EXPERIMENTAL METHODS FOR STUDY OF COSSERAT ELASTIC SOLIDS AND OTHER GENERALIZED ELASTIC CONTINUA

in
Continuum models for materials with micro-structure,
ed. H. Mühlhaus, J. Wiley, N. Y. Ch. 1, p. 1-22, (1995).

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Abstract

The behavior of solids can be represented by a variety of continuum theories. For example, Cosserat elasticity allows the points in the continuum to rotate as well as translate, and the continuum supports couple per unit area as well as force per unit area.

We examine experimental methods for determining the six Cosserat elastic constants of an isotropic elastic solid, or the six Cosserat relaxation functions of a Cosserat viscoelastic solid. We also consider other generalized continuum theories (including micromorphic elasticity, Cowin's void theory, and nonlocal elasticity). Ways of experimentally discriminating among various generalized continuum representations are presented. The applicability of Cosserat elasticity to cellular solids and fibrous composite materials is considered as is the application of related generalized continuum theories.

I Introduction

The classical theory of elasticity is presently used in engineering analyses of deformable objects at small strain. However there are other continuum theories for linear isotropic materials. Some have more freedom, and some have less freedom than classical elasticity. The various continuum theories are all mathematically self consistent. Therefore a discrimination among them is to be made by experiment.

It is the purpose of this article to explore the physical consequences of various continuum theories, and how these consequences may be used in the design of experiments to discriminate among the theories. The constitutive equations for several theories are presented, and some of the salient consequences of each theory are stated and discussed. Some of the causal physical mechanisms associated with each theory are briefly discussed. Experimental methods for evaluating materials as generalized continua are presented, with emphasis on Cosserat elasticity. The treatment is restricted to linearly elastic behavior; study of Cosserat plasticity and related issues is presented elsewhere in this volume. A discussion of experimental aspects of generalized continua is considered particularly appropriate in view of the fact that most of the work done thus far in generalized continuum mechanics has been theoretical.

II Constitutive Equations

Uniconstant Elasticity

The early uniconstant elasticity theory of Navier is based upon the assumption that forces act along the lines joining pairs of atoms and are proportional to changes in distance between them (Timoshenko, 1983). The constitutive equation is as follows.

$$\sigma_{kl} = G\, \varepsilon_{rr}\, \delta_{kl} + 2G\, \varepsilon_{kl} \qquad (1)$$

σ is stress, ε is strain, and G is an elastic constant, the shear modulus. This uniconstant theory was used by Navier, Cauchy, Poisson, and Lamé during the early days of the theory of elasticity. The theory contains *less* freedom than the classical theory of elasticity now in common use. There is no length scale in uniconstant elasticity.

Classical Elasticity

The constitutive equation for *classical* isotropic elasticity (Sokolnikoff, 1983; Fung, 1968), is as follows, in which there are the two independent elastic constants λ and G, the Lamé constants.

$$\sigma_{kl} = \lambda\varepsilon_{rr}\,\delta_{kl} + 2G\,\varepsilon_{kl} \qquad (2)$$

The Poisson's ratio $\nu = \lambda/2(\lambda + G)$ is restricted by energy considerations to have values in the range from -1 to 1/2. There is no length scale in classical elasticity.

Cosserat (micropolar) Elasticity

The Cosserat theory of elasticity (Cosserat, 1909) incorporates a local rotation of *points* as well as the translation assumed in classical elasticity; and a couple stress (a torque per unit area) as well as the force stress (force per unit area). The force stress is referred to simply as 'stress' in classical elasticity in which there is no other kind of stress. The idea of a couple stress can be traced to Voigt (1887,1894) during the formative period of the theory of elasticity. In more recent years, theories incorporating couple stresses were developed using the full capabilities of modern continuum mechanics (Ericksen and Truesdell, 1958; Grioli, 1960; Aero and Kuvshinskii, 1960; Toupin, 1962; Mindlin and Tiersten, 1962; Mindlin, 1965; Eringen, 1868; Nowacki, 1970). A survey of the interrelation between generalized continuum analysis and material defects, dislocations and other inhomogeneities was presented by Kunin (1982, 1983). Eringen (1968) incorporated micro-inertia and renamed Cosserat elasticity *micropolar* elasticity. Here we use the terms Cosserat and micropolar interchangeably. In the isotropic Cosserat solid there are six elastic constants, in contrast to the classical elastic solid in which there are two, and the uniconstant material in which there is one. The constitutive equations for a linear isotropic Cosserat elastic solid are, in the symbols of Eringen, (1968):

$$\sigma_{kl} = \lambda\varepsilon_{rr}\,\delta_{kl} + (2\mu + \kappa)\varepsilon_{kl} + \kappa\varepsilon_{klm}(r_m - \phi_m) \qquad (3)$$

$$m_{kl} = \alpha\,\phi_{r,r}\,\delta_{kl} + \beta\phi_{k,l} + \gamma\phi_{l,k} \qquad (4)$$

The usual summation convention for repeated indices is used throughout, as is the comma convention representing differentiation with respect to the coordinates. σ_{kl} is the force stress, which is a symmetric tensor in equations 1 and 2 but it is asymmetric in Eq. 3. m_{kl} is the couple stress, $\varepsilon_{kl} = (u_{k,l} + u_{l,k})/2$ is the small strain, u_k is the displacement, and e_{klm} is the permutation symbol. The microrotation ϕ_k in Cosserat elasticity is kinematically distinct from the macrorotation $r_k = (e_{klm}u_{m,l})/2$ obtained from the displacement gradient. Components of stress and couple stress on a differential element of a Cosserat solid, and the corresponding increments of force and moment on the structural elements of a real material are shown in Fig. 1.

In three dimensions, the isotropic Cosserat elastic solid requires six elastic constants λ, μ, α, β, γ, and κ for its description. A comparison of symbols used by various authors was presented by Cowin (1970a). The following technical constants derived from the tensorial constants are more beneficial in terms of physical insight. These are (Eringen, 1968; Gauthier and Jahsman, 1975):

Young's modulus $E = (2\mu+\kappa)(3\lambda+2\mu+\kappa)/(2\lambda+2\mu+\kappa)$,
shear modulus $G = (2\mu+\kappa)/2$,
Poisson's ratio $\nu = \lambda/(2\lambda+2\mu+\kappa)$,
characteristic length for torsion $l_t = [(\beta+\gamma)/(2\mu+\kappa)]^{1/2}$,
characteristic length for bending $l_b = [\gamma/2(2\mu+\kappa)]^{1/2}$,
coupling number $N = [\kappa/2(\mu+\kappa)]^{1/2}$, and
polar ratio $\Psi = (\beta+\gamma)/(\alpha+\beta+\gamma)$

When α, β, γ, κ vanish, the solid becomes classically elastic. The case N = 1 (its upper bound) is known as 'couple stress theory' (Mindlin and Tiersten, 1962; Cowin, 1970b). This corresponds to $\kappa \rightarrow \infty$, a situation which is permitted by energetic considerations, as is incompressibility in classical elasticity. The case $\kappa = 0$ corresponds to a decoupling of the rotational and translational degrees of freedom. Although some theoretical writers choose to solve problems for this case since the analysis is simpler, the limit $\kappa \rightarrow 0$ presents physical difficulties (Lakes, 1985a).

Void Elasticity

The theory of elastic materials with voids (Cowin, 1983) incorporates a change of volume fraction, rather than rotation, as an additional kinematical variable. The constitutive equations for the elastic case (no rate dependence) are as follows.

$$\sigma_{kl} = \lambda\varepsilon_{rr}\,\delta_{kl} + 2\mu\varepsilon_{kl} + \beta\phi\delta_{kl} \qquad (5)$$

$$h_k = \alpha\phi_{,k} \qquad (6)$$

$$g = -\xi\phi - \beta\varepsilon_{rr} \qquad (7)$$

with σ as the stress, h as the equilibrated stress vector, λ and μ as the classical Lamé elastic constants, g as the intrinsic equilibrated body force, ϕ as the change of volume fraction, and $\phi_{,k}$ as the gradient of the change of volume fraction. The change of volume fraction can be interpreted as a dilatation of *points* in the continuum.

Nonlocal Elasticity

In an isotropic *nonlocal solid*, the points can only undergo translational motion as in the classical case, but the stress at a point depends on the strain in a *region* near that point (Kröner, 1967, Eringen, 1972). The constitutive equation for stress σ_{ij} is, in terms of the position vector $\mathbf{x}$ of points in the solid,

$$\sigma_{ij}(\mathbf{x}) = \int_V \lambda(|\mathbf{x}' - \mathbf{x}|)\varepsilon_{rr}(\mathbf{x}')\,\delta_{ij} + 2\mu(|\mathbf{x}' - \mathbf{x}|)\varepsilon_{ij}(\mathbf{x}')\,dV(\mathbf{x}'). \qquad (8)$$

A simpler, alternative representation is (Eringen, 1981):

$$\sigma_{ij}(\mathbf{x}) = \int_V \alpha(|\mathbf{x}' - \mathbf{x}|)\,[\lambda\,\varepsilon_{rr}(\mathbf{x}')\,\delta_{ij} + 2\mu\varepsilon_{ij}(\mathbf{x}')]\,dV(\mathbf{x}'), \qquad (9)$$

with the nonlocal kernel $\alpha(|\mathbf{x}|)$ subject to

$$\int_V \alpha(|\mathbf{x}|)dV = 1 \qquad (10)$$

requiring the kernel to be a member of a Dirac delta sequence. So, in the limit of the nonlocal distance of influence or characteristic length 'a' becoming vanishingly small, Hooke's law (Eq. 2 for classical elasticity) is recovered.

An example of a finite range kernel is $\alpha(|x|) = \frac{1}{a}\,[1 - \frac{|x|}{a}]$, for $|x| < a$ (11)

$= 0$, for $|x| > a$.

An infinite range kernel is $\alpha(|x|) = \frac{1}{2a}\, e^{-|x|/a}$. (12)

A simple finite range kernel is $\alpha(|x|) = \frac{1}{2a}$, for $|x| < a$ (13)

$= 0$, for $|x| > a$.

Characteristic lengths can be defined in nonlocal elasticity, in terms of the effective range associated with a kernel.

Microstructure (micromorphic) Elasticity

In microstructure (Mindlin, 1965) or micromorphic (Eringen, 1968) elasticity the *points* in the continuum representation of the solid can deform microscopically as well as translate and rotate. There are 18 elastic constants in the isotropic case. The constitutive equations for isotropic microstructure elasticity are:

$$\tau_{pq} = \lambda\delta_{pq}\varepsilon_{ii} + 2\mu\varepsilon_{pq} + g_1\delta_{pq}\gamma_{ii} + g_2(\gamma_{pq} + \gamma_{qp}) \qquad (14)$$

$$\sigma_{pq} = g_1\delta_{pq}\gamma_{ii} + 2g_2\varepsilon_{pq} + b_1\delta_{pq}\gamma_{ii} + b_2\gamma_{pq} + + b_3\gamma_{qp} \qquad (15)$$

$$\mu_{pqr} = a_1(\xi_{iip}\delta_{qr} + \xi_{rii}\delta_{pq}) + a_2(\xi_{iiq}\delta_{pr} + \xi_{iri}\delta_{pq}) + a_3\xi_{iir}\delta_{pq} + a_4\xi_{pii}\delta_{qr} +$$
$$a_5(\xi_{qii}\delta_{pr} + \xi_{ipi}\delta_{qr}) + a_8\xi_{iqi}\delta_{pr} + a_{10}\xi_{pqr} + a_{11}(\xi_{rpq} + \xi_{qrp}) + a_{13}\xi_{prq} + a_{14}\xi_{qpr} + a_{15}\xi_{rqp} \qquad (16)$$

Here τ_{pq} is the symmetric Cauchy stress, σ_{pq} is the asymmetric relative stress, and μ_{pqr} is the double stress. The g's, b's, and a's as well as λ and μ are elastic constants. The antisymmetric part (with respect to the last two indices) of the double stress represents the couple stress of Cosserat elasticity. ε is strain, γ is the macro-deformation minus the micro-deformation, and ξ is the micro-gradient of micro-deformation, the antisymmetric part of which corresponds to the rotation gradient in Cosserat elasticity.

Microstructure elasticity includes Cosserat elasticity and the theory of voids as special cases. Classical elasticity is a special case of Cosserat elasticity and of void theory. Uniconstant elasticity is a special case of classical elasticity.

Viscoelastic materials

Time dependence or frequency dependence can be incorporated in any of the above constitutive equations by use of the correspondence principle of linear viscoelasticity. For classically viscoelastic materials, the transition is well known. Each elastic constant becomes a complex number in the viscoelastic case in the frequency domain. In the time domain, the constitutive equations assume a convolution form. Eringen (1967) has developed a viscoelastic version of the micropolar theory. The theory of voids as originally presented has a simple time dependence built in.

III Consequences of constitutive equations

The *uniconstant* theory predicts a Poisson's ratio of 1/4 for *all* materials. Since most common isotropic materials exhibit a Poisson's ratio close to 1/3, the uniconstant theory was rejected based on experimental measurements of Poisson's ratio. Decisive experiments were difficult to perform in the late 1800's, so the issue was not decided until well after the introduction of the theories.

In several segments below we refer to size effects. These size effects, and all other phenomena presented in this article, are within the framework of linear (but possibly nonclassical) elastic or viscoelastic behavior. Size effects in the current context refer to a non-classical dependence of the rigidity of an object upon one or more of its dimensions. This type of size effects are to be distinguished from size effects in the fracture behavior; fracture is a nonlinear process not considered here.

Classical elasticity is, according to its name, the currently accepted theory of elasticity. Several salient predictions are as follows.

(i) The rigidity of circular cylindrical bars of diameter d in tension goes as d^2; in bending and torsion, the rigidity goes as d^4.

(ii) Plane waves in an unbounded medium propagate without dispersion (the wave speed is independent of frequency) for shear waves and dilatational waves.

(iii) There is no length scale in classical elasticity, hence stress concentration factors for holes or inclusions in an infinite domain under a uniform stress field depend only on the shape of the inhomogeneity, not on its size.

(iv) Poisson's ratio ν can have values in the range $-1 \leq \nu \leq 1/2$.

Ordinary materials have a positive Poisson's ratio, so that the range from zero to 1/2 was considered for many years to be the physically acceptable range (Fung, 1968). Recently, a new class of cellular solids with a negative Poisson's ratio has been developed (Lakes, 1987), extending the range for ν to -0.7 and below. Further developments in negative Poisson's ratio materials are reviewed by Lakes (1993a).

Cosserat or *micropolar* elasticity has the following consequences.

(i) A size-effect is predicted in the torsion of circular cylinders of Cosserat elastic materials. Slender cylinders appear more stiff then expected classically (Gauthier and Jahsman, 1975); Fig. 2. A similar size effect is also predicted in the bending of plates (Gauthier and Jahsman, 1975) and of beams (Krishna Reddy and Venkatasubramanian, 1978); Fig. 2. No size effect is predicted in tension.

(ii) The stress concentration factor for a circular hole, is smaller than the classical value, and small holes exhibit less stress concentration than larger ones (Mindlin, 1963). Stress concentration around a rigid inclusion in an flexible medium is greater in a Cosserat solid than in a classical solid. Stress concentration near cracks and elliptic holes is reduced in comparison to classical predictions (Kim and Eringen, 1973; Itou, 1973; Sternberg and Muki, 1967; Ejike, 1969; Nakamura et al, 1984).

(iii) Dilatational waves propagate non-dispersively, i.e. with velocity independent of frequency, in an unbounded isotropic Cosserat elastic medium as in the classical case. Shear waves propagate dispersively in a Cosserat solid (Eringen, 1968). A new kind of wave associated with the micro-rotation is predicted to occur in Cosserat solids.

(iv) The mode structure of vibrating Cosserat bodies is modified from that of classical elastic bodies (Mindlin and Tiersten, 1975).

(v) The range in Poisson's ratio is from -1 to +0.5, the same as in the classical case (Gauthier and Jahsman, 1975).

(vi) In Cosserat solids which lack a center of symmetry, called noncentrosymmetric or chiral materials, qualitatively new phenomena are predicted. A rod under tensile load deforms in torsion (Lakes and Benedict, 1982). Wave speed for transverse circularly polarized waves depends on the sense of polarization. This leads to rotation of the principal plane of elliptically polarized transverse waves (Lakhtakia, Varadan, and Varadan, 1988; 1990). Examples of chiral materials include crystalline materials such as sugar which are chiral on an atomic scale, as well as composites with helical inclusions or spiraling fibers. Chirality has no mechanical effect in classical elasticity.

Void theory gives rise to the following.

(i) Size effects are predicted by Cowin and Nunziato (1983) in the bending of rods but not in torsion or tension.

(ii) The stress concentration factor for a hole in a planar region under tension is greater than the classical value (Cowin, 1984a).

(iii) Dilatational waves in an unbounded medium propagate dispersively (Puri and Cowin, 1985). There are two kinds of such waves. Shear waves exhibit no dispersion.

Microstructure (micromorphic) elasticity gives rise to the following.

(i) Dispersion of both dilatational waves and shear waves occurs in solids obeying microstructure elasticity (Mindlin, 1965). Cut-off frequencies for acoustic waves are predicted.

(ii) The stress concentration factor for a spherical cavity can be greater than the classical value (Bleustein, 1966)

Nonlocal elasticity gives rise to the following.

(i) The stress concentration near a crack is alleviated (Eringen, et al, 1977).

(ii) Dispersion of elastic waves is predicted (Eringen, 1972).

(iii) Size effects are predicted in tension and bending, as described below. For certain short range nonlocal behaviors, these size effects can be thought of as surface or 'skin' effects.

IV Further consequences of constitutive equations

In this section we present several new results which are relevant to experimental methods. A slab is considered in tension and in bending for a Cosserat solid and for a nonlocal solid.

Cosserat solid

Consider bending of an isotropic Cosserat elastic beam of rectangular section and width B and depth A, with the following displacement field u and microrotation field φ. R is the radius of curvature of the bent beam.

$$\{ u_x = -(1/2R)[z^2+\nu(x^2-y^2)],\ u_y = -\nu xy/R,\ u_z = xz/R \} \qquad (17)$$

$$\{ \varphi_x = 0 \quad \varphi_y = -z/R \quad \varphi_z = -\nu y/R \}$$

We use the semi-inverse method. The procedure is the same for Cosserat solids as it is for classical solids. The displacement field is assumed to be the same as for the classical elastic case, and the microrotation field is assumed to be the same as the classical macrorotation. Again, size effects in the rigidity occur. The bending rigidity ratio (ratio of rigidity of a Cosserat beam to that of its classical counterpart) for a particular rectangular cross section bar of sides A and B is found to be:

$$\Omega = RM/E(BA^3/12) = [1+24(l_b/A)^2(1-\nu)]. \qquad (18)$$

Size effects in the rigidity are predicted, in which slender bars (with $A \approx l_b$) are stiffer than expected classically. The displacement and rotation fields give rise to an exact solution only if $\beta/\gamma = -\nu$. If $\beta/\gamma \neq -\nu$, then they are valid if a system of couple stresses were applied to the lateral surface. The exact solution for the general case is not known. If these stresses are not applied, the displacement field will be different: the inclined lateral surfaces will exhibit some bulge rather than being straight as they are in the classical case, or in the Cosserat case with $\beta/\gamma = -\nu$. Hence Cosserat elasticity predicts a change of *shape* of the cross section of the bent beam, for $\beta/\gamma \neq -\nu$. The bending rigidity will also be different from the above.

The rigidity size effects in other situations such as bending of a plate or circular rod, or torsion of a circular rod, have a similar form. For example, Gauthier and Jahsman (1975) give, for cylindrical bending of a plate,

$$\Omega = 1 + 24\,\frac{l_b^2(1-\nu)}{h^2} \qquad (19)$$

with h as the plate thickness. The length parameter of Gauthier and Jahsman was converted into the characteristic length for bending defined above. In plate bending, the anticlastic curvature due to the Poisson effect is constrained, in contrast to beam bending. A similar kind of solution for bending of a void solid was constructed by Cowin, 1984b.

Nonlocal solid

Chirita (1978) has examined Saint Venant's problem in nonlocal elastic solids; this includes simple tension, bending, and torsion of a slender bar. Chirita writes the constitutive equation much as in Eqs. (8,9) but with x' called z', α called K, and extra terms outside the integral, corresponding to classical elasticity. For both tension and bending, the displacement field is predicted to be *identical* to the classical displacement field. So, nonlocal elasticity predicts no *shape* changes in comparison with classical elasticity for bending. Compare the bending of a Cosserat elastic bar, considered above. Chirita gives integral forms for the rigidities but not explicit forms or interpretation.

In the following, consider the origin of coordinates to be the center line of the slab, which has width W and breadth V >> W.

In **simple tension**, we have uniform strain so (away from a boundary),

$$\sigma(x) = E\varepsilon \int_{x-x'=-a}^{a} \alpha(x - x')\, dx' \qquad (20)$$

$$\sigma(x) = 1\, E\varepsilon, \text{ from Eq.10.} \qquad (21)$$

The nonlocal region of influence has dimensions ±a and is shown in Fig. 4. To first order there is no size effect in tension, however to second order there will be a size effect due to integration over part of the domain as a result of interception of a portion of the nonlocal influence by the boundaries. We remark that this effect has been neglected in prior treatments of crack problems in nonlocal elasticity (see Eringen, 1983).

Let us now consider the surface effect of interception of a portion of the kernel's region of influence. In the one dimensional slab geometry, the problem is tractable.

$$\sigma(x) = E\varepsilon \int_{x-x'=-a}^{a} \alpha(x - x')\, dx', \qquad \text{for} \quad -W/2 + a < x < W/2 - a \qquad (22)$$

$$\sigma(x) = E\varepsilon \int_{x-x'=-W/2-x}^{a} \alpha(x - x')\, dx', \qquad \text{for} \quad -W/2 + a > x \qquad (23)$$

$$\sigma(x) = E\varepsilon \int_{x-x'=-a}^{W/2-a} \alpha(x - x')\, dx', \qquad \text{for} \quad x > W/2 - a. \qquad (24)$$

Near the surface, only a portion of the kernel's region of influence is integrated over, hence contributes to the stress. Suppose that the kernel is positive definite throughout its range, as has been done in several analyses of stress around cracks. Then there is a surface layer of depth a in which the stress is *less* than $E\varepsilon$. Such a surface effect has a negligible effect on the rigidity if a << W, however it becomes progressively more important for thinner slabs. For such a kernel, the stiffness is *smaller* for a slender specimen than for a thick one. The tensile force is, for the constant kernel of Eq. 13,

$$F = \int\sigma\, dx\, dy = E\varepsilon\, V((W-2a)+ (2)3a/4) = E\varepsilon V(W-a/2). \qquad (25)$$

The stress drops to half its central value at the edge of the slab (Fig. 5), since half the nonlocal zone of influence then extends beyond the edge, where there is no material.

So the normalized rigidity Ω is, for $W \geq 2a$,

$$\Omega = [E\varepsilon V(W-a/2)]/E\varepsilon VW = 1 - a/2W. \qquad \mathbf{(26)}$$

The size effects above are of softening of small specimens, but that need not always be the case. For example consider the kernel

$$\alpha(|x|) = 2\delta(x) - \frac{1}{2a}\text{, for } |x| < a;\ \alpha(|x|) = 0\text{, for } |x| > a. \qquad (27)$$

The delta function represents classical behavior but the negative term gives a surface effect of the opposite sign as that considered above, hence a stiffening effect of thin specimens.

The above analysis is one dimensional. In two dimensions the nonlocal region of influence may be circular and in three dimensions it may be spherical. Computation of segments intercepted by such boundaries would be more complex, however we expect that, as in Eq. 26, the rigidity would depend on some fraction of a/W.

In **cylindrical bending** of a plate, $\varepsilon = gx$ and we consider one dimension only. We consider first x = 0.

$$\text{Then, } \sigma(0) = E \int_{x'=-a}^{a} \alpha(x - x') g x'\, dx'. \qquad (28)$$

$\sigma(0) = E \int_{x'=-a}^{a} \alpha(-x') g x'\, dx'$. If α is an even function, then the integral is zero. For example,

$$\sigma(0) = E \int_{x'=-a}^{a} (x'^2) g x'\, dx' = [(x'^4)/4]_{-a}^{a} = 0. \qquad (29)$$

So in cylindrical bending of a plate, in which the classical displacement field gives rise to strains varying in one direction, the one dimensional nonlocal theory predicts stresses at the origin identical to the classical values.

Consider now the stress distribution.

$$\sigma(x) = E \int_{x'=-\infty}^{\infty} \alpha(x - x')\, g x'\, dx' \qquad (30)$$

Suppose that the kernel is constant over a range as in Eq. 13.

$$\sigma(x) = E \int_{x-x'=-a}^{a} g x'/a\, dx' = E \int_{x'=x+a}^{x-a} g x'/a\, dx' \qquad (31)$$

$$\sigma(x) = \frac{gEx'^2}{2a} \Big[_{x+a}^{x-a} = \frac{gE}{2a} ((x-a)^2 - (x+a)^2) = \frac{gE}{2a} (x^2 - 2ax + a^2 - (x^2 + 2ax + a^2)) \qquad (32)$$

$$\sigma(x) = \frac{gE}{2a}(-2ax) = \mathbf{-gEx} \qquad (33)$$

There is *no* influence of nonlocality on the stress field (away from the free surfaces); the stress field is purely classical for this kernel. Consequently there are no size effects in rigidity except for those associated with surface phenomena, as was the case in tension. The surface effects will be similar to those for tension, provided that a<<W; so size effects in bending can consist of a softening effect of thin specimens for a positive definite kernel, or a stiffening effect of small specimens for a kernel which goes negative over part of its range.

Since the nonlocal theory is linear, we may consider a more general kernel as a superposition of functions of the form in Eq. 13, but with different values of a. Specifically, $\alpha_j(|x|) = \frac{1}{2a_j}$, for $|x| < a_j$, $\alpha(|x|) = 0$, for $|x| > a_j$. Then the superposed kernel is

$$\alpha_j(|x|) = \Sigma_j \frac{q_j}{2a_j}\text{, for } |x| < a_j,\ \alpha(|x|) = 0\text{, for } |x| > a_j\text{, with } \Sigma_j q_j = 1. \qquad (34)$$

A wider variety of functional forms of the kernel could be accommodated by passing to the limit as an integral. Since each component of such a superposition gives rise to a classical stress field away from the edges, then the superposed kernel as well gives rise to a classical stress field.

We remark that for a Poisson's ratio of zero, the displacement field for bending of a bar is identical to that for cylindrical bending of a plate (which we have considered), but for nonzero Poisson's ratio, strains in a bent bar are nonzero in all three coordinate directions.

Observe that the Cosserat bending equation differs from Eq. 26 for nonlocal size effects in that the latter has a linear term in the length scale ratio, for simple kernels. Consequently nonlocal and Cosserat solids can be distinguished by the functional form of the size effects. A comparison between predicted size effects is shown in Fig. 6. Observe that the Cosserat and nonlocal curves cross each other and have different shape.

V Physical causes of mechanical behavior

Continuum theories make no reference to structural features, however they are intended to represent physical solids which always have some form of structure.

The ultimate origin of elastic behavior is the electromagnetic force between atoms in a solid. The uniconstant theory was derived assuming that such interatomic forces were central forces along lines connecting pairs of atoms, and that the movement of atoms was affine. Since the Poisson's ratio for most materials differs experimentally from 1/4, it can be concluded that in most materials either the interatomic forces are non-central, or non-affine deformation occurs, or both.

The couple stresses in Cosserat and microstructure elasticity represent spatial averages of distributed moments per unit area, just as the ordinary (force) stress represents a spatial average of force per unit area. Such moments can occur as a result of the fact that the interatomic forces propagate further than one atomic spacing (Kröner, 1963). Such effects will occur in *all* solids, but the corresponding characteristic lengths would be of atomic scale and not amenable to macroscopic mechanical experiment. Moments may be also transmitted on a much larger scale through fibers in fiber-reinforced materials, or in the cell ribs or walls in cellular solids. The Cosserat characteristic lengths would then be associated with the physical size scales in the microstructure, and be sufficiently large to observe experimentally.

The nonlocal theory incorporates long range interactions between particles in a continuum model. Such long range interactions occur between charged atoms or molecules in a solid. Long range forces may also be considered to propagate along fibers or laminae in a composite material (Ilcewicz, et. al, 1981, 1981).

Analytical predictions of Cosserat characteristic lengths have been developed for a variety of structures. In fibrous composites, the characteristic length l may be the on the order of the spacing between fibers (Hlavacek, 1975); in cellular solids it may be comparable to the average cell size (Adomeit, 1967); in laminates it may be on the order of the lamination thickness (Herrmann and Achenbach, 1967). Structure, however, does not necessarily lead to Cosserat elastic effects. Composite materials containing elliptic or spherical inclusions are predicted to have a characteristic length of *zero* (Hlavacek, 1976; Berglund, 1982).

A schematic diagram of force increments upon ribs (in the structural view) corresponding to stress (in the continuum view) and moment increments corresponding to couple stress is shown in Fig. 1.

One may distinguish the continuum view from the structural view. The continuum view is useful for making engineering predictions and for visualizing global response of materials. The structural view is relevant to the underlying causes of the behavior. One may link these views by developing an analytical model of the material microstructure, and obtaining approximations (possibly by series expansions for local deformation fields) in order to obtain average values. Retention of only the lowest order terms in such analysis gives classical elasticity as a continuum representation. When higher order terms are retained, a generalized continuum representation (such as Cosserat elasticity) is obtained. In either case the predicted elastic constants are functions of the structure and properties of the constituents. This is how the microphysics is introduced.

V Experimental Procedures for Cosserat elastic solids

Methods based on size effects

The bending and torsion rigidities of a classically elastic rod are proportional to the fourth power of the diameter. In a thin classical elastic plate the bending rigidity is proportional to the third power of the thickness. The rigidity depends on size in a different way in Cosserat elastic materials as discussed above. Thin specimens are more rigid than would be expected classically. It is possible to determine one or more of the Cosserat elasticity constants by measurements of specimen rigidity vs size. This approach, which we call the method of size effects, has been used to experimentally determine Cosserat elastic constants. The method of size effects makes use of analytical solutions for the dependence of rigidity upon size. Most of these solutions have dealt with *isotropic* materials. Specifically, Gauthier and Jahsman (1975) demonstrated that no size effects occur in tension, so that E and ν are determined from a tension test as in the classical case. Size effects occur in torsion, and the isotropic Cosserat constants G, l_t, N, Ψ can be obtained from size effect data for torsion of rods of circular section. Bending of circular section rods of different size (Krishna Reddy and Venkatasubramanian, 1978) gives E, l_b, N. Cylindrical bending of a plate gives E and l_b, according to Gauthier and Jahsman (1975). Bending of a circular plate with a clamped edge depends on E, l_b , N, according to Ariman (1968). Gauthier and Jahsman (1976) show that bending of a curved bar depends on E and l_b. Park and Lakes (1987) show that size effects in torsion of a square cross section bar depend on G, l_t ,N.

Instrumentation capable of determining bending and/or torsion rigidity may be used in the method of size effects. Rigidity is to be determined over a considerable range, so it is necessary to make sure that parasitic errors such as those due to instrument friction are minimized or eliminated. Load may be applied electromagnetically or by dead weights. Cantilever bending is appropriate since there is no friction associated with dead weight loading. If dead weights are used in torsion, the pulleys used to redirect the load could introduce errors due to friction. Such errors would be more important for thin specimens and would obscure the size effects. Frictional errors could be eliminated by the use of air bearings. Deformation can be measured by holographic interferometry, other optical methods, strain gages, and free core LVDT's without friction error. Spring loaded LVDT's with sleeve bearings would by contrast introduce friction errors which are more problematical in thin specimens.

Size effect methods have been used by several authors to obtain only one Cosserat elastic constant. A particular form of the method of size effects was found to be useful by the present author. The rigidity of the same circular rod specimen was tested in both torsion and pure bending using the same apparatus, which makes use of electromagnetic torque generation and interferometric detection of angular displacement. Each specimen was then cut to a smaller size and the rigidities again determined. All six of the Cosserat elastic constants can be determined this way. Moreover, cross verification of the results is possible, in a manner similar to the measurement of E, G, and ν in classical elasticity, with verification of their interrelation.

Specimen preparation for size effect method

In the method of size effects a set of specimens of different diameter or thickness is used. If the characteristic lengths are small, thin specimens must be studied. Stiff materials, such as bone and the stiffer polymer foams, may be successfully cut on a lathe with conventional cutting tools. Circularly cylindrical specimens thinner than about 3 mm in diameter down to 0.2 - 0.5 mm are prepared on a lathe by an abrasive machining technique. The lathe is operated at high speed and a strip of abrasive cloth is applied to the surface using a small force. Rectangular section specimens may be cut with a low speed saw, and the surfaces polished with graded abrasives. Flexible polymer foams can be cut into circular cylinders by use of a coring tool driven by a power drill. The coring tool consists of a metal tube with a sharpened end and thin walls. Rectangular section specimens of foam can be cut from these by compressing the foam between platens and cutting with a scalpel.

Since surface damage to the specimen would cause a softening size effect, the opposite of that expected in Cosserat solids, considerable care should be taken to avoid surface damage. In the case of cellular solids, a layer of damaged or incomplete cells has been shown to cause such a softening size effect by Brezny and Green (1990). To minimize the effect of surface damage, removal of the damaged layer by polishing is recommended if it is possible.

Data reduction for size effect method

Data analysis in the method of size effects makes use of the exact analytical solutions for the geometry used in the experiments. For torsion of a circular cylinder (Gauthier and Jahsman, 1975), the ratio of rigidity to its classical value is

$$\Omega = 1 + 6(l_t/r)^2 \,[1 - 4\Psi X/3)/(1 - \Psi X)], \qquad (35)$$

with r as the rod radius, $\Psi = (\beta + \gamma)/(\alpha + \beta + \gamma)$ and $X = I_1(pr)/pr\, I_0(pr)$, and $p^2 = 2\kappa/(\alpha + \beta + \gamma)$. I_1 and I_0 are the modified Bessel functions of the first kind. A special case of interest, referred to as 'couple stress elasticity', is for N = 1 ($\kappa \rightarrow \infty$) in which the [] bracket in equation 4 becomes unity. If the rod diameter is large, the corresponding rigidity ratio is $\Omega \approx 1 + 6(l_t/r)^2$.

For bending (Krishna Reddy and Venkatasubramanian, 1978) of a circular section rod of radius r, the rigidity ratio Ω is

$$\Omega = 1 + 8(l_b/r)^2 \,(1 - (\beta/\gamma)^2) + \mathbf{[}(8N^2(\beta/\gamma + \nu)^2 / (\zeta(\delta a) + N^2(1 - \nu))(1 + \nu)\mathbf{]} \qquad (36)$$

with $\zeta(\delta r) = (\delta r)^2\, [(\delta r\, I_0(\delta r) - I_1(\delta r)) / (\delta r\, I_0(\delta r) - 2I_1(\delta r))]$, and $\delta = N/l_b$.

Data may be plotted as Ω vs radius; however one may also plot rigidity divided by the square of the diameter vs. the square of the diameter for torsion (Fig. 2) and bending (Fig. 3). Such plots are useful since the characteristic lengths can be extracted from the intercept of the extrapolated straight portion of the curve upon the ordinate. They are not stress strain curves.

In classical elasticity, the torsion size effect plot is a straight line through the origin with slope proportional to the shear modulus G. The Young's modulus E is obtained from the slope in the bending case. Comparison of experimental plots and theoretical curves permit the determination of the Cosserat elastic constants. The characteristic lengths are obtained by intercepts as described above, and as indicated in the figures, either graphically or by numerical procedures. The shape of the torsion plot is then used to extract the coupling number N. A large value of N (the upper bound is 1) leads to a large apparent stiffening for slender specimens. The structure of the torsion plot in the vicinity of the origin is used to determine Ψ; this is difficult since it requires very thin specimens. In our laboratory, a numerical algorithm has been used to minimize the mean-square deviation between the experimental data and the theoretical graphs, to extract the elastic constants.

For some combinations of elastic constants, the apparent modulus tends to infinity as the bar or plate size goes to zero. Large stiffening effects might be seen in composite materials consisting of very stiff fibers or laminae in a compliant matrix. However, infinite stiffening effects are unphysical. For very slender specimens, it is likely that a continuum theory more general than Cosserat elasticity; or use of a discrete structural model, would be required to deal with the observed phenomena.

In the bending of long rods, one can invoke Saint Venant's principle in order to eliminate the need to consider end effects. For that reason, we consider rod bending to be a more attractive experimental modality than plate bending. Moreover, the same rod can be used for bending, torsion, and tension experiments.

As for the cross sectional shape of the rod, circular sections currently have the advantage that the available analytical solutions are exact and not excessively complex. However square cross section rods can also be studied and the results interpreted using the analysis presented by Park and Lakes (1987).

Field methods

Discrimination among generalized continuum theories can also be accomplished by examining the distribution of strain in deformed objects. This is in contrast to the above method of size effects in which the rigidity, a global quantity, is measured. For example, Park and Lakes (1987) presented analysis of the distribution of strain in a bar of rectangular cross section under torsion. The surface strain does not vanish at the corner of the cross section, in contrast to the case of classical elasticity. A screening method based on this prediction was developed by Lakes, et al (1975): a holographic image of a small notch in the corner discloses any motion of the notch under torsion. Such motion would occur in a Cosserat solid but not in a classical one. A lecture demonstration based on displacement of a corner notch was presented by Lakes (1985b). Another field method involves measuring the distribution of strain around a stress concentrator such as a circular hole, for which analytical solutions are available in classical elasticity (Sokolnikoff, 1983; Fung, 1968) and for Cosserat elasticity (Eringen, 1968, Cowin, 1970; Mindlin, 1963).

Wave methods

The propagation of stress waves can be used as a probe into the constitutive equation governing materials with structure. Plane waves in an unbounded classically elastic material propagate without dispersion: their speed is independent of frequency. The elastic moduli can be extracted from the speeds of transverse and longitudinal waves and from the material density. Dispersion of waves is predicted to occur in Cosserat solids (Eringen, 1968), in micromorphic/ microstructure solids (Mindlin, 1964), in nonlocal solids (Eringen, 1972), and in void solids (Puri and Cowin, 1985). Measurement of wave dispersion could be used to determine generalized continuum characteristics of a material. A wave method was used by Gauthier (1982) to examine the particulate composite which had appeared classical in the size effect studies of Gauthier and Jahsman (1975). The micro-inertial characteristics were determined, and it was found that N^2 = 0.0039, so small that the static behavior would indeed appear classical.

A drawback of wave methods is that wave dispersion also arises from viscoelasticity of materials. Therefore wave methods are most suitable if the wave attenuation due to viscoelasticity is small enough to be neglected. By contrast, the method of size effects can be made independent of viscoelasticity by conducting all measurements at the same time following loading, or at the same frequency. If a material exhibits Cosserat viscoelasticity, the time or frequency dependence of the six Cosserat coefficients can be extracted from size effect plots generated at different times or frequencies. This cannot be done in wave methods, since the wavelength which governs the strain gradient cannot be decoupled from the frequency. However wave methods can be used for large scale structures such as layered rock formations, which are too large to study in the laboratory.

Resolution of the characteristic length l

In all methods there is a limit to the smallest value of characteristic length l which can be resolved. In the method of size effects, preparation of very thin specimens can present difficulties. In field methods the specimen can be large but the effects of generalized continuum mechanics depends on strain gradients, and measurement of strain in the presence of the large strain gradients required to reveal a small l is difficult. One can examine the strain at the corner of a square section bar in torsion, as a null experiment, but again the resolution is limited by the fact that strain can be measured only over a nonzero length. In wave methods, the resolution of small l requires waves of high frequency, but in many materials the attenuation of stress waves becomes large at sufficiently high frequency.

VI Experimental Results: a review

Cosserat elastic constants

The results of some published experimental studies of materials as Cosserat solids are presented in Table 1. The relationship between Cosserat characteristic lengths and the structure size is evident. Characteristic lengths on the millimeter scale are not observed unless there are corresponding structural features of a similar size scale. For example polymethyl methacrylate (PMMA) is an amorphous polymer for which the relevant structure scale is atomic and molecular.

It was used as a control in experiments upon bone (Yang and Lakes, 1981), which is a natural composite in which the largest structural elements are large fibers up to 250 µm in diameter. No macroscopic evidence of Cosserat elasticity was expected or found in PMMA. Although structure appears to be necessary to produce Cosserat elastic effects, it is not sufficient. Particle reinforced composites exhibit Cosserat characteristic lengths of zero (Gauthier and Jahsman, 1975). That observation is in harmony with micromechanical analysis (Hlavacek, 1976; Berglund, 1982) which predicts characteristic lengths of zero. We remark that the syntactic foam which was found to be nearly classical, is composed of glass micro-balloons in an epoxy matrix, a structure which is particulate in nature.

Several authors have used 'couple stress theory' for interpretation. This corresponds to a Cosserat solid for which N = 1. Since the characteristic lengths are defined differently, the characteristic length in Cosserat elasticity is $\sqrt{3}$ times the length in couple stress theory. Results have been converted to the Cosserat form in Table 1. In the study of the PVC foam, a resonant thickness shear size effect approach was used. Viscoelasticity could be a confounding variable here since as the layer thickness was reduced, the resonant frequency increased. In viscoelastic materials, stiffness increases with frequency even if they are classical. The graphite was nonlinear in its stress-strain relation, but most of the others were studied in the linear domain. The studies of bone and dense polyurethane and syntactic foams incorporated error analysis from which a meaningful discrimination between Cosserat and classical behavior was achieved.

Cosserat viscoelasticity has been studied in human bone. In a Cosserat viscoelastic solid, the characteristic lengths as well as the stiffnesses can depend on time or frequency. The torsional characteristic length in bone was a factor of 1.6 larger under quasistatic conditions equivalent to a frequency of about 0.1 Hz (Yang and Lakes (1981) than at 32 kHz (Lakes, 1982). This was attributed to the viscoelastic attributes of the cement substance between the large fibers (osteons) in bone.

Study of predictive power

The experimental determination of Cosserat elastic constants is useful if it permits one to predict stresses and strains under conditions which differ from those in the experiments used to find the elastic constants. Several experimental tests of predictive power have been conducted. Cosserat elastic constants (based on isotropic theory) for bone, which is actually anisotropic, were used to predict the strain distribution around a hole in a strip under tension, and the results were compared with experiment. Reasonable agreement was found by Lakes and Yang, (1983) even though the anisotropic solution was not available. The same Cosserat elastic constants, derived from size effect studies, were used to predict strain distributions in a square bar under torsion. Comparison with experimental results by Park and Lakes (1986) was favorable. In this case, anisotropy is less of a problem, since the same elastic constants l_t and N are relevant in this geometry as in the torsion size effect study. Consequently, with the specimen aligned the same way in both cases, the same elastic constants appear in both cases.

In torsion of a Cosserat square cross section bars a small notch in the corner of the cross section is predicted to displace as the bar is twisted. The displacement should be zero in a classical solid, since the stress is zero at the corner (Park and Lakes, 1985). Holographic studies were conducted, and corner notch displacement was zero in solid polymethyl methacrylate (PMMA), but was nonzero in dense polyurethane foam (Lakes, et al, 1985) which was shown by size effect studies to be Cosserat elastic (Lakes, 1986). Similar displacements were easily observed visually in large cell foams (Lakes, 1986) which had been identified as Cosserat elastic (Lakes, 1983).

Interpretation via void, nonlocal, and microstructure theories

If we attempt to interpret the size effect results in Table 1 with the void theory of Cowin and Nunziato (1983), there arises the difficulty that the theory predicts size effects in bending but not in torsion. Among the materials studied, none exhibits a size effect only in bending. We conclude that a material describable by the void theory has not yet been found. Materials with a small volume fraction of voids have not, however been studied in this context.

As for nonlocal elasticity, some cases of wave dispersion have been interpreted via that theory by Ilcewicz et al (1981, 1985). In a particle board composite, the characteristic length was found to be about 0.3 mm, and that value was linked to the fracture toughness.

As for microstructure elasticity, little comparison has been made with experiment since few analytical solutions are available for this theory. However, wave dispersion and cut-off frequencies were observed in dynamic studies of foams, including foams with negative Poisson's ratios (Chen and Lakes, 1989), and the results interpreted in view of microstructure elasticity. Wave dispersion has been observed in several other structured materials (Sutherland and Lingle, 1972; Kinra, et. al, 1980), without interpretation via generalized continuum mechanics.

Studies of fibrous composites

The fracture strength of graphite epoxy plates with holes depends on the size of the hole (Karlak, 1977). Moreover the strain around small holes and notches in fibrous composites well below the yield point is smaller than expected classically (Whitney and Nuismer, 1974; Daniel, 1978), while for large holes, the strain field follows classical predictions (Rowlands et al, 1973). Further results are given in a review by Awerbuch and Madhukar (1985). Such results are in harmony with the predictions of generalized continuum mechanics. However, thus far in the fibrous composites community, it has been fashionable to interpret results of this kind in terms of *ad hoc* criteria rather than to use generalized continua. One such criterion involves attempting to predict fracture by calculating the average stress in a region near a stress raiser, rather than using the actual maximum stress.

VII Discussion

A variety of experimental procedures are capable of revealing non-classical aspects of the behavior of materials and of determining the elastic constants according to generalized continuum theories. The procedure which has been the most used is the method of size effects. Since the ratio of surface area to volume of a specimen varies with its size, the experimenter is wise to take care with the method of cutting, since surface damage can also generate size effects.

Continuum theories are available with a range of complexity. The degree of complexity which is appropriate depends on what kinds of experiments are done and how carefully we wish to examine material response. Size effects predicted by Cosserat theory can involve apparent torsional or bending stiffness tending to infinity as the specimen thickness tends to zero. This is unphysical unless the material is a composite with one constituent very much stiffer than the others. It is possible that the additional freedom of the microstructure / micromorphic theory would be required for a better description. However the problems of torsion and bending in that theory have not to the writer's knowledge, been solved.

Experimental work reveals metals, an amorphous polymer, and several particulate composites to behave essentially classically. Cosserat elastic constants have been found for several cellular and fibrous materials. The characteristic lengths are on the order of the largest structure size as expected theoretically. However in some cellular materials, the characteristic length can exceed the cell size, in contrast to theory which predicts it should be smaller. There is the intriguing possibility that tough materials could be created with characteristic lengths significantly larger than the structure size (Lakes, 1993b).

VIII Conclusions

1. Determination of Cosserat elastic constants can be achieved by the method of size effects.
2. Size effects for Cosserat and nonlocal elastic solids are predicted to differ.
3. Polycrystalline and particulate type material microstructures behave classically or nearly classically.
4. Several cellular solids, exhibit behavior consistent with Cosserat elasticity.
5. The continuum theory of voids does not adequately describe the microelastic effects observed in cellular solids of solid volume fraction less than 0.5.

Acknowledgment

Partial support by the NASA/ Boeing ATCAS program is gratefully acknowledged.

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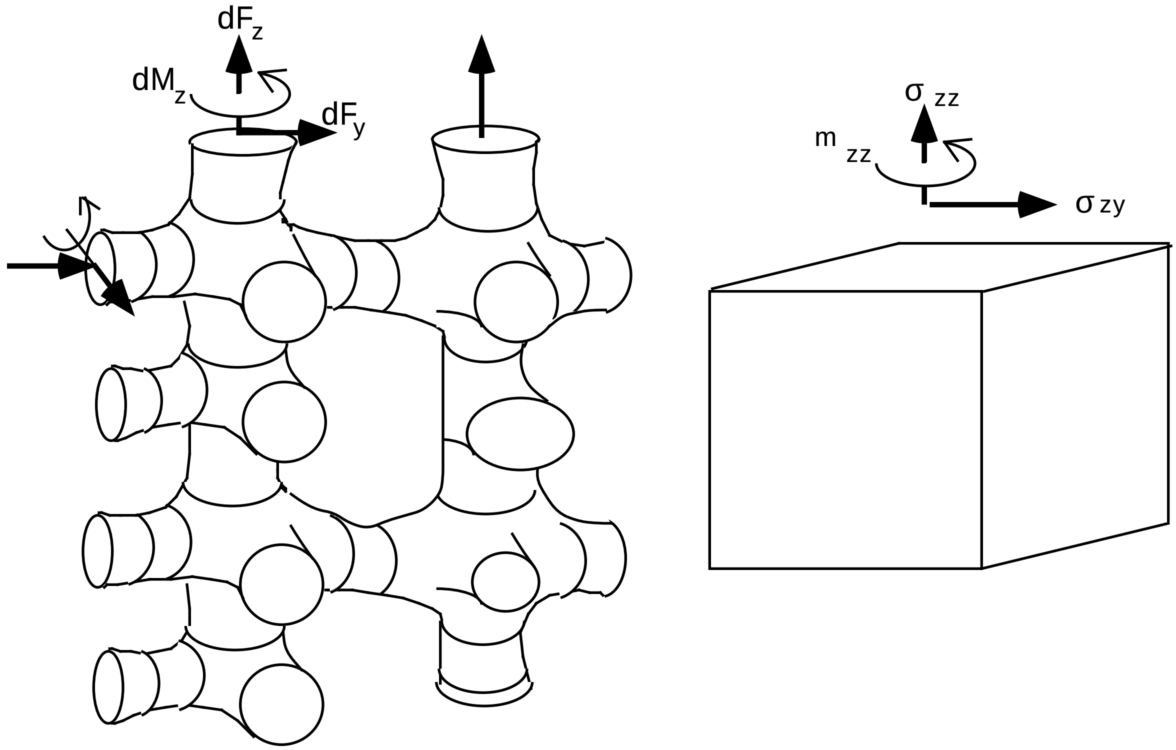
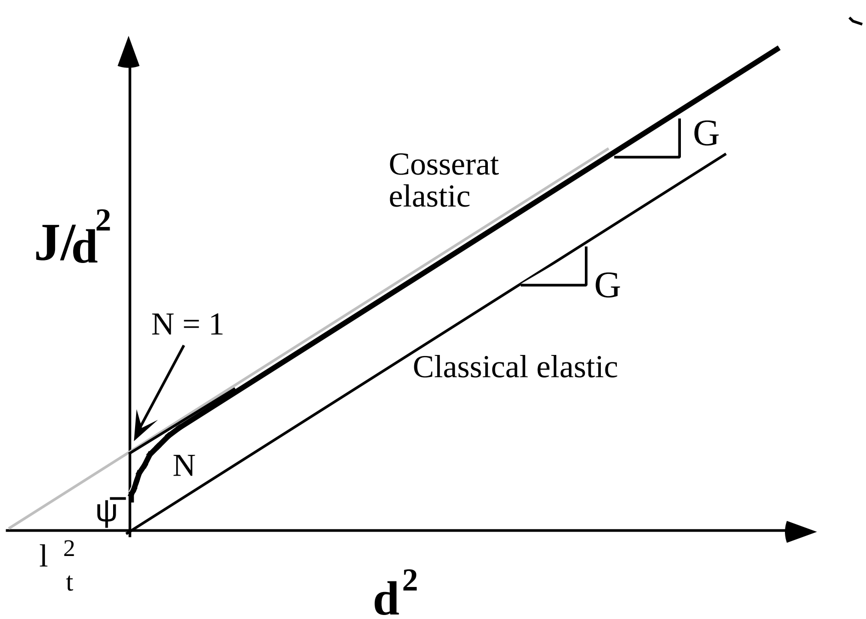
Figures

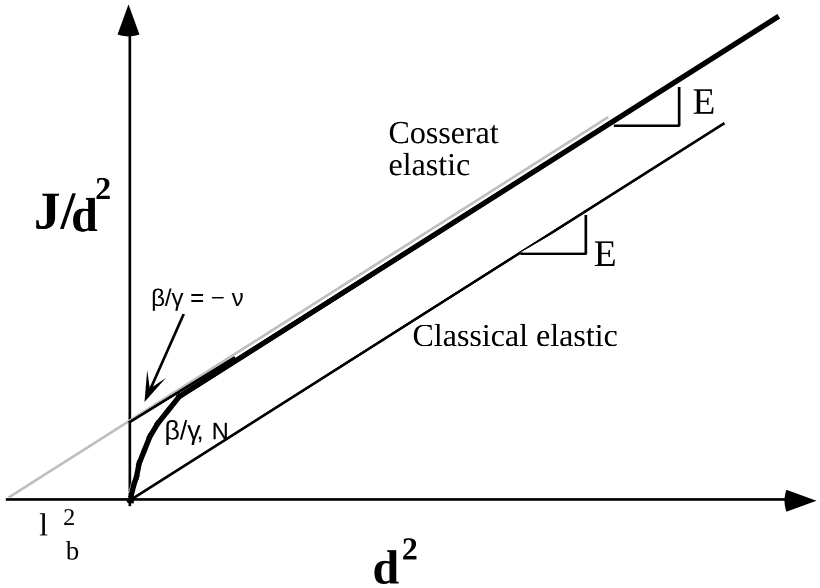


1 Left, structural view of increments of force and moment upon structural elements in a cellular solid.

Right, continuum view of stress and couple stress on a differential element.



2 Determination of Cosserat elastic constants from size effect data in torsion of a circular cylindrical rod. Rigidity/ diameter squared vs diameter squared.



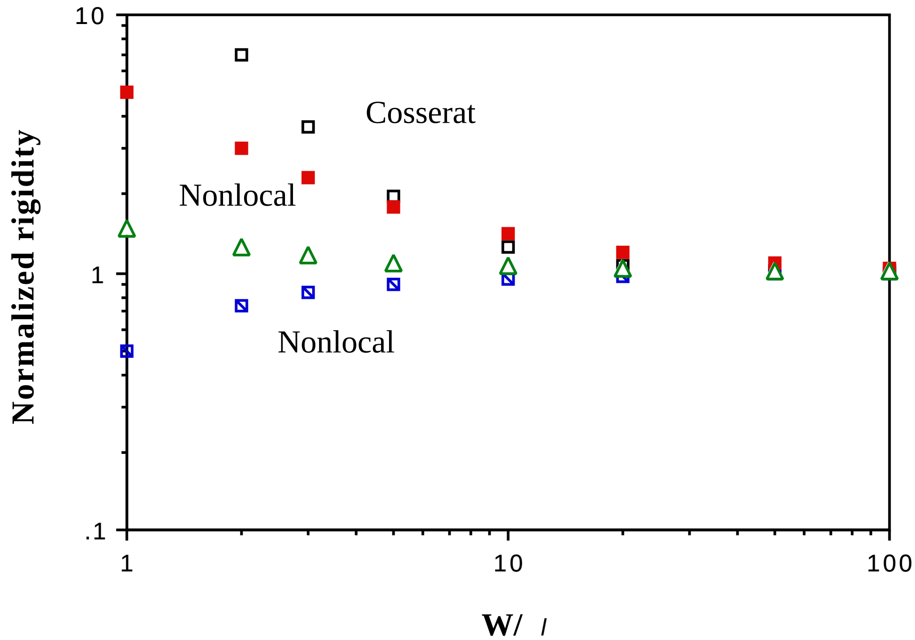
3 Determination of Cosserat elastic constants from size effect data in bending of a circular cylindrical rod. Rigidity/ diameter squared vs diameter squared.

Please see original publication

4 Slab geometry and nonlocal region of influence (over distance a) when that region is entirely within the slab dimensions, from -W/2 to W/2.

Please see original publication

5 Distribution of stress for a nonlocal slab in tension, for a constant kernel over range [-a,a].



6 Comparison of size effects expected in a Cosserat solid and in a nonlocal solid. Normalized rigidity vs slab width. Nonlocal curves are for several kernels. As slab width becomes smaller, Cosserat solid always becomes more rigid, but nonlocal solid can become more or less rigid, depending on the kernel. Observe cross-over of Cosserat and nonlocal curve for a stiffening type nonlocal kernel.

Table 1
Classical and Cosserat elastic technical constants of materials

	Elastic constants Classical			Cosserat				Structure	
Material	E(MPa)	G(MPa)	ν	l_t(mm)	l_b(mm)	N^2	Ψ	Size	Comment
1 Aluminum	73000	--	--	--	<0.03	*	--		Bend plate
2 Aluminum	69000	--	--	--	<0.05	*	--	@0.05mm	Bend plate
3 Steel	212000	--	--	--	<0.05	*	--	@0.05mm	Bend plate
4 KNO_3	@36000	--	--	@6x10^{-8}	@0.03			atomic	Waves
5 Foam, PVC	--	2.8	--	0.95	--	*	--	@1mm	Resonance shear thickness
6 Epoxy/ -- aluminum particle	7000	--	0	--	--	--		@1.4mm	Torsion, rod Classical
7 PMMA	--	1000	--	@0		*	--	@0.1nm	Torsion, rod
8 Human bone	12000	4000	--	0.22	0.45	≥0.5	1.5	@0.2mm	Bend, torsion Anisotropic
9 Graphite, H237	4500	--	0.06	1.6	2.8	*	--	@1.6mm	Bend bar
10 Foam, 0.6 PS	1.1	0.07	3.8	5.0	0.09	1.5		@1mm	Bend, torsion of rods
11 Foam, dense polyurethane	300	104	0.4	0.62	0.33	0.04	1.5	@0.18mm	Bend, torsion of rods
12 Foam, syntactic	2758	1033	0.34	0.065	0.032	0.1	1.5	@0.15mm	Bend, torsion Nearly classical

*: Interpretation based on couple stress theory for which N = 1 (its upper bound) by assumption.
All except waves were done by a size effect approach.
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