*Stochastic Orders and Decision Under Risk*
IMS Lecture Notes – Monograph Series (1991)

# EXPECTATION INEQUALITIES FROM CONVEX GEOMETRY

BY RICHARD A. VITALE*
*University of Connecticut*

By making use of ideas from convex geometry, it is possible to derive novel inequalities for certain expectations. This is illustrated with reference to the Brunn-Minkowski inequality and the theory of zonoids.

**1. Introduction.** Among the various aspects of convex geometry, one which has a long and rich history is the study of inequalities. For a survey and an extensive bibliography, see Burago and Zalgaller (1988). Besides being of idepenent interest, many of these inequalities have been applied elsewhere. It is worth recalling that the second volume of Beckenbach and Bellman (1961) was to have been based on certain of these inequalities involving so-called mixed volumes. There have not, in fact, been many applications to probability and statistics, although there have been notable exceptions. The application of the Brunn-Minowski inequality to multivariate densities by Anderson (1955) is one example. Similarly, the fifty year old van der Waerden permanent conjecture was resolved by Egorychev (1981) by means which were originally developed by Alexandrov (see Burago and Zalgaller) (one should note the related, but *ad hoc* attack by Falikman, 1981). These tools have also been used by Stanley (1981) to resolve certain combinatorial questions. As indicated by these successes, it seems worthwhile to look for other connections between convex geometry and problems of a stochastic nature. The purpose here is to survey some possibilities, the very last section of the paper devoted explicitly to a novel stochastic ordering. We shall keep the discussion informal and largely omit proofs, which appear elsewhere.

In the next section we present notation and preliminaries. Section 3 is devoted to a general form of the Brunn-Minkowski inequality and its relation to Anderson's inequality. Section 4 discusses inequalities for random determinants. The last section treats an open question on the nature of a certain class of convex bodies.

---

* Supported in part by NSF Grant DMS 8603944.

AMS 1980 Subject Classification: Primary 60D05; Secondary 60E15, 52A22

Key words and phrases: Brunn-Minkowski inequality, convex body, random set, zonoid.

**2. Notation and Preliminaries.** The setting will be $d$-dimensional Euclidean space $R^d$ equipped with its usual inner product $\langle \cdot, \cdot \rangle$, norm $\| \cdot \|$, closed unit ball $B$, and unit sphere $S^{d-1}$. $\mathcal{K}$ will stand for the class of nonempty compact, convex subsets of $R^d$; we call such figures *bodies*. Distance between bodies is given by the Hausdorff metric

$$\rho(K, L) = \inf\{\epsilon > 0 \mid K \subseteq L + \epsilon B, L \subseteq K + \epsilon B\}.$$

Here "+" means vector addition of sets, and we mean by $\alpha x$ the set $\{\alpha x \mid x \in K\}$.

Associated with each $K \in \mathcal{K}$ is its support function $h_K : S^{d-1} \to R^1$ given by

$$h_K(u) = \max\{\langle u, y \rangle \mid y \in K\}.$$

The following properties hold:

$$\begin{aligned} &h_{\alpha K}(\cdot) = \alpha h_K(\cdot), \qquad \alpha > 0 \\ &h_{K+L}(\cdot) = h_K(\cdot) + h_L(\cdot) \\ &h_K \leq h_L \Leftrightarrow K \subseteq L \\ &\rho(K, L) = \max\{|h_K(u) - h_L(u)| \mid \|u\| = 1\} (\equiv \|h_K - h\|_\infty). \end{aligned}$$

Accordingly, $h_K$ may be regarded as an analytical surrogate for $K$. For further background, see Eggleston (1969) and Guggenheimer (1977).

A *random set* $X$ is a (Borel measurable) map from a probability space $(\Omega, A, P)$ to $\mathcal{K}$. Its norm $\|X\|$, volume $\mathrm{vol}(X)$, and other common functionals are usual random variables.

The *expectation* of a random set $X$ is defined under the assumption $E\|X\| < \infty$. In this case, $E|h_X(u)| < \infty$ for each $u \in S^{d-1}$ and $EX \in \mathcal{K}$ is then given implicitly by

$$h_{EX}(u) \equiv Eh_X(u) \quad u \in S^{d-1}.$$

This set-valued expectation has appeared in different settings, as for example, in a strong law of large numbers (Artstein and Vitale, 1975): if $X_1, X_2, \cdots$ is an iid sequence of random sets with $E\|X_1\| < \infty$, then the sequence of averages $\bar{X}_n = \frac{1}{n}(X_1 + \cdots + X_n)$ converges a.s. in the Hausdorff metric to $EX_1$.

**3. Brunn-Minkowski Inequality.** The classical Brunn-Minkowski inequality asserts that for two bodies $K$ and $L$ and $0 \leq \lambda \leq 1$

$$\mathrm{vol}^{\frac{1}{d}}(\lambda K + (1 - \lambda)L) \geq \lambda \,\mathrm{vol}^{\frac{1}{d}}(K) + (1 - \lambda)\mathrm{vol}^{\frac{1}{d}}(L) \tag{3.1}$$

(Eggleston, 1969). It figured in the original proof of Anderson's (1955) inequality and in turn can be reinterpreted in probabilistic terms. We shall recast Anderson's result in these new terms.

Anderson's result can be understood as a multivariate generalization of the following observation for a symmetric unimodal density on the line: one maximizes the integral of the density over all intervals of a fixed length by choosing that interval which is centered at the origin. In $R^d$, a density is *unimodal* if $\{x \in R^d \mid f(x) \geq z\}$ is convex for each $z$.

THEOREM 3.1 (ANDERSON, 1955). *Let $f : R^d \to R^1$ be a symmetric unimodal density on $R^d$, and suppose that $A$ is a symmetric, convex set in $R^d$. Then for any fixed $x_0 \neq 0$*

$$\int_{A+\theta x_0} f(x)dx \tag{3.2}$$

*is a decreasing function of the positive parameter $\theta$.*

This can be proved as follows from a somewhat more general framework. First, observe that (3.1) can be regarded as inequality regarding expectations, set-valued on the left, scalar on the right: let $X$ be a random set which takes the values $K$ and $L$ with probabilities $\lambda$ and $1-\lambda$ respectively. Then (3.1) can be rewritten

$$\mathrm{vol}^{\frac{1}{d}}(EX) \geq E\ \mathrm{vol}^{\frac{1}{d}}(X). \tag{3.3}$$

In fact, the following holds.

THEOREM 3.2 (Vitale, 1990). *Let $X$ be any $d$-dimensional random set with $E\|X\| < \infty$. Then (3.3) holds.*

PROOF. We use the strong law of large numbers. Note that, by induction, (3.1) can be extended to any finite number of summand sets. Suppose then that $X_1, X_2, \cdots$ is an iid sequence with each $X_i$ distributed like $X$. Then

$$\mathrm{vol}^{\frac{1}{d}}\left(\frac{1}{n}(X_1 + \cdots + X_n)\right) \geq \frac{1}{n}\sum_{i=1}^{d} \mathrm{vol}^{\frac{1}{d}}(X_i).$$

By the Kolmogorov strong law, the sum on the right converges a.s. as $n \to \infty$ to $E\ \mathrm{vol}^{\frac{1}{d}}(X)$. On the left, the argument set converges a.s. to $EX$ by the strong law for sets and, by the continuity of the volume functional, the entire expression converges to $\mathrm{vol}^{\frac{1}{d}}(EX)$.

A second, auxiliary inequality for which we omit the proof is also of interest.

THEOREM 3.3 (Vitale, 1990). *Let $X$ be a random $d$-dimensional set and let $K \in \mathcal{K}$ with $X \cap K \neq \phi$ a.s. Then*

$$\mathrm{vol}^{\frac{1}{d}}(EX \cap K) \geq E\ \mathrm{vol}^{\frac{1}{d}}(X \cap K). \tag{3.4}$$

To argue Anderson's inequality, observe that it is enough to do it for $f \equiv I_B$, the indicator function of a symmetric convex set, in which case (3.2) is equivalent to

$$\mathrm{vol}(B \cap (A + \theta x_0)) \text{ is a decreasing function of } \theta. \tag{3.5}$$

Let $Y$ be a random set which takes the (non-empty) values $B \cap (A + x_0)$ and $B \cap (A - x_0)$ with probabilities $\lambda$ and $1 - \lambda$ respectively. Then (3.3) reads

$$\begin{aligned}\mathrm{vol}^{\frac{1}{d}}(EY) \geq E\ \mathrm{vol}^{\frac{1}{d}}(Y) &= \lambda\ \mathrm{vol}^{\frac{1}{d}}(B \cap (A + x_0)) \\ &\quad + (1 - \lambda)\ \mathrm{vol}^{\frac{1}{d}}(B \cap (A - x_0)) \\ &\geq \mathrm{vol}^{\frac{1}{d}}(B \cap (A + x_0))\end{aligned}$$

or

$$\mathrm{vol}(EY) \geq \mathrm{vol}(B \cap (A + x_0)). \tag{3.6}$$

Observe as well that $Y = X \cap B$ where $X$ is a random set which takes the values $A + x_0$ and $A - x_0$ with probabilities $\lambda$ and $1 - \lambda$ respectively. Note that $EX = \lambda(A + x_0) + (1 - \lambda)(A - x_0) = A + \theta x_0$ where, without loss of generality we assume $\theta$ lies in $(0, 1)$. By (3.4) we have

$$\mathrm{vol}^{\frac{1}{d}}(EX \cap B) \geq \mathrm{vol}^{\frac{1}{d}}(EY)$$

or

$$\mathrm{vol}((A + \theta x_0) \cap B) \geq \mathrm{vol}(EY). \tag{3.7}$$

Together with (3.6) this implies that

$$\mathrm{vol}(B \cap (A + \theta x_0)) \geq \mathrm{vol}(B \cap (A + x_0))$$

for arbitrary $\theta \in (0.1)$, which is equivalent to (3.5).

The same machinery can be used to derive Mudholkar's (1966) generalization of Anderson's inequality. For this and discussion of the non-convex case (motivated by Eaton, 1984), see Vitale (1990).

**4. Zonoids and Random Determinants.** Random determinants arise in a variety of areas and are extensively surveyed in Girko (1988). In that reference, some eight methods are described for treating moments of random determinants. To that list, one can add an approach from convex geometry for expected *absolute* determinants (eads).

The point of departure is the observation that a *zonotype*, or sum of line segments, say

$$\overline{Oy}_1 + \cdots + \overline{Oy}_n \subseteq R^d \qquad y_i \in R^d,$$

has volume given by

$$\sum_{i_1 < i_2 < \cdots < i_d} |\det[y_{i_1}, \cdots, y_{i_d}]|. \tag{4.1}$$

Using this and the strong law for random sets, it is possible to deduce the following representation (see Vitale, 1991, for this and other results of this section).

THEOREM 4.1. *Let $Y, Y_1, \cdots, Y_d$ be iid random d-vectors with $E\|Y\| < \infty$, and let $M_Y$ stand for the matrix with respective columns $Y_1, Y_2, \cdots, Y_d$. Then*

$$E|\det M_Y| = d! \operatorname{vol}(E\overline{OY}). \tag{4.2}$$

The expected set $E\overline{OY}$ is a *zonoid* in that it is a limit (in the Hausdorff metric) of zonotopes. Its support function is

$$h_{E\overline{OY}}(u) = Eh_{\overline{OY}}(u) = E\langle u, Y\rangle_+ \tag{4.3}$$

($a_+ \equiv \max\{0, a\}$). The theory and applications of zonoids are extremely far-ranging. Bolker (1969) and Schneider and Weil (1983) are two excellent surveys.

By examining $E\overline{OY}$, it is possible to use (4.2) to produce a bound for $E|\det M_Y|$. The classical Hadamard determinantal inequality provides

$$|\det M_Y| \leq \|Y_1\| \cdot \|Y_2\| \cdots \|Y_d\| \tag{4.4}$$

from which

$$E|\det M_Y| \leq (E\|Y\|)^d.$$

By making use of the Urysohn inequality from convex geometry (Burago and Zalgaller, 1988), this can be improved.

THEOREM 4.2. *Let $Y$ be a random d-vector with $E\|Y\| < \infty$. Then*

$$E|\det M_Y| \leq \alpha_d (E\|Y\|)^d \tag{4.5}$$

*where $\alpha_d$ $d = 1, 2, \cdots$ is a sequence of universal constants such that $(\alpha_d)^{\frac{1}{d}} \to e^{-1/2}$.*

The improvement over the Hadamard bound is due in part to the fact that (4.5) does not rely on an a.s. bound such as (4.4). A further improvement is possible by making use of an inequality of Lutwak (1975):

THEOREM 4.3. *Under the same conditions as the last theorem,*

$$E|\det M_Y| \leq d!\, \beta_d w_{-d}$$

where $\beta_d$ is the volume of the $d$-ball and $w_{-d}$ is the $(-d)$*-mean width of* $E\overline{OY}$ $(w_{-d} \equiv E\{E[|\langle U, Y\rangle| \mid Y]^{-d}\}^{-\frac{1}{d}}$, *where* $U$ *is uniform on* $S^{d-1}$*).*

In another vein, (4.2) can be used to provide comparison results, which rely on comparing the bodies $E\overline{OY}$ and $E\overline{OY}'$ for random variables $Y$ and $Y'$ of different distribution.

The following result reflects an examination of the support functions of $E\overline{OY}$ and $E\overline{OY}'$ under the condition that the first body is contained in a translate of the second.

THEOREM 4.4. *For two random d-vectors* $Y$ *and* $Y'$ *of finite mean,*

$$E|\det M_Y| \leq E|\det M_{Y'}|$$

*if there is some vector* $a \in R^d$ *such that* $E|\langle u, Y\rangle| \leq E|\langle u, Y'\rangle| + \langle u, a\rangle$, $\forall u \in S^{d-1}$.

A second result shows that spreading out a distribution in a certain way increases the ead.

THEOREM 4.5. *If* $Y$ *and* $Y'$ *are two independent random d-vectors of finite mean,* $EY' = 0$, *then*

$$E|\det M_Y| \leq E|\det M_{Y+Y'}|.$$

This is reminiscent of a corollary to Theorem 2 of Anderson (1955) which asserts a similar property of densities.

Finally, we mention a result which seems to be a direct analogue of Anderson's result discussed in Section 3.

THEOREM 4.6. *Suppose that* $x_0 \in R^d$, $x_0 \neq 0$, *is fixed and that* $Y$ *is a random d-vector symmetrically distributed about 0 and with* $E\|Y\| < \infty$. *Then* $E|\det M_{Y+\theta x_0}|$ *is an increasing function of the positive parameter* $\theta$.

This result suggests a definition of a spatial median which may have interesting properties.

**4. Generalized Zonoids.** An open question, which is important to several branches of mathematics, is the extent of the class of so-called *generalized zonoids* (Schneider and Weil, 1983). Loosely speaking, a generalized zonoid

$K^*$ is the "difference" of two ordinary zonoids $K$ and $L$, or, more precisely, in terms of support functions

$$h_{K^*}(u) = h_K(u) - h_L(u) \qquad u \in S^{d-1}.$$

Every zonoid is a generalized zonoid since one can take $L = \{O\}$ and observe that then $K^* = K$. The question is, what other bodies can arise? By making use of the defining properties of support functions (extending their domain to all of $R^d$) and the representation (4.3), the question can be reformulated as follows:

*Open Problem.* For which pairs of random vectors $Y_1$ and $Y_2$ in $R^d$ is it the cast that

$$E|\langle x, Y_1\rangle| - E|\langle x, Y_2\rangle|$$

is a convex function of $x \in R^d$?

Evidently this condition can be thought of as an ordering of random vectors (which apparently has not been investigated previously). Certain easy examples (e.g. $Y_2 = \theta Y_1$ $|\theta| < 1$) can be read off, but an alternate characterization which holds generally does not seem to be easily available.

## REFERENCES

ANDERSON,, T. W. (1955). The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities. *Proc. Amer. Math. Soc.* **6**, 170–176.

ARTSTEIN, Z. and VITALE, R. A. (1975). A strong law of large numbers for random compact sets. *Ann. Probab.* **3**, 879–882.

BECKENBACH, E. F. and BELLMAN, R. (1961). *Inequalities.* Springer-Verlag, New York.

BOLKER, E. (1969). A class of convex bodies. *Trans. Amer. Math. Soc.* **145**, 323–346.

BURAGO, YU. D. and ZALGALLER, V. A. (1988). *Geometric Inequalities.* Springer-Verlag, New York.

EATON, M. L. (1984). On group induced orderings, monotone functions, and convolution theorems. In *Inequalities in Statistics* (Y. L. Tong, ed.), Institute of Mathematical Statistics, 4–12.

EGGLESTON, H. G. (1969). *Convexity.* Cambridge University Press, New York.

EGORYCHEV, G. P. (1981). The solution of the van der Waerden problem for permanents. *Dokl. Akad. Nauk. SSR* **258**, 1041–1044.

FALIKMAN, D. I. (1981). A proof of van der Waerden's conjecture on the permanent of a doubly stochastic matrix. *Mat. Zametki* **29**, 931–938.

GIRKO, V. L. (1988). Random determinants. *J. Soviet Math.* **42**, 1731–1765.

GUGGENHEIMER, H. (1977). *Applicable Geometry – Global and Local Convexity*. Krieger, New York.

LUTWAK, E. (1975). A general Bieberbach inequality. *Math. Proc. Camb. Phil. Soc.* **78**, 493–495.

MUDHOLKAR, G. S. (1966). The integral of an invariant unimodal function over an invariant convex set – an inequality and applications. *Proc. Amer. Math. Soc.* **17**, 1327–1333.

SCHNEIDER, R. and WEIL, W. (1983). Zonoids and related topics. In *Convexity and Its Applications* (P. M. Gruber, J. M. Wills, eds.). Birkhauser, Boston.

STANLEY, R. P. (1981). Two combinatorial applications of the Aleksandrov-Fenchel inequalities. *J. Comb. Theory, Series A*, **31**, 56–65.

VITALE, R. A. (1990). The Brunn-Minkowski inequality for random sets. *J. Multivariate Anal.* **33**, 286–293.

VITALE, R. A. (1991). Expected absolute random determinants and zonoids. *Ann. Appl. Probab.*, to appear.

DEPARTMENT OF STATISTICS
UNIVERSITY OF CONNECTICUT
STORRS, CONNECTICUT 06269