

EXPANSION OF ITERATED STRATONOVICH STOCHASTIC INTEGRALS BASED ON GENERALIZED MULTIPLE FOURIER SERIES

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Abstract. The article is devoted to expansions of iterated Stratonovich stochastic integrals of multiplicities 1-4 on the base of the method of generalized multiple Fourier series. We prove the mean-square convergence of expansions in the case of Legendre polynomials as well as in the case of trigonometric functions. The considered expansions contain only one passage to the limit in contrast to its existing analogues. This property is very convenient for the mean-square approximation of iterated stochastic integrals. It is well-known that a prospective approach to numerical solving of Itô stochastic differential equations being adequate mathematical models of dynamical systems of various physical nature is one based on stochastic analogue of Taylor formula for the solutions to these equations. The iterated stochastic Stratonovich integrals are parts of so-called Taylor-Stratonovich expansion being one of the aforementioned stochastic analogues of Taylor formula. This is why the results of the paper can be applied to constructing strong numerical methods of convergence orders 1.0, 1.5 and 2.0 for Itô stochastic differential equations. The method of generalized multiple Fourier series does not require a partitioning of the integration interval for iterated stochastic Stratonovich integrals. This feature is essential since the mentioned integration interval is small playing a role of the integration step in numerical methods for Itô stochastic differential equations.

Keywords: iterated Stratonovich stochastic integral, multiple Fourier series, Legendre polynomial, expansion, mean-square convergence.

Mathematics Subject Classification: 60H05

1. INTRODUCTION

Suppose that we are given a fixed probability space $(\Omega, \mathcal{F}, \mathbf{P})$, a non-decreasing set of σ -algebras $\{\mathcal{F}_t, t \in [0, T]\}$ on this space and a \mathcal{F}_t -measurable for all $t \in [0, T]$ m -dimensional Wiener process \mathbf{f}_t with independent components $\mathbf{f}_t^{(i)}$, $i = 1, \dots, m$, and the process $\mathbf{f}_{t+\Delta} - \mathbf{f}_t$ is independent of the events in the σ -algebra \mathcal{F}_t for all $t \geq 0$, $\Delta > 0$. We assume that the σ -algebra \mathcal{F} is complete with respect to the measure \mathbf{P} , while the σ -algebra \mathcal{F}_0 contains all events of zero probability.

We consider a Itô stochastic differential equation (SDE)

$$\mathbf{x}_t = \mathbf{x}_0 + \int_0^t \mathbf{a}(\mathbf{x}_\tau, \tau) d\tau + \int_0^t B(\mathbf{x}_\tau, \tau) d\mathbf{f}_\tau, \quad \mathbf{x}_0 = \mathbf{x}(0, \omega), \quad \omega \in \Omega. \quad (1)$$

Here \mathbf{x}_τ , $\tau \in [0, T]$, is a n -dimensional random process being a strong solution of Itô SDE (1), the second integral in the right hand side (1) is treated as a Itô stochastic integral, $\mathbf{a} :$

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$\mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}^n$, $B : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}^{n \times m}$ are deterministic functions, for the right hand side in (1) is well-defined and satisfies standard conditions of existence and uniqueness of a strong solution \mathbf{x}_τ , $\tau \in [0, T]$, to Itô SDE (1) [1], \mathbf{x}_0 and $\mathbf{f}_\tau - \mathbf{f}_0$, $\tau > 0$, are assumed to be independent and \mathbf{x}_0 is a n -dimensional F_0 -measurable random variable obeying $M\{|\mathbf{x}_0|^2\} < \infty$, M is the expectation operator.

It is known [2]–[4] that one of the prospective approaches to a numerical integration of Itô SDE is one based on stochastic analogues of the Taylor formula for solutions to these equations. This approach employs a finite discretization of a time variable and supposes a numerical modelling of the solution to Itô SDE at discrete times by means of stochastic analogue of Taylor formula.

An important feature of the stochastic analogues of Taylor formula [2]–[11] for solutions to Itô SDE is that they involve so-called Itô or Stratonovich iterated stochastic integrals (ISI), which are functionals of the components of the Wiener process and have a complicated structure.

In one of the most general form, the mentioned Itô and Stratonovich ISIs read as

$$J[\psi^{(k)}]_{T,t} = \int_t^T \psi_k(t_k) \dots \int_t^{t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \dots d\mathbf{w}_{t_k}^{(i_k)}, \quad (2)$$

$$J^*[\psi^{(k)}]_{T,t} = \int_t^{*T} \psi_k(t_k) \dots \int_t^{*t_2} \psi_1(t_1) d\mathbf{w}_{t_1}^{(i_1)} \dots d\mathbf{w}_{t_k}^{(i_k)}, \quad (3)$$

where $\psi_l(\tau)$, $l = 1, \dots, k$, are continuous on the segment $[t, T]$ deterministic functions, \mathbf{w}_τ is a random vector with $m+1$ components of the form: $\mathbf{w}_\tau^{(i)} = \mathbf{f}_\tau^{(i)}$ as $i = 1, \dots, m$ and $\mathbf{w}_\tau^{(0)} = \tau$, the quantities i_1, \dots, i_k takes values $0, 1, \dots, m$, $\mathbf{f}_\tau^{(i)}$, $i = 1, \dots, m$ are independent standard Wiener processes, k is the multiplicity of ISI. In (2) and (3), and also in what follows, to simplify the writing, instead of $J[\psi^{(k)}]_{T,t}^{i_1 \dots i_k}$ and $J^*[\psi^{(k)}]_{T,t}^{i_1 \dots i_k}$ we write $J[\psi^{(k)}]_{T,t}$ and $J^*[\psi^{(k)}]_{T,t}$, respectively.

We note that classical stochastic analogues of Taylor formula, so-called Taylor-Itô and Taylor-Stratonovich expansions [2]–[6] involve respectively Itô and Stratonovich ISIs of form (2) and (3) as $\psi_1(\tau), \dots, \psi_k(\tau) \equiv 1$ and $i_1, \dots, i_k = 0, 1, \dots, m$.

Transformed analogues of the above expansions, so-called unified Taylor-Itô and Taylor-Stratonovich expansions [7], [8] involve respectively Itô and Stratonovich ISIs of form (2) and (3) as $\psi_l(\tau) \equiv (t - \tau)^{q_l}$, $\tau \in [t, T]$ and $i_l = 1, \dots, m$, $q_l = 0, 1, 2, \dots$, $l = 1, \dots, k$.

In view of the said above, the systems of Itô and Stratonovich ISIs play an exceptionally important role in resolving the problem on numerical integration of Itô SDEs.

At first glance, it could seem that ISIs can be approximated by iterated integral sums. However, such approach supposes a partitioning of the integration interval $[t, T]$ for ISI and its length $T - t$ is already rather small since it serves as an integration step in numerical methods for Itô SDE. As numerical experiments show [9], this leads one to unacceptable computational costs.

In [3], it was proposed to employ mean-square converging trigonometric Fourier expansions for Wiener processes, by which ISI is constructed. In [3], this method was employed to obtain expansions for Itô ISI of form (2) as $k = 2$ and $\psi_1(\tau), \psi_2(\tau) \equiv 1$, $i_1, i_2 = 0, 1, \dots, m$.

An attempt to develop this idea for Stratonovich ISI of form (3) as $k = 3$ and $\psi_1(\tau), \psi_2(\tau), \psi_3(\tau) \equiv 1$, $i_1, i_2, i_3 = 0, 1, \dots, m$ was made in [2], [10].

In [11] there was proposed a more general method of mean-square approximation of Stratonovich ISI of form (3) based on generalized iterated Fourier series, which allows one to employ complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_2([t, T])$, due to its features, the method used in [3] admits the application of only trigonometric basis functions.

The methods employing the Fourier series and being proposed in [2]–[4], [10], [11] turned out to be essentially more effective for a mean-square approximation of Stratonovich and Itô ISIs than the methods based on integral sums [9]. However, the Fourier methods considered in [2]–[4], [10], [11] lead one to iterated series of standard Gaussian random variables, in which the passage to the limit is made iteratively. This is opposite to the multiple series, in which the passage to the limit is made just once. This fact is essential and gives rise to a series of restrictions for the application of the methods from [2]–[4], [10], [11] to ISIs of form (2) and (3) of multiplicity 3 and higher, here we mean at least three-multiple integration over Wiener processes in ISI.

In [9], there was proposed a method of mean-square approximation for Itô ISI of form (2) (see Theorem 1 in what follows) based on multiple (not iterated) generalized Fourier series over various complete orthonormal systems of basis functions in the space $L_2([t, T]^k)$. As a result, in the mentioned method, the passage to the limit is made just once that ensures a correct choice of the lengths for the sequences of standard Gaussian random variables needed for constructing an approximation of Itô ISI. Moreover, the Fourier method provides new opportunities for estimating and exact calculation of mean-square errors in approximations for Itô ISI [9].

The present paper is devoted to an adaption of the Fourier method [9] of expansion of Itô ISI of form (2) to Stratonovich ISI of form (3). In the work we show that the expansions for Stratonovich ISIs of form (3) obtained by the method in [9] turn out to be essentially simpler and having no additional rather complicated terms in comparison with its analogues obtained first in [9] for Itô ISI of form (2).

2. FORMULATION OF MAIN RESULTS

We provide a formulation of Fourier method [9].

Let $\{\phi_j(x)\}_{j=0}^{\infty}$ be a complete orthonormal system of functions in the space $L_2([t, T])$, and $\psi_1(\tau), \dots, \psi_k(\tau)$ be deterministic functions continuous on $[t, T]$. We introduce the following function:

$$K(t_1, \dots, t_k) = \psi_1(t_1) \dots \psi_k(t_k) \mathbf{1}_{\{t_1 < \dots < t_k\}}, \quad t_1, \dots, t_k \in [t, T], \quad k \geq 2, \quad (4)$$

and $K(t_1) = \psi_1(t_1)$, $t_1 \in [t, T]$, where $\mathbf{1}_{\{A\}}$ is the indicator of the set A .

The function $K(t_1, \dots, t_k)$ is piece-wise continuous in the hyper-cube $[t, T]^k$ and this is why a multiple Fourier series of the function $K(t_1, \dots, t_k) \in L_2([t, T]^k)$ converges in the hyper-cube $[t, T]^k$ in the square-mean sense, that is,

$$\lim_{p_1, \dots, p_k \rightarrow \infty} \int_{[t, T]^k} \left(K(t_1, \dots, t_k) - \sum_{j_1=0}^{p_1} \dots \sum_{j_k=0}^{p_k} C_{j_k \dots j_1} \prod_{l=1}^k \phi_{j_l}(t_l) \right)^2 dt_1 \dots dt_k = 0, \quad (5)$$

where

$$C_{j_k \dots j_1} = \int_{[t, T]^k} K(t_1, \dots, t_k) \prod_{l=1}^k \phi_{j_l}(t_l) dt_1 \dots dt_k \quad (6)$$

and the Parseval identity holds:

$$\int_{[t, T]^k} K^2(t_1, \dots, t_k) dt_1 \dots dt_k = \lim_{p_1, \dots, p_k \rightarrow \infty} \sum_{j_1=0}^{p_1} \dots \sum_{j_k=0}^{p_k} (C_{j_k \dots j_1})^2.$$

We consider a partition $\{\tau_j\}_{j=0}^N$ of the segment $[t, T]$ such that

$$t = \tau_0 < \dots < \tau_N = T, \quad \Delta_N = \max_{0 \leq j \leq N-1} \Delta \tau_j \rightarrow 0 \quad \text{as } N \rightarrow \infty, \quad \Delta \tau_j = \tau_{j+1} - \tau_j. \quad (7)$$

Theorem 1. [9] *Let $\{\phi_j(x)\}_{j=0}^\infty$ be a complete orthonormal system of continuous functions in the space $L_2([t, T])$, and $\psi_i(\tau)$, $i = 1, 2, \dots, k$, are continuous on the segment $[t, T]$ functions. Then Itô ISI $J[\psi^{(k)}]_{T,t}$ of form (2) is expanded into a multiple series converging in the mean-square sense:*

$$J[\psi^{(k)}]_{T,t} = \text{l.i.m.}_{p_1, \dots, p_k \rightarrow \infty} \sum_{j_1=0}^{p_1} \dots \sum_{j_k=0}^{p_k} C_{j_k \dots j_1} \left(\prod_{l=1}^k \zeta_{j_l}^{(i_l)} \right. \\ \left. - \text{l.i.m.}_{N \rightarrow \infty} \sum_{(l_1, \dots, l_k) \in G_k} \phi_{j_{l_1}}(\tau_{l_1}) \Delta \mathbf{w}_{\tau_{l_1}}^{(i_1)} \dots \phi_{j_{l_k}}(\tau_{l_k}) \Delta \mathbf{w}_{\tau_{l_k}}^{(i_k)} \right), \quad (8)$$

where l.i.m. is the limit in the mean-square sense,

$$G_k = H_k \setminus L_k, \quad H_k = \{(l_1, \dots, l_k) : l_1, \dots, l_k = 0, 1, \dots, N-1\},$$

$$L_k = \{(l_1, \dots, l_k) : l_1, \dots, l_k = 0, 1, \dots, N-1, l_g \neq l_r \ (g \neq r), \ g, r = 1, \dots, k\},$$

and

$$\zeta_j^{(i)} = \int_t^T \phi_j(s) d\mathbf{w}_s^{(i)}$$

are independent standard Gaussian random variables for different i or j if $i \neq 0$, and $\Delta \mathbf{w}_{\tau_j}^{(i)} = \mathbf{w}_{\tau_{j+1}}^{(i)} - \mathbf{w}_{\tau_j}^{(i)}$, $i = 0, 1, \dots, m$, $\{\tau_j\}_{j=0}^N$ is a partition of the segment $[t, T]$ satisfying condition (7).

It is easy to show that particular cases (8) as $k = 1, \dots, 4$ are written as

$$J[\psi^{(1)}]_{T,t} = \text{l.i.m.}_{p_1 \rightarrow \infty} \sum_{j_1=0}^{p_1} C_{j_1} \zeta_{j_1}^{(i_1)}, \quad (9)$$

$$J[\psi^{(2)}]_{T,t} = \text{l.i.m.}_{p_1, p_2 \rightarrow \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \mathbf{1}_{\{i_1=i_2 \neq 0\}} \mathbf{1}_{\{j_1=j_2\}} \right), \quad (10)$$

$$J[\psi^{(3)}]_{T,t} = \text{l.i.m.}_{p_1, \dots, p_3 \rightarrow \infty} \sum_{j_1=0}^{p_1} \dots \sum_{j_3=0}^{p_3} C_{j_3 j_2 j_1} \left(\zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \right. \\ \left. - \mathbf{1}_{\{i_1=i_2 \neq 0\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3 \neq 0\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} - \mathbf{1}_{\{i_1=i_3 \neq 0\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \right), \quad (11)$$

$$J[\psi^{(4)}]_{T,t} = \text{l.i.m.}_{p_1, \dots, p_4 \rightarrow \infty} \sum_{j_1=0}^{p_1} \dots \sum_{j_4=0}^{p_4} C_{j_4 \dots j_1} \left(\prod_{l=1}^4 \zeta_{j_l}^{(i_l)} - \mathbf{1}_{\{i_1=i_2 \neq 0\}} \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \right. \\ - \mathbf{1}_{\{i_1=i_3 \neq 0\}} \mathbf{1}_{\{j_1=j_3\}} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_1=i_4 \neq 0\}} \mathbf{1}_{\{j_1=j_4\}} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \\ - \mathbf{1}_{\{i_2=i_3 \neq 0\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} - \mathbf{1}_{\{i_2=i_4 \neq 0\}} \mathbf{1}_{\{j_2=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} \\ - \mathbf{1}_{\{i_3=i_4 \neq 0\}} \mathbf{1}_{\{j_3=j_4\}} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} + \mathbf{1}_{\{i_1=i_2 \neq 0\}} \mathbf{1}_{\{j_1=j_2\}} \mathbf{1}_{\{i_3=i_4 \neq 0\}} \mathbf{1}_{\{j_3=j_4\}} \\ \left. + \mathbf{1}_{\{i_1=i_3 \neq 0\}} \mathbf{1}_{\{j_1=j_3\}} \mathbf{1}_{\{i_2=i_4 \neq 0\}} \mathbf{1}_{\{j_2=j_4\}} + \mathbf{1}_{\{i_1=i_4 \neq 0\}} \mathbf{1}_{\{j_1=j_4\}} \mathbf{1}_{\{i_2=i_3 \neq 0\}} \mathbf{1}_{\{j_2=j_3\}} \right). \quad (12)$$

We are in position to formulate the main results of the work, which show that analogues of expansions (10)–(12) obtained for Stratonovich ISI of form (3) turn out to be essentially simpler than expansions (10)–(12).

Theorem 2. *Let $\{\phi_j(x)\}_{j=0}^\infty$ be a complete orthonormalized system of Legendre polynomials or a system of trigonometric functions in the space $L_2([t, T])$. Assume also that a function $\psi_2(\tau)$*

is continuously differentiable on the segment $[t, T]$, and a function $\psi_1(\tau)$ is twice continuously differentiable on this segment. Then the Stratonovich ISI $J^*[\psi^{(2)}]_{T,t}$ of multiplicity 2 of form (3) with $i_1, i_2 = 1, \dots, m$ satisfies the following converging in the square-mean sense expansion

$$J^*[\psi^{(2)}]_{T,t} = \text{l.i.m.}_{p_1, p_2 \rightarrow \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)},$$

where the notations are the same as in Theorem 1.

Theorem 3. Let $\{\phi_j(x)\}_{j=0}^{\infty}$ be a complete orthonormal system of Legendre polynomials or a system of trigonometric functions in the space $L_2([t, T])$. Assume also that a function $\psi_2(s)$ is continuously differentiable on the segment $[t, T]$, and function $\psi_1(s), \psi_3(s)$ are twice continuously differentiable on this segment. Then the Stratonovich ISI $J^*[\psi^{(3)}]_{T,t}$ of multiplicity 3 of form (3) with $i_1, i_2, i_3 = 1, \dots, m$ satisfies the following converging in the square-mean sense expansion

$$J^*[\psi^{(3)}]_{T,t} = \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_1, j_2, j_3=0}^p C_{j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}, \quad (13)$$

where the notations are the same as in Theorem 1.

Theorem 4. Let $\{\phi_j(x)\}_{j=0}^{\infty}$ be a complete orthonormal system of Legendre polynomials or a system of trigonometric functions in the space $L_2([t, T])$. Assume also that $\psi_1(s), \dots, \psi_4(s) \equiv 1$. Then the Stratonovich ISI $J^*[\psi^{(4)}]_{T,t}$ of multiplicity 4 of form (3) with $i_1, i_2, i_3, i_4 = 0, 1, \dots, m$ satisfies the following converging in the square-mean sense expansion

$$J^*[\psi^{(4)}]_{T,t} = \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_1, j_2, j_3, j_4=0}^p C_{j_4 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)}, \quad (14)$$

where the notations are the same as in Theorem 1.

3. PROOF OF THEOREM 2

In view of the standard relation of Stratonovich and Itô integrals [2], the identity holds

$$J^*[\psi^{(2)}]_{T,t} = J[\psi^{(2)}]_{T,t} + \frac{1}{2} \mathbf{1}_{\{i_1=i_2 \neq 0\}} \int_t^T \psi_1(t_1) \psi_2(t_1) dt_1 \quad (15)$$

with probability 1.

According to (10), (15), Theorem 2 will be proved if we show that

$$\frac{1}{2} \int_t^T \psi_1(t_1) \psi_2(t_1) dt_1 = \sum_{j_1=0}^{\infty} C_{j_1 j_1}. \quad (16)$$

We consider the function

$$K^*(t_1, t_2) = K(t_1, t_2) + \frac{1}{2} \mathbf{1}_{\{t_1=t_2\}} \psi_1(t_1) \psi_2(t_1), \quad (17)$$

where $t_1, t_2 \in [t, T]$, and $K(t_1, t_2)$ is of form (4) as $k = 2$.

We expand the function $K^*(t_1, t_2)$ into the Fourier series on the interval (t, T) with respect to the variable t_1 for a fixed t_2 :

$$K^*(t_1, t_2) = \sum_{j_1=0}^{\infty} C_{j_1}(t_2) \phi_{j_1}(t_1), \quad t_1 \neq t, \quad t_1 \neq T, \quad (18)$$

where

$$C_{j_1}(t_2) = \int_t^T K^*(t_1, t_2) \phi_{j_1}(t_1) dt_1 = \psi_2(t_2) \int_t^{t_2} \psi_1(t_1) \phi_{j_1}(t_1) dt_1.$$

Identity (18) is satisfied at each point in the interval (t, T) with respect to the variable t_1 for a fixed t_2 thanks to a piece-wise smoothness of the function $K^*(t_1, t_2)$ in the variable t_1 [12]–[14]. We also note that according the well-known properties of the Fourier series [12]–[14], series (18) converges as $t_1 = t$ and $t_1 = T$. While obtaining (18), we have also employed the fact [12]–[14] that the right hand side of (18) converges as $t_1 = t_2$, which is a point of a finite jump of the function $K^*(t_1, t_2)$, to the quantity

$$\frac{1}{2} (K^*(t_2 - 0, t_2) + K^*(t_2 + 0, t_2)) = \frac{1}{2} \psi_1(t_2) \psi_2(t_2) = K^*(t_2, t_2).$$

The function $C_{j_1}(t_2)$ is continuously differentiable. We expand it into the Fourier series on the interval (t, T) :

$$C_{j_1}(t_2) = \sum_{j_2=0}^{\infty} C_{j_2 j_1} \phi_{j_2}(t_2), \quad t_2 \neq t, \quad t_2 \neq T, \quad (19)$$

where $C_{j_2 j_1}$ is of form (6) as $k = 2$, while identity (19) is satisfied at each point in the interval (t, T) , the right hand side of (19) converges as $t_2 = t$, $t_2 = T$ [12]–[14].

We substitute (19) into (18):

$$K^*(t_1, t_2) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} C_{j_2 j_1} \phi_{j_1}(t_1) \phi_{j_2}(t_2), \quad (t_1, t_2) \in (t, T)^2. \quad (20)$$

It is easy to see that letting $t_1 = t_2$ in (20), we obtain:

$$\frac{1}{2} \psi_1(t_1) \psi_2(t_1) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} C_{j_2 j_1} \phi_{j_1}(t_1) \phi_{j_2}(t_1). \quad (21)$$

By means (21) we can formally write:

$$\begin{aligned} \frac{1}{2} \int_t^T \psi_1(t_1) \psi_2(t_1) dt_1 &= \int_t^T \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} C_{j_2 j_1} \phi_{j_1}(t_1) \phi_{j_2}(t_1) dt_1 \\ &= \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \int_t^T C_{j_2 j_1} \phi_{j_1}(t_1) \phi_{j_2}(t_1) dt_1 \\ &= \lim_{p_1 \rightarrow \infty} \lim_{p_2 \rightarrow \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \int_t^T \phi_{j_1}(t_1) \phi_{j_2}(t_1) dt_1 \\ &= \lim_{p_1 \rightarrow \infty} \lim_{p_2 \rightarrow \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \mathbf{1}_{\{j_1=j_2\}} \\ &= \lim_{p_1 \rightarrow \infty} \lim_{p_2 \rightarrow \infty} \sum_{j_1=0}^{\min\{p_1, p_2\}} C_{j_2 j_1} = \sum_{j_1=0}^{\infty} C_{j_1 j_1}. \end{aligned} \quad (22)$$

In what follows, by C , K , C_0 , K_0 , C_1 , K_1 , ... we denote various constants.

Let us clarify how we pass from the first line in (22) to the second one, other arguing in (22) follows the orthonormality of the functions $\phi_j(s)$ on the segment $[t, T]$. We have:

$$\begin{aligned} \left| \int_t^T \sum_{j_1=0}^{\infty} C_{j_1}(t_1) \phi_{j_1}(t_1) dt_1 - \sum_{j_1=0}^{p_1} \int_t^T C_{j_1}(t_1) \phi_{j_1}(t_1) dt_1 \right| &\leq \int_t^T |\psi_2(t_1) G^{p_1}(t_1)| dt_1 \\ &\leq C \int_t^T |G^{p_1}(t_1)| dt_1, \end{aligned} \quad (23)$$

where

$$G^p(\tau) \stackrel{\text{def}}{=} \sum_{j=p+1}^{\infty} \int_t^{\tau} \psi_1(s) \phi_j(s) ds \phi_j(\tau).$$

We consider the case of Legendre polynomials. We have:

$$|G^{p_1}(t_1)| = \frac{1}{2} \left| \sum_{j_1=p_1+1}^{\infty} (2j_1 + 1) \int_{-1}^{z(t_1)} \psi_1(u(y)) P_{j_1}(y) dy P_{j_1}(z(t_1)) \right|, \quad (24)$$

where

$$u(y) = \frac{T-t}{2}y + \frac{T+t}{2}, \quad z(s) = \left(s - \frac{T+t}{2} \right) \frac{2}{T-t}, \quad (25)$$

and $\{P_j(s)\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials in the space $L_2([-1, 1])$.

Throughout the paper, for a rational q we let

$$J_q(t, T) \stackrel{\text{def}}{=} \int_t^T \frac{ds}{(1-z^2(s))^q}, \quad I_q(x) \stackrel{\text{def}}{=} \int_{-1}^x \frac{dy}{(1-y^2)^q}, \quad f_q(x) \stackrel{\text{def}}{=} \frac{1}{(1-z^2(x))^q}.$$

It follows from (24) and the formula [12]

$$P'_{j+1}(x) - P'_{j-1}(x) = (2j+1)P_j(x), \quad j = 1, 2, \dots, \quad (26)$$

where the prime denotes the derivative with respect to x , that

$$\begin{aligned} |G^{p_1}(t_1)| &= \frac{1}{2} \left| \sum_{j_1=p_1+1}^{\infty} \left((P_{j_1+1}(z(t_1)) - P_{j_1-1}(z(t_1))) \psi_1(t_1) \right. \right. \\ &\quad \left. \left. - \frac{T-t}{2} \int_{-1}^{z(t_1)} (P_{j_1+1}(y) - P_{j_1-1}(y)) \psi'_1(u(y)) dy \right) P_{j_1}(z(t_1)) \right| \\ &\leq C_0 \left| \sum_{j_1=p_1+1}^{\infty} (P_{j_1+1}(z(t_1)) P_{j_1}(z(t_1)) - P_{j_1-1}(z(t_1)) P_{j_1}(z(t_1))) \right| \\ &\quad + \frac{T-t}{4} \left| \sum_{j_1=p_1+1}^{\infty} \left(\psi'_1(t_1) \left(\frac{P_{j_1+2}(z(t_1)) - P_{j_1}(z(t_1))}{2j_1+3} \right. \right. \right. \\ &\quad \left. \left. - \frac{P_{j_1}(z(t_1)) - P_{j_1-2}(z(t_1))}{2j_1-1} \right) - \frac{T-t}{2} \int_{-1}^{z(t_1)} \left(\frac{P_{j_1+2}(y) - P_{j_1}(y)}{2j_1+3} \right. \right. \\ &\quad \left. \left. - \frac{P_{j_1}(y) - P_{j_1-2}(y)}{2j_1-1} \right) \psi''_1(u(y)) dy \right) P_{j_1}(z(t_1)) \right|, \end{aligned} \quad (27)$$

where ψ'_1, ψ''_1 are the derivatives of the function $\psi_1(s)$ with respect to the variable $u(y)$.

By (27) and the estimate for the Legendre polynomials [12]

$$|P_n(y)| < \frac{K}{\sqrt{n+1}(1-y^2)^{1/4}}, \quad y \in (-1, 1), \quad n \in \mathbb{N}, \quad (28)$$

for $t_1 \in (t, T)$ we obtain:

$$\begin{aligned} |G^{p_1}(t_1)| &< C_0 \left| \lim_{n \rightarrow \infty} \sum_{j_1=p_1+1}^n (P_{j_1+1}(z(t_1))P_{j_1}(z(t_1)) - P_{j_1-1}(z(t_1))P_{j_1}(z(t_1))) \right| \\ &\quad + C_1 \sum_{j_1=p_1+1}^{\infty} \frac{1}{j_1^2} \left(f_{\frac{1}{2}}(t_1) + I_{\frac{1}{4}}(z(t_1))f_{\frac{1}{4}}(t_1) \right) \\ &< C_0 \left| \lim_{n \rightarrow \infty} (P_{n+1}(z(t_1))P_n(z(t_1)) - P_{p_1}(z(t_1))P_{p_1+1}(z(t_1))) \right| \\ &\quad + C_1 \sum_{j_1=p_1+1}^{\infty} \frac{1}{j_1^2} \left(f_{\frac{1}{2}}(t_1) + C_2 f_{\frac{1}{4}}(t_1) \right) \\ &< C_3 \lim_{n \rightarrow \infty} \left(\frac{1}{n} + \frac{1}{p_1} \right) f_{\frac{1}{2}}(t_1) + C_1 \sum_{j_1=p_1+1}^{\infty} \frac{1}{j_1^2} \left(f_{\frac{1}{2}}(t_1) + C_2 f_{\frac{1}{4}}(t_1) \right) \\ &\leq C_4 \left(\left(\frac{1}{p_1} + \sum_{j_1=p_1+1}^{\infty} \frac{1}{j_1^2} \right) f_{\frac{1}{2}}(t_1) + \sum_{j_1=p_1+1}^{\infty} \frac{1}{j_1^2} f_{\frac{1}{4}}(t_1) \right) \leq \frac{K}{p_1} \left(f_{\frac{1}{2}}(t_1) + f_{\frac{1}{4}}(t_1) \right), \end{aligned} \quad (29)$$

where we have employed the inequality:

$$\sum_{j_1=p_1+1}^{\infty} \frac{1}{j_1^2} \leq \int_{p_1}^{\infty} \frac{dx}{x^2} = \frac{1}{p_1}. \quad (30)$$

It follows from (23) and (29) that

$$\left| \int_t^T \sum_{j_1=0}^{\infty} C_{j_1}(t_1) \phi_{j_1}(t_1) dt_1 - \sum_{j_1=0}^{p_1} \int_t^T C_{j_1}(t_1) \phi_{j_1}(t_1) dt_1 \right| < \frac{K_1}{p_1} \left(I_{\frac{1}{2}}(1) + I_{\frac{1}{4}}(1) \right) \rightarrow 0 \quad \text{as } p_1 \rightarrow \infty.$$

Hence,

$$\begin{aligned} \frac{1}{2} \int_t^T \psi_1(t_1) \psi_2(t_1) dt_1 &= \int_t^T \sum_{j_1=0}^{\infty} C_{j_1}(t_1) \phi_{j_1}(t_1) dt_1 = \sum_{j_1=0}^{\infty} \int_t^T C_{j_1}(t_1) \phi_{j_1}(t_1) dt_1 \\ &= \sum_{j_1=0}^{\infty} \int_t^T \sum_{j_2=0}^{\infty} C_{j_2 j_1} \phi_{j_2}(t_1) \phi_{j_1}(t_1) dt_1 \\ &= \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \int_t^T C_{j_2 j_1} \phi_{j_2}(t_1) \phi_{j_1}(t_1) dt_1 = \sum_{j_1=0}^{\infty} C_{j_1 j_1}. \end{aligned} \quad (31)$$

In (31) we have employed the fact that the Fourier-Legendre series

$$\sum_{j_2=0}^{\infty} C_{j_2 j_1} \phi_{j_2}(t_1)$$

of a smooth function $C_{j_1}(t_1)$ converges uniformly to this function on each segment $[t + \varepsilon, T - \varepsilon]$ for all $\varepsilon > 0$, converges to this function at each point (t, T) and converges to $C_{j_1}(t + 0)$ and

$C_{j_1}(T - 0)$ as $t_1 = t$ and $t_1 = T$, respectively [12], [14]. This completes the proof of relation (16) for the Legendre polynomials.

Let $\{\phi_j(x)\}_{j=0}^{\infty}$ be a complete orthonormal system of trigonometric functions in the space $L_2([t, T])$. We have:

$$\begin{aligned}
& \left| \int_t^T \sum_{j_1=0}^{\infty} C_{j_1}(t_1) \phi_{j_1}(t_1) dt_1 - \sum_{j_1=0}^{p_1} \int_t^T C_{j_1}(t_1) \phi_{j_1}(t_1) dt_1 \right| \\
&= \left| \int_t^T \sum_{j_1=p_1+1}^{\infty} \psi_2(t_1) \phi_{j_1}(t_1) \int_t^{t_1} \psi_1(\theta) \phi_{j_1}(\theta) d\theta dt_1 \right| \\
&= \frac{2}{T-t} \left| \int_t^T \psi_2(t_1) \sum_{j_1=p_1+1}^{\infty} \left(\int_t^{t_1} \psi_1(s) \sin \frac{2\pi j_1(s-t)}{T-t} ds \sin \frac{2\pi j_1(t_1-t)}{T-t} \right. \right. \\
&\quad \left. \left. + \int_t^{t_1} \psi_1(s) \cos \frac{2\pi j_1(s-t)}{T-t} ds \cos \frac{2\pi j_1(t_1-t)}{T-t} \right) dt_1 \right| \\
&= \frac{1}{\pi} \left| \int_t^T \left(\psi_1(t) \psi_2(t_1) \sum_{j_1=p_1+1}^{\infty} \frac{1}{j_1} \sin \frac{2\pi j_1(t_1-t)}{T-t} \right. \right. \\
&\quad \left. \left. + \frac{T-t}{2\pi} \psi_2(t_1) \sum_{j_1=p_1+1}^{\infty} \frac{1}{j_1^2} \left(\psi_1'(t_1) - \psi_1'(t) \cos \frac{2\pi j_1(t_1-t)}{T-t} \right. \right. \right. \\
&\quad \left. \left. - \int_t^{t_1} \sin \frac{2\pi j_1(s-t)}{T-t} \psi_1''(s) ds \sin \frac{2\pi j_1(t_1-t)}{T-t} \right. \right. \\
&\quad \left. \left. - \int_t^{t_1} \cos \frac{2\pi j_1(s-t)}{T-t} \psi_1''(s) ds \cos \frac{2\pi j_1(t_1-t)}{T-t} \right) \right) dt_1 \right| \\
&\leq C_1 \left| \int_t^T \psi_2(t_1) \sum_{j_1=p_1+1}^{\infty} \frac{1}{j_1} \sin \frac{2\pi j_1(t_1-t)}{T-t} dt_1 \right| + \frac{C_2}{p_1} \\
&= C_1 \left| \sum_{j_1=p_1+1}^{\infty} \frac{1}{j_1} \int_t^T \psi_2(t_1) \sin \frac{2\pi j_1(t_1-t)}{T-t} dt_1 \right| + \frac{C_2}{p_1},
\end{aligned} \tag{32}$$

where the last step is implied by the uniform convergence of the series $\sum_{j_1=1}^{\infty} \frac{1}{j_1} \sin \frac{2\pi j_1(t_1-t)}{T-t}$

by Dirichlet-Abel test [13].

By (32) we get:

$$\begin{aligned}
& \left| \int_t^T \sum_{j_1=p_1+1}^{\infty} \psi_2(t_1) \phi_{j_1}(t_1) \int_t^{t_1} \psi_1(\theta) \phi_{j_1}(\theta) d\theta dt_1 \right| \\
&\leq C_3 \left| \sum_{j_1=p_1+1}^{\infty} \frac{1}{j_1^2} \left(\psi_2(T) - \psi_2(t) - \int_t^T \cos \frac{2\pi j_1(s-t)}{T-t} \psi_2'(s) ds \right) \right| + \frac{C_2}{p_1} \leq \frac{K}{p_1}.
\end{aligned} \tag{33}$$

The further consideration of this case is similar to the proof of relation (16) for the Legendre polynomials. The proof of Theorem 2 is complete.

4. PROOF OF THEOREM 3

First we consider the case of Legendre polynomials. It follows from formula (11) with $p_1 = p_2 = p_3 = p$ and standard relations between stochastic Itô and Stratonovich integrals [2] that Theorem 3 is true provided

$$\text{l.i.m.}_{p \rightarrow \infty} \sum_{j_1=0}^p \sum_{j_3=0}^p C_{j_3 j_1 j_1} \zeta_{j_3}^{(i_3)} = \frac{1}{2} \int_t^T \psi_3(s) \int_t^s \psi_2(s_1) \psi_1(s_1) ds_1 d\mathbf{f}_s^{(i_3)}, \quad (34)$$

$$\text{l.i.m.}_{p \rightarrow \infty} \sum_{j_1=0}^p \sum_{j_3=0}^p C_{j_3 j_3 j_1} \zeta_{j_1}^{(i_1)} = \frac{1}{2} \int_t^T \psi_3(s) \psi_2(s) \int_t^s \psi_1(s_1) d\mathbf{f}_{s_1}^{(i_1)} ds, \quad (35)$$

$$\text{l.i.m.}_{p \rightarrow \infty} \sum_{j_1=0}^p \sum_{j_3=0}^p C_{j_1 j_3 j_1} \zeta_{j_3}^{(i_2)} = 0. \quad (36)$$

Let us prove (34). By Theorem 1 for $k = 1$, see also (9), we get:

$$\frac{1}{2} \int_t^T \psi_3(s) \int_t^s \psi_2(s_1) \psi_1(s_1) ds_1 d\mathbf{f}_s^{(i_3)} = \frac{1}{2} \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_3=0}^p \tilde{C}_{j_3} \zeta_{j_3}^{(i_3)},$$

where

$$\tilde{C}_{j_3} = \int_t^T \phi_{j_3}(s) \psi_3(s) \int_t^s \psi_2(s_1) \psi_1(s_1) ds_1 ds.$$

We have:

$$\begin{aligned} E_p &\stackrel{\text{def}}{=} \mathbb{M} \left(\left(\sum_{j_1=0}^p \sum_{j_3=0}^p C_{j_3 j_1 j_1} \zeta_{j_3}^{(i_3)} - \frac{1}{2} \sum_{j_3=0}^p \tilde{C}_{j_3} \zeta_{j_3}^{(i_3)} \right)^2 \right) \\ &= \mathbb{M} \left(\left(\sum_{j_3=0}^p \left(\sum_{j_1=0}^p C_{j_3 j_1 j_1} - \frac{1}{2} \tilde{C}_{j_3} \right) \zeta_{j_3}^{(i_3)} \right)^2 \right) = \sum_{j_3=0}^p \left(\sum_{j_1=0}^p C_{j_3 j_1 j_1} - \frac{1}{2} \tilde{C}_{j_3} \right)^2 \\ &= \sum_{j_3=0}^p \left(\sum_{j_1=0}^p \int_t^T \psi_3(s) \phi_{j_3}(s) \int_t^s \psi_2(s_1) \phi_{j_1}(s_1) \int_t^{s_1} \psi_1(s_2) \phi_{j_1}(s_2) ds_2 ds_1 ds \right. \\ &\quad \left. - \frac{1}{2} \int_t^T \psi_3(s) \phi_{j_3}(s) \int_t^s \psi_1(s_1) \psi_2(s_1) ds_1 ds \right)^2 \\ &= \sum_{j_3=0}^p \left(\int_t^T \psi_3(s) \phi_{j_3}(s) \int_t^s \left(\sum_{j_1=0}^p \psi_2(s_1) \phi_{j_1}(s_1) \int_t^{s_1} \psi_1(s_2) \phi_{j_1}(s_2) ds_2 \right. \right. \\ &\quad \left. \left. - \frac{1}{2} \psi_1(s_1) \psi_2(s_1) \right) ds_1 ds \right)^2. \end{aligned} \quad (37)$$

Letting $t_1 = t_2 = s_1$ in (18), we obtain that for each $s_1 \in (t, T)$:

$$\sum_{j_1=0}^{\infty} \psi_2(s_1) \phi_{j_1}(s_1) \int_t^{s_1} \psi_1(s_2) \phi_{j_1}(s_2) ds_2 = \frac{1}{2} \psi_1(s_1) \psi_2(s_1). \quad (38)$$

It follows from (37) and (38) that

$$E_p = \sum_{j_3=0}^p \left(\int_t^T \psi_3(s) \phi_{j_3}(s) \int_t^s \sum_{j_1=p+1}^{\infty} \psi_2(s_1) \phi_{j_1}(s_1) \int_t^{s_1} \psi_1(s_2) \phi_{j_1}(s_2) ds_2 ds_1 ds \right)^2. \quad (39)$$

By (39) and (29) we obtain

$$\begin{aligned} E_p &< C_1 \sum_{j_3=0}^p \left(\int_t^T |\phi_{j_3}(s)| \frac{1}{p} \left(I_{\frac{1}{2}}(z(s)) + I_{\frac{1}{4}}(z(s)) \right) ds \right)^2 < \frac{C_2}{p^2} \sum_{j_3=0}^p \left(\int_t^T |\phi_{j_3}(s)| ds \right)^2 \\ &\leq \frac{C_2(T-t)}{p^2} \sum_{j_3=0}^p \int_t^T \phi_{j_3}^2(s) ds = \frac{C_3 p}{p^2} \rightarrow 0 \end{aligned}$$

as $p \rightarrow \infty$. This completes the proof of (34).

We proceed to proving (35). By Itô formula we find

$$\frac{1}{2} \int_t^T \psi_3(s) \psi_2(s) \int_t^s \psi_1(s_1) d\mathbf{f}_{s_1}^{(i_1)} ds = \frac{1}{2} \int_t^T \psi_1(s_1) \int_{s_1}^T \psi_3(s) \psi_2(s) ds d\mathbf{f}_{s_1}^{(i_1)}$$

with probability 1.

Applying Theorem 1 with $k = 1$ and (9), we get:

$$\frac{1}{2} \int_t^T \psi_1(s) \int_s^T \psi_3(s_1) \psi_2(s_1) ds_1 d\mathbf{f}_s^{(i_1)} = \frac{1}{2} \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_1=0}^p C_{j_1}^* \zeta_{j_1}^{(i_1)},$$

where

$$C_{j_1}^* = \int_t^T \psi_1(s) \phi_{j_1}(s) \int_s^T \psi_3(s_1) \psi_2(s_1) ds_1 ds. \quad (40)$$

We have

$$\begin{aligned} E_p' &\stackrel{\text{def}}{=} \mathbf{M} \left(\left(\sum_{j_1=0}^p \sum_{j_3=0}^p C_{j_3 j_3 j_1} \zeta_{j_1}^{(i_1)} - \frac{1}{2} \sum_{j_1=0}^p C_{j_1}^* \zeta_{j_1}^{(i_1)} \right)^2 \right) = \\ &= \mathbf{M} \left(\left(\sum_{j_1=0}^p \left(\sum_{j_3=0}^p C_{j_3 j_3 j_1} - \frac{1}{2} C_{j_1}^* \right) \zeta_{j_1}^{(i_1)} \right)^2 \right) = \sum_{j_1=0}^p \left(\sum_{j_3=0}^p C_{j_3 j_3 j_1} - \frac{1}{2} C_{j_1}^* \right)^2, \end{aligned} \quad (41)$$

$$\begin{aligned} C_{j_3 j_3 j_1} &= \int_t^T \psi_3(s) \phi_{j_3}(s) \int_t^s \psi_2(s_1) \phi_{j_3}(s_1) \int_t^{s_1} \psi_1(s_2) \phi_{j_1}(s_2) ds_2 ds_1 ds \\ &= \int_t^T \psi_1(s_2) \phi_{j_1}(s_2) \int_{s_2}^T \psi_2(s_1) \phi_{j_3}(s_1) \int_{s_1}^T \psi_3(s) \phi_{j_3}(s) ds ds_1 ds_2. \end{aligned} \quad (42)$$

By (40)–(42) we find:

$$E'_p = \sum_{j_1=0}^p \left(\int_t^T \psi_1(s_2) \phi_{j_1}(s_2) \int_{s_2}^T \left(\sum_{j_3=0}^p \psi_2(s_1) \phi_{j_3}(s_1) \int_{s_1}^T \psi_3(s) \phi_{j_3}(s) ds - \frac{1}{2} \psi_3(s_1) \psi_2(s_1) \right) ds_1 ds_2 \right)^2. \quad (43)$$

Let us show that for all $s_1 \in (t, T)$, the identity holds:

$$\sum_{j_3=0}^{\infty} \psi_2(s_1) \phi_{j_3}(s_1) \int_{s_1}^T \psi_3(s) \phi_{j_3}(s) ds = \frac{1}{2} \psi_2(s_1) \psi_3(s_1). \quad (44)$$

We denote

$$K_1^*(t_1, t_2) = \psi_2(t_1) \psi_3(t_2) \mathbf{1}_{\{t_1 < t_2\}} + \frac{1}{2} \mathbf{1}_{\{t_1 = t_2\}} \psi_2(t_1) \psi_3(t_1), \quad t_1, t_2 \in [t, T].$$

We expand the function $K_1^*(t_1, t_2)$ into the Fourier-Legendre series in the interval (t, T) with respect to the variable t_2 assuming that t_1 is fixed:

$$K_1^*(t_1, t_2) = \sum_{j_3=0}^{\infty} \psi_2(t_1) \int_{t_1}^T \psi_3(s) \phi_{j_3}(s) ds \phi_{j_3}(t_2), \quad (t_2 \neq t, \quad t_2 \neq T). \quad (45)$$

Letting $t_1 = t_2 = s_1$ in (45), we obtain (44), see also the proof of formula (38).

By (43) and (44) we obtain:

$$E'_p = \sum_{j_1=0}^p \left(\int_t^T \psi_1(s_2) \phi_{j_1}(s_2) \int_{s_2}^T \sum_{j_3=p+1}^{\infty} \psi_2(s_1) \phi_{j_3}(s_1) \int_{s_1}^T \psi_3(s) \phi_{j_3}(s) ds ds_1 ds_2 \right)^2. \quad (46)$$

Arguing as in the proof of estimate (29), for a twice continuously differentiable function $\psi_3(s)$ we obtain the following inequality:

$$\left| \sum_{j_3=p+1}^{\infty} \phi_{j_3}(s_1) \int_{s_1}^T \psi_3(s) \phi_{j_3}(s) ds \right| < \frac{C}{p} \left(f_{\frac{1}{2}}(s_1) + f_{\frac{1}{4}}(s_1) \right), \quad (47)$$

where $s_1 \in (t, T)$. The rest of the proof of (35) is similar to the proof of (34).

We proceed to proving (36). We have

$$E''_p \stackrel{\text{def}}{=} \mathbf{M} \left(\left(\sum_{j_1=0}^p \sum_{j_3=0}^p C_{j_1 j_3 j_1} \zeta_{j_3}^{(i_2)} \right)^2 \right) = \sum_{j_3=0}^p \left(\sum_{j_1=0}^p C_{j_1 j_3 j_1} \right)^2, \quad (48)$$

$$\begin{aligned} C_{j_1 j_3 j_1} &= \int_t^T \psi_3(s) \phi_{j_1}(s) \int_t^s \psi_2(s_1) \phi_{j_3}(s_1) \int_t^{s_1} \psi_1(s_2) \phi_{j_1}(s_2) ds_2 ds_1 ds \\ &= \int_t^T \psi_2(s_1) \phi_{j_3}(s_1) \int_t^{s_1} \psi_1(s_2) \phi_{j_1}(s_2) ds_2 \int_{s_1}^T \psi_3(s) \phi_{j_1}(s) ds ds_1. \end{aligned} \quad (49)$$

We substitute (49) into (48):

$$E''_p = \sum_{j_3=0}^p \left(\int_t^T \psi_2(s_1) \phi_{j_3}(s_1) \sum_{j_1=0}^p \int_t^{s_1} \psi_1(\theta) \phi_{j_1}(\theta) d\theta \int_{s_1}^T \psi_3(s) \phi_{j_1}(s) ds ds_1 \right)^2. \quad (50)$$

Let $\tilde{K}(t_1, t_2) = \psi_1(t_1)\mathbf{1}_{\{t_1 < t_2\}}$, $t_1, t_2 \in [t, T]$. We expand the function $\tilde{K}(t_1, t_2)$ into the Fourier-Legendre series on the interval (t, T) with respect to the variable t_1 for a fixed t_2 :

$$\tilde{K}(t_1, t_2) = \sum_{j_1=0}^{\infty} \int_t^{t_2} \psi_1(s)\phi_{j_1}(s)ds \phi_{j_1}(t_1), \quad t_1 \neq t_2. \quad (51)$$

Employing (51), we obtain:

$$\begin{aligned} \sum_{j_1=0}^p \int_t^{s_1} \psi_1(\theta)\phi_{j_1}(\theta)d\theta \int_{s_1}^T \psi_3(s)\phi_{j_1}(s)ds &= \int_{s_1}^T \psi_3(s) \left(\sum_{j_1=0}^p \phi_{j_1}(s) \int_t^{s_1} \psi_1(\theta)\phi_{j_1}(\theta)d\theta \right) ds \\ &= \int_{s_1}^T \psi_3(s) \left(\sum_{j_1=0}^{\infty} \phi_{j_1}(s) \int_t^{s_1} \psi_1(\theta)\phi_{j_1}(\theta)d\theta - \sum_{j_1=p+1}^{\infty} \phi_{j_1}(s) \int_t^{s_1} \psi_1(\theta)\phi_{j_1}(\theta)d\theta \right) ds \\ &= \int_{s_1}^T \psi_3(s)\psi_1(s)\mathbf{1}_{\{s < s_1\}}ds - \int_{s_1}^T \psi_3(s) \sum_{j_1=p+1}^{\infty} \phi_{j_1}(s) \int_t^{s_1} \psi_1(\theta)\phi_{j_1}(\theta)d\theta ds \\ &= - \int_{s_1}^T \psi_3(s) \sum_{j_1=p+1}^{\infty} \phi_{j_1}(s) \int_t^{s_1} \psi_1(\theta)\phi_{j_1}(\theta)d\theta ds. \end{aligned} \quad (52)$$

We substitute (52) into (50):

$$\begin{aligned} E_p'' &= \sum_{j_3=0}^p \left(\int_t^T \psi_2(s_1)\phi_{j_3}(s_1) \int_{s_1}^T \psi_3(s) \sum_{j_1=p+1}^{\infty} \phi_{j_1}(s) \int_t^{s_1} \psi_1(\theta)\phi_{j_1}(\theta)d\theta ds ds_1 \right)^2 \\ &= \sum_{j_3=0}^p \left(\lim_{N \rightarrow \infty} \sum_{l=0}^{N-1} \psi_2(u_l^*)\phi_{j_3}(u_l^*) \int_{u_l^*}^T \psi_3(s) \sum_{j_1=p+1}^{\infty} \phi_{j_1}(s) \int_t^{u_l^*} \psi_1(\theta)\phi_{j_1}(\theta)d\theta ds \Delta u_l \right)^2 \\ &= \sum_{j_3=0}^p \left(\lim_{N \rightarrow \infty} \sum_{l=0}^{N-1} \psi_2(u_l^*)\phi_{j_3}(u_l^*) \sum_{j_1=p+1}^{\infty} \int_{u_l^*}^T \psi_3(s)\phi_{j_1}(s)ds \int_t^{u_l^*} \psi_1(\theta)\phi_{j_1}(\theta)d\theta \Delta u_l \right)^2, \end{aligned} \quad (53)$$

where

$$t = u_0 < u_1 < \dots < u_N = T, \quad \Delta u_l = u_{l+1} - u_l,$$

u_l^* is the point of the minimum of the function $(1 - (z(s))^2)^{-\alpha}$ ($0 < \alpha < 1$) in the interval $[u_l, u_{l+1}]$, and

$$\max_{0 \leq l \leq N-1} \Delta u_l \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty, \quad l = 0, 1, \dots, N-1.$$

The last step in (53) is made on the base of the uniform convergence of the Fourier-Legendre series for the function $\tilde{K}(s, u_l^*)$ on the segment $[u_l^* + \varepsilon, T - \varepsilon]$ for all $\varepsilon > 0$ since $\tilde{K}(s, u_l^*) \equiv 0$ as $s \in [u_l^*, T]$ [12], [14].

We have

$$\begin{aligned}
\int_t^x \psi_1(s) \phi_{j_1}(s) ds &= \frac{\sqrt{(T-t)(2j_1+1)}}{2} \int_{-1}^{z(x)} P_{j_1}(y) \psi(u(y)) dy \\
&= \frac{\sqrt{T-t}}{2\sqrt{2j_1+1}} \left((P_{j_1+1}(z(x)) - P_{j_1-1}(z(x))) \psi_1(x) \right. \\
&\quad \left. - \frac{T-t}{2} \int_{-1}^{z(x)} ((P_{j_1+1}(y) - P_{j_1-1}(y)) \psi_1'(u(y))) dy \right), \tag{54}
\end{aligned}$$

where $x \in (t, T)$, $j_1 \geq p+1$, $z(x)$ and $u(y)$ are given by identities (25), ψ_1' is the derivative of the function $\psi_1(s)$ with respect to the variable $u(y)$.

We note that in (54) we have employed the well-known property of the Legendre polynomials: $P_{j+1}(-1) = -P_j(-1)$, $j = 0, 1, 2, \dots$ and (26).

By (28) and (54) we find:

$$\left| \int_t^x \psi_1(s) \phi_{j_1}(s) ds \right| < \frac{C}{j_1} \left(f_{\frac{1}{4}}(x) + C_1 \right), \quad x \in (t, T). \tag{55}$$

Similar to (55) and in view of the identities $P_j(1) = 1$, $j = 0, 1, 2, \dots$, for the integral similar to one in the left hand side of (55) but with the integration limits x and T we obtain an estimate of form (55). Combining estimate (55) and its analogue for the integral with the integration limits x and T , we get:

$$\left| \int_t^x \psi_1(s) \phi_{j_1}(s) ds \int_x^T \psi_3(s) \phi_{j_1}(s) ds \right| < \frac{K}{j_1^2} \left(f_{\frac{1}{2}}(x) + K_1 \right), \quad x \in (t, T). \tag{56}$$

We estimate the right hand side in (53) by employing (56):

$$\begin{aligned}
E_p'' &\leq C \sum_{j_3=0}^p \left(\lim_{N \rightarrow \infty} \sum_{l=0}^{N-1} |\phi_{j_3}(u_l^*)| \sum_{j_1=p+1}^{\infty} \frac{1}{j_1^2} \left(f_{\frac{1}{2}}(u_l^*) + K_1 \right) \Delta u_l \right)^2 \\
&\leq \frac{C_1}{p^2} \sum_{j_3=0}^p \left(\lim_{N \rightarrow \infty} \sum_{l=0}^{N-1} \left(f_{\frac{3}{4}}(u_l^*) + K_1 f_{\frac{1}{4}}(u_l^*) \right) \Delta u_l \right)^2 \\
&\leq \frac{C_1}{p^2} \sum_{j_3=0}^p \left(\lim_{N \rightarrow \infty} \left(J_{\frac{3}{4}}(t, T) + K_1 J_{\frac{1}{4}}(t, T) \right) \right)^2 = \frac{C_1}{p^2} \sum_{j_3=0}^p \left(J_{\frac{3}{4}}(t, T) + K_1 J_{\frac{1}{4}}(t, T) \right)^2 \\
&= \frac{C_1(T-t)^2 p}{4p^2} \left(I_{\frac{3}{4}}(1) + K_1 I_{\frac{1}{4}}(1) \right)^2 \leq \frac{C_2}{p} \rightarrow 0 \quad \text{as } p \rightarrow \infty.
\end{aligned} \tag{57}$$

This proves (36) and this completes the proof of Theorem 3 for the Legendre polynomials.

We proceed to proving Theorem 3 for trigonometric functions. Similar to inequality (33) we obtain:

$$\left| \int_{s_2}^T \sum_{j_3=p+1}^{\infty} \psi_2(s_1) \phi_{j_3}(s_1) \int_{s_1}^T \psi_3(s) \phi_{j_3}(s) ds ds_1 \right| \leq \frac{K_1}{p}, \tag{58}$$

where s_2 is fixed. Employing (33) and (39), we obtain:

$$\begin{aligned}
E_p &\leq K \sum_{j_3=0}^p \left(\int_t^T \left| \int_t^s \sum_{j_1=p+1}^{\infty} \psi_2(s_1) \phi_{j_1}(s_1) \int_t^{s_1} \psi_1(s_2) \phi_{j_1}(s_2) ds_2 ds_1 \right| ds \right)^2 \\
&= K \sum_{j_3=0}^p \left(\lim_{N \rightarrow \infty} \sum_{l=0}^{N-1} \left| \int_t^{u_l^*} \sum_{j_1=p+1}^{\infty} \psi_2(s_1) \phi_{j_1}(s_1) \int_t^{s_1} \psi_1(s_2) \phi_{j_1}(s_2) ds_2 ds_1 \right| \Delta u_l \right)^2 \\
&\leq K \sum_{j_3=0}^p \left(\lim_{N \rightarrow \infty} \sum_{l=0}^{N-1} \frac{K_1}{p} \Delta u_l \right)^2 \leq \frac{K_2}{p^2} \sum_{j_3=0}^p (T-t)^2 \leq \frac{C}{p} \rightarrow 0 \quad \text{as } p \rightarrow \infty,
\end{aligned} \tag{59}$$

where

$$\begin{aligned}
t = u_0 < u_1 < \dots < u_N = T, \quad \Delta u_l = u_{l+1} - u_l, \quad u_l^* \in [u_l, u_{l+1}], \\
\max_{0 \leq l \leq N-1} \Delta u_l \rightarrow 0 \quad \text{as } N \rightarrow \infty, \quad l = 0, 1, \dots, N-1.
\end{aligned}$$

In the same way, employing (58) and (46), we obtain that $E'_p \rightarrow 0$ as $p \rightarrow \infty$.

It is easy to see that in the considered case, the estimate holds:

$$\left| \int_t^x \psi_1(s) \phi_{j_1}(s) ds \int_x^T \psi_3(s) \phi_{j_1}(s) ds \right| < \frac{K}{j_1^2}, \quad j_1 \neq 0. \tag{60}$$

It follows from (53) and (60) that

$$\begin{aligned}
E''_p &\leq K_1 \sum_{j_3=0}^p \left(\lim_{N \rightarrow \infty} \sum_{l=0}^{N-1} \sum_{j_1=p+1}^{\infty} \left| \int_{u_l^*}^T \psi_3(s) \phi_{j_1}(s) ds \int_t^{u_l^*} \psi_1(\theta) \phi_{j_1}(\theta) d\theta \right| \Delta u_l \right)^2 \\
&\leq K_2 \sum_{j_3=0}^p \left(\lim_{N \rightarrow \infty} \sum_{l=0}^{N-1} \sum_{j_1=p+1}^{\infty} \frac{1}{j_1^2} \Delta u_l \right)^2 \leq \frac{K_2}{p^2} \sum_{j_3=0}^p (T-t)^2 \leq \frac{C}{p} \rightarrow 0 \quad \text{as } p \rightarrow \infty.
\end{aligned}$$

Here we have employed the same notations as in (59). This completes the proof of Theorem 3.

5. PROOF OF THEOREM 4

It follows from (12) that

$$\begin{aligned}
\text{l.i.m.}_{p \rightarrow \infty} \sum_{j_1, j_2, j_3, j_4=0}^p C_{j_4 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} &= J[\psi^{(4)}]_{T,t} + \mathbf{1}_{\{i_1=i_2 \neq 0\}} A_1^{(i_3 i_4)} \\
&\quad + \mathbf{1}_{\{i_1=i_3 \neq 0\}} A_2^{(i_2 i_4)} + \mathbf{1}_{\{i_1=i_4 \neq 0\}} A_3^{(i_2 i_3)} \\
&\quad + \mathbf{1}_{\{i_2=i_3 \neq 0\}} A_4^{(i_1 i_4)} + \mathbf{1}_{\{i_2=i_4 \neq 0\}} A_5^{(i_1 i_3)} \\
&\quad + \mathbf{1}_{\{i_3=i_4 \neq 0\}} A_6^{(i_1 i_2)} - \mathbf{1}_{\{i_1=i_2 \neq 0\}} \mathbf{1}_{\{i_3=i_4 \neq 0\}} B_1 \\
&\quad - \mathbf{1}_{\{i_1=i_3 \neq 0\}} \mathbf{1}_{\{i_2=i_4 \neq 0\}} B_2 \\
&\quad - \mathbf{1}_{\{i_1=i_4 \neq 0\}} \mathbf{1}_{\{i_2=i_3 \neq 0\}} B_3,
\end{aligned} \tag{61}$$

where $J[\psi^{(4)}]_{T,t}$ is of form (2) as $\psi_1(s), \dots, \psi_4(s) \equiv 1$ and $i_1, \dots, i_4 = 0, 1, \dots, m$,

$$A_1^{(i_3 i_4)} = \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_3, j_1=0}^p C_{j_4 j_3 j_1 j_1} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)},$$

$$\begin{aligned}
A_2^{(i_2 i_4)} &= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_3, j_2=0}^p C_{j_4 j_3 j_2 j_3} \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)}, \\
A_3^{(i_2 i_3)} &= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_3, j_2=0}^p C_{j_4 j_3 j_2 j_4} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)}, \\
A_4^{(i_1 i_4)} &= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_3, j_1=0}^p C_{j_4 j_3 j_3 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)}, \\
A_5^{(i_1 i_3)} &= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_3, j_1=0}^p C_{j_4 j_3 j_4 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)}, \\
A_6^{(i_1 i_2)} &= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_3, j_2, j_1=0}^p C_{j_3 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)}, \\
B_1 &= \lim_{p \rightarrow \infty} \sum_{j_1, j_4=0}^p C_{j_4 j_4 j_1 j_1}, \quad B_2 = \lim_{p \rightarrow \infty} \sum_{j_4, j_3=0}^p C_{j_3 j_4 j_3 j_4}, \\
B_3 &= \lim_{p \rightarrow \infty} \sum_{j_4, j_3=0}^p C_{j_4 j_3 j_3 j_4}.
\end{aligned}$$

Interchanging the integration order in Riemann integrals and employing Theorem 1 for $k = 2$, see (10), relation (16), Parseval identity and Itô formula, we obtain:

$$\begin{aligned}
A_1^{(i_3 i_4)} &= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_3, j_1=0}^p \frac{1}{2} \int_t^T \phi_{j_4}(s) \int_t^s \phi_{j_3}(s_1) \left(\int_t^{s_1} \phi_{j_1}(s_2) ds_2 \right)^2 ds_1 ds \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \\
&= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_3=0}^p \frac{1}{2} \int_t^T \phi_{j_4}(s) \int_t^s \phi_{j_3}(s_1) \sum_{j_1=0}^p \left(\int_t^{s_1} \phi_{j_1}(s_2) ds_2 \right)^2 ds_1 ds \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \\
&= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_3=0}^p \frac{1}{2} \int_t^T \phi_{j_4}(s) \int_t^s \phi_{j_3}(s_1) \left((s_1 - t) - \sum_{j_1=p+1}^{\infty} \left(\int_t^{s_1} \phi_{j_1}(s_2) ds_2 \right)^2 \right) ds_1 ds \\
&\quad \cdot \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \\
&= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_3=0}^p \frac{1}{2} \int_t^T \phi_{j_4}(s) \int_t^s \phi_{j_3}(s_1) (s_1 - t) ds_1 ds \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} - \Delta_1^{(i_3 i_4)} \\
&= \frac{1}{2} \int_t^T \int_t^s (s_1 - t) d\mathbf{w}_{s_1}^{(i_3)} d\mathbf{w}_s^{(i_4)} \\
&\quad + \frac{1}{2} \mathbf{1}_{\{i_3=i_4 \neq 0\}} \lim_{p \rightarrow \infty} \sum_{j_3=0}^p \int_t^T \phi_{j_3}(s) \int_t^s \phi_{j_3}(s_1) (s_1 - t) ds_1 ds - \Delta_1^{(i_3 i_4)} \\
&= \frac{1}{2} \int_t^T \int_t^s \int_t^{s_1} ds_2 d\mathbf{w}_{s_1}^{(i_3)} d\mathbf{w}_s^{(i_4)} + \frac{1}{4} \mathbf{1}_{\{i_3=i_4 \neq 0\}} \int_t^T (s_1 - t) ds_1 - \Delta_1^{(i_3 i_4)}
\end{aligned} \tag{62}$$

with probability 1, where

$$\begin{aligned}\Delta_1^{(i_3 i_4)} &= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_3, j_4=0}^p a_{j_4 j_3}^p \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)}, \\ a_{j_4 j_3}^p &= \frac{1}{2} \int_t^T \phi_{j_4}(s) \int_t^s \phi_{j_3}(s_1) \sum_{j_1=p+1}^{\infty} \left(\int_t^{s_1} \phi_{j_1}(s_2) ds_2 \right)^2 ds_1 ds.\end{aligned}\tag{63}$$

We consider $A_2^{(i_2 i_4)}$:

$$\begin{aligned}A_2^{(i_2 i_4)} &= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_3, j_2=0}^p \left(\frac{1}{2} \int_t^T \phi_{j_4}(s) \left(\int_t^s \phi_{j_3}(s_1) ds_1 \right)^2 \int_t^s \phi_{j_2}(s_1) ds_1 ds \right. \\ &\quad - \frac{1}{2} \int_t^T \phi_{j_4}(s) \int_t^s \phi_{j_2}(s_1) \left(\int_t^{s_1} \phi_{j_3}(s_2) ds_2 \right)^2 ds_1 ds \\ &\quad \left. - \frac{1}{2} \int_t^T \phi_{j_4}(s) \int_t^s \phi_{j_2}(s_3) \left(\int_{s_3}^s \phi_{j_3}(s_1) ds_1 \right)^2 ds_3 ds \right) \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} \\ &= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_2=0}^p \left(\frac{1}{2} \int_t^T \phi_{j_4}(s) (s-t) \int_t^s \phi_{j_2}(s_1) ds_1 ds \right. \\ &\quad - \frac{1}{2} \int_t^T \phi_{j_4}(s) \int_t^s \phi_{j_2}(s_1) (s_1-t) ds_1 ds \\ &\quad \left. - \frac{1}{2} \int_t^T \phi_{j_4}(s) \int_t^s \phi_{j_2}(s_3) (s-t+t-s_3) ds_3 ds \right) \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)} \\ &\quad - \Delta_2^{(i_2 i_4)} + \Delta_1^{(i_2 i_4)} + \Delta_3^{(i_2 i_4)} \\ &= -\Delta_2^{(i_2 i_4)} + \Delta_1^{(i_2 i_4)} + \Delta_3^{(i_2 i_4)}\end{aligned}\tag{64}$$

with probability 1, where

$$\begin{aligned}\Delta_2^{(i_2 i_4)} &= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_2=0}^p b_{j_4 j_2}^p \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)}, \\ \Delta_3^{(i_2 i_4)} &= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_2=0}^p c_{j_4 j_2}^p \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)}, \\ b_{j_4 j_2}^p &= \frac{1}{2} \int_t^T \phi_{j_4}(s) \sum_{j_3=p+1}^{\infty} \left(\int_t^s \phi_{j_3}(s_1) ds_1 \right)^2 \int_t^s \phi_{j_2}(s_1) ds_1 ds, \\ c_{j_4 j_2}^p &= \frac{1}{2} \int_t^T \phi_{j_4}(s) \int_t^s \phi_{j_2}(s_3) \sum_{j_3=p+1}^{\infty} \left(\int_{s_3}^s \phi_{j_3}(s_1) ds_1 \right)^2 ds_3 ds.\end{aligned}$$

We consider $A_5^{(i_1 i_3)}$:

$$\begin{aligned}
A_5^{(i_1 i_3)} &= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_3, j_1=0}^p \int_t^T \phi_{j_1}(s_3) \int_{s_3}^T \phi_{j_4}(s_2) \int_{s_2}^T \phi_{j_3}(s_1) \int_{s_1}^T \phi_{j_4}(s) ds ds_1 ds_2 ds_3 \\
&\quad \cdot \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} \\
&= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_3, j_1=0}^p \int_t^T \phi_{j_1}(s_3) \int_{s_3}^T \phi_{j_4}(s) \int_{s_3}^s \phi_{j_3}(s_1) \int_{s_3}^{s_1} \phi_{j_4}(s_2) ds_2 ds_1 ds ds_3 \\
&\quad \cdot \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} \\
&= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_3, j_1=0}^p \left(\frac{1}{2} \int_t^T \phi_{j_1}(s_3) \left(\int_{s_3}^T \phi_{j_4}(s) ds \right)^2 \int_{s_3}^T \phi_{j_3}(s) ds ds_3 \right. \\
&\quad \left. - \frac{1}{2} \int_t^T \phi_{j_1}(s_3) \int_{s_3}^T \phi_{j_3}(s) \left(\int_{s_3}^s \phi_{j_4}(s_1) ds_1 \right)^2 ds ds_3 \right. \\
&\quad \left. - \frac{1}{2} \int_t^T \phi_{j_1}(s_3) \int_{s_3}^T \phi_{j_3}(s_2) \left(\int_{s_2}^T \phi_{j_4}(s_1) ds_1 \right)^2 ds_2 ds_3 \right) \\
&\quad \cdot \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} \\
&= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_3, j_1=0}^p \left(\frac{1}{2} \int_t^T \phi_{j_1}(s_3) (T - s_3) \int_{s_3}^T \phi_{j_3}(s) ds ds_3 \right. \\
&\quad \left. - \frac{1}{2} \int_t^T \phi_{j_1}(s_3) \int_{s_3}^T \phi_{j_3}(s) (s - s_3) ds ds_3 \right. \\
&\quad \left. - \frac{1}{2} \int_t^T \phi_{j_1}(s_3) \int_{s_3}^T \phi_{j_3}(s_2) (T - s_2) ds_2 ds_3 \right) \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)} \\
&\quad - \Delta_4^{(i_1 i_3)} + \Delta_5^{(i_1 i_3)} + \Delta_6^{(i_1 i_3)} \\
&= -\Delta_4^{(i_1 i_3)} + \Delta_5^{(i_1 i_3)} + \Delta_6^{(i_1 i_3)}
\end{aligned} \tag{65}$$

where

$$\begin{aligned}
\Delta_4^{(i_1 i_3)} &= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_3, j_1=0}^p d_{j_3 j_1}^p \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)}, \\
\Delta_5^{(i_1 i_3)} &= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_3, j_1=0}^p e_{j_3 j_1}^p \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)}, \\
d_{j_3 j_1}^p &= \frac{1}{2} \int_t^T \phi_{j_1}(s_3) \sum_{j_4=p+1}^{\infty} \left(\int_{s_3}^T \phi_{j_4}(s) ds \right)^2 \int_{s_3}^T \phi_{j_3}(s) ds ds_3,
\end{aligned}$$

$$\begin{aligned}
e_{j_3 j_1}^p &= \frac{1}{2} \int_t^T \phi_{j_1}(s_3) \int_{s_3}^T \phi_{j_3}(s) \sum_{j_4=p+1}^{\infty} \left(\int_{s_3}^s \phi_{j_4}(s_1) ds_1 \right)^2 ds ds_3, \\
\Delta_6^{(i_1 i_3)} &= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_3, j_1=0}^p f_{j_3 j_1}^p \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)}, \\
f_{j_3 j_1}^p &= \frac{1}{2} \int_t^T \phi_{j_1}(s_3) \int_{s_3}^T \phi_{j_3}(s_2) \sum_{j_4=p+1}^{\infty} \left(\int_{s_2}^T \phi_{j_4}(s_1) ds_1 \right)^2 ds_2 ds_3 \\
&= \frac{1}{2} \int_t^T \phi_{j_3}(s_2) \sum_{j_4=p+1}^{\infty} \left(\int_{s_2}^T \phi_{j_4}(s_1) ds_1 \right)^2 \int_t^{s_2} \phi_{j_1}(s_3) ds_3 ds_2.
\end{aligned}$$

Moreover,

$$\begin{aligned}
A_3^{(i_2 i_3)} + A_5^{(i_2 i_3)} &= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_3, j_2=0}^p (C_{j_4 j_3 j_2 j_4} + C_{j_4 j_3 j_4 j_2}) \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \\
&= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_3, j_2=0}^p \int_t^T \phi_{j_4}(s) \int_t^s \phi_{j_3}(s_1) \int_t^{s_1} \phi_{j_2}(s_2) \int_t^{s_1} \phi_{j_4}(s_3) ds_3 ds_2 ds_1 ds \\
&\quad \cdot \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \\
&= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_3, j_2=0}^p \int_t^T \phi_{j_3}(s_1) \int_t^{s_1} \phi_{j_2}(s_2) \int_t^{s_1} \phi_{j_4}(s_3) ds_3 ds_2 \int_{s_1}^T \phi_{j_4}(s) ds ds_1 \\
&\quad \cdot \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \\
&= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_3, j_2=0}^p \left(\int_t^T \phi_{j_3}(s_1) \int_t^{s_1} \phi_{j_2}(s_2) \int_t^{s_1} \phi_{j_4}(s_3) ds_3 \int_{s_1}^T \phi_{j_4}(s) ds ds_2 ds_1 \right. \\
&\quad \left. - \int_t^T \phi_{j_3}(s_1) \int_t^{s_1} \phi_{j_2}(s_2) \left(\int_{s_1}^T \phi_{j_4}(s) ds \right)^2 ds_2 ds_1 \right) \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \\
&= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_3, j_2=0}^p \int_t^T \phi_{j_3}(s_1) \int_t^{s_1} \phi_{j_2}(s_2) \left((T - s_1) - \sum_{j_4=0}^p \left(\int_{s_1}^T \phi_{j_4}(s_3) ds_3 \right)^2 \right) ds_2 ds_1 \\
&\quad \cdot \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \\
&= 2\Delta_6^{(i_2 i_3)}
\end{aligned}$$

with probability 1. This is why

$$\begin{aligned}
A_3^{(i_2 i_3)} &= 2\Delta_6^{(i_2 i_3)} - A_5^{(i_2 i_3)} \\
&= \Delta_4^{(i_2 i_3)} - \Delta_5^{(i_2 i_3)} + \Delta_6^{(i_2 i_3)}
\end{aligned} \tag{66}$$

with probability 1.

We consider $A_4^{(i_1 i_4)}$:

$$\begin{aligned}
A_4^{(i_1 i_4)} &= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_3, j_1=0}^p \int_t^T \phi_{j_4}(s) \int_t^s \phi_{j_1}(s_3) \int_{s_3}^s \phi_{j_3}(s_2) \int_{s_2}^s \phi_{j_3}(s_1) ds_1 ds_2 ds_3 ds \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} \\
&= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_1=0}^p \frac{1}{2} \int_t^T \phi_{j_4}(s) \int_t^s \phi_{j_1}(s_3) \sum_{j_3=0}^p \left(\int_{s_3}^s \phi_{j_3}(s_2) ds_2 \right)^2 ds_3 ds \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} \\
&= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_1=0}^p \frac{1}{2} \int_t^T \phi_{j_4}(s) \int_t^s \phi_{j_1}(s_3) (s - s_3) ds_3 ds \zeta_{j_1}^{(i_1)} \zeta_{j_4}^{(i_4)} - \Delta_3^{(i_1 i_4)} \\
&= \frac{1}{2} \int_t^T \int_t^s (s - s_3) d\mathbf{w}_{s_3}^{(i_1)} d\mathbf{w}_s^{(i_4)} + \frac{1}{2} \mathbf{1}_{\{i_1=i_4 \neq 0\}} \lim_{p \rightarrow \infty} \sum_{j_4=0}^p \int_t^T \phi_{j_4}(s) \int_t^s \phi_{j_4}(s_3) (s - s_3) ds_3 ds - \Delta_3^{(i_1 i_4)} \\
&= \frac{1}{2} \int_t^T \int_t^{s_2} \int_t^{s_1} d\mathbf{w}_s^{(i_1)} ds_1 d\mathbf{w}_{s_2}^{(i_4)} + \frac{1}{2} \mathbf{1}_{\{i_1=i_4 \neq 0\}} \left(\sum_{j_4=0}^{\infty} \int_t^T (s - t) \phi_{j_4}(s) \int_t^s \phi_{j_4}(s_3) ds_3 ds \right. \\
&\quad \left. - \sum_{j_4=0}^{\infty} \int_t^T \phi_{j_4}(s) \int_t^s (s_3 - t) \phi_{j_4}(s_3) ds_3 ds \right) - \Delta_3^{(i_1 i_4)} = \frac{1}{2} \int_t^T \int_t^{s_2} \int_t^{s_1} d\mathbf{w}_s^{(i_1)} ds_1 d\mathbf{w}_{s_2}^{(i_4)} - \Delta_3^{(i_1 i_4)}
\end{aligned} \tag{67}$$

with probability 1. We consider $A_6^{(i_1 i_2)}$:

$$\begin{aligned}
A_6^{(i_1 i_2)} &= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_3, j_2, j_1=0}^p \int_t^T \phi_{j_1}(s_3) \int_{s_3}^T \phi_{j_2}(s_2) \int_{s_2}^T \phi_{j_3}(s_1) \int_{s_1}^T \phi_{j_3}(s) ds ds_1 ds_2 ds_3 \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \\
&= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_1, j_2=0}^p \frac{1}{2} \int_t^T \phi_{j_1}(s_3) \int_{s_3}^T \phi_{j_2}(s_2) \sum_{j_3=0}^p \left(\int_{s_2}^T \phi_{j_3}(s) ds \right)^2 ds_2 ds_3 \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \\
&= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_1, j_2=0}^p \frac{1}{2} \int_t^T \phi_{j_1}(s_3) \int_{s_3}^T \phi_{j_2}(s_2) (T - s_2) ds_2 ds_3 \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \Delta_6^{(i_1 i_2)} \\
&= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_1, j_2=0}^p \frac{1}{2} \int_t^T \phi_{j_2}(s_2) (T - s_2) \int_t^{s_2} \phi_{j_1}(s_3) ds_3 ds_2 \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} - \Delta_6^{(i_1 i_2)} \\
&= \frac{1}{2} \int_t^T (T - s_2) \int_t^{s_2} d\mathbf{w}_{s_3}^{(i_1)} d\mathbf{w}_{s_2}^{(i_2)} \\
&\quad + \frac{1}{2} \mathbf{1}_{\{i_1=i_2 \neq 0\}} \sum_{j_2=0}^{\infty} \int_t^T \phi_{j_2}(s_2) (T - s_2) \int_t^{s_2} \phi_{j_2}(s_3) ds_3 ds_2 - \Delta_6^{(i_1 i_2)} \\
&= \frac{1}{2} \int_t^T \int_t^{s_1} \int_t^{s_2} d\mathbf{w}_s^{(i_1)} d\mathbf{w}_{s_2}^{(i_2)} ds_1 + \frac{1}{4} \mathbf{1}_{\{i_1=i_2 \neq 0\}} \int_t^T (T - s_2) ds_2 - \Delta_6^{(i_1 i_2)}
\end{aligned} \tag{68}$$

with probability 1.

We proceed to B_1 , B_2 , B_3 . We have:

$$\begin{aligned}
B_1 &= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_1, j_4=0}^p \frac{1}{2} \int_t^T \phi_{j_4}(s) \int_t^s \phi_{j_4}(s_1) \left(\int_t^{s_1} \phi_{j_1}(s_2) ds_2 \right)^2 ds_1 ds \\
&= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4=0}^p \frac{1}{2} \int_t^T \phi_{j_4}(s) \int_t^s \phi_{j_4}(s_1) (s_1 - t) ds_1 ds - \lim_{p \rightarrow \infty} \sum_{j_4=0}^p a_{j_4 j_4}^p \\
&= \frac{1}{4} \int_t^T (s_1 - t) ds_1 - \lim_{p \rightarrow \infty} \sum_{j_4=0}^p a_{j_4 j_4}^p.
\end{aligned} \tag{69}$$

The next formula is

$$\begin{aligned}
B_2 &= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_3=0}^p \left(\frac{1}{2} \int_t^T \phi_{j_3}(s_3) \left(\int_t^{s_3} \phi_{j_4}(s) ds \right)^2 \int_t^{s_3} \phi_{j_3}(s_1) ds_1 ds_3 \right. \\
&\quad - \frac{1}{2} \int_t^T \phi_{j_3}(s_1) \int_t^{s_1} \phi_{j_3}(s_2) \left(\int_t^{s_2} \phi_{j_4}(s_3) ds_3 \right)^2 ds_2 ds_1 \\
&\quad \left. - \frac{1}{2} \int_t^T \phi_{j_3}(s_1) \int_t^{s_1} \phi_{j_3}(s) \left(\int_s^{s_1} \phi_{j_4}(s_2) ds_2 \right)^2 ds ds_1 \right) \\
&= \sum_{j_3=0}^p \frac{1}{2} \int_t^T \phi_{j_3}(s_3) (s_3 - t) \int_t^{s_3} \phi_{j_3}(s_1) ds_1 ds_3 - \lim_{p \rightarrow \infty} \sum_{j_3=0}^p b_{j_3 j_3}^p \\
&\quad - \sum_{j_3=0}^p \frac{1}{2} \int_t^T \phi_{j_3}(s_1) \int_t^{s_1} (s_2 - t) \phi_{j_3}(s_2) ds_2 ds_1 + \lim_{p \rightarrow \infty} \sum_{j_3=0}^p a_{j_3 j_3}^p \\
&\quad - \sum_{j_3=0}^p \frac{1}{2} \int_t^T \phi_{j_3}(s_1) \int_t^{s_1} \phi_{j_3}(s) (s_1 - t + t - s) ds ds_1 + \lim_{p \rightarrow \infty} \sum_{j_3=0}^p c_{j_3 j_3}^p \\
&= \lim_{p \rightarrow \infty} \sum_{j_3=0}^p a_{j_3 j_3}^p + \lim_{p \rightarrow \infty} \sum_{j_3=0}^p c_{j_3 j_3}^p - \lim_{p \rightarrow \infty} \sum_{j_3=0}^p b_{j_3 j_3}^p.
\end{aligned} \tag{70}$$

Moreover,

$$\begin{aligned}
B_2 + B_3 &= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_3=0}^p (C_{j_3 j_4 j_3 j_4} + C_{j_3 j_4 j_4 j_3}) \\
&= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_3=0}^p \int_t^T \phi_{j_3}(s) \int_t^s \phi_{j_4}(s_1) \int_t^{s_1} \phi_{j_4}(s_2) \int_t^{s_1} \phi_{j_3}(s_3) ds_3 ds_2 ds_1 ds \\
&= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_3=0}^p \int_t^T \phi_{j_4}(s_1) \int_t^{s_1} \phi_{j_4}(s_2) \int_t^{s_1} \phi_{j_3}(s_3) ds_3 ds_2 \int_{s_1}^T \phi_{j_3}(s) ds ds_1
\end{aligned}$$

$$\begin{aligned}
&= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_4, j_3=0}^p \left(\int_t^T \phi_{j_4}(s_1) \int_t^{s_1} \phi_{j_4}(s_3) \int_t^T \phi_{j_3}(s_2) ds_2 \int_{s_1}^T \phi_{j_3}(s) ds ds_3 ds_1 \right. \\
&\quad \left. - \int_t^T \phi_{j_4}(s_1) \int_t^{s_1} \phi_{j_4}(s_3) \left(\int_{s_1}^T \phi_{j_3}(s) ds \right)^2 ds_3 ds_1 \right) \\
&= \sum_{j_4=0}^{\infty} \int_t^T \phi_{j_4}(s_1) (T - s_1) \int_t^{s_1} \phi_{j_4}(s_3) ds_3 ds_1 \\
&\quad - \sum_{j_4=0}^{\infty} \int_t^T \phi_{j_4}(s_1) (T - s_1) \int_t^{s_1} \phi_{j_4}(s_3) ds_3 ds_1 + 2 \lim_{p \rightarrow \infty} \sum_{j_4=0}^p f_{j_4 j_4}^p = 2 \lim_{p \rightarrow \infty} \sum_{j_4=0}^p f_{j_4 j_4}^p.
\end{aligned}$$

This is why

$$B_3 = 2 \lim_{p \rightarrow \infty} \sum_{j_3=0}^p f_{j_3 j_3}^p - \lim_{p \rightarrow \infty} \sum_{j_3=0}^p a_{j_3 j_3}^p - \lim_{p \rightarrow \infty} \sum_{j_3=0}^p c_{j_3 j_3}^p + \lim_{p \rightarrow \infty} \sum_{j_3=0}^p b_{j_3 j_3}^p. \quad (71)$$

Substituting relations (62)–(71) into (61), we obtain

$$\begin{aligned}
&\text{l.i.m.}_{p \rightarrow \infty} \sum_{j_1, j_2, j_3, j_4=0}^p C_{j_4 j_3 j_2 j_1} \zeta_{j_1}^{(i_1)} \zeta_{j_2}^{(i_2)} \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} = J[\psi^{(4)}]_{T,t} \\
&\quad + \frac{1}{2} \mathbf{1}_{\{i_1=i_2 \neq 0\}} \int_t^T \int_t^s \int_t^{s_1} ds_2 d\mathbf{w}_{s_1}^{(i_3)} d\mathbf{w}_s^{(i_4)} \\
&\quad + \frac{1}{2} \mathbf{1}_{\{i_2=i_3 \neq 0\}} \int_t^T \int_t^{s_2} \int_t^{s_1} d\mathbf{w}_s^{(i_1)} ds_1 d\mathbf{w}_{s_2}^{(i_4)} + \frac{1}{2} \mathbf{1}_{\{i_3=i_4 \neq 0\}} \int_t^T \int_t^{s_1} \int_t^{s_2} d\mathbf{w}_s^{(i_1)} d\mathbf{w}_{s_2}^{(i_2)} ds_1 \\
&\quad + \frac{1}{4} \mathbf{1}_{\{i_1=i_2 \neq 0\}} \mathbf{1}_{\{i_3=i_4 \neq 0\}} \int_t^T \int_t^{s_1} ds_2 ds_1 + R = J^*[\psi^{(4)}]_{T,t} + R
\end{aligned} \quad (72)$$

with probability 1, where

$$\begin{aligned}
R &= - \mathbf{1}_{\{i_1=i_2 \neq 0\}} \Delta_1^{(i_3 i_4)} + \mathbf{1}_{\{i_1=i_3 \neq 0\}} \left(-\Delta_2^{(i_2 i_4)} + \Delta_1^{(i_2 i_4)} + \Delta_3^{(i_2 i_4)} \right) \\
&\quad + \mathbf{1}_{\{i_1=i_4 \neq 0\}} \left(\Delta_4^{(i_2 i_3)} - \Delta_5^{(i_2 i_3)} + \Delta_6^{(i_2 i_3)} \right) - \mathbf{1}_{\{i_2=i_3 \neq 0\}} \Delta_3^{(i_1 i_4)} \\
&\quad + \mathbf{1}_{\{i_2=i_4 \neq 0\}} \left(-\Delta_4^{(i_1 i_3)} + \Delta_5^{(i_1 i_3)} + \Delta_6^{(i_1 i_3)} \right) - \mathbf{1}_{\{i_3=i_4 \neq 0\}} \Delta_6^{(i_1 i_2)} \\
&\quad + \mathbf{1}_{\{i_1=i_2 \neq 0\}} \mathbf{1}_{\{i_3=i_4 \neq 0\}} \lim_{p \rightarrow \infty} \sum_{j_3=0}^p a_{j_3 j_3}^p \\
&\quad - \mathbf{1}_{\{i_1=i_3 \neq 0\}} \mathbf{1}_{\{i_2=i_4 \neq 0\}} \left(\lim_{p \rightarrow \infty} \sum_{j_3=0}^p a_{j_3 j_3}^p + \lim_{p \rightarrow \infty} \sum_{j_3=0}^p c_{j_3 j_3}^p - \lim_{p \rightarrow \infty} \sum_{j_3=0}^p b_{j_3 j_3}^p \right) \\
&\quad - \mathbf{1}_{\{i_1=i_4 \neq 0\}} \mathbf{1}_{\{i_2=i_3 \neq 0\}} \\
&\quad \cdot \left(2 \lim_{p \rightarrow \infty} \sum_{j_3=0}^p f_{j_3 j_3}^p - \lim_{p \rightarrow \infty} \sum_{j_3=0}^p a_{j_3 j_3}^p - \lim_{p \rightarrow \infty} \sum_{j_3=0}^p c_{j_3 j_3}^p + \lim_{p \rightarrow \infty} \sum_{j_3=0}^p b_{j_3 j_3}^p \right).
\end{aligned} \quad (73)$$

It follows from (72) and (73) that we shall complete the proof of Theorem 4 if

$$\Delta_k^{(ij)} = 0 \quad \text{with probability 1 and} \quad (74)$$

$$\sum_{j_3=0}^p a_{j_3 j_3}^p \rightarrow 0, \quad \sum_{j_3=0}^p b_{j_3 j_3}^p \rightarrow 0, \quad \sum_{j_3=0}^p c_{j_3 j_3}^p \rightarrow 0, \quad \sum_{j_3=0}^p f_{j_3 j_3}^p \rightarrow 0 \quad \text{as } p \rightarrow \infty,$$

where $k = 1, 2, \dots, 6$, $i, j = 0, 1, \dots, m$.

We consider the case of Legendre polynomials. We are going to prove that $\Delta_1^{(i_3 i_4)} = 0$ with probability 1. We have:

$$\begin{aligned} & \mathbb{M} \left(\left(\sum_{j_3, j_4=0}^p a_{j_4 j_3}^p \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \right)^2 \right) \\ &= \sum_{j'_3=0}^p \sum_{j_3=0}^{j'_3-1} \left(2a_{j_3 j_3}^p a_{j'_3 j'_3}^p + \left(a_{j_3 j'_3}^p \right)^2 + 2a_{j_3 j'_3}^p a_{j'_3 j_3}^p + \left(a_{j'_3 j_3}^p \right)^2 \right) + 3 \sum_{j'_3=0}^p \left(a_{j'_3 j'_3}^p \right)^2 \end{aligned} \quad (75)$$

$$= \left(\sum_{j_3=0}^p a_{j_3 j_3}^p \right)^2 + \sum_{j'_3=0}^p \sum_{j_3=0}^{j'_3-1} \left(a_{j_3 j'_3}^p + a_{j'_3 j_3}^p \right)^2 + 2 \sum_{j'_3=0}^p \left(a_{j'_3 j'_3}^p \right)^2, \quad i_3 = i_4 \neq 0,$$

$$\mathbb{M} \left(\left(\sum_{j_3, j_4=0}^p a_{j_4 j_3}^p \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \right)^2 \right) = \sum_{j_3, j_4=0}^p \left(a_{j_4 j_3}^p \right)^2, \quad i_3 \neq i_4, \quad i_3 \neq 0, \quad i_4 \neq 0, \quad (76)$$

$$\mathbb{M} \left(\left(\sum_{j_3, j_4=0}^p a_{j_4 j_3}^p \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \right)^2 \right) = \begin{cases} (T-t) \sum_{j_4=0}^p \left(a_{j_4, 0}^p \right)^2 & \text{as } i_3 = 0, i_4 \neq 0 \\ (T-t) \sum_{j_3=0}^p \left(a_{0, j_3}^p \right)^2 & \text{as } i_4 = 0, i_3 \neq 0 \\ (T-t)^2 \left(a_{00}^p \right)^2 & \text{as } i_3 = i_4 = 0. \end{cases} \quad (77)$$

We consider the case $i_3 = i_4 \neq 0$:

$$\begin{aligned} a_{j_4 j_3}^p &= \frac{(T-t)^2 \sqrt{(2j_4+1)(2j_3+1)}}{32} \int_{-1}^1 P_{j_4}(y) \int_{-1}^y P_{j_3}(y_1) \sum_{j_1=p+1}^{\infty} (2j_1+1) \left(\int_{-1}^{y_1} P_{j_1}(y_2) dy_2 \right)^2 dy_1 dy \\ &= \frac{(T-t)^2 \sqrt{(2j_4+1)(2j_3+1)}}{32} \\ &\quad \cdot \int_{-1}^1 P_{j_3}(y_1) \sum_{j_1=p+1}^{\infty} \frac{1}{2j_1+1} (P_{j_1+1}(y_1) - P_{j_1-1}(y_1))^2 \int_{y_1}^1 P_{j_4}(y) dy dy_1 \\ &= \frac{(T-t)^2 \sqrt{2j_3+1}}{32 \sqrt{2j_4+1}} \int_{-1}^1 P_{j_3}(y_1) (P_{j_4-1}(y_1) - P_{j_4+1}(y_1)) \\ &\quad \cdot \sum_{j_1=p+1}^{\infty} \frac{1}{2j_1+1} (P_{j_1+1}(y_1) - P_{j_1-1}(y_1))^2 dy_1, \quad j_4 \neq 0, \\ a_{j_4 j_3}^p &= \frac{(T-t)^2 \sqrt{2j_3+1}}{32} \cdot \int_{-1}^1 P_{j_3}(y_1) (1-y_1) \sum_{j_1=p+1}^{\infty} \frac{1}{2j_1+1} (P_{j_1+1}(y_1) - P_{j_1-1}(y_1))^2 dy_1, \quad j_4 = 0. \end{aligned}$$

By (28) and the estimate $|P_{j_4-1}(y) - P_{j_4+1}(y)| \leq 2$, $y \in [-1, 1]$ we obtain

$$|a_{j_4 j_3}^p| \leq \frac{C_0}{\sqrt{j_4}} \sum_{j_1=p+1}^{\infty} \frac{1}{j_1^2} I_{\frac{3}{4}}(1) \leq \frac{C_1}{p\sqrt{j_4}}, \quad j_4 \neq 0, \quad (78)$$

$$|a_{0, j_3}^p| \leq C \sum_{j_1=p+1}^{\infty} \frac{1}{j_1^2} I_{\frac{3}{4}}(1) \leq \frac{C_1}{p}, \quad |a_{00}^p| \leq C \sum_{j_1=p+1}^{\infty} \frac{1}{j_1^2} I_{\frac{1}{2}}(1) \leq \frac{C_1}{p}. \quad (79)$$

Taking (75)–(79) into consideration, we write

$$\begin{aligned} \mathbb{M} \left(\left(\sum_{j_3, j_4=0}^p a_{j_4 j_3}^p \zeta_{j_3}^{(i_3)} \zeta_{j_4}^{(i_4)} \right)^2 \right) &= \left(a_{00}^p + \sum_{j_3=1}^p a_{j_3 j_3}^p \right)^2 + \sum_{j'_3=1}^p \left(a_{0, j'_3}^p + a_{j'_3, 0}^p \right)^2 \\ &\quad + \sum_{j'_3=1}^p \sum_{j_3=1}^{j'_3-1} \left(a_{j_3 j'_3}^p + a_{j'_3 j_3}^p \right)^2 + 2 \left(\sum_{j'_3=1}^p \left(a_{j'_3 j'_3}^p \right)^2 + (a_{00}^p)^2 \right) \\ &\leq K_0 \left(\frac{1}{p} + \frac{1}{p} \sum_{j_3=1}^p \frac{1}{\sqrt{j_3}} \right)^2 + \frac{K_1}{p} + K_2 \sum_{j'_3=1}^p \sum_{j_3=1}^{j'_3-1} \frac{1}{p^2} \left(\frac{1}{\sqrt{j'_3}} + \frac{1}{\sqrt{j_3}} \right)^2 \\ &\leq K_0 \left(\frac{1}{p} + \frac{1}{p} \int_0^p \frac{dx}{\sqrt{x}} \right)^2 + \frac{K_1}{p} + \frac{K_3}{p} \sum_{j_3=1}^p \frac{1}{j_3} \\ &\leq K_0 \left(\frac{1}{p} + \frac{2}{\sqrt{p}} \right)^2 + \frac{K_1}{p} + \frac{K_3}{p} \left(1 + \int_1^p \frac{dx}{x} \right) \\ &\leq \frac{K_4}{p} + \frac{K_3 (\ln p + 1)}{p} \rightarrow 0 \quad \text{as } p \rightarrow \infty, \quad i_3 = i_4 \neq 0. \end{aligned}$$

A similar result for the cases (76), (77) is implied by estimates (78), (79). This is why

$$\Delta_1^{(i_3 i_4)} = 0 \quad \text{with probability 1.} \quad (80)$$

It is easy to see that the formulae

$$\Delta_2^{(i_2 i_4)} = 0, \quad \Delta_4^{(i_1 i_3)} = 0, \quad \Delta_6^{(i_1 i_3)} = 0 \quad \text{with probability 1} \quad (81)$$

can be obtained similar to the proof of relation (80). Moreover, by estimates (78), (79) we obtain

$$\lim_{p \rightarrow \infty} \sum_{j_3=0}^p a_{j_3 j_3}^p = 0. \quad (82)$$

Similar to the proof of (82), we find:

$$\lim_{p \rightarrow \infty} \sum_{j_3=0}^p b_{j_3 j_3}^p = 0, \quad \lim_{p \rightarrow \infty} \sum_{j_3=0}^p f_{j_3 j_3}^p = 0. \quad (83)$$

We consider $\Delta_3^{(i_2 i_4)}$:

$$\Delta_3^{(i_2 i_4)} = \Delta_4^{(i_2 i_4)} + \Delta_6^{(i_2 i_4)} - \Delta_7^{(i_2 i_4)} = -\Delta_7^{(i_2 i_4)} \quad (84)$$

with probability 1, where

$$\begin{aligned} \Delta_7^{(i_2 i_4)} &= \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_2, j_4=0}^p g_{j_4 j_2}^p \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)}, \\ g_{j_4 j_2}^p &= \int_t^T \phi_{j_4}(s) \int_t^s \phi_{j_2}(s_1) \sum_{j_1=p+1}^{\infty} \left(\int_{s_1}^T \phi_{j_1}(s_2) ds_2 \int_s^T \phi_{j_1}(s_2) ds_2 \right) ds_1 ds \\ &= \sum_{j_1=p+1}^{\infty} \int_t^T \phi_{j_4}(s) \int_s^T \phi_{j_1}(s_2) ds_2 \int_t^s \phi_{j_2}(s_1) \int_{s_1}^T \phi_{j_1}(s_2) ds_2 ds_1 ds. \end{aligned} \quad (85)$$

Identity (85) is implied by the estimate:

$$|g_{j_4 j_2}^p| \leq K \sum_{j_1=p+1}^{\infty} \frac{1}{j_1^2} \int_{-1}^1 \frac{I_{\frac{1}{2}}(y)}{(1-y^2)^{\frac{1}{2}}} dy \leq \frac{K_1}{p}.$$

We observe that

$$g_{j_4 j_4}^p = \sum_{j_1=p+1}^{\infty} \frac{1}{2} \left(\int_t^T \phi_{j_4}(s) \int_s^T \phi_{j_1}(s_2) ds_2 ds \right)^2, \quad (86)$$

$$g_{j_4 j_2}^p + g_{j_2 j_4}^p = \sum_{j_1=p+1}^{\infty} \int_t^T \phi_{j_4}(s) \int_s^T \phi_{j_1}(s_2) ds_2 ds \int_t^T \phi_{j_2}(s) \int_s^T \phi_{j_1}(s_2) ds_2 ds \quad (87)$$

and moreover, as $j_4, j_2 \leq p$,

$$\begin{aligned} g_{j_4 j_2}^p &= \frac{(T-t)^2 \sqrt{(2j_4+1)(2j_2+1)}}{16} \sum_{j_1=p+1}^{\infty} \frac{1}{2j_1+1} \\ &\quad \cdot \int_{-1}^1 P_{j_4}(y_1) (P_{j_1-1}(y_1) - P_{j_1+1}(y_1)) \int_{-1}^{y_1} P_{j_2}(y) (P_{j_1-1}(y) - P_{j_1+1}(y)) dy dy_1. \end{aligned}$$

By the orthonormality of the Legendre polynomials we obtain

$$\begin{aligned} g_{j_4 j_2}^p + g_{j_2 j_4}^p &= \frac{(T-t)^2 \sqrt{(2j_4+1)(2j_2+1)}}{16} \sum_{j_1=p+1}^{\infty} \frac{1}{2j_1+1} \\ &\quad \cdot \int_{-1}^1 P_{j_4}(y_1) (P_{j_1-1}(y_1) - P_{j_1+1}(y_1)) dy_1 \int_{-1}^1 P_{j_2}(y) (P_{j_1-1}(y) - P_{j_1+1}(y)) dy \\ &= \frac{(T-t)^2 (2p+1)}{16} \frac{1}{2p+3} \left(\int_{-1}^1 P_p^2(y_1) dy_1 \right)^2 \cdot \begin{cases} 1 & \text{as } j_2 = j_4 = p \\ 0 & \text{otherwise} \end{cases} \\ &= \frac{(T-t)^2}{4(2p+3)(2p+1)} \cdot \begin{cases} 1 & \text{as } j_2 = j_4 = p \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \quad (88)$$

$$g_{j_4 j_4}^p = \frac{1}{2} (g_{j_4 j_2}^p + g_{j_2 j_4}^p) \Big|_{j_2=j_4} = \frac{(T-t)^2}{8(2p+3)(2p+1)} \cdot \begin{cases} 1 & \text{as } j_4 = p, \\ 0 & \text{otherwise.} \end{cases} \quad (89)$$

By (75), (88) and (89) we obtain

$$\begin{aligned} \mathbb{M}\left(\left(\sum_{j_2, j_4=0}^p g_{j_4 j_2}^p \zeta_{j_2}^{(i_2)} \zeta_{j_4}^{(i_4)}\right)^2\right) &= \left(\sum_{j_3=0}^p g_{j_3 j_3}^p\right)^2 + \sum_{j'_3=0}^p \sum_{j_3=0}^{j'_3-1} \left(g_{j_3 j'_3}^p + g_{j'_3 j_3}^p\right)^2 + 2 \sum_{j'_3=0}^p \left(g_{j'_3 j'_3}^p\right)^2 \\ &\leq \left(\frac{(T-t)^2}{8(2p+3)(2p+1)}\right)^2 + 0 + 2 \left(\frac{(T-t)^2}{8(2p+3)(2p+1)}\right)^2 \rightarrow 0 \end{aligned}$$

as $p \rightarrow \infty$, $i_2 = i_4 \neq 0$.

We proceed to the case $i_2 \neq i_4$, $i_2 \neq 0$, $i_4 \neq 0$. It is easy to see that

$$g_{j_4 j_2}^p = \int_t^T \phi_{j_4}(s) \int_t^s \phi_{j_2}(s_1) F^p(s, s_1) ds_1 ds = \int_{[t, T]^2} K_p(s, s_1) \phi_{j_4}(s) \phi_{j_2}(s_1) ds_1 ds$$

is the Fourier coefficients in the double Fourier-Legendre series of the function

$$K_p(s, s_1) = \mathbf{1}_{\{s_1 < s\}} F^p(s, s_1), \quad (90)$$

where

$$F^{p, n}(s, s_1) \stackrel{\text{def}}{=} \sum_{j_1=p+1}^n \int_{s_1}^T \phi_{j_1}(s_2) ds_2 \int_s^T \phi_{j_1}(s_2) ds_2, \quad F^{p, \infty}(s, s_1) \stackrel{\text{def}}{=} F^p(s, s_1).$$

In this case, the Parseval identity reads as

$$\lim_{p_1 \rightarrow \infty} \sum_{j_4, j_2=0}^{p_1} (g_{j_4 j_2}^p)^2 = \int_{[t, T]^2} (K_p(s, s_1))^2 ds_1 ds = \int_t^T \int_t^s (F^p(s, s_1))^2 ds_1 ds. \quad (91)$$

By (28) we obtain:

$$\begin{aligned} \left| \int_{s_1}^T \phi_{j_1}(\theta) d\theta \right| &= \frac{1}{2} \sqrt{2j_1 + 1} \sqrt{T-t} \left| \int_{z(s_1)}^1 P_{j_1}(y) dy \right| \\ &= \frac{\sqrt{T-t}}{2\sqrt{2j_1+1}} |P_{j_1-1}(z(s_1)) - P_{j_1+1}(z(s_1))| \leq \frac{K}{j_1} f_{\frac{1}{4}}(s_1), \quad s_1 \in (t, T). \end{aligned} \quad (92)$$

Employing (30) and (92), we find:

$$(F^p(s, s_1))^2 \leq \frac{C}{p^2} f_{\frac{1}{2}}(s) f_{\frac{1}{2}}(s_1), \quad s, s_1 \in (t, T). \quad (93)$$

It follows from (93) that $|F^p(s, s_1)| \leq K/p$ in the domain

$$D_\varepsilon = \{(s, s_1) : s \in [t + \varepsilon, T - \varepsilon], s_1 \in [t + \varepsilon, s]\},$$

where $\varepsilon > 0$ is a sufficiently small fixed number. Then the uniform convergence holds:

$$F^{-1, p}(s, s_1) \rightarrow F^{-1}(s, s_1) \quad (94)$$

on the domain D_ε as $p \rightarrow \infty$. By the continuity of the left hand side in (94) we get the continuity of the limiting function in the right hand side in (94) on the set D_ε . Employing this fact and

(93), we obtain:

$$\begin{aligned}
\int_t^T \int_t^s (F^p(s, s_1))^2 ds_1 ds &= \lim_{\varepsilon \rightarrow +0} \int_{t+\varepsilon}^{T-\varepsilon} \int_{t+\varepsilon}^s (F^p(s, s_1))^2 ds_1 ds \\
&\leq \frac{C}{p^2} \lim_{\varepsilon \rightarrow +0} \int_{t+\varepsilon}^{T-\varepsilon} f_{\frac{1}{2}}(s) \int_{t+\varepsilon}^s f_{\frac{1}{2}}(s_1) ds_1 ds = \frac{C}{p^2} \int_t^T f_{\frac{1}{2}}(s) \int_t^s f_{\frac{1}{2}}(s_1) ds_1 ds \\
&= \frac{C(T-t)^2}{4p^2} \int_{-1}^1 \frac{I_{\frac{1}{2}}(y)}{(1-y^2)^{\frac{1}{2}}} dy < \frac{K_1}{p^2}.
\end{aligned} \tag{95}$$

By (95) and (91) we get:

$$0 \leq \sum_{j_2, j_4=0}^p (g_{j_4 j_2}^p)^2 \leq \lim_{p_1 \rightarrow \infty} \sum_{j_2, j_4=0}^{p_1} (g_{j_4 j_2}^p)^2 = \sum_{j_2, j_4=0}^{\infty} (g_{j_4 j_2}^p)^2 \leq \frac{K_1}{p^2} \rightarrow 0 \tag{96}$$

as $p \rightarrow \infty$. This completes the case $i_2 \neq i_4$, $i_2 \neq 0$, $i_4 \neq 0$.

It is easy to obtain a similar result for the cases $i_2 = 0$, $i_4 \neq 0$; $i_4 = 0$, $i_2 \neq 0$; and $i_2 = 0$, $i_4 = 0$. Then $\Delta_7^{(i_2 i_4)} = 0$ and $\Delta_3^{(i_2 i_4)} = 0$ with probability 1.

We consider $\Delta_5^{(i_1 i_3)}$:

$$\Delta_5^{(i_1 i_3)} = \Delta_4^{(i_1 i_3)} + \Delta_6^{(i_1 i_3)} - \Delta_8^{(i_1 i_3)} \quad \text{with probability 1,}$$

where

$$\Delta_8^{(i_1 i_3)} = \text{l.i.m.}_{p \rightarrow \infty} \sum_{j_3, j_1=0}^p h_{j_3 j_1}^p \zeta_{j_1}^{(i_1)} \zeta_{j_3}^{(i_3)}, \quad h_{j_3 j_1}^p = \int_t^T \phi_{j_1}(s_3) \int_{s_3}^T \phi_{j_3}(s) F^p(s_3, s) ds ds_3.$$

Similar to the above arguing, we obtain that $\Delta_8^{(i_1 i_3)} = 0$ with probability 1. Here we employ the function $K_p(s, s_3) = \mathbf{1}_{\{s_3 < s\}} F^p(s_3, s)$ and the relation

$$h_{j_3 j_1}^p = \int_{[t, T]^2} K_p(s, s_3) \phi_{j_1}(s_3) \phi_{j_3}(s) ds ds_3, \quad i_1 \neq i_3, \quad i_1 \neq 0, \quad i_3 \neq 0.$$

In the case $i_1 = i_3 \neq 0$, for $h_{j_1 j_1}^p$ and $h_{j_3 j_1}^p + h_{j_1 j_3}^p$, we employ the right hand sides of the formulae (86) and (87), respectively, in which we replace j_1 by j_4 and j_2 by j_3 , respectively.

Let us show that

$$\lim_{p \rightarrow \infty} \sum_{j_3=0}^p c_{j_3 j_3}^p = 0. \tag{97}$$

We have:

$$c_{j_3 j_3}^p = f_{j_3 j_3}^p + d_{j_3 j_3}^p - g_{j_3 j_3}^p. \tag{98}$$

Similar to the second identity in (83) we obtain

$$\lim_{p \rightarrow \infty} \sum_{j_3=0}^p d_{j_3 j_3}^p = 0.$$

It follows from (89) that

$$0 \leq \lim_{p \rightarrow \infty} \sum_{j_3=0}^p g_{j_3 j_3}^p \leq \lim_{p \rightarrow \infty} \frac{(T-t)^2}{8(2p+3)(2p+1)} = 0,$$

that is, identity (97) holds. This completes the proof of identities (74) for the case of Legendre polynomials.

We consider a trigonometric case. According to (63),

$$a_{j_4 j_3}^p = \frac{1}{2} \int_t^T \phi_{j_3}(s_1) \sum_{j_1=p+1}^{\infty} \left(\int_t^{s_1} \phi_{j_1}(s_2) ds_2 \right)^2 \int_{s_1}^T \phi_{j_4}(s) ds ds_1.$$

Moreover,

$$\begin{aligned} \left| \int_t^{s_1} \phi_j(s_2) ds_2 \right| &\leq \frac{K_0}{j}, \quad j \neq 0, & \int_{s_1}^T \phi_0(s) ds &= \frac{T-s_1}{\sqrt{T-t}}, \\ |a_{j_4 j_3}^p| &\leq \frac{K_1}{j_4} \sum_{j_1=p+1}^{\infty} \frac{1}{j_1^2} \leq \frac{K_1}{p j_4} \quad (j_4 \neq 0), & |a_{0, j_3}^p| &\leq \frac{K_1}{p}. \end{aligned} \quad (99)$$

By (75)–(77) and (99) we obtain that $\Delta_1^{(i_3 i_4)} = 0$ with probability 1. In the same way, $\Delta_2^{(i_2 i_4)} = 0$, $\Delta_4^{(i_1 i_3)} = 0$, $\Delta_6^{(i_1 i_3)} = 0$ with probability 1 and

$$\lim_{p \rightarrow \infty} \sum_{j_3=0}^p a_{j_3 j_3}^p = 0, \quad \lim_{p \rightarrow \infty} \sum_{j_3=0}^p b_{j_3 j_3}^p = 0, \quad \lim_{p \rightarrow \infty} \sum_{j_3=0}^p f_{j_3 j_3}^p = 0.$$

We consider $\Delta_3^{(i_2 i_4)}$. In this case, as $i_2 = i_4 \neq 0$, we employ (84)–(87) to get:

$$\int_t^T \phi_{j_4}(s) \int_s^T \phi_{j_1}(s_2) ds_2 ds = \frac{\sqrt{2}\sqrt{T-t}}{2\pi j_1} \int_t^T \begin{cases} \phi_{j_4}(s) \left(1 - \cos \frac{2\pi j_1(s-t)}{T-t} \right) ds, \\ \phi_{j_4}(s) \left(-\sin \frac{2\pi j_1(s-t)}{T-t} \right) ds, \end{cases}$$

where $j_1 \geq p+1$, $j_4 = 0, 1, \dots, p$. By the orthonormality of the trigonometric functions we obtain:

$$\int_t^T \phi_{j_4}(s) \int_s^T \phi_{j_1}(s_2) ds_2 ds = \frac{\sqrt{2}(T-t)}{2\pi j_1} \begin{cases} 1 \text{ or } 0 & \text{as } j_4 = 0 \\ 0 & \text{otherwise,} \end{cases} \quad j_1 \geq p+1. \quad (100)$$

It follows from (100) and (85)–(87) that

$$g_{j_4 j_2}^p + g_{j_2 j_4}^p = \sum_{j_1=p+1}^{\infty} \frac{(T-t)^2}{2\pi^2 j_1^2} \begin{cases} 1 \text{ or } 0 & \text{as } j_2 = j_4 = 0 \\ 0 & \text{otherwise} \end{cases} \leq \frac{K_1}{p}, \quad (101)$$

$$g_{j_4 j_4}^p = \sum_{j_1=p+1}^{\infty} \frac{(T-t)^2}{4\pi^2 j_1^2} \begin{cases} 1 \text{ or } 0 & \text{as } j_4 = 0 \\ 0 & \text{otherwise} \end{cases} \leq \frac{K_1}{p}. \quad (102)$$

By (101), (102) and (75) we obtain $\Delta_7^{(i_2 i_4)} = 0$ and $\Delta_3^{(i_2 i_4)} = 0$ with probability 1 as $i_2 = i_4 \neq 0$. Similar to the polynomial case, $\Delta_7^{(i_2 i_4)} = 0$ and $\Delta_3^{(i_2 i_4)} = 0$ with probability 1 as $i_2 \neq i_4$, $i_2 \neq 0$, $i_4 \neq 0$. The same arguing shows that $\Delta_5^{(i_1 i_3)} = 0$ with probability 1.

Taking into considerations (98) and the relations

$$\lim_{p \rightarrow \infty} \sum_{j_3=0}^p f_{j_3 j_3}^p = \lim_{p \rightarrow \infty} \sum_{j_3=0}^p d_{j_3 j_3}^p = 0$$

implied by the estimates

$$|f_{jj}^p| + |d_{jj}^p| \leq \frac{K_1}{pj}, \quad |f_{00}^p| + |d_{00}^p| \leq \frac{K_1}{p},$$

we obtain

$$\lim_{p \rightarrow \infty} \sum_{j_3=0}^p c_{j_3 j_3}^p = - \lim_{p \rightarrow \infty} \sum_{j_3=0}^p g_{j_3 j_3}^p, \quad 0 \leq \lim_{p \rightarrow \infty} \sum_{j_3=0}^p g_{j_3 j_3}^p \leq \lim_{p \rightarrow \infty} \frac{K_1}{p} = 0.$$

Thus, we arrive to (97) in the trigonometric case. This completes the proof of relations (74) in the trigonometric case and the proof of Theorem 4.

6. CONCLUSION

The results obtained in Theorems 2–4 can be applied to the realization of strong [2], [4] numerical methods of convergence order 1.0 (Milstein method [3]), of convergence orders 1.5 and 2.0 for Itô SDE of form (1) (case of a multi-dimensional Wiener process and a function $B(\mathbf{x}, t)$ depending not only on t , but also on \mathbf{x}) based on Taylor–Stratonovich expansions [2], [6], [8]. It should be noted that the set of Stratonovich ISIs of multiplicities 1–4 of form (3) employed in constructing of the mentioned numerical methods is universal for both explicit one-step numerical methods and for implicit multi-step and finite-difference (of Runge-Kutta type) modifications.

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