $\mathcal{C}(I)^*$  coincides with the space of finite signed Borel measures on  $I$  by the Riesz theorem. Then  $X$  can be seen as a  $\mathcal{C}(I)$ -valued Gaussian random vector and the covariance operator  $C : \mathcal{C}(I)^* \rightarrow \mathcal{C}(I)$  takes the form

$$\begin{aligned} Cu(t) &= \langle \delta_t, Cu \rangle = \langle C\delta_t, u \rangle \\ &= \langle EX_t X, u \rangle = \int_I EX_t X_s du(s). \end{aligned} \quad (3.1)$$

**Corollary 1.** Let  $(f_j)_{j \geq 1} \in \mathcal{C}(I)^{\mathbb{N}}$ .

(a) If

$$EX_s X_t = \sum_{j=1}^{\infty} f_j(s) f_j(t) \text{ for every } s, t \in I$$

then  $(f_j)$  is admissible for  $X$ .

(b) If

$$\sum_{j=1}^{\infty} f_j(t)^2 < \infty \text{ for every } t \in I$$

and if the process  $Y$  with  $Y_t = \sum_{j=1}^{\infty} \xi_j f_j(t)$  has a pathwise continuous modification  $X$ , then  $(f_j)$  is admissible for  $X$  and  $X = \sum_{j=1}^{\infty} \xi_j f_j$  a.s.

**Proof.** (a) For  $u \in G := \text{span} \{\delta_t : t \in I\}$ ,  $u = \sum_{i=1}^m \alpha_i \delta_{t_i}$  we have

$$\langle u, Cu \rangle = \sum_{i=1}^m \sum_{k=1}^m \alpha_i \alpha_k EX_{t_i} X_{t_k}$$

and

$$\sum_{j=1}^n \langle u, f_j \rangle^2 = \sum_{i=1}^m \sum_{k=1}^m \alpha_i \alpha_k \sum_{j=1}^n f_j(t_i) f_j(t_k)$$

so that

$$\sum_{j=1}^{\infty} \langle u, f_j \rangle^2 = \langle u, Cu \rangle.$$

Since  $G$  is a separating subspace of  $\mathcal{C}(I)^*$  the assertion follows from Lemma 1.

(b) Notice that  $\sum \xi_j f_j(t)$  converges a.s. in  $\mathbb{R}$  and  $Y$  is a centered Gaussian process. Hence  $X$  is centered Gaussian. Since

$$EX_s X_t = EY_s Y_t = \sum_{j=1}^{\infty} f_j(s) f_j(t) \text{ for every } s, t \in I,$$

the assertion follows from (a). □

>Factorizations of  $C$  can be obtained as follows. For Hilbert spaces  $K_i$ , let  $\oplus_{i=1}^m K_i$  denote the Hilbertian (or  $l_2$ -)direct sum.

**Lemma 3.** For  $i \in \{1, \dots, m\}$ , let  $K_i$  be a real separable Hilbert space. Assume the representation

$$EX_s X_t = \sum_{i=1}^m (g_s^i, g_t^i)_{K_i}, s, t \in I$$

for vectors  $g_t^i \in K_i$ . Then

$$S : \oplus_{i=1}^m K_i \rightarrow \mathcal{C}(I), Sk(t) := \sum_{i=1}^m (g_t^i, k_i)_{K_i}$$

is a linear continuous operator,  $(\ker S)^\perp = \overline{\text{span}}\{(g_t^1, \dots, g_t^m) : t \in I\}$  and  $C = SS^*$ .

**Proof.** Let  $K := \oplus_{i=1}^m K_i$  and  $g_t := (g_t^1, \dots, g_t^m)$ . Then  $EX_s X_t = (g_s, g_t)_K$  and  $Sk(t) = (g_t, k)_K$ . First, observe that

$$\sup_{t \in I} \|g_t\|_K \leq \|C\|^{1/2} < \infty.$$

Indeed, for every  $t \in I$ , by (3.1),

$$\|g_t\|_K^2 = EX_t^2 = \langle \delta_t, C \delta_t \rangle \leq \|C\|.$$

The function  $Sk$  is continuous for  $k \in \text{span}\{g_s : s \in I\}$ . This easily implies that  $Sk$  is continuous for every  $k \in \overline{\text{span}}\{g_s : s \in I\}$  and thus for every  $k \in K$ .  $S$  is obviously linear and

$$\|Sk\| = \sup_{t \in I} |(g_t, k)_K| \leq \|C\|^{1/2} \|k\|_K.$$

Finally,  $S^*(\delta_t) = g_t$  so that

$$SS^* \delta_t(s) = Sg_t(s) = EX_s X_t = C \delta_t(s)$$

for every  $s, t \in I$ . Consequently, for every  $u \in \mathcal{C}(I)^*$ ,  $t \in I$ ,

$$\begin{aligned} SS^*u(t) &= \langle SS^*u, \delta_t \rangle = \langle u, SS^* \delta_t \rangle \\ &= \langle u, C \delta_t \rangle = \langle Cu, \delta_t \rangle = Cu(t) \end{aligned}$$

and hence  $C = SS^*$ . □

**EXAMPLE** Let  $K$  be the first Wiener chaos, that is  $K = \overline{\text{span}}\{X_t : t \in I\}$  in  $L^2(P)$  and  $g_t = X_t$ . Then  $Sk = EkX$  and  $S$  is injective. If for instance  $X = W$  (Brownian motion) and  $I = [0, T]$ , then

$$K = \left\{ \int_0^T f(s) dW_s : f \in L^2([0, T], dt) \right\}.$$

We derive from the preceding lemma and Proposition 1 the following corollary.

**Corollary 2.** Assume the situation of Lemma 3. Let  $(e_j^i)_j$  be a Parseval frame sequence in  $K_i$  satisfying  $\{g_t^i : t \in I\} \subset \overline{\text{span}} \{e_j^i : j = 1, 2, \dots\}$ . Then,  $(S_i(e_j^i))_{1 \leq i \leq m, j}$  is admissible for  $X$ , where  $S_i k(t) = (g_t^i, k)_{K_i}$ .

The next corollary implies the well known fact that the Karhunen-Loève expansion of  $X$  in some Hilbert space  $L^2(I, \mu)$  already converges uniformly in  $t \in I$ . It appears as special case of Proposition 2.

**Corollary 3.** Let  $\mu$  be a finite Borel measure on  $I$  with  $\text{supp}(\mu) = I$  and let  $V : \mathcal{C}(I) \rightarrow L^2(I, \mu)$  denote the natural (injective) embedding. Let  $(g_j)_{j \geq 1}$  be admissible for  $V(X)$  in  $L^2(I, \mu)$ . Then there exists a sequence  $(f_j)_{j \geq 1}$  in  $\mathcal{C}(I)$  which is admissible for  $X$  such that  $V(f_j) = g_j$  for every  $j$ .

The admissibility feature is stable under tensor products. For  $i \in \{1, \dots, d\}$ , let  $I_i$  be a compact metric space and  $X^i = (X_t^i)_{t \in I_i}$  a continuous centered Gaussian process. Set  $I := \prod_{i=1}^d I_i$  and let  $X = (X_t)_{t \in I}$  be a continuous centered Gaussian process with covariance function

$$EX_s X_t = \prod_{i=1}^d EX_{s_i}^i X_{t_i}^i, s, t \in I. \quad (3.2)$$

For instance,  $X := \otimes_{i=1}^d X^i$  satisfies (3.2) provided  $X^1, \dots, X^d$  are independent. For real separable Hilbert spaces  $K_i$ , let  $\widehat{\otimes}_{i=1}^d K_i$  denote the  $d$ -fold Hilbertian tensor product.

**Proposition 3.** For  $i \in \{1, \dots, d\}$ , let  $(f_j^i)_{j \geq 1}$  be an admissible sequence for  $X^i$  in  $\mathcal{C}(I_i)$ . Then

$$(\otimes_{i=1}^d f_{j_i}^i)_{\underline{j}=(j_1, \dots, j_d) \in \mathbb{N}^d}$$

is admissible for  $X$  with covariance (3.2) in  $\mathcal{C}(I)$ . Furthermore, if  $C_{X^i} = S_i S_i^*$ ,  $S_i : K_i \rightarrow \mathcal{C}(I_i)$  is a factorization of  $C_{X^i}$ , then  $\widehat{\otimes}_{i=1}^d S_i : \widehat{\otimes}_{i=1}^d K_i \rightarrow \mathcal{C}(I)$  provides a factorization of  $C_X$ .

**Proof.** For  $i \in \{1, \dots, d\}$ , let  $K_i$  be a real separable Hilbert space and  $(e_j^i)_j$  an orthonormal basis of  $K_i$ . Then  $(\otimes_{i=1}^d e_{j_i}^i)_{\underline{j}}$  is an orthonormal basis of  $K := \widehat{\otimes}_{i=1}^d K_i$ .

If  $C_{X^i} = S_i S_i^*$ ,  $S_i : K_i \rightarrow \mathcal{C}(I_i)$  is a factorization of  $C_{X^i}$ , set  $g_t^i := S_i^* \delta_t$ ,  $t \in I_i$ . Then  $EX_s^i X_t^i = (g_s^i, g_t^i)_{K_i}$  and hence, by (3.2)

$$EX_s X_t = \prod_{i=1}^d (g_{s_i}^i, g_{t_i}^i)_{K_i} = (\otimes_{i=1}^d g_{s_i}^i, \otimes_{i=1}^d g_{t_i}^i)_K, s, t \in I.$$

Consequently, by Lemma 3

$$U : K \rightarrow \mathcal{C}(I), Uk(t) = (\otimes_{i=1}^d g_{t_i}^i, k)_K$$

provides a factorization of  $C_X$ . Since

$$\begin{aligned} U(\otimes_{i=1}^d e_{j_i}^i)(t) &= \prod_{i=1}^d (g_{t_i}^i, e_{j_i}^i)_{K_i} \\ &= \prod_{i=1}^d S_i e_{j_i}^i(t_i) = \otimes_{i=1}^d (S_i e_{j_i}^i)(t) \\ &= (\otimes_{i=1}^d S_i)(\otimes_{i=1}^d e_{j_i}^i)(t), t \in I, \end{aligned}$$

we obtain  $U = \otimes_{i=1}^d S_i$  and thus  $\widehat{\otimes}_{i=1}^d S_i$  provides a factorization of  $C_X$ .

If  $(f_j^i)_{j \geq 1}$  is admissible for  $X^i$ , then by Theorem 2 assuming now that  $K_i$  is infinite dimensional, there is a factorization  $C_{X^i} = T_i T_i^*$ ,  $T_i : K_i \rightarrow \mathcal{C}(I_i)$  such that  $T_i(e_j^i) = f_j^i$  for every  $j$ . Since

$\otimes_{i=1}^d T_i : K \rightarrow \mathcal{C}(I)$  provides a factorization of  $C_X$  as shown above and  $(\otimes_{i=1}^d T_i)(\otimes_{i=1}^d e_{j_i}^i) = \otimes_{i=1}^d f_{j_i}^i$ , it follows from Proposition 1 that  $(\otimes_{i=1}^d f_{j_i}^i)_{j \in \mathbb{N}^d}$  is admissible for  $X$ .  $\square$

**Comments on optimal expansions.** For  $n \in \mathbb{N}$ , let

$$l_n(X) := \inf \left\{ E \left\| \sum_{j=n}^{\infty} \xi_j f_j \right\| : (f_j)_{j \geq 1} \in \mathcal{C}(I)^{\mathbb{N}} \text{ admissible for } X \right\}. \quad (3.3)$$

Rate optimal solutions of the  $l_n(X)$ -problem are admissible sequences  $(f_j)$  for  $X$  in  $\mathcal{C}(I)$  such that

$$E \left\| \sum_{j=n}^{\infty} \xi_j f_j \right\| \approx l_n(X) \quad \text{as } n \rightarrow \infty.$$

For  $I = [0, T]^d \subset \mathbb{R}^d$ , consider the covariance operator  $R = R_X$  of  $X$  on  $L^2(I, dt)$  given by

$$R : L^2(I, dt) \rightarrow L^2(I, dt), Rk(t) = \int_I EX_s X_t k(s) ds. \quad (3.4)$$

Using (3.1) we have  $R_X = VC_X V^*$ , where  $V : C(I) \rightarrow L^2(I, dt)$  denotes the natural (injective) embedding. The choice of Lebesgue measure on  $I$  is the best choice for our purposes (see (A1)). Let  $\lambda_1 \geq \lambda_2 \geq \dots > 0$  be the ordered nonzero eigenvalues of  $R$  (each written as many times as its multiplicity).

**Proposition 4.** Let  $I = [0, T]^d$ . Assume that the eigenvalues of  $R$  satisfy

$$(A1) \quad \lambda_j \geq c_1 j^{-2\vartheta} \log(1+j)^{2\gamma} \text{ for every } j \geq 1 \text{ with } \vartheta > 1/2, \gamma \geq 0 \text{ and } c_1 > 0$$

and that  $X$  admits an admissible sequence  $(f_j)$  in  $\mathcal{C}(I)$  satisfying

$$(A2) \quad \|f_j\| \leq c_2 j^{-\vartheta} \log(1+j)^\gamma \text{ for every } j \geq 1 \text{ with } c_2 < \infty,$$

(A3)  $f_j$  is  $a$ -Hölder-continuous and  $[f_j]_a \leq c_3 j^b$  for every  $j \geq 1$  with  $a \in (0, 1], b \in \mathbb{R}$  and  $c_3 < \infty$ , where

$$[f]_a = \sup_{s \neq t} \frac{|f(s) - f(t)|}{|s - t|^a}$$

(and  $|t|$  denotes the  $l_2$ -norm of  $t \in \mathbb{R}^d$ ).

Then

$$l_n(X) \approx n^{-(\vartheta - \frac{1}{2})} (\log n)^{\gamma + \frac{1}{2}} \quad \text{as } n \rightarrow \infty \quad (3.5)$$

and  $(f_j)$  is rate optimal.

**Proof.** The lower estimate in (3.5) follows from (A1) (see [8], Proposition 4.1) and from (A2) and (A3) follows

$$E \left\| \sum_{j=n}^{\infty} \xi_j f_j \right\| \leq c_4 n^{-(\vartheta - \frac{1}{2})} (\log(1+n))^{\gamma + \frac{1}{2}}$$

for every  $n \geq 1$ , (see [13], Theorem 1). □

Concerning assumption (A3) observe that we have by (2.2) and (3.1) for  $h \in H = H_X, s, t \in I$ ,

$$h(t) = \langle \delta_t, h \rangle = (h, C\delta_t)_H$$

and

$$\|C(\delta_s - \delta_t)\|_H^2 = \langle \delta_s - \delta_t, C(\delta_s - \delta_t) \rangle = E|X_s - X_t|^2$$

so that

$$\begin{aligned} |h(s) - h(t)| &= |(h, C(\delta_s - \delta_t))_H| \\ &\leq \|h\|_H \|C(\delta_s - \delta_t)\|_H \\ &= \|h\|_H (E|X_s - X_t|^2)^{1/2}. \end{aligned} \tag{3.6}$$

Consequently, since admissible sequences are contained in the unit ball of  $H$ , (A3) is satisfied with  $b = 0$  provided  $I \rightarrow L^2(P), t \mapsto X_t$  is  $\alpha$ -Hölder-continuous.

The situation is particularly simple for Gaussian sheets.

**Corollary 4.** *Assume that for  $i \in \{1, \dots, d\}$ , the continuous centered Gaussian process  $X^i = (X_t^i)_{t \in [0, T]}$  satisfies (A1) - (A3) for some admissible sequence  $(f_j^i)_{j \geq 1}$  in  $\mathcal{C}([0, T])$  with parameters  $\vartheta_i, \gamma_i, a_i, b_i$  such that  $\gamma_i = 0$  and let  $X = (X_t)_{t \in I}, I = [0, T]^d$  be the continuous centered Gaussian sheet with covariance (3.2). Then*

$$l_n(X) \approx n^{-(\vartheta - \frac{1}{2})} (\log n)^{\vartheta(m-1) + \frac{1}{2}} \tag{3.7}$$

with  $\vartheta = \min_{1 \leq i \leq d} \vartheta_i$  and  $m = \text{card}\{i \in \{1, \dots, d\} : \vartheta_i = \vartheta\}$  and a decreasing arrangement of  $(\otimes_{i=1}^d f_j^i)_{j \in \mathbb{N}^d}$  is rate optimal for  $X$ .

**Proof.** In view of Lemma 1 in [13] and Proposition 3, the assertions follow from Proposition 4. □

EXAMPLES The subsequent examples may serve as illustrations.

• Let  $W = (W_t)_{t \in [0, T]}$  be a standard Brownian motion. Since  $EW_s W_t = s \wedge t = \int_0^T 1_{[0, s]}(u) 1_{[0, t]}(u) du$ , the (injective) operator

$$S : L^2([0, T], dt) \rightarrow \mathcal{C}([0, T]), \quad Sk(t) = \int_0^t k(s) ds$$

provides a factorization of  $C_W$  so that we can apply Corollary 2. The orthonormal basis  $e_j(t) = \sqrt{2/T} \cos(\pi(j - 1/2)t/T), j \geq 1$  of  $L^2([0, T], dt)$  yields the admissible sequence

$$f_j(t) = S e_j(t) = \frac{\sqrt{2T}}{\pi(j - 1/2)} \sin\left(\frac{\pi(j - 1/2)t}{T}\right), \quad j \geq 1 \tag{3.8}$$

for  $W$  (Karhunen-Loève basis of  $H_W$ ) and  $e_j(t) = \sqrt{2/T} \sin(\pi j t/T)$  yields the admissible sequence

$$g_j(t) = \frac{\sqrt{2T}}{\pi j} (1 - \cos(\frac{\pi j t}{T})), \quad j \geq 1.$$

Then

$$\begin{aligned} f_j^1(t) &= \frac{1}{\sqrt{2}} f_j(t) = \frac{\sqrt{T}}{\pi(j-1/2)} \sin\left(\frac{\pi(j-1/2)t}{T}\right), \quad j \geq 1 \\ f_j^2(t) &= \frac{1}{\sqrt{2}} g_j(t) = \frac{\sqrt{T}}{\pi j} (1 - \cos\left(\frac{\pi j t}{T}\right)), \quad j \geq 1 \end{aligned} \quad (3.9)$$

is a Parseval frame for  $H_W$  and hence admissible for  $W$ . The trigonometric basis  $e_0(t) = 1/\sqrt{T}$ ,  $e_{2j}(t) = \sqrt{2/T} \cos(2\pi j t/T)$ ,  $e_{2j-1}(t) = \sqrt{2/T} \sin(2\pi j t/T)$  of  $L^2([0, T], dt)$  yields the admissible sequence

$$\begin{aligned} f_0(t) &= \frac{t}{\sqrt{T}}, f_{2j}(t) = \frac{\sqrt{T}}{\sqrt{2}\pi j} \sin\left(\frac{2\pi j t}{T}\right), \\ f_{2j-1}(t) &= \frac{\sqrt{T}}{\sqrt{2}\pi j} (1 - \cos\left(\frac{2\pi j t}{T}\right)), \quad j \geq 1 \end{aligned} \quad (3.10)$$

(Paley-Wiener basis of  $H_W$ ). By Proposition 4, all these admissible sequences for  $W$  (with  $f_{2j} := f_j^1, f_{2j-1} := f_j^2$ , say in (3.9)) are rate optimal.

Assume that the wavelet system  $2^{j/2}\psi(2^j \cdot -k)$ ,  $j, k \in \mathbb{Z}$  is an orthonormal basis (or only a Parseval frame) for  $L^2(\mathbb{R}, dt)$ . Then the restrictions of these functions to  $[0, T]$  clearly provide a Parseval frame for  $L^2([0, T], dt)$  so that the sequence

$$f_{j,k}(t) = S(2^{j/2}\psi(2^j \cdot -k))(t) = 2^{-j/2} \int_{-k}^{2^j t - k} \psi(u) du, \quad j, k \in \mathbb{Z}$$

is admissible for  $W$ . If  $\psi \in L^1(\mathbb{R}, dt)$  and  $\Psi(x) := \int_{-\infty}^x \psi(u) du$ , then this admissible sequence takes the form

$$f_{j,k}(t) = 2^{-j/2} (\Psi(2^j t - k) - \Psi(-k)), \quad j, k \in \mathbb{Z}. \quad (3.11)$$

• We consider the Dzaparidze-van Zanten expansion of the *fractional Brownian motion*  $X = (X_t)_{t \in [0, T]}$  with Hurst index  $\rho \in (0, 1)$  and covariance function

$$EX_s X_t = \frac{1}{2} (s^{2\rho} + t^{2\rho} - |s - t|^{2\rho}).$$

These authors discovered in [5] for  $T = 1$  a time domain representation

$$EX_s X_t = (g_s^1, g_t^1)_K + (g_s^2, g_t^2)_K$$

with  $K = L^2([0, 1], dt)$  and kernels  $g_t^i \in L^2([0, 1], dt)$ . Hence by Lemma 3, the operator

$$S : L^2([0, 1], dt) \oplus L^2([0, 1], dt) \rightarrow \mathcal{C}([0, 1]), \quad S(k_1, k_2)(t) = \int_0^1 g_t^1(s) k_1(s) ds + \int_0^1 g_t^2(s) k_2(s) ds$$

provides a factorization of  $C_X$  so that for every pair of orthonormal bases  $(e_j^1)_{j \geq 1}$  and  $(e_j^2)_{j \geq 1}$  of  $L^2([0, 1], dt)$ ,

$$f_j^i(t) = \int_0^1 g_t^i(s) e_j^i(s) ds, \quad j \geq 1, \quad i = 1, 2$$

is admissible in  $\mathcal{C}([0, 1])$  for  $X$ . By Corollary 2, this is a consequence of the above representation of the covariance function (and needs no extra work). Then Dzaparidze and van Zanten [5] could calculate  $f_j^i$  explicitly for the Fourier-Bessel basis of order  $-\rho$  and  $1 - \rho$ , respectively and arrived at the admissible family in  $\mathcal{C}([0, 1])$

$$\begin{aligned} f_j^1(t) &= \frac{c_\rho \sqrt{2}}{|J_{1-\rho}(x_j)| x_j^{\rho+1}} \sin(x_j t), j \geq 1 \\ f_j^2(t) &= \frac{c_\rho \sqrt{2}}{|J_{-\rho}(y_j)| y_j^{\rho+1}} (1 - \cos(y_j t)), j \geq 1 \end{aligned}$$

where  $J_\nu$  denotes the Bessel function of the first kind of order  $\nu$ ,  $0 < x_1 < x_2 < \dots$  are the positive zeros of  $J_{-\rho}$ ,  $0 < y_1 < y_2 < \dots$  the positive zeros of  $J_{1-\rho}$  and  $c_\rho^2 = \Gamma(1 + 2\rho) \sin(\pi\rho)/\pi$ . Consequently, by self-similarity of  $X$ , the sequence

$$\begin{aligned} f_j^1(t) &= \frac{T^\rho c_\rho \sqrt{2}}{|J_{1-\rho}(x_j)| x_j^{\rho+1}} \sin\left(\frac{x_j t}{T}\right), j \geq 1 \\ f_j^2(t) &= \frac{T^\rho c_\rho \sqrt{2}}{|J_{-\rho}(y_j)| y_j^{\rho+1}} (1 - \cos\left(\frac{y_j t}{T}\right)), j \geq 1 \end{aligned} \tag{3.12}$$

in  $\mathcal{C}([0, T])$  is admissible for  $X$ . Using Lemma 1, one can deduce (also without extra work)

$$EX_s X_t = \sum_{j=1}^{\infty} f_j^1(s) f_j^1(t) + \sum_{j=1}^{\infty} f_j^2(s) f_j^2(t)$$

uniformly in  $(s, t) \in [0, T]^2$ . Rate optimality of (3.12) (using an arrangement like  $f_{2j} := f_j^1, f_{2j-1} := f_j^2$ ) is shown in [6] based on the work [8] and is also an immediate consequence of Proposition 4 since

$$x_j \sim y_j \sim \pi j, J_{1-\rho}(x_j) \sim J_{-\rho}(y_j) \sim \frac{\sqrt{2}}{\pi} j^{-1/2}$$

(see [5]), and the eigenvalues satisfy  $\lambda_j \sim c j^{-(1+2\rho)}$  as  $j \rightarrow \infty$  (see [3], [12]).

In the ordinary Brownian motion case  $\rho = 1/2$ , (3.12) coincides with (3.9). The interesting extension of (3.10) to fractional Brownian motions is discussed in [7] and extensions of the wavelet expansion (3.11) can be found in [1], [14].

• Let  $X = (X_t)_{t \in [0, T]}$  be *Brownian bridge* with covariance

$$EX_s X_t = s \wedge t - \frac{st}{T} = \int_0^T (1_{[0, s]}(u) - \frac{s}{T})(1_{[0, t]}(u) - \frac{t}{T}) du.$$

By Lemma 3, the operator

$$S : L^2([0, T], dt) \rightarrow \mathcal{C}([0, T]), Sk(t) = \int_0^t k(s) ds - \frac{t}{T} \int_0^T k(s) ds$$



provides a factorization of  $C_X$  and  $\ker S = \text{span}\{1_{[0,T]}\}$ . The choice  $e_j(t) = \sqrt{2/T} \cos(\pi jt/T)$ ,  $j \geq 1$  of an orthonormal basis of  $(\ker S)^\perp$  yields admissibility of

$$f_j(t) = S e_j(t) = \frac{\sqrt{2T}}{\pi j} \sin\left(\frac{\pi jt}{T}\right), j \geq 1 \quad (3.13)$$

for  $X$  (Karhunen-Loève basis of  $H_X$ ). By Proposition 4, this sequence is rate optimal.

• One considers the *stationary Ornstein-Uhlenbeck process* as the solution of the Langevin equation

$$dX_t = -\alpha X_t dt + \sigma dW_t, t \in [0, T]$$

with  $X_0$  independent of  $W$  and  $N(0, \frac{\sigma^2}{2\alpha})$ -distributed,  $\sigma > 0, \alpha > 0$ . It admits the explicit representation

$$X_t = e^{-\alpha t} X_0 + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} dW_s$$

and

$$EX_s X_t = \frac{\sigma^2}{2\alpha} e^{-\alpha|s-t|} = \frac{\sigma^2}{2\alpha} e^{-\alpha(s+t)} + \sigma^2 e^{-\alpha(s+t)} \int_0^{s \wedge t} e^{2\alpha u} du.$$

Thus the (injective) operator

$$S : R \oplus L^2([0, T], dt) \rightarrow \mathcal{C}([0, T]), S(c, k)(t) = \frac{c\sigma}{\sqrt{2\alpha}} e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} k(s) ds$$

provides a factorization of  $C_X$  so that for every Parseval frame  $(e_j)_{j \geq 1}$  for  $L^2([0, T], dt)$ , the functions

$$f_0(t) = \frac{\sigma}{\sqrt{2\alpha}} e^{-\alpha t}, f_j(t) = \sigma \int_0^t e^{-\alpha(t-s)} e_j(s) ds, j \geq 1 \quad (3.14)$$

provide an admissible sequence for  $X$ . For instance the choice of the orthonormal basis  $\sqrt{2/T} \cos(\pi(j-1/2)t/T)$ ,  $j \geq 1$  implies that (3.14) is rate optimal. This follows from Lemma 1 in [13] and Proposition 4.

Another representation is given by the Lamperti transformation  $X = V(W)$  for the linear continuous operator

$$V : \mathcal{C}([0, e^{2\alpha T}]) \rightarrow \mathcal{C}([0, T]), Vx(t) = \frac{\sigma}{\sqrt{2\alpha}} e^{-\alpha t} x(e^{2\alpha t}).$$

The admissible sequence  $(f_j)$  in  $\mathcal{C}([0, e^{2\alpha T}])$  for  $(W_t)_{t \in [0, e^{2\alpha T}]}$  from (4.1) yields the admissible sequence

$$\tilde{f}_j(t) = V f_j(t) = \frac{\sigma}{\sqrt{\alpha} \pi (j-1/2)} e^{\alpha(T-t)} \sin(\pi(j-1/2)e^{-2\alpha(T-t)}), j \geq 1 \quad (3.15)$$

for  $X$ . By Proposition 4, the sequence (3.15) is rate optimal.

• Sheet versions can be deduced from Proposition 3 and Corollary 4 (and need no extra work).

## 4 Optimal expansion of a class of stationary Gaussian processes

### 4.1 Expansion of fractional Ornstein-Uhlenbeck processes

The fractional Ornstein-Uhlenbeck process  $X^\rho = (X_t^\rho)_{t \in \mathbb{R}}$  of index  $\rho \in (0, 2)$  is a continuous stationary centered Gaussian process having the covariance function

$$EX_s^\rho X_t^\rho = e^{-\alpha|s-t|^\rho}, \alpha > 0. \quad (4.1)$$

We derive explicit optimal expansions of  $X^\rho$  for  $\rho \leq 1$ . Let

$$\gamma^\rho : \mathbb{R} \rightarrow \mathbb{R}, \gamma^\rho(t) = e^{-\alpha|t|^\rho}$$

and for a given  $T > 0$ , set

$$\beta_0(\rho) := \frac{1}{2T} \int_{-T}^T \gamma^\rho(t) dt, \beta_j(\rho) := \frac{1}{T} \int_{-T}^T \gamma^\rho(t) \cos(\pi jt/T) dt, j \geq 1. \quad (4.2)$$

**Theorem 3.** *Let  $\rho \in (0, 1]$ . Then  $\beta_j(\rho) > 0$  for every  $j \geq 0$  and the sequence*

$$f_0 = \sqrt{\beta_0(\rho)}, \quad f_{2j} = \sqrt{\beta_j(\rho)} \cos(\pi jt/T), \quad f_{2j-1}(t) = \sqrt{\beta_j(\rho)} \sin(\pi jt/T), j \geq 1 \quad (4.3)$$

is admissible for  $X^\rho$  in  $\mathcal{C}([0, T])$ . Furthermore,

$$l_n(X^\rho) \approx n^{-\rho/2} (\log n)^{1/2} \quad \text{as } n \rightarrow \infty$$

and the sequence (4.3) is rate optimal.

**Proof.** Since  $\gamma^\rho$  is of bounded variation (and continuous) on  $[-T, T]$ , it follows from the Dirichlet criterion that its (classical) Fourier series converges pointwise to  $\gamma^\rho$  on  $[-T, T]$ , that is using symmetry of  $\gamma^\rho$ ,

$$\gamma^\rho(t) = \beta_0(\rho) + \sum_{j=1}^{\infty} \beta_j(\rho) \cos(\pi jt/T), t \in [-T, T].$$

Thus one obtains the representation

$$EX_s X_t = \gamma^\rho(s-t) = \beta_0(\rho) + \sum_{j=1}^{\infty} \beta_j(\rho) [\cos(\pi js/T) \cos(\pi jt/T) + \sin(\pi js/T) \sin(\pi jt/T)], s, t \in [0, T]. \quad (4.4)$$

This is true for every  $\rho \in (0, 2)$ . If  $\rho = 1$ , then integration by parts yields

$$\beta_0(1) = \frac{1 - e^{-\alpha T}}{\alpha T}, \quad \beta_j(1) = \frac{2\alpha T(1 - e^{-\alpha T}(-1)^j)}{\alpha^2 T^2 + \pi^2 j^2}, j \geq 1. \quad (4.5)$$

In particular, we obtain  $\beta_j(1) > 0$  for every  $j \geq 0$ . If  $\rho \in (0, 1)$ , then  $\gamma_{[0, \infty)}^\rho$  is the Laplace transform of a suitable one-sided strictly  $\rho$ -stable distribution with Lebesgue-density  $q_\rho$ . Consequently, for

$j \geq 1$ ,

$$\begin{aligned}
\beta_j(\rho) &= \frac{2}{T} \int_0^T e^{-at^\rho} \cos(\pi jt/T) dt \\
&= \int_0^\infty \frac{2}{T} \int_0^T e^{-tx} \cos(\pi jt/T) dt q_\rho(x) dx \\
&= \int_0^\infty \frac{2xT(1 - e^{-xT}(-1)^j)}{x^2T^2 + \pi^2j^2} q_\rho(x) dx.
\end{aligned} \tag{4.6}$$

Again,  $\beta_j(\rho) > 0$  for every  $j \geq 0$ . It follows from (4.4) and Corollary 1(a) that the sequence  $(f_j)_{j \geq 0}$  defined in (4.3) is admissible for  $X^\rho$  in  $\mathcal{C}([0, T])$ .

Next we investigate the asymptotic behaviour of  $\beta_j(\rho)$  as  $j \rightarrow \infty$  for  $\rho \in (0, 1)$ . The spectral measure of  $X^\rho$  still for  $\rho \in (0, 2)$  is a symmetric  $\rho$ -stable distribution with continuous density  $p_\rho$  so that

$$\begin{aligned}
\gamma^\rho(t) &= \int_{\mathbb{R}} e^{itx} p_\rho(x) dx \\
&= 2 \int_0^\infty \cos(tx) p_\rho(x) dx, \quad t \in \mathbb{R}
\end{aligned}$$

and the spectral density satisfies the high-frequency condition

$$p_\rho(x) \sim c(\rho)x^{-(1+\rho)} \quad \text{as } x \rightarrow \infty \tag{4.7}$$

where

$$c(\rho) = \frac{\alpha \Gamma(1 + \rho) \sin(\pi\rho/2)}{\pi}.$$

Since by the Fourier inversion formula

$$p_\rho(x) = \frac{1}{\pi} \int_0^\infty \gamma^\rho(t) \cos(tx) dt, \quad x \in \mathbb{R},$$

we obtain for  $j \geq 1$ ,

$$\begin{aligned}
\beta_j(\rho) &= \frac{2}{T} \int_0^T \gamma^\rho(t) \cos(\pi jt/T) dt \\
&= \frac{2}{T} \left( \int_0^\infty \gamma^\rho(t) \cos(\pi jt/T) dt - \int_T^\infty \gamma^\rho(t) \cos(\pi jt/T) dt \right) \\
&= \frac{2\pi}{T} p_\rho(\pi j/T) - \frac{2}{T} \int_T^\infty \gamma^\rho(t) \cos(\pi jt/T) dt
\end{aligned}$$

Integrating twice by parts yields

$$\int_T^\infty \gamma^\rho(t) \cos(\pi jt/T) dt = O(j^{-2})$$

for any  $\rho \in (0, 2)$  so that for  $\rho \in (0, 1)$

$$\begin{aligned} \beta_j(\rho) &\sim \frac{2\pi}{T} p_\rho(\pi j/T) \\ &\sim \frac{2\pi T^\rho c(\rho)}{(\pi j)^{1+\rho}} \\ &= \frac{2\alpha T^\rho \Gamma(1+\rho) \sin(\pi\rho/2)}{(\pi j)^{1+\rho}} \quad \text{as } j \rightarrow \infty. \end{aligned} \tag{4.8}$$

We deduce from (4.5) and (4.8) that the admissible sequence (4.3) satisfies the conditions (A2) and (A3) from Proposition 4 with parameters  $\vartheta = (1+\rho)/2$ ,  $\gamma = 0$ ,  $a = 1$  and  $b = (1-\rho)/2$ . Furthermore, by Theorem 3 in Rosenblatt [16], the asymptotic behaviour of the eigenvalues of the covariance operator of  $X^\rho$  on  $L^2([0, T], dt)$  (see (3.4)) for  $\rho \in (0, 2)$  is as follows:

$$\lambda_j \sim \frac{2T^{1+\rho} \pi c(\rho)}{(\pi j)^{1+\rho}} \quad \text{as } j \rightarrow \infty. \tag{4.9}$$

Therefore, the remaining assertions follow from Proposition 4.  $\square$

Here are some comments on the above theorem.

First, note that the admissible sequence (4.3) is not an orthonormal basis for  $H = H_{X^\rho}$  but only a Parseval frame at least in case  $\rho = 1$ . In fact, it is well known that for  $\rho = 1$ ,

$$\|h\|_H^2 = \frac{1}{2}(h(0)^2 + h(T)^2) + \frac{1}{2\alpha} \int_0^T (h'(t)^2 + \alpha^2 h(t)^2) dt$$

so that e.g.

$$\|f_{2j-1}\|_H^2 = \frac{1 - e^{-\alpha T} (-1)^j}{2} < 1.$$

A result corresponding to Theorem 3 for fractional Ornstein-Uhlenbeck sheets on  $[0, T]^d$  with covariance structure

$$EX_s X_t = \prod_{i=1}^d e^{-\alpha_i |s_i - t_i|^{\rho_i}}, \quad \alpha_i > 0, \rho_i \in (0, 1] \tag{4.10}$$

follows from Corollary 4.

As a second comment, we wish to emphasize that, unfortunately, in the nonconvex case  $\rho \in (1, 2)$  it is not true that  $\beta_j(\rho) \geq 0$  for every  $j \geq 0$  so that the approach of Theorem 3 no longer works. In fact, starting again from

$$\beta_j(\rho) = \frac{2\pi}{T} p_\rho(\pi j/T) - \frac{2}{T} \int_T^\infty \gamma^\rho(t) \cos(\pi j t/T) dt,$$

three integrations by parts show that

$$\begin{aligned} \beta_j(\rho) &= \frac{2\pi}{T} p_\rho(\pi j/T) + \frac{(-1)^{j+1} 2T\alpha\rho e^{-\alpha T^\rho} T^{\rho-1}}{(\pi j)^2} + O(j^{-3}) \\ &= \frac{(-1)^{j+1} 2T\alpha\rho e^{-\alpha T^\rho} T^{\rho-1}}{(\pi j)^2} + O(j^{-(1+\rho)}), \quad j \rightarrow \infty. \end{aligned}$$

This means that for any  $T > 0$ , the  $2T$ -periodic extension of  $\gamma|_{[-T, T]}^\rho$  is not nonnegative definite for  $\rho \in (1, 2)$  in contrast to the case  $\rho \in (0, 1]$ .

## 4.2 Expansion of stationary Gaussian processes with convex covariance function

As concerns admissibility, it is interesting to observe that the convex function  $\gamma(t) = e^{-\alpha t^\rho}$ ,  $\rho \in (0, 1]$  in Theorem 3 can be replaced by any convex on  $(0, \infty)$  function  $\gamma$  going to zero at infinity. Some additional natural assumptions related on its regularity at 0 or the rate of decay of its spectral density at infinity then provide the optimality.

Namely, let  $X = (X)_{t \in \mathbb{R}}$  be a continuous stationary centered Gaussian process with  $EX_s X_t = \gamma(s - t)$ ,  $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ . Then  $\gamma$  is continuous, symmetric and nonnegative definite. Set

$$\beta_0 := \frac{1}{2T} \int_{-T}^T \gamma(t) dt, \quad \beta_j := \frac{1}{T} \int_{-T}^T \gamma(t) \cos(\pi j t / T) dt, \quad j \geq 1.$$

Then the extension of Theorem 3 reads as follows.

**Theorem 4.** (a) *ADMISSIBILITY.* Assume that the function  $\gamma$  is convex, positive on  $(0, \infty)$  and  $\gamma(\infty) := \lim_{t \rightarrow \infty} \gamma(t) = 0$ . Then  $X$  admits a spectral density  $p$ ,  $\beta_j \geq 0$  for every  $j \geq 0$  and the sequence defined by

$$f_0 = \sqrt{\beta_0}, \quad f_{2j}(t) = \sqrt{\beta_j} \cos(\pi j t / T), \quad f_{2j-1}(t) = \sqrt{\beta_j} \sin(\pi j t / T), \quad j \geq 1 \quad (4.11)$$

is admissible for the continuous process  $X$  in  $\mathcal{C}([0, T])$ .

*OPTIMALITY.* Assume furthermore that one of the following two conditions is satisfied by  $\gamma$  for some  $\delta \in (1, 2]$ :

( $A_\delta$ )  $\gamma \in L^1([T, \infty))$ , the spectral density  $p \in L^2(\mathbb{R}, dx)$  and the high-frequency condition

$$p(x) \sim cx^{-\delta} \text{ as } x \rightarrow \infty,$$

or

( $B_\delta$ )  $\gamma(t) = \gamma(0) - a|t|^{\delta-1} + b(t)$ ,  $|t| \leq T$  with  $a > 0$  and  $b : [-T, T] \rightarrow \mathbb{R}$  is a  $(\delta + \eta)$ -Hölder function for some  $\eta > 0$ , null at zero.

Then,

$$l_n(X) \approx n^{-(\delta-1)/2} (\log n)^{1/2} \quad \text{as } n \rightarrow \infty$$

and the sequence (4.11) is rate optimal.

**Remarks.** • Assumption ( $A_\delta$ ) is related to the spectral density whereas ( $B_\delta$ ) is related to the (right) regularity of  $\gamma$  at 0. In fact, these assumptions are somewhat similar since the rate of decay of  $p$  is closely related to the right regularity of  $\gamma$  at 0. So, in practice it mainly depends on which quantity is straightforwardly available for a given process.

• The notion of  $\beta$ -Hölder regularity of a function  $g$  means that the  $[\beta]$  first derivatives of  $g$  do exist on  $[-T, T]$  and that  $g^{([\beta])}$  is  $\beta - [\beta]$ -Hölder. In fact, in [16], the notion is still a bit more general ( $g^{([\beta])}$  is assumed to  $\beta - [\beta]$ -Hölder in a  $L^1(dt)$ -sense).

**Proof.** (a) The function  $\gamma$  is positive, convex over  $(0, \infty)$  and  $\gamma(\infty) = 0$  so that  $\gamma$  is in fact a Polya-type function. Hence its right derivative  $\gamma'$  is non-decreasing with  $\gamma'(\infty) = 0$ , the spectral measure of  $X$  admits a Lebesgue-density  $p$  and

$$\gamma(t) = \int_{(0, \infty)} \left(1 - \frac{|t|}{s}\right)^+ d\nu(s)$$

for all  $t \in \mathbb{R}$ , where  $\nu$  is a finite Borel measure on  $(0, \infty)$  with mass  $\gamma(0)$  (see [10], Theorems 4.3.1 and 4.3.3 for details). Therefore, using Fubini's theorem, it is enough to show the positivity of the numbers  $\beta_j$  for functions of the type  $\gamma(t) = (1 - \frac{|t|}{s})^+, s \in (0, \infty)$ . But in this case an integration by parts yields

$$\beta_0 = \frac{T \wedge s}{T} \left(1 - \frac{T \wedge s}{2s}\right) \geq 0 \text{ and } \beta_j = \frac{2T}{s(\pi j)^2} (1 - \cos(\pi j(T \wedge s)/T)) \geq 0, \quad j \geq 1.$$

Now one proceeds along the lines of the proof of Theorem 3. Since  $\gamma$  is of bounded variation on  $[-T, T]$ , the representation (4.4) of  $EX_s X_t$  is true with  $\beta_j(\rho)$  replaced by  $\beta_j$  so that the sequence  $(f_j)_{j \geq 0}$  is admissible for  $X$  in  $\mathcal{C}([0, T])$ .

(b) Assume  $(A_\delta)$ . Using  $\gamma \in L^1(\mathbb{R}, dt)$  and the Fourier inversion formula, one gets for  $j \geq 1$

$$\beta_j = \frac{2\pi}{T} p(\pi j/T) - \frac{2}{T} \int_T^\infty \gamma(t) \cos(\pi j t/T) dt.$$

Since  $\gamma(\infty) = \gamma'(\infty) = 0$ , integrating twice by parts yields

$$\int_T^\infty \gamma(t) \cos(\pi j t/T) dt = O(j^{-2}),$$

hence

$$\beta_j = O(j^{-\delta}) \quad \text{as } j \rightarrow \infty$$

in view of  $\delta \leq 2$ . Furthermore, the assumption  $p \in L^2(\mathbb{R}, dx)$  and the high-frequency condition yield

$$\lambda_j \sim c_1 j^{-\delta} \quad \text{as } j \rightarrow \infty$$

for an appropriate constant  $c_1 \in (0, \infty)$  (see [16]). Now, one derives from Proposition 4 the remaining assertions.

Assume  $(B_\delta)$ . An integration by parts (using that  $\gamma$  is even) yields

$$\beta_j = -\frac{2}{\pi j} \int_0^T \sin\left(\frac{\pi j t}{T}\right) \gamma'(t) dt$$

where  $\gamma'$  denotes the right derivative of  $\gamma$ . Consequently, for  $\varepsilon \in (0, T]$ , using that  $\gamma' \leq 0$  and is non-decreasing,

$$\beta_j = |\beta_j| \leq \frac{2}{\pi j} (\gamma(0) - \gamma(\varepsilon)) + \frac{\gamma(0) - \gamma(\varepsilon)}{\varepsilon} \frac{4T}{(\pi j)^2}$$

At this stage, Assumption  $(B_\delta)$  yields

$$\beta_j \leq C \left( \frac{\varepsilon^{\delta-1}}{j} + \frac{\varepsilon^{\delta-2}}{j^2} \right)$$

where  $C$  is positive real constant. Setting  $\varepsilon = T/j$ , implies as expected  $\beta_j = O(j^{-\delta})$ .

On the other hand, calling upon Theorem 2 in [16], shows that the eigenvalues  $\lambda_j$  of the covariance operator on  $L^2([0, T], dt)$  (indexed in decreasing order) satisfy

$$\lambda_j \sim 2\kappa \Gamma(\delta) \left( \frac{T}{\pi j} \right)^\delta \quad \text{as } j \rightarrow \infty$$

where  $\kappa = -a \cos\left(\frac{\pi}{2}\delta\right) > 0$  and  $\Gamma$  denotes the Gamma function. One concludes as above.  $\square$

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