EXPECTATION INEQUALITIES ASSOCIATED WITH PROPHET PROBLEMS

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ABSTRACT

Applications of the original prophet inequalities of Krengel and Sucheston are made to problems of order selection, non-measurable stop rules, look-ahead stop rules, and iterated maps of random variables. Also, proofs are given of two results of Hill and Hordijk concerning optimal orderings of uniform and exponential distributions.

§1. INTRODUCTION

Universal inequalities comparing the two functionals

$$M = M(X_1, X_2, \ldots) = E(\sup_n X_n)$$

and

$$V = V(X_1, X_2, \ldots) = \sup\{EX_t\colon t \text{ is a stop rule for } X_1, X_2, \ldots\}$$

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of sequences of random variables are called "prophet inequalities" because of the natural interpretation of M as the value to a prophet, or player with complete foresight, in an optimal stopping problem involving random variables $X_1, X_2, \ldots$. First discovered by Krengel and Sucheston [22, 23], these inequalities have been the subject of a number of recent investigations (e.g., [1, 2, 4-15, 17-21, 24-27]).

In §2, the applications of prophet inequalities to inequalities involving functionals other than M or V are given, with attention focused on the fundamental prophet inequality [23]

(1) If $X_1, X_2, \ldots$ are independent and nonnegative, then $M \le 2V$, and this bound is sharp.

(Analogous applications of other prophet inequalities to similar problems are left to the reader.)

Section 3 contains proofs of two optimal-ordering results of Hill and Hordijk [11].

§2. APPLICATIONS OF PROPHET INEQUALITIES

The initial discovery and application of prophet inequalities such as (1) were made by Krengel and Sucheston in conjunction with investigations of semi-amarts and processes with finite value [22, 23]. In this section, other applications of the basic inequality (1) are given to several optimal-stopping problems and an iterated map problem.

For the main application theorem, which follows immediately from (1), let

$$U = U(X_1, X_2, \ldots)$$

be any (real-valued) functional of $X_1, X_2, \ldots$. (More formally, U is a function from C, the set of infinite sequences of probability distributions, to the real numbers. In practice, U is usually Borel measurable, with C endowed with the product topology induced by the total-variation norm topology on the space of probability distributions.)

<u>Theorem 2.1</u>. Let $X_1, X_2, \ldots$ be independent nonnegative random variables. Then

(i) $V \le U$ implies $U \le 2V$; and

(ii) $U \le M$ implies $M \le 2U$.

<u>Proof</u>. Immediate from (1). □

<u>Application to Order Selection</u>

Let U_S be the value of the sequence $X_1, X_2, \ldots$ to a player free to choose the order of observation of the random variables, as well as the time of stopping, that is,

$$U_S = U_S(X_1, X_2, \ldots) = \sup\{V(X_{\pi(1)}, X_{\pi(2)}, \ldots): \pi \text{ is a permutation of } 1,2,\ldots\}.$$

(For a formal definition, including stochastic permutaions π, see [9].)

<u>Corollary 2.2</u>. Let $X_1, X_2, \ldots$ be independent nonnegative random variables. Then

(i) (Hill [9]) $U_S \le 2V$; and

(ii) $M \le 2U_S$.

Moreover, the bound in (i) is sharp.

(Whether or not the constant "2" in (ii) is a sharp bound is not known to the author.) Inequality (i) says that a player may never do better than double his expected value by rearranging the order of a given sequence of random variables. Inequality (ii) is immediate from (1) and the fact that $U_S \ge V$; only the question of its sharpness is of interest.

Application to Use of Non-Measurable Stop Rules

Let U_N be the value of the sequence $X_1, X_2, \ldots$ to a player free to use non-measurable stop rules, i.e., integer-valued functions s for which $\{s = j\}$ can be any (not necessarily measurable) function of $X_1, \ldots, X_j$. That is, U_N is the functional

$$U_N = U_N(X_1, X_2, \ldots)$$
$$= \sup_s \{EX_s : s \text{ is a "non-measurable" stop rule}\}.$$

(For a formal definition, see [16].)

Corollary 2.3. Let $X_1, X_2, \ldots$ be independent nonnegative random variables. Then

(i) (Hill and Pestien [16]) $U_N \le 2V$; and

(ii) $M \le 2U_N$.

Moreover both bounds are sharp.

Proof. The inequalities follow immediately from (1); the sharpness of (i) is in [16]. To see that the bound in (ii) is sharp, let X_1 be constant +1, and let X_2 be a "long shot" [12] given by $P(X_2 = \varepsilon^{-1}) = \varepsilon = 1 - P(X_2 = 0)$. Then $M = 2-\varepsilon$, and $U_N = U_S = 1$. □

Application to "Look-Ahead" Stop Rules

Let $U_{A,k}$ be the value of the sequence $X_1, X_2, \ldots$ to a player free to use stop rules s which allow looking ahead k steps (i.e., integer-valued measurable functions satisfying $\{s = j\} \in \sigma(X_1, \ldots, X_{j+k})$, so

$$U_{A,k} = U_{A,k}(X_1, X_2, \ldots)$$
$$= \sup\{EX_s : s \text{ is a k-step "look-ahead" stop rule}\}.$$

Corollary 2.4. Let $X_1, X_2, \ldots$ be independent nonnegative random variables, and let k be a positive integer. Then

(i) $U_{A,k} \le 2V$; and

(ii) $M \le 2U_{A,k}$.

Moreover, both bounds are sharp.

Proof. The inequalities follow immediately from (1). To see that (i) is sharp, let X_1 = constant +1, $X_2 = \ldots = X_{k+1}$ = constant 0, and let X_{k+2} be a "long shot" with $P(X_{k+2} = \varepsilon^{-1}) = \varepsilon = 1 - P(X_{k+2} = 0)$; then $U_{A,k} = 2-\varepsilon$ and $V = 1$. To see that (i) is sharp, let $X_1 \equiv +1$, $X_2 \equiv \ldots \equiv X_{k+2} = 0$, and let X_{k+3} be the "long shot" random variable just described; then $M = 2-\varepsilon$ and $U_{A,k} = 1$. ☐

Thus (i) says that a player able to look k steps into the future never has optimal expected return more than twice that of a player who cannot look ahead, and (ii) says that a prophet's optimal expected return is never more than twice that of a player who may look a *fixed* number of steps into the future. On the other hand, for a fixed sequence of random variables, it is clear that

$$\lim_{k\to\infty} U_{A,k}(X_1,X_2,\ldots) = M(X_1,X_2,\ldots).$$

Application to Iterated Maps

Let $\phi(X,Y)$ and $\psi(X,Y)$ be the random variables $\phi(X,Y) = \max\{X,Y\}$ and $\psi(X,Y) = \max\{X,EY\}$, and define the random variables $\phi_n(X_n,\ldots,X_1)$ and $\psi_n(X_n,\ldots,X_1)$ inductively by

$$\phi_2(X_2,X_1) = \phi(X_2,X_1), \quad \text{and}$$
$$\phi_k(X_k,\ldots,X_1) = \phi(X_k,\phi_{k-1}(X_{k-1},\ldots,X_1))$$

and

$$\psi_2(X_2,X_1) = \psi(X_2,X_1), \quad \text{and}$$
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Application to Use of Non-Measurable Stop Rules

Let U_N be the value of the sequence $X_1, X_2, \ldots$ to a player free to use non-measurable stop rules, i.e., integer-valued functions s for which $\{s = j\}$ can be any (not necessarily measurable) function of $X_1, \ldots, X_j$. That is, U_N is the functional

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$$\lim_{k\to\infty} U_{A,k}(X_1,X_2,\dots) = M(X_1,X_2,\dots).$$

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Let $\phi(X,Y)$ and $\psi(X,Y)$ be the random variables $\phi(X,Y) = \max\{X,Y\}$ and $\psi(X,Y) = \max\{X,EY\}$, and define the random variables $\phi_n(X_n,\dots,X_1)$ and $\psi_n(X_n,\dots,X_1)$ inductively by

$$\phi_2(X_2,X_1) = \phi(X_2,X_1), \quad \text{and}$$
$$\phi_k(X_k,\dots,X_1) = \phi(X_k,\phi_{k-1}(X_{k-1},\dots,X_1)$$

and

$$\psi_2(X_2,X_1) = \psi(X_2,X_1), \quad \text{and}$$
$$\psi_k(X_k,\dots,X_1) = \psi(X_k,\psi_{k-1}(X_{k-1},\dots,X_1).$$

Then $E(\max\{X_1,\ldots,X_n\}) = E(\phi_n(X_n,\ldots,X_1))$, and $V(X_1,\ldots,X_n) = E(\psi_n(X_n,\ldots,X_1))$, so the finite version of (1) may be restated as

$$(2) \qquad E[\phi_n(X_n,\ldots,X_1)] \le 2E[\psi_n(X_n,\ldots,X_1)].$$

<u>Corollary 2.5</u>. Let $X_1,X_2,\ldots$ be independent nonnegative random variables, let $g(X,Y)$ be such that $g \ge \psi$, and $g(X,Y) \ge g(X,\hat{Y})$ if $Y \ge \hat{Y}$ a.e. Define $g_n(X_n,\ldots,X_1)$ inductively by $g_2(X_2,X_1) = g(X_2,X_1)$ and $g_k(X_k,\ldots,X_1) = g(X_k,g_{k-1}(X_{k-1},\ldots,X_1))$. Then

$$(3) \qquad E(\max\{X_1,\ldots,X_n\}) \le 2E[g_n(X_n,\ldots,X_1)].$$

<u>Proof</u>. Follows easily by (2) and induction. □

The iterated maps g_n need not closely resemble ordinary stopping theory functions, for example consider $g(X,Y) = \max\{X, \|Y\|_p\}$ for $p > 1$, or $g(X,Y) = (\max\{X,Y\} + \max\{X,EY\})/2$. Inequality (3) corresponds to the inequality $M \le 2U$ in Theorem 2.1; the analog of (3) corresponding to $U \le 2V$ is also possible under similar hypotheses.

§3. PROOFS OF TWO RESULTS IN ORDER SELECTION

The purpose of this section is to give proofs of two results, both concerning optimal stopping with order selection, which appear in [11] without proof.

<u>Theorem 3.1</u> (4.6(ii) of [11]). Let $\alpha_1,\alpha_2,\ldots$ be a sequence of non-increasing positive numbers, and let $X_1,X_2,\ldots$ be independent random variables with distributions uniform on $[0,\alpha_1],[0,\alpha_2],\ldots$ respectively. Then

$$V(X_1,X_2,\ldots) = \sup\{V(X_{\pi(1)},X_{\pi(2)},\ldots): \pi \text{ is a permutation of } \mathbb{N}\}.$$

Proof (due to Hordijk and Hill). The proof will be an application of Proposition 4.5 of [11]. By renormalizing, it suffices to show

(4) $$V(X_1, X_\alpha, c) \geq V(X_\alpha, X_1, c)$$
for all $\alpha \in (0,1)$ and all $c \in \mathbb{R}$.

For a random variable T with values in {1,2,3}, let $R_T(X,Y,c) = X$ if $T = 1$; $= Y$ if $T = 2$; and $= c$ if $T = 3$. Also, let $T(X,Y,c) = 1$ if $X > E(\max\{Y,c\})$; $= 2$ if $X \leq E(\max\{Y,c\})$ and $Y > c$; and $= 3$ otherwise. Letting X and Y be i.i.d. $U[0,1]$, by Lemma 2.1 of [3] it follows that (4) is equivalent to

(5) $$E[R_{T(X,\alpha Y,c)}(X,\alpha Y,c) - R_{T(\alpha Y,X,c)}(\alpha Y,X,c)] \geq 0.$$

To see (5), first observe that

(6) $$E[R_{T(X,\alpha Y,c)}(X,\alpha Y,c) \mid X \in [0,\alpha]] = E[R_{T(\alpha X,\alpha Y,c)}(\alpha X,\alpha Y,c)],$$

since the distribution of X given $X \in [0,\alpha]$ is uniform on $[0,\alpha]$, that is, has the same distribution as αX.

Next calculate

(7) $$E[R_{T(\alpha Y,X,c)}(\alpha Y,X,c) \mid X \in [0,\alpha]] = E[R_{T(\alpha Y,X,c)}(\alpha Y,\alpha X,c)] \leq E[R_{T(\alpha Y,\alpha X,c)}(\alpha Y,\alpha X,c)],$$

where the first equality follows as in (6), and the inequality since $T(\alpha Y,\alpha X,c)$ is the optimal stop rule (by Lemma 2.1 of [3]) for $(\alpha Y,\alpha X,c)$. Together (6) and (7) imply

(8) $$E[R_{T(X,\alpha Y,c)}(X,\alpha Y,c) - R_{T(\alpha Y,X,c)}(\alpha Y,X,c) \mid X \in [0,\alpha]] \geq 0 \text{ a.s.}$$

Similarly, conditioning on $X \in (\alpha,1]$ and using the fact that given $X \in (\alpha,1]$, the conditional distribution of X is *uniform* on $(\alpha,1]$, one has the following two relations:

(9) $$E[R_{T(X,\alpha Y,c)}(X,\alpha Y,c)\,|\,X_1 \in (\alpha,1]] = E[\max\{Z,c\}],$$

and

(10) $$\begin{aligned} E[R_{T(\alpha Y,X,c)}(\alpha Y,X,c)\,|\,X \in (\alpha,1]] &\\ &= E[R_{T(\alpha Y,X,c)}(\alpha Y,Z,c)] \\ &\le E[R_{T(\alpha Y,Z,c)}(\alpha Y,Z,c)] \\ &= E[\max\{Z,c\}], \end{aligned}$$

where Z is uniform $(\alpha,1]$ (and independent of Y,X).

From (9) and (10) follows the inequality corresponding to (8) given that $X \in (\alpha,1]$, which together with (8) yields (5) and completes the proof. □

Theorem 3.2 (4.6(iii) of [11]). Let $\alpha_1,\alpha_2,\ldots$ be a sequence of non-increasing positive numbers, and let $X_1,X_2,\ldots$ be independent exponentially distributed random variables with means $\alpha_1,\alpha_2,\ldots$ respectively. Then

$$V(X_1,X_2,\ldots) = \sup\{V(X_{\pi(1)},X_{\pi(2)},\ldots : \pi \text{ is a permutation of } \mathbb{N}\}.$$

Proof (due to Chris Klaassen). By Proposition 4.5 of [11] and renormalizing, it suffices to show

(11) $$xe^{-\frac{c}{x}-\frac{e^{-c}}{x}} + e^{-c} \ge e^{-xe^{-\frac{c}{x}}-c} + xe^{-\frac{c}{x}}$$

for all $x \ge 1$ and all $c \ge 0$.

Substituting $y = e^{-c}$ and $\alpha = 1/x$, it suffices to show

(12) $$\psi_\alpha(y) = -(1-e^{-\alpha y}) + \alpha y^{1-\alpha}(1-e^{-\frac{y^\alpha}{\alpha}}) \ge 0$$

for all $\alpha \in [0,1]$ and all $y \in [0,1]$.

Since $\psi_0(y) \equiv \psi_1(y) \equiv 0$, fix $\alpha \in (0,1)$. Let $F(y) = 1-e^{-\alpha y}$ and $G(y) = 1-e^{-(y^\alpha/\alpha)}$. Then

$$\psi_\alpha(y) = -F(y) + \alpha y^{1-\alpha}G(y); \quad \text{and} \tag{13}$$

$$\psi'_\alpha(y) = \alpha\{F(y) + [(1-\alpha)y^{-\alpha}-1]G(y)\}. \tag{14}$$

Since F and G are non-negative on $[0,1]$, and $(1-\alpha)y^{-\alpha} - 1 \geq 0$ for $y \in I_1 = (0,(1-\alpha)^{1/\alpha}]$, from (14) it follows that $\psi'_\alpha(y) \geq 0$ for $y \in I_1$. Since $\psi_\alpha(0) = 0$ this implies $\psi_\alpha(y) \geq 0$ for all $y \in I_1$.

If $\psi_\alpha(\hat{y}) \leq 0$ for some $\hat{y} \in I_2 = ((1-\alpha)^{1/\alpha},1]$, then (13) would imply

$$G(\hat{y}) \leq \frac{1}{\alpha}\hat{y}^{\alpha-1}F(\hat{y}). \tag{15}$$

Since $(1-\alpha)\hat{y}^{-\alpha}-1 < 0$ on I_2, it follows from (14) and (15) with $\chi(\hat{y}) = \alpha + (1-\alpha)\hat{y}^{-1} - \hat{y}^{\alpha-1}$ that

$$\psi'_\alpha(\hat{y}) \geq F(\hat{y})\chi(\hat{y}). \tag{16}$$

Since $\chi(1) = 0$ and $\chi'(\hat{y}) = (1-\alpha)\hat{y}^{-2}(\hat{y}^\alpha-1) \leq 0$, $\psi_\alpha(\hat{y}) \leq 0$ implies $\psi'_\alpha(\hat{y}) \geq 0$ for $\hat{y} \in I_2$. But the continuity of ψ_α then implies $\psi_\alpha \geq 0$ for all $y \in I_2$ also, which establishes (12), completing the proof. □

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REFERENCES

[1] A. Brunel and U. Krengel. Parier avec un prophete dans le cas d'un processus sous-additif, C. R. Acad. Sci., Series A 288 (1979) 57-60.

[2] A. Brunel and U. Krengel. Betting with a prophet, preprint (1985).

[3] Y. Chow, H. Robbins, and D. Siegmund. Great Expectations: The Theory of Optimal Stopping, Houghton Mifflin, Boston (1971).

[4] D. C. Cox and R. Kertz. Prophet regions and sharp inequalities for p^{th} absolute moments of martingales, J. Multivariate Anal. 18 (1986) 242-273.

[5] D. Darling, T. Liggett and H. Taylor, Optimal stopping for partial sums, Ann. Math. Stat. 43 (1972) 1363-1368.

[6] J. Elton and R. Kertz. Comparison of stop rule and maximum expectations for finite sequences of exchangeable random variables, preprint (1985).

[7] D. Gilat. A prophet inequality with order selection for two independent random variables, preprint (1985).

[8] D. Gilat. On the best order of observation in optimal stopping problems, J. Appl. Prob., to appear.

[9] T. Hill. Prophet inequalities and order selection in optimal stopping problems, Proc. Amer. Math. Soc. 88 (1983) 131-137.

[10] T. Hill. Prophet inequalities for averages of independent non-negative random variables, Math. Z. 192 (1986) 427-436.

[11] T. Hill and A. Hordijk. Selection of order of observation in optimal stopping problems, J. Appl. Prob. 22 (1985) 177-184.

[12] T. Hill and R. Kertz. Ratio comparisons of supremum and stop rule expectations, Z. Wahrsch. Verw. Gebiete 56 (1981) 283-285.

[13] T. Hill and R. Kertz. Additive comparisons of stop rule and supremum expectations of uniformly bounded independent random variables, Proc. Amer. Math. Soc. 83 (1981) 582-585.

[14] T. Hill and R. Kertz. Comparisons of stop rule and supremum expectations of i.i.d. random variables, Ann. Prob. 10 (1982) 336-345.

[15] T. Hill and R. Kertz. Stop rule inequalities for uniformly bounded sequences of random variables, Trans. Amer. Math. Soc. 278 (1983) 197-207.

[16] T. Hill and V. Pestien. The advantage of using non-measurable stop rules, Ann. Prob. 11 (1983) 442-450.

[17] D. Kennedy. An extension of the prophet inequality, in Lecture Notes in Control and Information Science 61, 104-110, Springer, Berlin (1984).

[18] D. Kennedy, Optimal stopping of independent random variables and maximizing prophets, Ann. Prob. 13 (1985) 566-571.

[19] D. Kennedy. Prophet-type inequalities for multi-choice optimal stopping, preprint (1986).

[20] R. Kertz. Stop rule and supremum expectations of i.i.d. random variables: a complete comparison by conjugate duality, J. Multivariate Anal. 19 (1981) 88-112.

[21] R. Kertz. Comparison of optimal value and constrained maxima expectations for independent random variables, Adv. Appl. Prob. 18 (1986) 311-340.

[22] U. Krengel and L. Sucheston. Semiamarts and finite values, Bull. Amer. Math. Soc. 83 (1977) 745-747.

[23] U. Krengel and L. Sucheston. On semiamarts, amarts, and processes with finite values, Adv. Appl. Prob. 4 (1978) 197-266.

[24] U. Krengel and L. Sucheston. Prophet compared to gambler: an inequality for transforms of processes, to appear in Ann. Prob.

[25] Y. Rinott and E. Samuel-Cahn. Comparisons of optimal stopping values and prophet inequalities for negatively dependent random variables and random replacement schemes, preprint (1986).

[26] E. Samuel-Cahn. Comparison of threshold stop rules and maximum for independent nonnegative random variables, Ann. Prob. 12 (1984) 1213-1216.

[27] E. Samuel-Cahn. Prophet inequalities for threshold rules for independent random variables, to appear in Proc. 4th Purdue Symposium on Decision Theory.