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EXPANSIONS FOR THE DISTRIBUTION AND QUANTILES OF A REGULAR FUNCTIONAL OF THE EMPIRICAL DISTRIBUTION WITH APPLICATIONS TO NONPARAMETRIC CONFIDENCE INTERVALS

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Let $T(\cdot)$ be a suitably regular functional on the space of distribution functions, F, on R^s. A method is given for obtaining the derivatives of T at F. This is used to obtain asymptotic expansions for the distribution and quantiles of $T(F_n)$ where F_n is the empirical distribution of a random sample of size n from a distribution F with an absolutely continuous component. One- and two-sided confidence intervals for $T(F)$ are given of level $1-\alpha+O(n^{-j/2})$ for any given j. Examples include approximate nonparametric confidence intervals for the mean and variance of a distribution on R.

1. Introduction. In a previous paper (Withers, 1980a) the author considered an asymptotically normal random variable Y_n whose cumulants have power series expansions in $n^{-1/2}$; the Edgeworth and Cornish-Fisher expansions for the distribution and quantiles of Y_n were re-expressed as power series in $n^{-1/2}$ in terms of the coefficients of the cumulant expansions.

In Section 3 the cumulants of $T(F_n)$ are expanded as power series in n^{-1} where $T(\cdot)$ is a real functional on $\mathscr{F}_s$, the space of distribution functions on R^s, and F_n is the empirical distribution of a random sample $X_1, \cdots, X_n$ from a distribution F on R^s; the cumulant coefficients needed for the distribution and quantiles of $n^{1/2}(T(F_n) - T(F))$ to within $O(n^{-2})$ are given in terms of the derivatives of T at F. Conditions for the validity of these expansions have been given by Bhattacharya and Ghosh (1978) for the case where $T(F)$ is a function of $\int \mathbf{f}\, dF$ for some function $\mathbf{f}$ in R^k. (However, they do not give explicit expressions for the expansions, nor do they consider the quantiles.)

Section 4 illustrates these results for $T(F_n)$ the sample variance and the one-sample Student t-statistic.

Section 5 extends these results to $T_n(F_n)$ where $T_n(\cdot)$ has an expansion in powers of $n^{-1/2}$. This is then applied to obtain nonparametric confidence intervals for $T(F)$ with error $O(n^{-j/2})$ for any given j. These intervals are illustrated for $T(F)$ the mean and variance of $h(X_1)$ for any given function $h(\cdot)$.

We begin with a method for obtaining the derivatives of $T(F)$.

2. Functional derivatives. In this section we specialise the notion of a functional derivative used by von Mises (1947), and give a rule for differentiating a functional derivative.

Let $x, x_1, x_2, \cdots$ be arbitrary points in R^s, let G, H be arbitrary distributions on R^s, and let $T(\cdot)$ be a real functional defined on the space of such distributions. Von Mises defined the ith derivative of $T(\cdot)$ at $(x_1, \cdots x_i, H)$ as any symmetric function $T_H^{(i)}(x_1, \cdots, x_i)$ such that for all distributions G on R^s,

$$g^{(i)}(0) = \int \cdots \int T_H^{(i)}(x_1, \cdots, x_i) \textstyle\prod_{j=1}^{i} d\{G(x_j) - H(x_j)\}, \qquad (2.1)$$

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where $g(\varepsilon) = T(H + \varepsilon(G - H))$, $0 \le \varepsilon \le 1$, and $g^{(i)}(\varepsilon)$ is the ordinary ith derivative of $g(\varepsilon)$ with respect to ε.

When $g^{(r+1)}(\cdot)$ is continuous, the rth order Taylor series expansion of $g(1)$ about $g(0)$ is then

$$T(G) - T(H) = \sum_{i=1}^{r} \int \cdots \int T_H^{(i)}(x_1, \cdots, x_i) \prod_{j=1}^{i} d\{G(x_j) - H(x_j)\}/i! + \Delta_r(G, H), \tag{2.2}$$

where

$$\Delta_r(G, H) = \int \cdots \int T_{H_G}^{(r+1)}(x_1, \cdots, x_{r+1}) \prod_{j=1}^{r+1} d\{G(x_j) - H(x_j)\}/(r+1)!$$

and $H_G(x) = H(x) + \lambda\{G(x) - H(x)\}$ and λ is some constant in $[0, 1]$ depending on (G, H, T). We shall make $T_H^{(i)}(\cdot)$ unique by imposing the constraint

$$\mathscr{I}_{jH} T_H^{(i)} = 0, \quad 1 \le j \le i, \tag{2.3}$$

where $\mathscr{I}_{jH}$ is the operation of averaging with respect to H over the jth argument:

$$\mathscr{I}_{jH} f(x_1, \cdots, x_i) = \int f(x_1, \cdots, x_{j-1}, y, x_{j+1}, \cdots, x_i)\, dH(y). \tag{2.4}$$

Thus, $G(x_j) - H(x_j)$ in (2.1) and (2.2) can be replaced by $G(x_j)$. If for a given H there does exist a function $T_H^{(i)}(\cdot)$ satisfying (2.1) for all distributions G on R^s, then the unique ith functional derivative of $T(H)$ is obtained by operating on it with $\{1 - \mathscr{I}_{jH}, 0 \le j \le i\}$ and $\mathscr{S}_i$, the operation of symmetrising a function of i arguments.

For example, if $T(H) = \int (x)_0^2\, dH(x)$ where $(x)_0 = x - \int z\, dH(z)$, then for any constants a, b, c, $T_H^{(2)}(x, y) = -(x)_0(y)_0 + (x)_0 a + (y)_0 b + c$ satisfies (2.1) and $(1 - \mathscr{I}_{1H})(1 - \mathscr{I}_{2H}) T_H^{(2)}(x, y) = -(x)_0(y)_0$ satisfies (2.3) as well.

The first derivative $T_H^{(1)}(x)$ is most easily calculated using

LEMMA 2.1. *$T_H^{(i)}(x_1, \cdots, x_i)$ is the coefficient of $\lambda_1 \cdots \lambda_i \varepsilon^i$ in the Taylor series expansion of $g(\varepsilon)$ about $\varepsilon = 0$ when*

$$G = \sum_{j=1}^{i} \lambda_j \delta_{x_j}, \quad \text{where } \sum_{j=1}^{i} \lambda_j = 1, \tag{2.5}$$

and $\{\lambda_j\}$ are otherwise arbitrary positive weights, and δ_x puts mass 1 at x.

(The proof is obvious.) However, the higher derivatives are more easily calculated using the following rule for differentiating derivatives:

THEOREM 2.1. *Suppose that the $i + 1$st derivative of $T(\cdot)$ at $(x_1, \cdots, x_{i+1}, H)$ exists. Then the first derivative at (x_{i+1}, H) of $T_H^{(i)}(x_1, \cdots, x_i)$—considered as a functional of H—is*

$$(T_H^{(i)}(x_1, \cdots, x_i))^{(1)}(x_{i+1}) = T_H^{(i+1)}(x_1, \cdots, x_{i+1}) - \sum_{r=1}^{i} T_H^{(i)}\langle x_1, \cdots, x_{i+1}\rangle_r, \tag{2.6}$$

where $\langle\ \rangle_r$ indicates that the rth argument is dropped: for example,

$$(T_H^{(2)}(x, y))^{(1)}(z) = T_H^{(3)}(x, y, z) - T_H^{(2)}(x, z) - T_H^{(2)}(y, z).$$

PROOF. $S(H) = T_H^{(i)}(x_1, \cdots, x_i)$ is the coefficient of $\lambda_1 \cdots \lambda_i \zeta^i$ in $T((1 - \zeta)H + \zeta G)$ for G given by (2.5). $S_H^{(1)}(x_{i+1})$ = coefficient of ε in $S((1 - \varepsilon)H + \varepsilon\delta_{x_{i+1}})$ = coefficient of $\varepsilon\zeta^i\lambda_1 \cdots \lambda_i$ in $T(M)$, where $M - H = (-\zeta - \varepsilon + \zeta\varepsilon)H + L$ and $L = (1 - \zeta)\varepsilon\delta_{x_{i+1}} + \zeta G$. Denoting the ith term in (2.2) as $\nabla_i(G - H)$, we have

$$T(M) = T(H) + \sum_{i}^{r+1} \nabla_k/k! + \Delta_{r+1}(M, H)$$

where $\nabla_k = \nabla_k(M - H) = \nabla_k(L)$ by (2.3). The coefficient of ε in $\prod_1^k L(y_j)$ is $\sum_{l=1}^k (\zeta^{l-1} - \zeta^l)\delta_{x_{r+1}}(y_h) \prod_{j=1,j\neq l}^k \sum_{p=1}^i \lambda_p \delta_{x_p}(y_j)$, so that the coefficient of $\varepsilon\zeta^i$ in $T(M)$ is $A - B$, where A = coefficient of ζ^{k-1} in ∇_k with $k = i + 1$ and B = coefficient of ζ^k in ∇_k with $k = i$. Now take the coefficients of $\lambda_1 \cdots \lambda_i$ in A and B. □

3. Expansions for the moments, cumulants, and distribution of $T(F_n)$. In Corollary 3.3 of Withers (1980a) it was shown that if Y_n is a r.v. with rth cumulant having the formal expansion

(3.1) $$K_r(Y_n) \approx n^{r/2} \sum_{i=r-1}^{\infty} A_{ri} n^{-i}, \quad r \geq 1, \quad \text{where } A_{10} = 0, A_{21} = 1,$$

then $P_n(x) = P(Y_n \leq x)$ has the formal "reduced Cornish-Fisher" expansions

(3.2) $$P_n(x) \approx \Phi(x) - \phi(x) \sum_{r=1}^{\infty} n^{-r/2} h_r(x),$$

(3.3) $$\Phi^{-1}(P_n(x)) \approx x - \sum_{r=1}^{\infty} n^{-r/2} f_r(x),$$

(3.4) $$P_n^{-1}(\Phi(x)) \approx x + \sum_{r=1}^{\infty} n^{-r/2} g_r(x),$$

where Φ, ϕ are the distribution and density of the standard normal r.v. and h_r, f_r, g_r are certain polynomials of degree $3r - 1$, $r + 1$ and $r + 1$ respectively, odd for r even and even for r odd. For example, if He_i is the ith Hermite polynomial ($He_0(x) = 1$, $He_1(x) = x$, $He_2(x) = x^2 - 1, \cdots$) then

(3.5) $$h_1 = f_1 = g_1 = A_{11} + A_{32}He_2/6, \quad \text{and}$$
$$h_2 = (A_{11}^2 + A_{22})He_1/2 + (4A_{11}A_{32} + A_{43})He_3/24 + A_{32}^2 He_5/72,$$

and

(3.6) $$\begin{aligned} h_3 = A_{12} &+ (A_{11}^3 + 3A_{11}A_{22} + A_{33})He_2/6 \\ &+ (10A_{11}^2 A_{32} + 5A_{11}A_{43} + 10A_{22}A_{32} + A_{54})He_4/120 \\ &+ (2A_{11}A_{32}^2 + A_{32}A_{43})He_6/144 + A_{32}^3 He_8/1296. \end{aligned}$$

In this section we formally expand the cumulants and moments of $T(F_n)$ in the form

(3.7) $$K_r(T(F_n)) \approx \sum_{i=r-1}^{\infty} a_{ri} n^{-i}, \quad E(T(F_n) - T(F))^r \approx \sum_{j \geq r/2} a'_{rj} n^{-j}, \quad r \geq 1,$$

where $a_{10} = T(F)$ and $a_{21} = \int T_F^{(1)}(x)^2 dF(x) = \sigma_T(F)^2$ say.

It follows that (3.1) is formally satisfied by

(3.8) $$Y_n = n^{1/2}\{T(F_n) - T(F)\}/\sigma_T(F)$$

with

(3.9) $$A_{ri} = a_{21}^{-r/2} a_{ri}, \quad (r, i) \neq (1, 0).$$

Expressions for the coefficients $\{a_{ri}\}$ needed to calculate h_j, f_j, g_j for $1 \leq j \leq 3$ are given in terms of the derivatives of T at F. We use the notation

$$[1, 12^3] = \int\int T_{x_1} T_{x_1 x_2}^3 \prod_1^2 dF(x_j), \quad [1, 2, 3^2, 123] = \int\int\int T_{x_1} T_{x_2} T_{x_3}^2 T_{x_1 x_2 x_3} \prod_1^3 dF(x_j),$$

and so forth, where $T_{x_1 \cdots x_i} = T_F^{(i)}(x_1, \cdots, x_i)$.

THEOREM 3.1. *The cumulants and moments of $T(F_n)$ have formal expansions of the type* (3.7). *The cumulant coefficients $\{a_{ri}\}$ needed to compute the polynomials $\{h_j(x), f_j(x), g_j(x), 1 \leq j \leq 3\}$ in* (3.2) – (3.4) *for the distribution of $T(F_n)$ are given formally in terms* *of the derivatives of T by* $a_{10} = T(F)$, $a_{21} = [1^2]$,

for $j = 1$: $a_{11} = \tfrac{1}{2}[11]$, $a_{32} = [1^3] + 3[1, 2, 12]$,

for $j = 2$: $a_{22} = [1, 11] + \tfrac{1}{2}[12^2] + [1, 122]$, $a_{43} = [1^4] - 3[1^2]^2 + 12[1, 2^2, 12] + 12[1, 2, 13, 23] + 4[1, 2, 3, 123]$,

for $j = 3$: $a_{12} = [111]/6 + [1122]/8$, $a_{33} = 3[1^2, 11]/2 - 3[1^2][11]/2 - 3[1, 2, 12] + 3[1, 12, 22] + 3[1, 12^2] + 3[1, 2, 122] + 3[1^2, 122]/2 + 3[1, 2, 1233]/2 + [12, 23, 31] + 3[1, 12, 233] + 3[1, 23, 123]$, $a_{54} = [1^5] - 10[1^2][1^3] - 60[1, 2, 12][1^2] + 20[1, 2^3, 12] + 15[1^2, 2^2, 12] + 60[1, 2, 3, 12, 23] + 60[1, 2^2, 13, 23] + 30[1, 2, 3^2, 123] + 5[1, 2, 3, 4, 1234] + 60[1, 2, 34, 13, 24] + 60[1, 2, 3, 14, 234]$.

Also, the leading moment coefficients are given by

$a'_{11} = a_{11}$, $a'_{12} = a_{12}$, $a'_{21} = a_{21}$, $a'_{22} = a_{22} + a_{11}^2$, $a'_{32} = a_{32} + 3a_{21}a_{11}$, $a'_{33} = a_{33} + 3a_{21}a_{12} + a_{11}^3 + 3a_{11}a_{22}$, $a'_{42} = 3a_{21}^2$, $a'_{43} = a_{43} + 4a_{11}a_{32} + 6a_{21}a_{22} + 6a_{21}a_{11}^2$, $a'_{53} = 10a_{21}(a_{32} + 3a_{21}a_{11})$, $a'_{54} = a_{54} + 5a_{43}a_{11} + 10a_{33}a_{21} + 10a_{32}(a_{22} + a_{11}^2) + 30a_{22}a_{21}a_{11} + 15a_{21}^2a_{12} + 10a_{21}a_{11}^3$.

See the remark on page 438 of Bhattacharya and Ghosh (1978) concerning the validity of (3.7)

The proof of the theorem will require two lemmas. The first of these gives the joint moments of the empirical distribution, $F_n(x) = n^{-1}\sum_{j=1}^{n} 1(X_j \le x)$, where $1(\cdot)$ is the indicator function. Define

$$\mu_i = \mu_i(x_1, \cdots, x_i) = E\prod_{j=1}^{i}\{1(X_1 \le x_j) - F(x_j)\}. \tag{3.10}$$

For example, $\mu_2 = F(\min_{j=1}^{2} x_j) - F(x_1)F(x_2)$, where $\min_{j=1}^{r} x_j$ denotes the vector $(\min_{j=1}^{r}(x_j)_1, \cdots, \min_{j=1}^{r}(x_j)_s)'$. For integers $i_1, i_2, \cdots$ let $I_j = i_1 + i_2 + \cdots + i_j, j \ge 1$, and let $\mu_{i_1\cdot i_2\cdots i_p} = \mu_{i_1\cdot i_2\cdots i_p}(x_1, x_2, \cdots, x_{I_p})$ denote

$$K\mathscr{S}_{I_p}\mu_{i_1}(x_1, \cdots, x_{i_1})\mu_{i_2}(x_{i_1+1}, \cdots, x_{I_2}) \cdots \mu_{i_p}(x_{(I_{p-1}+1)}, \cdots, x_{I_p}),$$

where $\mathscr{S}_i$ is the symmetrising operator and K is the integer such that on allowing for the symmetry of each μ_i, no two terms are the same and each term has coefficient one. For example,

$$\mu_{2.2} = \mu_2(x_1, x_2)\mu_2(x_3, x_4) + \mu_2(x_1, x_3)\mu_2(x_2, x_4) + \mu_2(x_2, x_3)\mu_2(x_1, x_4).$$

(If $a_1 < a_2 < \cdots < a_q$ and a^R denotes $a \cdot a \cdots\cdot a$ (R times), then $\mu_{a_1^{R_1}\cdots a_q^{R_q}}$ has $(\sum_1^q a_iR_i)!/\prod_1^q (a_i!^{R_i}R_i!)$ terms. For example, $\mu_{2\cdot 2\cdot 3}$ has 105 terms.)

LEMMA 3.1. *The pth joint central moment of the empirical distribution is*

$$E\prod_{i=1}^{p}\{F_n(x_i) - F(x_i)\} = n^{-p}\sum_{1\le j\le p/2} n(n-1)\cdots(n-j+1)\sum_p \mu_{i_1\cdots i_j}, \tag{3.11}$$

where $\sum_p$ *denotes summation over all integers* $(i_1, \cdots, i_j)$ *that add to p and satisfy* $2 \le i_1 \le i_2 \le \cdots \le i_j$.

Its proof is straightforward. This lemma directly implies

LEMMA 3.2. *Let* $\mathscr{A}_{i_1\cdots i_p}$ *denote the operation of integrating with respect to the signed measure* $\mu_{i_1\cdots i_p}$: *that is, for f a function of* I_p *variables* $\mathscr{A}_{i_1\cdots i_p} f = \int f\, d\mu_{i_1\cdots i_p}$. *For* $\sum_p$ *as in Lemma* 3.1, *let* $\mathscr{B}_{p\cdot j} = \sum_p \mathscr{A}_{i_1\cdots i_j}$. *Then the random integral operator* $\mathscr{J}_{pn} = \prod_{i=1}^{p}(\mathscr{J}_{iF_n} -$

$\mathscr{I}_{iF}$), $p \geq 1$, *has mean*

$$E\mathscr{J}_{pn} = n^{-p} \textstyle\sum_{1\leq j\leq p/2} n(n-1) \cdots (n-j+1)\, \mathscr{B}_{p\cdot j} = \sum_{p/2\leq j\leq p-1} n^{-j} \mathscr{L}_{p\cdot j}, \tag{3.12}$$

$$\text{where for} \quad p_1 = \begin{cases} p/2, & p \text{ even} \\ (p+1)/2, & p \text{ odd} \end{cases} \quad p_2 = \begin{cases} p/2, & p \text{ even} \\ (p-1)/2, & p \text{ odd} \end{cases}$$

$$\mathscr{L}_{p\cdot p_1} = \mathscr{B}_{p\cdot p_2}, \quad \mathscr{L}_{p\cdot p_1+1} = -c_{1\cdot p_2}\mathscr{B}_{p\cdot p_2} + \mathscr{B}_{p\cdot p_2-1},$$

$$\mathscr{L}_{p\cdot p_1+2} = c_{2\cdot p_2}\mathscr{B}_{p\cdot p_2} - c_{1\cdot p_2}\mathscr{B}_{p\cdot p_2-1} + \mathscr{B}_{p\cdot p_2-2}, \cdots,$$

$$\mathscr{L}_{p\cdot p-1} = (-)^{p_2-1}(p_2-1)!\, \mathscr{B}_{p\cdot p_2} + \cdots + \mathscr{B}_{p\cdot 1},$$

and

$$c_{j\cdot k} = \textstyle\sum i_1 i_2 \cdots i_j, \text{ summed over } 1 \leq i_1 < i_2 < \cdots < i_j < k.$$

PROOF OF THEOREM 3.1. When $G = F_n$, $H = F$ and $r = \infty$, the Taylor Series expansion (2.2) may be written in the notation of Lemma 3.2 as $T(F_n) - T(F) = \mathscr{J}_{1n}Q_1 + \mathscr{J}_{2n}Q_2 + \cdots\cdots$, where $Q_i = T_F^{(i)}/i!$. Using the shorthand $Q_1^2 = Q_1(x_1)Q_1(x_2)$, $Q_1Q_2 = Q_1(x_1)Q_2(x_2, x_3)$, and so forth, we can write

$$\{T(F_n) - T(F)\}^p \approx \textstyle\sum_{q=p}^{\infty} \mathscr{J}_{qn} C_{qp}(\{Q_i\}),$$

where $C_{qp}(\{Q_i\})$ is the coefficient of ε^q in the expansion of $(\sum_{i=1}^{\infty} \varepsilon^i Q_i)^p$ given by Lemma A2 of Withers (1982b). Applying (3.11), we see that (3.7) holds with

$$a'_{pj} = \textstyle\sum_{q=j+1}^{2j} \mathscr{L}_{q\cdot j} C_{qp}(\{Q_i\}). \tag{3.13}$$

For example, $a'_{21} = \mathscr{A}_2 Q_1^2$, $a'_{11} = \mathscr{A}_2 Q_2$, and $a'_{32} = \mathscr{A}_3 Q_1^3 + \mathscr{A}_{2\cdot 2} 3Q_1^2 Q_2$. Because of (2.3), in calculating these expressions we may replace $F(x_j)$ in (3.10) by zero. Calculation yields $\mathscr{A}_{2\cdot 2} T_1^2 T_2 = [1^2][11] + 2[1, 2, 12]$, and so forth.

Substitution into (3.13) and the relations between the moments and cumulants completes the proof. □

COROLLARY 3.1. *Let $f: R^s \to R^k$ and $H: R^k \to R$ be functions such that for some integer $I \geq 3$ derivatives of H of order I are continuous in a neighborhood of $\mu = \mu(F) = \int f\, dF$, $\int |f|^I dF < \infty$, and* $\limsup_{|t|\to\infty} |\int e^{it'f}\, dF| < 1$ *for t in R^k. Set*

$$l^{i_1\cdots i_j} = \frac{\partial}{\partial \mu_{i_1}} \cdots \frac{\partial}{\partial \mu_{i_j}} H(\mu), \quad \mu^{i_1\cdots i_j} = \int (f_{i_1} - \mu_{i_1}) \cdots (f_{i_k} - \mu_{i_j})\, dF,$$

and suppose $\sigma^2 = \sum l^i l^j \mu^{ij} \neq 0$, where $\sum$ sums over $\{1 \leq i, j \leq k\}$. Set $Y_n = n^{1/2}\{T(F_n) - T(F)\}/\sigma$, where $T(F) = H(\mu(F))$. Then there exist polynomials $\{h_r\}: R \to R$ such that

$$\sup_x |P(Y_n \leq x) - \Phi(x) + \phi(x) \textstyle\sum_1^{I-2} n^{-r/2} h_r(x)| = o(n^{-(I-2)/2}). \tag{3.14}$$

For $1 \leq r \leq I - 2 \leq 3$, h_r is given by (3.5), (3.6), (3.9) *and the expressions for $a_{21}, \cdots, a_{43}$ in Theorem* 3.1, *where now* $[1, 11] = l^{a_1} l^{b_1 c_1} \mu^{a_1 b_1 c_1}$,

$$[1, 23, 123] = l^{a_1} l^{b_2 c_3} l^{d_1 e_2 f_3} \mu^{a_1 d_1} \mu^{b_2 e_2} \mu^{c_3 f_3},$$

$$[1, 12^2] = [1, 12, 12] = l^{a_1} l^{b_1 c_2} l^{d_1 e_2} \mu^{a_1 b_1 d_1} \mu^{c_2 e_2},$$

and so forth, with summation over repeated suffixes implied.

PROOF. The first part follows from Theorem 2(b) of Bhattacharya and Ghosh (1978).

The second part follows by checking their (1.14) against the expressions above, obtained formally from Theorem 3.1 and the fact that

$$(3.15)\qquad T_F^{(i)}(x_1, \cdots, x_i) = \sum \mu_{p_1x_1} \cdots \mu_{p_ix_i} l^{p_1\cdots p_i}, \quad \mu_{ix} = f_i(x) - \int f_i\, dF;$$

but this amounts to checking that (2.2) with $r = I - 1$ reduces to the Taylor expansion for $H(\mu(F) + \varepsilon)$, where $\varepsilon = \mu(G) - \mu(F)$. (Their $n^{-1/2}W_n'$ is essentially the Taylor expansion (2.2).) □

4. EXAMPLES. Let $h(\cdot)$ be a given real function on R^s, and $V_j = h(X_j)$, $1 \le j \le n$. Set $\mu = \mu(F) = EV_1 = \int h\, dF$; $\mu_i = \int \{h(x) - \mu\}^i\, dF(x) = E(V_1 - \mu)^i$, $i \ge 1$; $\lambda_i = \mu_2^{-i/2}\mu_i$.

EXAMPLE 1. $T(F) = \mu(F)$. Here $\mu_x = \mu_F^{(1)}(x) = h(x) - \mu(F)$, and higher derivatives vanish. Substitution yields the Edgeworth expansion.

EXAMPLE 2. $T(F) = \mu_2(F)$. Here $\mu_{2F}^{(1)}(x) = \mu_x^2 - \mu_2(F)$, $\mu_{2F}^{(2)}(x, y) = -2\mu_x\mu_y$. Since $T(G)$ is "quadratic" in G, higher derivatives vanish. Substitution yields $a_{21} = \mu_4 - \mu_2^2$, $a_{10} = \mu_2$; for $r = 1$,

$$A_{11} = -(\lambda_4 - 1)^{-1/2}, \quad A_{32} = (\lambda_4 - 1)^{-3/2}(\lambda_6 - 3\lambda_4 + 2 - 6\lambda_3^2);$$

for $r = 2$,

$$A_{22} = (\lambda_4 - 1)^{-1}(4 - 2\lambda_4), \quad A_{43} = (\lambda_4 - 1)^{-2}(\lambda_8 - 4\lambda_6 + 12\lambda_4 - 3\lambda_4^2 - 24\lambda_5\lambda_3 + 96\lambda_3^2 - 6);$$

$$A_{12} = 0, \quad A_{33} = (\lambda_4 - 1)^{-3/2}(-3\lambda_6 + 21\lambda_4 - 26 + 18\lambda_3^2),$$

and for $r = 3$,

$$A_{54} = (\lambda_4 - 1)^{-5/2}(\lambda_{10} - 5\lambda_8 - 40\lambda_7\lambda_3 - 10\lambda_6\lambda_4 + 20\lambda_6 - 30\lambda_5^2 + 480\lambda_5\lambda_3 + 360\lambda_4\lambda_3^2 + 30\lambda_4^2 - 60\lambda_4 - 1560\lambda_3^2 + 24).$$

As a check these were derived independently from Church (1925) and also from $K(2^r)$, $1 \le r \le 5$ of Fisher (1929). Hsu (1945) gave conditions for the validity of this expansion for the distribution of $\mu_2(F_n)$ without giving any of its terms.

EXAMPLE 3. Student's one-sample t-statistic is $t_{n-1} = (n - 1)^{1/2}S(F_n)$, where $S(G) = \mu_2(G)^{-1/2}\{\mu(G) - \mu(F)\}$.

Differentiating S, then substituting G = F, we obtain $S_x = \mu_2^{-1/2}\mu_x$, $S_{xy} = -\mu_2^{-3/2}\mathscr{S}_2\mu_x(\mu_y^2 - \mu_2)$, and so forth. Substitution yields

$a_{21} = 1$, $a_{10} = 0$, $a_{11} = -\lambda_3/2$, $a_{32} = -2\lambda_3$, $a_{22} = 7\lambda_3^2/4 + 3$, $a_{43} = -2\lambda_4 + 12\lambda_3^2 + 12$, $a_{12} = -25\lambda_3/16 + 3\lambda_5/8 - 15\lambda_3\lambda_4/16$, $a_{33} = -123\lambda_3/4 + 3\lambda_5 - 15\lambda_3\lambda_4/4 - 83\lambda_3^3/8$, and $a_{54} = -180\lambda_3 + 6\lambda_5 + 20\lambda_3\lambda_4 - 105\lambda_3^3$.

So, for example, under the conditions of Corollary 3.1 with $I = 6$, the distribution of Student's t-statistic in its standardized form is

$$P(n^{1/2}S(F_n) \le x) = \Phi(x) - \phi(x)\textstyle\sum_1^3 n^{-r/2}h_r(x) + O(n^{-2}),$$

where

$$h_1(x) = -(2x^2 + 1)\lambda_3/6, \quad h_2(x) = -\lambda_4(x^3 - 3x)/12 + \lambda_3^2(x^5 + 2x^3 - 3x)/18 + x^3/2,$$

and

$$h_3(x) = \lambda_3(3 + 6x^2 - 12x^4 - 8x^6)/48 + \lambda_5(1 + 8x^2 + 2x^4)/40 + \lambda_3\lambda_4(-15 - 90x^2 - 30x^4 + 4x^6)/144 + \lambda_3^3(105 + 525x^2 + 210x^4 - 28x^6 - 8x^8)/1296.$$

The $\{a_{ij}\}$ above were checked against the expansion on page 214 of Geary (1947), who gives an alternate expansion for the density of t_{n-1}, as does Davis (1976).

5. Nonparametric confidence intervals. A frequently encountered approximate nonparametric confidence interval is that based on the approximation $n^{1/2}T_{(0)}(F_n) \sim \mathcal{N}(0, 1)$, where

$$T_{(0)}(F_n) = \{T(F_n) - T(F)\}/\sigma_T(F_n), \tag{5.1}$$

the Studentised statistic. That is, a one-sided confidence interval for $T(F)$ of level $1 - \tilde{\alpha} \approx 1 - \alpha = \Phi(x)$ is given by re-arranging

$$A^-: n^{1/2}T_{(0)}(F_n) \leq x \quad \text{or} \quad A^+: n^{1/2}T_{(0)}(F_n) \geq -x,$$

and a two-sided confidence interval for $T(F)$ of level $1 - \tilde{\alpha} \approx 1 - \alpha = 2\Phi(x) - 1$ is given by re-arranging

$$A^{\pm}: n^{1/2} \mid T_{(0)}(F_n)| \leq x.$$

According to (3.2) the *error* $1 - \tilde{\alpha} - (1 - \alpha)$ of these approximate confidence intervals has magnitude $n^{-1/2}$ in the one-sided case and n^{-1} in the two-sided case. (The reason the magnitude is not merely $n^{-1/2}$ in the two-sided case is that h_1 is an even function.)

The main result of this section, Theorem 5.1, shows that for any $j \geq 1$ their errors can be reduced to magnitude $n^{-j/2}$. This is done by adding to $T_{(0)}(F_n)$ successive correction terms of the form $n^{-i/2}T_{(i)}(F_n)$, where $T_{(i)}$ depends on x, defined in terms of α as above. Thus we need to consider statistics of the form $T_n(F_n)$ where

$$T_n(G) \approx \textstyle\sum_{i=0}^{\infty} n^{-i/2}T_{(i)}(G), \quad G \text{ in } \mathcal{F}_s. \tag{5.2}$$

EXAMPLE. For $T(F) = \mu(F)$ and $\{\mu_i\}$ as in Section Four,

$$P(n^{1/2}T_{(0)}(F_n) \leq x) = \Phi(x) - n^{-1/2}\phi(x)h_1(x) + O(n^{-1}),$$

where $h_1(x) = -\mu_3\mu_2^{-3/2}(1 + 2x^2)/6$. Hence for $T_{(2)}(F) = -h_1(x)$

$$P(n^{1/2}T_{(0)}(F_n) + n^{-1/2}T_{(2)}(F_n) \leq x) = \Phi(x) + O(n^{-1}),$$

under suitable conditions.

To obtain an expansion for the distribution of $T_n(F_n)$ we first need to expand its cumulants. This may be done in terms of $(a_{ri})_j = (a_{ri})_{j_0 \cdots j_q}$, the coefficient of $\lambda_0^{j_0} \cdots \lambda_q^{j_q}$ in a_{ri} of (3.7) when $T(\cdot) = \sum_{i=0}^{r} \lambda_i T_{(i)}(\cdot)$, $\{\lambda_i\}$ are arbitrary numbers, and $j_0 + j_1 + \cdots + j_r = r$.

From Withers (1982a) we have

LEMMA 5.1. *The rth cumulant at $T_n(F_n)$ has the formal expansion*

$$K_r(T_n(F_n)) \approx \textstyle\sum_{k \geq 2r-2} b_{rk} n^{-j/2}, \text{ where } b_{rk} = \sum_{r-1 \leq i \leq k/2} \sum_{(k-2i)} (a_{ri})_j \tag{5.3}$$

and for $j = (j_0 \cdots j_q)$, $\sum_{(q)}$ sums over $\{j_0 + \cdots + j_q = r, 0.j_0 + 1.j_1 + \cdots + qj_q = q\}$. In particular,

$$b_{10} = (a_{10})_1, \quad b_{11} = (a_{10})_{01}, \quad b_{22} = (a_{21})_2, \tag{5.4}$$

$$b_{12} = (a_{11})_1 + (a_{10})_{001}, \quad b_{23} = (a_{21})_{11}, \quad b_{34} = (a_{32})_3, \tag{5.5}$$

$$\begin{cases} b_{13} = (a_{11})_{01} + (a_{10})_{0001}, \quad b_{24} = (a_{22})_2 + (a_{21})_{02} + (a_{21})_{101}, \\ b_{35} = (a_{32})_{21}, \quad b_{46} = (a_{43})_4, \end{cases} \tag{5.6}$$

and

$$\begin{cases} b_{14} = (a_{12})_1 + (a_{11})_{001} + (a_{10})_{00001}, \quad b_{25} = (a_{22})_{11} + (a_{21})_{011} + (a_{21})_{1001}, \\ b_{36} = (a_{33})_3 + (a_{32})_{12} + (a_{32})_{201}, \; b_{47} = (a_{43})_{31}, \quad b_{58} = (a_{54})_5. \square \end{cases} \tag{5.7}$$

Note that $(a_{ij})_i = a_{ij}(T_{(0)})$, $(a_{ij})_{0i} = a_{ij}(T_{(1)})$ and so forth, where $a_{ij}(T) = a_{ij}$. Define

$[1_0^2, 1_1] = \int T_{(0)x_1}^2 T_{(1)x_1} dF(x_1)$, $[1_1, 122_0] = \int\int T_{(1)x_1} T_{(0)x_1x_2x_2} dF(x_1)\, dF(x_2)$, and so forth. The cumulant coefficients $\{b_{rk}\}$ may now be expressed directly in terms of the derivatives of $\{T_{(i)}\}$ at F using Theorem 3.1 and the following corollary of it.

COROLLARY 5.1. *The expressions for* $\{a_{ri}\}$ *in Theorem* 3.1 *imply*

$$(a_{ij})_i = (a_{ij})(T_{(0)}), \quad (a_{21})_{11} = 2[1_0, 1_1],$$
$$(a_{32})_{21} = 3[1_0^2, 1_1] + 3[1_0, 2_0, 12_1] + 6[1_0, 2_1, 12_0],$$
$$(a_{22})_{11} = [1_0, 11_1] + [1_1, 11_0] + [12_0, 12_1] + [1_0, 122_1] + [1_1, 122_0].$$
$$(a_{43})_{31} = 4[1_0^3, 1_1] - 12[1_0^2][1_0, 1_1] + 12[1_0, 2_0^2, 12_1] + 24[1_0, 2_0, 2_1, 12_0]$$
$$+ 12[2_0^2, 1_1, 12_0] + 24[1_0, 2_0, 13_0, 23_1] + 24[1_0, 2_1, 13_0, 23_0]$$
$$+ 4[1_0, 2_0, 3_0, 123_1] + 12[1_0, 2_0, 3_1, 123_0]. \square$$

For example, $(a_{21})_{11} = 2[1_0, 1_1]$ implies $(a_{21})_{101} = 2[1_0, 1_2]$.

From (5.3) it follows that the rth cumulant of

$$Y_n = n^{1/2} b_{22}^{-1/2}\{T_n(F_n) - b_{10} - b_{11}n^{-1/2}\} \tag{5.8}$$

has the formal expansion

$$K_r(Y_n) \approx n^{r/2} \textstyle\sum_{i=2r-2}^{\infty} B_{ri} n^{-i/2} \quad \text{with} \quad B_{10} = B_{11} = 0, \quad B_{21} = 1,$$

and

$$B_{ri} = b_{22}^{-r/2} b_{ri}, \quad (r, i) \neq (1, 0) \text{ or } (1, 1),$$

so that for F absolutely continuous and suitably regular, by Withers (1980a), the distribution and quantiles of Y_n have formal Edgeworth type expansions of the type (3.2)–(3.4), where $\{h_r, f_r, g_r\}$ are polynomials given in Theorem 3.1 and Corollary 3.1 of Withers (1980a), in terms of $\{B_{ri}\}$. In particular we have

COROLLARY 5.2. *The distribution of* Y_n *given by* (5.2), (5.8) *has formal expansions of the form* (3.2)–(3.4) *where*

$$h_1 = f_1 = g_1 = B_{12} + B_{23}He_1/2 + B_{34}He_2/6,$$

and

$$h_2 = B_{13} + (B_{24} + B_{12}^2)He_1/2 + (B_{35} + 3B_{12}B_{23})He_2/6 + (B_{46} + 4B_{12}B_{34} + 3B_{23}^2)He_3/24 + B_{23}B_{34}He_4/12 + B_{34}^2He_5/72$$

and g_r *has the form* $g_r(x) = (a_{21})_2^{-1/2}\{T_{(r+1)}(F) - T_{x(r+1)}(F)\}$ *with* $T_{x(r+1)}$ *determined by* x, $T_{(0)}, \cdots, T_{(r)}$.

In particular when $T_{(1)}(G) \equiv 0$ *and* $(a_{21})_2 = 1$, *then*

$$T_{x(2)}(F) = -(a_{11})_1 - (a_{32})_3(x^2 - 1)/6,$$
$$T_{x(3)}(F) = -\{(a_{22})_2 + (a_{21})_{101}\}x/2 + (a_{32})_3^2(2x^3 - 5x)/36 - (a_{43})_4(x^3 - 3x)/24,$$
$$T_{x(4)}(F) = -(a_{12})_1 - (a_{11})_{001} - (a_{21})_{1001}x/2 - \{(a_{33})_3 + (a_{32})_{201}\}(x^2 - 1)/6$$
$$+ \{(a_{22})_2 + (a_{21})_{101}\}(a_{32})_3(x^2 - 1)/6 - (a_{54})_5(x^4 - 6x^2 + 3)/120$$
$$+ (a_{32})_3(a_{43})_4(x^4 - 5x^2 + 2)/24 - (a_{32})_3^3(12x^4 - 53x^2 + 17)/324.$$

The next two theorems given formal results. Sufficient conditions for their validity are left until Corollaries 5.3, 5.4. We first give for any prescribed $j \geq 1$ a one-sided approximate confidence interval whose error has magnitude $n^{-j/2}$.

THEOREM 5.1. *Let $T(\cdot)$ be a real functional on $\mathscr{F}_s$ not depending on F. There exist real functionals $\{q_r(G, x)$ on $\mathscr{F}_s \times \mathscr{R}$, $r \geq 1\}$, not depending on F, such that for $j \geq 1$,*

(5.9) $P(V_{jn}(F_n, x) \leq T(F)) = \Phi(x) + O(n^{-j/2})$ *for all suitably regular (T, F),*

where $V_{jn}(G, x) = T(G) + \sum_{r=1}^{j} n^{-r/2} q_r(G, x)$.

The first three are given by $q_1(F, x) = -[1^2]^{1/2}x$,

$$q_2(F, x) = -[11]/2 + [1^2]^{-1}\{[1^3](1 + 2x^2) + 3[1, 2, 12](1 + x^2)\}/6,$$

$$\begin{aligned} q_3(F, x) = {} & [1^2]^{1/2}(x + x^3)/2 + [1^2]^{-1/2}(4[1, 11] + [12^2] + 2[1, 122])x/4 \\ & -[1^2]^{-3/2}\{[1^4](5x + 3x^3) + 6[1, 2^2, 12](5x + 2x^3) + 6[1, 2, 13, 23](3x + x^3) \\ & + 2[1, 2, 3, 123](3x + x^3)\}/12 + [1^2]^{-5/2}\{[1^3]^2(23x + 16x^3) \\ & + 48[1^3][1, 2, 12](2x + x^3) + 18[1, 2, 12]^2(5x + 2x^3)\}/72. \end{aligned}$$

PROOF. Apply Corollary 5.2 with $T_{(0)}$ given by (5.1), $T_{(r)} = T_{x(r)}$ if $1 < r \leq j$ and $T_{(r)} = 0$ otherwise. Setting $q_1(F, x) = -[1^2]^{1/2}x$ and $q_r(F, x) = [1^2]^{1/2}T_{(r)}(F)$ otherwise we have $Y_n \leq x$ if and only if $V_{jn}(F_n, x) \leq T(F)$. The expressions needed for $\{T_{x(i)}\}$ are $(a_{21})_2 = 1$,

$$(a_{11})_1 = [1^2]^{-1/2}[11]/2 - [1^2]^{-3/2}([1^3] + 2[1, 2, 12])/2,$$

$$(a_{32})_3 = -[1^2]^{-3/2}(2[1^3] + 3[1, 2, 12]),$$

and so forth. Further details, including $q_4(F, x)$ are given in Withers (1980b). □

From Theorem 5.1 we see that lower and upper confidence intervals of level $\Phi(x) + O(n^{-j/2})$ are given by

$$A^- : V_{jn}(F_n, x) \leq T(F), \text{ and } A^+ : T(F) \leq V_{jn}(F_n, -x),$$

respectively, and that a two-sided confidence interval of level $2\Phi(x) - 1 + O(n^{-j/2})$ is given by

$$A^+ : V_{jn}(F_n, x) \leq T(F) \leq V_{jn}(F_n, -x).$$

The next result gives more detail about the errors in their levels:

$$P(A^-) - \Phi(x) = e_{jn}(x), \quad \text{say,} \quad P(A^+) - \Phi(x) = -e_{jn}(-x),$$

and

$$P(A^\pm) - (2\Phi(x) - 1) = e^\pm_{jn}(x), \text{ say.}$$

THEOREM 5.2. *For (T, F) suitably regular,*

(5.10) $$e_{jn}(x) = n^{-j/2}\phi(x)[1^2]^{-1/2}q_{j+1}(F, x) + O(n^{-(j+1)/2})$$

and

(5.11) $$e^\pm_{jn}(x) = \begin{cases} 2n^{-j/2}\phi(x)[1^2]^{-1/2}q_{j+1}(F, x) + O(n^{-(j+2)/2}), j \text{ even}, \\ O(n^{-(j+1)/2}), j \text{ odd}. \end{cases}$$

PROOF. $e_{jn}(x) = P_{jn}(x) - \Phi(x)$, *where* $P_{jn}(x) = P(Y_n \leq x)$,

$$Y_n = \textstyle\sum_0^j n^{-r/2}T_{(r)}(F_n), \text{ and } \{T_{(r)}\} \text{ are given in the proof of Theorem 5.1.}$$

By Corollary 5.1, $P_{jn}^{-1}(\Phi(x)) \approx x + \sum_j^\infty n^{-r/2}g_r(x)$, where $g_j(x) = -[1^2]^{-1/2}q_{j+1}(F, x)$ is even for j odd and odd for j even. Finally, one uses $e^\pm_{jn}(x) = e_{jn}(x) - e_{jn}(-x) + \varepsilon$, where

$$|\varepsilon| \leq P(V_{jn}(F_n, x) \geq V_{jn}(F_n, -x)) = \begin{cases} 0, & 1 \leq j \leq 2, x > 0 \\ O(e^{-nK}), & j \geq 3, x > 0, \end{cases}$$

where K is arbitrary. More details on $e_{jn}(x)$ are given in Withers (1980b). □

The following examples use the notation of Section 4.

EXAMPLE 1. For $T(F) = \mu(F)$ one obtains $q_1(F, x) = -\mu_2^{1/2}x$,

$$q_2(F, x) = \mu_2^{-1}\mu_3(1 + 2x^2)/6,$$

$$q_3(F, x) = \mu_2^{1/2}(x + x^3)/2 - \mu_2^{-3/2}\mu_4(5x + 3x^3)/12 + \mu_2^{-5/2}\mu_3^2(23x + 16x^3)/72,$$

$$q_4(F, x) = \mu_2^{-1}\mu_3(-19 - 19x^2 + 36x^4)/12 + \mu_2^{-2}\mu_5(27 + 86x^2 + 24x^4)/120$$
$$- \mu_2^{-3}\mu_3\mu_4(14 + 55x^2 + 18x^4)/36 + \mu_2^{-4}\mu_3^3(110 + 529x^2 + 192x^4)/648.$$

These results were reported in Withers (1982a), except for q_4.

EXAMPLE 2. For $T(F) = \sigma^2(F)$ one obtains $q_1(F, x) = -(\mu_4 - \mu_2^2)^{1/2}x$,

$$q_2(F, x) = \mu_2 + (\mu_4 - \mu_2^2)^{-1}\{(\mu_6 - 3\mu_4\mu_2 + 2\mu_2^3)(1 + 2x^2)/6 - \mu_3^2(1 + x^2)\},$$

$$q_3(F, x) = (\mu_4 - \mu_2^2)^{1/2}(-3x + x^3)/2 + (\mu_4 - \mu_2^2)^{-1/2}\mu_2^2 x - (\mu_4 - \mu_2^2)^{-3/2}$$
$$\{(\mu_8 - 4\mu_6\mu_2 + 6\mu_4\mu_2^2 - 3\mu_2^4)(5x + 3x^3)/12 - \mu_5\mu_3(5x + 2x^3)$$
$$+ 2\mu_3^2\mu_2(8x + 3x^3)\}$$
$$+(\mu_4 - \mu_2^2)^{-5/2}\{(\mu_6 - 3\mu_4\mu_2 + 2\mu_2^3)^2(23x + 16x^3)/72$$
$$- 4(\mu_6 - 3\mu_4\mu_2 + 2\mu_2^3)\mu_3^2(2x + x^3)/3 + \mu_3^4(5x + 2x^3)\}.$$

REMARK. Regularity conditions for this section are obtained by noting that the results of Bhattacharya and Ghosh (1978) remain valid if H is allowed to depend on n and continuity of a derivative of H is replaced by continuity of a derivative of $H_{(n)} = H$ uniformly in n, and

(5.12) $$\sigma_n^2 = \sigma^2 \text{ is bounded away from } 0.$$

For example, setting $l_{\iota_1 \cdots \iota_k}(H) = l^{\iota_1 \cdots \iota_k}$ as defined in Corollary 3.1, and $a_{ij}(H) = a_{ij}$ as given by Corollary 3.1, we have

COROLLARY 5.3. *Suppose that the conditions for Corollary 3.1 hold for $H = (H_0, \cdots, H_J)$ a function from R^k to R^{J+1} with the condition $\sigma^2 \neq 0$ replaced by $\sigma_0^2 \neq 0$ where $\sigma_0^2 = \sum l_\iota(H_0)l_j(H_0)\mu^{\iota j}$. Set $H_{(n)} = \sum_0^J n^{-j/2}H_j$, $T_n(F) = H_{(n)}(\mu(F))$ and $Y_n = n^{1/2}\{T_n(F_n) - H_0(\mu(F)) - n^{-1/2}H_1(\mu(F))\}/\sigma_0$. Then there exists polynomials $\{h_r\} : R \to R$ such that* (3.14) *holds. In particular h_1, h_2 are given by Corollary* 5.2, (5.4)–(5.6), *and $(a_{ij})_\iota = a_{ij}(H_0)$, $(a_{ij})_{0\iota} = a_{ij}(H_1)$, and so forth, and*

$$(a_{21})_{11} = 2 \sum l_\iota(H_0)l_j(H_1)\mu^{\iota j},$$
$$(a_{32})_{21} = 3 \sum l_\iota(H_0)l_j(H_0)l_k(H_1)\mu^{\iota j k} + 3 \sum l_\iota(H_0)l_j(H_0)l_{km}(H_1)\mu^{\iota k}\mu^{jm}$$
$$+ 6 \sum l_\iota(H_0)l_j(H_1)l_{km}(H_0)\mu^{\iota k}\mu^{jm}.$$

PROOF. From the above remark it follows that (3.14) holds with Y_n replaced by $n^{1/2}\{T_n(F_n) - T_n(F)\}/\sigma_0 = Y_n - \delta_n$ and h_r replaced by h_{rn} depending on n through $H_{(n)}$, where $\delta_n = n^{1/2} \sum_2^J n^{-j/2}H_j(\mu(F))/\sigma_0$. Now replace x by $x - \delta_n$, expand, and use Corollary 5.1 and (3.15). □

Similarly, using Remark 1.1 of Bhattacharya and Ghosh (1978) and the form of $q_r(F, x)$ in Theorem 5.1, we have

COROLLARY 5.4. *Suppose for some $J \geq 1$ and $I \geq 2$ that $f: R^s \to R^k$ and $H: R^k \to R$ satisfy $\int |f|^{I(J+1)} dF < \infty$, the derivatives of H of order $I + J$ are continuous in a neighborhood of $\mu = \mu(F) = \int f\, dF$, and* $\limsup_{|t| \to \infty} |\int e^{it'g}\, dF| < 1$, *where $g = \{f_{\iota_1}, f_{\iota_1} f_{\iota_2},$*

$\cdots, f_{i_1} \cdots f_{i_{J+1}}\}$, and $i_1, \cdots i_{J+1}$ range over $1, \cdots k$ and linearly dependent components are excluded. Suppose also that σ^2 of Corollary 3.1 is non-zero. Then there exist functions $\{q_r(G, x), r \geq 1\}$ on $\mathscr{F}_s \times R$ such that for x in R

$$| P(V_{Jn}(F_n, x) \leq T(F)) - \Phi(x) | = \begin{cases} o(n^{-(J-1)/2}) & \text{if } I = J+1 \\ O(n^{-J/2}) & \text{if } I = J+2; \end{cases}$$

where $V_{Jn}(G, x) = T(G) + \sum_{r=1}^{J} n^{-r/2} q_r(G, x)$, and $T(G) = H(\mu(G))$. The first three q_r are given by Theorem 5.1 in terms of $[1^2], \cdots, [1, 2, 3, 123]$ which are given in Corollary 3.1.

Moreover, the above conditions with $J = j$, $I = J + 2$, $J + 3$ imply (5.10), (5.11) hold with the remainder terms replaced by $o(n^{-j/2})$, $O(n^{-(j+1)/2})$ respectively, while the above conditions with $I = J + 4$, $J = j$ imply (5.11). □

As a final remark we note that the asymptotic expansions (3.2)–(3.4) are usually divergent: for example the Edgeworth expansion for the distribution of the standardised sample mean may diverge if $\int e^{x^2/4}\, dF(x) = \infty$ (Cramer, 1946, page 223). However, the moment and cumulant expansions (3.7) converge for a large class satisfying $E | Y_n |^r < \infty$, including $T(F) = H(\mu(F))$ with H a polynomial on R^k and $\mu(F): \mathscr{F}_s \to R^k$ of the form $\int f\, dF$.

ADDENDUM. Since this paper was written we have noticed that Sargan (1976) has independently given expressions for h_1, h_2 of Corollary 3.1 His h_2 contains the following errors: in the expression for $H(g)$ on page 425, in line 1, $-\alpha_9/2$ should be $+\alpha_9/2$, and line 3 should read $x(\alpha_2 + 12\alpha_{10} + 4\alpha_6 + 12\alpha_8 + 2\alpha_4(\alpha_1 + 3\alpha_3))$. An advantage of our method is that an error such as the omission of the term $6\alpha_3\alpha_4$ from the factor of I_4 in the 10th to last line on page 425 of Sargan (1976) is immediately detectable since $[1^3]$ only occurs in $h_2(x)$ via a_{32}.

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