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Experimental Results for Almost Global Asymptotic and Locally Exponential Stabilization of the Natural Equilibria of a 3D Pendulum

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Abstract— **Feedback controllers that guarantee asymptotic stabilization of either the hanging equilibrium or the inverted equilibrium of the 3D pendulum are presented and experimentally evaluated using the Triaxial Attitude Control Testbed (TACT). Asymptotic stabilization results obtained previously are shown to guarantee almost global response properties, but the local response properties near the equilibrium show non-exponential convergence. These feedback controllers are modified to guarantee almost global response properties and improved local response near the equilibrium. Experimental results illustrate the closed loop time responses for various feedback controllers.**

I. Introduction

Pendulum models have provided a rich source of examples that have motivated and illustrated many recent developments in nonlinear dynamics and in nonlinear control [1]. An overview of pendulum control problems was given in [2], which provides motivation for the importance of such control problems. Much of the published research treats 1D planar pendulum models or 2D spherical pendulum models or some multibody version of these. In [3], we summarized much of this published research, emphasizing papers that treat control issues, and we introduced a new 3D pendulum model.

This paper provides experimental verification of stabilization results for a 3D rigid pendulum presented in [4]. In [4], we studied stabilization problems for a 3D rigid pendulum defined in terms of the reduced attitude. The reduced attitude is the attitude of the pendulum, modulo rotations about the vertical. In other words, two attitudes have identical reduced attitudes if they differ only by a rotation about the vertical.

A 3D rigid pendulum is supported at a pivot. The pivot is assumed to be frictionless and inertially fixed. The rigid body is asymmetric and the location of the center of mass is distinct from the location of the pivot. Forces that arise from uniform and constant gravity act on the rigid body. Three independent control moments are assumed to act on the rigid body. The 3D pendulum has two natural equilibria, namely *hanging* and *inverted* equilibria. The hanging equilibrium represents the case for which the center of mass lies below the pivot point, whereas the inverted equilibrium represents the case for which the center of mass lies above the pivot point.

We follow the development and notation introduced in [4]. In particular, the formulation of the model depends on construction of a Euclidean coordinate frame fixed to the rigid body with origin at the pivot and an inertial Euclidean coordinate frame with origin at the pivot. Without loss of generality, we assume that the inertial coordinate frame is selected so that the first two axes lie in a horizontal plane and the "positive" third axis points down. The relevant mathematical model is expressed in terms of the angular velocity vector and the reduced attitude vector of the rigid body. The reduced attitude vector is a unit vector in the direction of gravity, expressed in the body fixed coordinate frame. The control problem treated in this paper is global stabilization of an equilibrium of the 3D pendulum defined by zero angular velocity and a reduced attitude vector that corresponds to either the hanging or the inverted equilibrium configuration.

The Triaxial Attitude Control Testbed (TACT), shown in Figure 1, is used to perform experiments on global stabilization of the hanging and the inverted equilibrium. The TACT is an experimental testbed at the University of Michigan, created to study problems in attitude dynamics and control for a rigid body having three rotational degrees of freedom. It has been described in detail in [5] with mathematical models given in [6].

The main contribution of this paper is its summary and experimental verification of results for almost global stabilization of the hanging and the inverted equilibrium of a 3D rigid pendulum and the development of an improved controller that, in addition to global stability, also locally exponentially stabilizes the inverted equilibrium. These results provide almost global asymptotic stabilization in a direct way using a single nonlinear controller, in contrast to a switched control strategy used in [7] for the swing up problem of a spherical pendulum. We also present experimental results on the swing up problem for a 3D pendulum using our experimental testbed.

Thus, experiments verify the stabilization theory presented in [4] for the hanging equilibrium and for the inverted equilibrium. We present a slight modification of the controller in

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[4], and we show that it results in improved performance of the closed-loop. The theory presented for exponential stabilization is then verified by experiments performed on our experimental testbed.

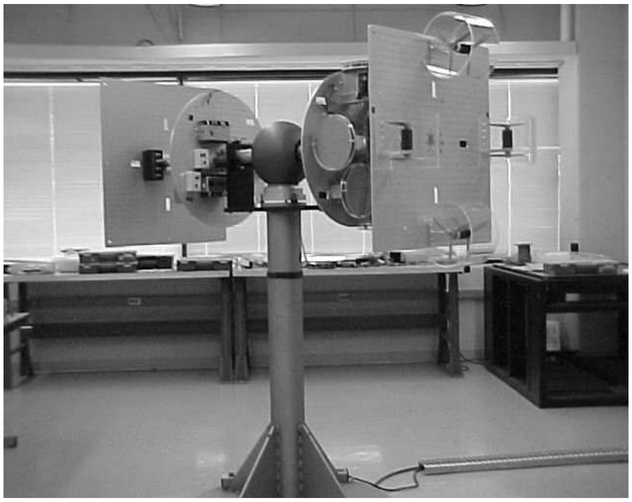

Fig. 1. Triaxial Attitude Control Testbed (TACT)

II. Background

In [4], we studied almost global asymptotic stabilization of the hanging and inverted equilibrium, based on feedback of angular velocity and the reduced attitude vector. We also proposed two controllers that almost globally asymptotically stabilize the hanging and the inverted equilibrium.

By almost global asymptotic stabilization of an equilibrium, we mean that for every initial condition in the phase-space contained in an open and dense set and whose complement is a set of Lebesgue measure zero, the solution converges to this equilibrium. Thus, the domain of attraction of the equilibrium is the whole of the phase-space, excluding a closed set of Lebesgue measure zero.

As shown in [3], the control model for the fully actuated 3D pendulum is given by

$$\begin{cases} J\dot{\omega} = J\omega \times \omega + mg\rho \times \Gamma + u, \\ \dot{\Gamma} = \Gamma \times \omega, \end{cases} \tag{1}$$

where, $\omega \in \mathbb{R}^3$, $\Gamma \in S^2$, and $u \in \mathbb{R}^3$. The phase-space for this system is given by $T\mathrm{SO}(3)/S^1 \simeq S^2 \times \mathbb{R}^3$.

In [4] we proposed a family of controllers that asymptotically stabilize the hanging equilibrium using angular velocity feedback. We state the result for completeness. Let $\Psi : \mathbb{R}^3 \mapsto \mathbb{R}^3$ be a smooth function such that

$$\epsilon_1 \|x\|^2 \leq x^{\mathrm{T}}\Psi(x) \leq \alpha(\|x\|), \quad \forall x \in \mathbb{R}^3, \tag{2}$$

where $\epsilon_1 > 0$, and $\alpha(\cdot)$ is a class-$\mathcal{K}$ function. Choose

$$u = -\Psi(\omega), \tag{3}$$

where $\Psi(\cdot)$ satisfies (2). Then, (3) renders the hanging equilibrium of a 3D pendulum almost globally asymptotically stable.

In [4], we also proposed controllers that asymptotically stabilize the hanging or the inverted equilibrium using both angular velocity and reduced attitude feedback.

Let $\Phi : [0, 1) \mapsto \mathbb{R}$ be a C^1 monotonically increasing function such that $\Phi(0) = 0$ and $\Phi(x) \to \infty$ as $x \to 1$. Let $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth function satisfying (2). Consider a class of controllers given by

$$u = \mathbf{K}(\Gamma)(\Gamma_d \times \Gamma) - \Psi(\omega), \tag{4}$$

where $\mathbf{K}(\Gamma) = \left[\Phi'\left(\frac{1}{4}(\Gamma_d^{\mathrm{T}}\Gamma - 1)^2\right)(1 - \Gamma_d^{\mathrm{T}}\Gamma) - \Gamma_d^{\mathrm{T}}\Gamma_h mg\|\rho\|\right]$ and Γ_d is equal to $\Gamma_h \triangleq \frac{\rho}{\|\rho\|}$ or $\Gamma_i \triangleq -\frac{\rho}{\|\rho\|}$. Then (4) renders the equilibrium Γ_d (either $\Gamma_d = \Gamma_h$ or $\Gamma_d = \Gamma_i$) almost globally asymptotically stable.

III. Implementation of the controller

For experimental implementation on the TACT [5], we choose

$$\Phi(x) = -k \ln(1 - x),$$

where $k > 0$, and $\Psi(x) = Px$, where P is a positive definite matrix. The resulting control law (4) is given by

$$u = -P\omega - mg\rho \times \Gamma + k\frac{\Gamma_d^{\mathrm{T}}\Gamma - 1}{1 - \frac{1}{4}(\Gamma_d^{\mathrm{T}}\Gamma - 1)^2}(\Gamma \times \Gamma_d). \tag{5}$$

Next, denote $\mu = mg\|\rho\|$. Thus, for stabilization of the hanging equilibrium, we obtain the controller

$$u = -P\omega - \mu(\Gamma_h \times \Gamma) + k\frac{\Gamma_h^{\mathrm{T}}\Gamma - 1}{1 - \frac{1}{4}(\Gamma_h^{\mathrm{T}}\Gamma - 1)^2}(\Gamma \times \Gamma_h), \tag{6}$$

and for stabilization of the inverted equilibrium, we obtain the controller

$$u = -P\omega + \mu(\Gamma_i \times \Gamma) + k\frac{\Gamma_i^{\mathrm{T}}\Gamma - 1}{1 - \frac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2}(\Gamma \times \Gamma_i). \tag{7}$$

The scalar parameter μ needs to be estimated.

IV. Estimation of μ

Note that it is difficult to measure the mass m of the TACT and the moment arm length $\|\rho\|$ for the TACT. However, it is much simpler to estimate μ alone. It equals the gravitational moment on the TACT when the vector from the pivot point to the center of mass lies in the horizontal plane.

To estimate μ, we start with the TACT in the hanging equilibrium position. The value for Γ at this position is Γ_h. Next, we command the pitch-axis thrusters to deliver a constant thrust so that the TACT is in equilibrium with a pitch-angle θ that can be measured, as shown in Figure 2. In Figure 2, C is the pivot point, O is the center of mass, AB and HG are the square plates and EF is the pipe that passes through the air bearing. Note that ρ is the vector from the pivot point to the center of mass and the thrust by the fans is applied to the plate HG.

Since we know the mapping from the voltage to torque, we know the applied torque τ. Then it is clear from Figure 2 that $|\tau| = mg\|\rho \sin\theta\| = \mu|\sin\theta|$. Thus, $\mu = |\tau|/|\sin\theta|$. Since τ and θ are measured, μ can be estimated. With the

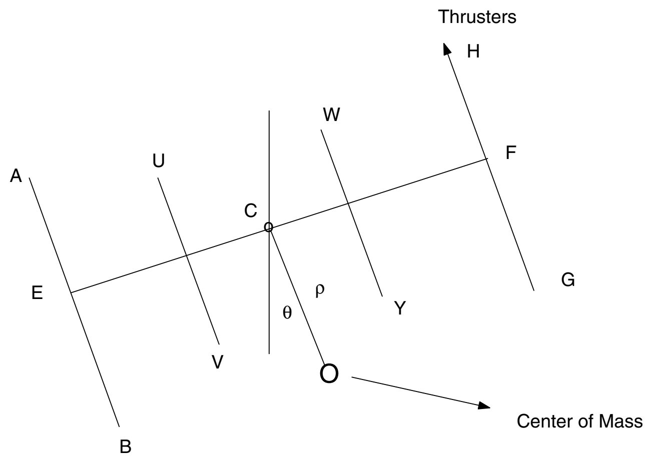

Fig. 2. A schematic of the TACT when it pitches due to a known applied torque

above procedure, we compute the value of μ for various thrust values. These values are presented in Table I. The maximum torque applied to TACT during the tests is $\tau_{\max} = 0.4$ N-m, since for higher pitch axis torque the TACT collides with the supporting pillar.

TABLE I

ESTIMATION OF μ FOR VARIOUS CONSTANT THRUST VALUES

Test Case	Torque (τ) [N-m]	θ [Deg]	$\mu = \tau/\sin\theta$ [kg-m^2-s^{-2}]
(1)	0.1	9.3592	0.6149
(2)	0.2	20.4815	0.5716
(3)	0.3	32.9207	0.5520
(4)	0.4	46.3972	0.5524

From Table I, the average value of μ was computed to be $\mu_{\text{avg}} = 0.5727$ kg-m^2-s^{-2}. Thus, we have the parameter required to implement the proposed control laws.

V. EXPERIMENTS ON STABILIZATION OF THE HANGING EQUILIBRIUM

In this section, we present experimental results for two almost global controllers for stabilization of the hanging equilibrium. The first one is based on angular velocity feedback alone and the other is based on angular velocity and reduced attitude feedback. The angular velocity feedback is a simple dissipation controller given by $u = -P\omega$, where P is a positive definite matrix gain. The hanging equilibrium corresponds to $\Gamma_h = [0\ 0\ 1]^{\mathrm{T}}$ and $\omega = 0$.

In all the experiments, we chose two sets of gains. These are $P = \mathrm{diag}(5, 20, 10)$ and $P = \mathrm{diag}(5, 5, 10)$. These gains were obtained after performing some trial experiments to find one that results in a good performance of the controlled system, in terms of convergence of the error to zero. These same gains were used later in the reduced attitude feedback experiments to contrast the performance of the angular velocity feedback controller with the angular velocity and reduced attitude feedback controller. However, we present results corresponding to the control gain $P = \mathrm{diag}(5, 5, 10)$ only.

A. Angular Velocity Feedback

First, we present the results obtained for the angular velocity feedback controller. The plots for the reduced attitude vector Γ and angular velocities for the controller with $P = \mathrm{diag}(5, 5, 10)$ are presented in Figures 3 and 4. As can be seen, $\Gamma \to [0\ 0\ 1]^{\mathrm{T}} = \Gamma_h$ and $\omega \to 0$.

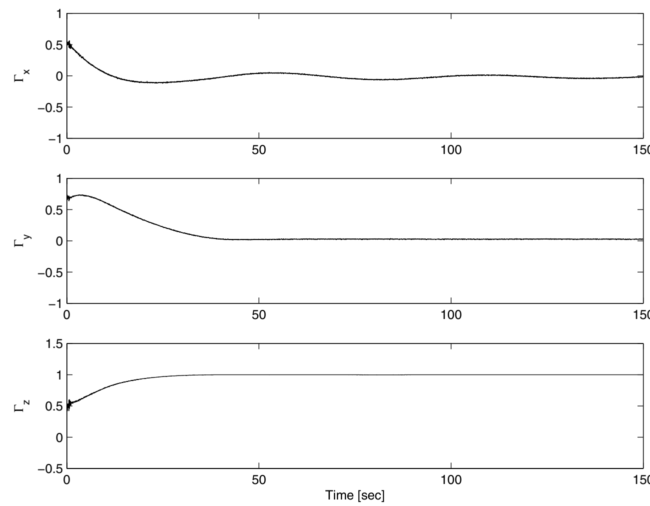

Fig. 3. Plot of the Γ vector for $P = \mathrm{diag}(5, 5, 10)$ in angular velocity feedback controller

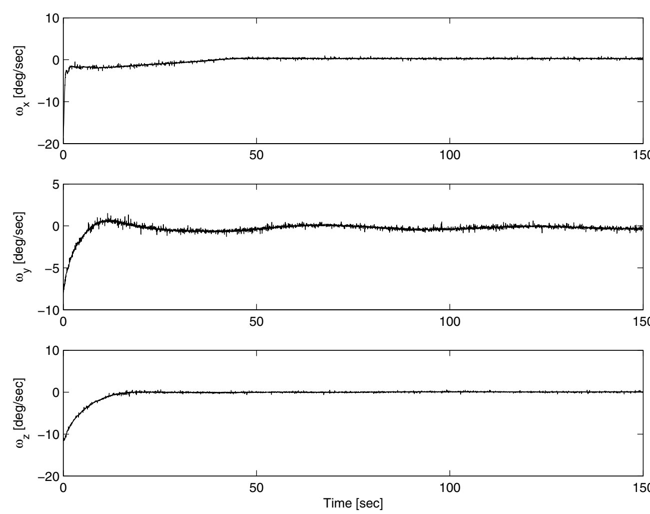

Fig. 4. Plot of the angular velocities for $P = \mathrm{diag}(5, 5, 10)$ in angular velocity feedback controller

B. Angular Velocity and Reduced Attitude Feedback

Next, we present experimental results obtained for the angular velocity and reduced attitude feedback controller. To contrast the performance with the angular velocity feedback controller, we chose the same value of P and chose two gain values for k namely, $k = 30$ and $k = 50$. The gains for

k were chosen by trial and error to obtain reasonable error convergence rates while avoiding excessively high torques. We present results for $k = 30$.

The plots for the reduced attitude vector Γ and angular velocities for the controller with $P = \mathrm{diag}(5, 5, 10)$ and $k = 30$ are presented in figures 5 and 6. Note that the initial transients are faster than the pure angular velocity feedback case. This is because of the potential that is created by the control. However, the final convergence rate is still not fast. As will be seen in the next section, this occurs due to the fact that the closed-loop system is not exponentially stable.

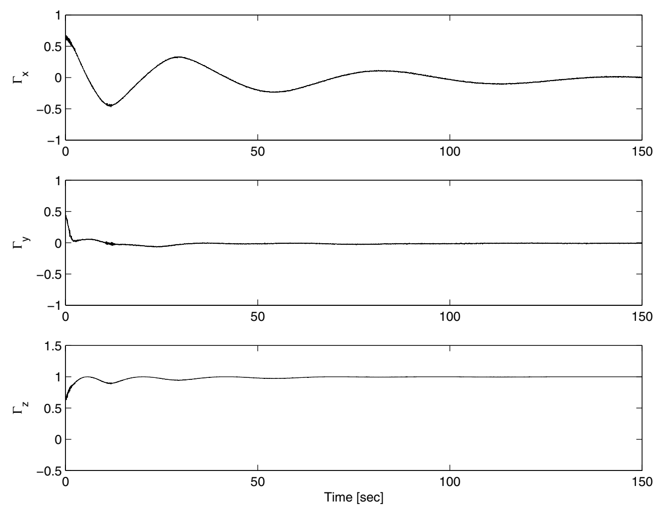

Fig. 5. Plot of the Γ vector for $P = \mathrm{diag}(5, 5, 10)$ and $k = 30$ in angular velocity and reduced attitude feedback controller

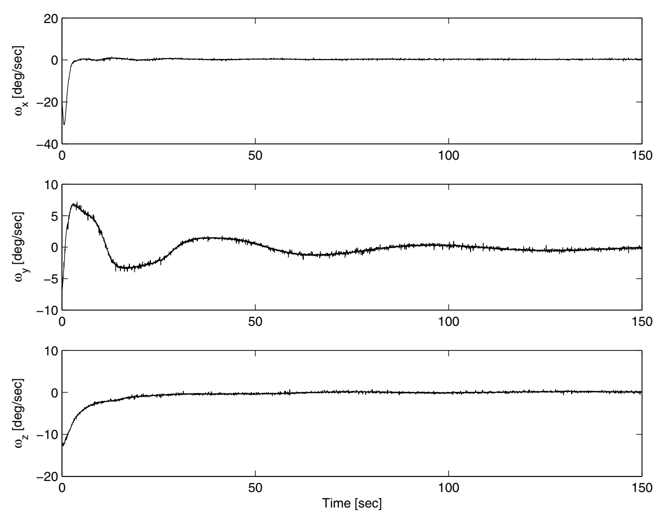

Fig. 6. Plot of the angular velocities for $P = \mathrm{diag}(5, 5, 10)$ and $k = 30$ in angular velocity and reduced attitude feedback controller

VI. Local Exponential Convergence for the Angular Velocity and Reduced Attitude Feedback Controller

From the results presented in the previous section, it is clear that even though the initial transients are faster for the angular velocity and reduced attitude feedback controller than for the angular velocity feedback controller, the asymptotic convergence is still slow, since the controller (4) is not locally exponentially stable. Thus, to improve the performance, we modify the controller to obtain local exponential stability.

We now present a modified controller which improves the local performance of the closed-loop system giving local exponential stability in addition to almost global asymptotic stability. We present the analysis for the inverted equilibrium; the analysis is similar for the hanging equilibrium.

Consider the modified controller

$$u = -\Psi(\omega) + \mathbf{K}(\Gamma)[\Gamma_i \times \Gamma]. \tag{8}$$

where $\mathbf{K}(\Gamma) = \left[\Phi'\left(\frac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2\right)(1 - \Gamma_i^{\mathrm{T}}\Gamma) + \boldsymbol{\kappa}\right]$, $\boldsymbol{\kappa} > \mu$ and $\Psi'(0)$ is positive definite and symmetric. Thus, $\mathbf{K}(\Gamma)$ is a *nonlinear* gain, such that $\mathbf{K}(\Gamma) \geq \mathbf{K}(\Gamma_i) = \boldsymbol{\kappa} > \mu$. Furthermore, $\mathbf{K}(\Gamma) \to \infty$ as $\Gamma \to \Gamma_h$.

We next show that this controller almost globally asymptotically stabilizes the inverted equilibrium with the domain of attraction given by $\left(\mathbb{R}^3 \times S^2 \backslash \{\Gamma_h\}\right)$. Furthermore, the controller improves the performance of the closed-loop system by locally exponentially stabilizing the inverted equilibrium.

The proof of almost global asymptotic stability of the controller (8) is similar to the proof of Theorem 4, in [8]. However, the Lyapunov function is modified corresponding to the new controller in (8). Consider the modified Lyapunov function on $\mathbb{R}^3 \times S^2$

$$\mathcal{V}(\omega, \Gamma) = \frac{\omega^{\mathrm{T}} J \omega}{2} + 2\Phi\left(\frac{(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2}{4}\right) + (\boldsymbol{\kappa} - \mu)(1 - \Gamma_i^{\mathrm{T}}\Gamma). \tag{9}$$

Note that $\mathcal{V}(0, \Gamma_i) = 0$ and that $\mathcal{V}$ is a proper function on $\mathbb{R}^3 \times S^2$. Next, direct computations reveal that $\dot{\mathcal{V}} = -\omega^{\mathrm{T}}\Psi(\omega)$, which is negative semidefinite. The rest of the proof proceeds as in [8].

To see that the modified controller also gives local exponential convergence, we linearize the nonlinear closed-loop equations given by (1) and (8). Since $\dim\left[T\mathrm{SO}(3)/S^1\right] = 5$, the linearized system should evolve on $\mathbb{R}^5$.

VII. Linearization of the Closed-Loop System

First, note that the closed-loop can be written as

$$\begin{cases} J\dot{\omega} = J\omega \times \omega + \Bigg[\Phi'\left(\dfrac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2\right)(1 - \Gamma_i^{\mathrm{T}}\Gamma)\Gamma_i \\ \qquad\qquad + (\boldsymbol{\kappa} - \mu)\Bigg](\Gamma_i \times \Gamma) - \Psi(\omega), \\ \dot{\Gamma} = \Gamma \times \omega. \end{cases} \tag{10}$$

As in [8], we consider a perturbation in Γ and ω in terms of $\Delta\Theta$, where $\Delta\Theta$ represents a perturbation of the rotation matrix in exponential coordinates. Then, the linearization of

(10) in terms of the full attitude is given as

$$\begin{cases} J\Delta\dot{\omega} = -(\boldsymbol{\kappa} - \mu)\dfrac{\widehat{\rho}}{\|\rho\|}\widehat{\Gamma}_i\Delta\Theta - \Psi'(0)\Delta\omega, \\ \Delta\dot{\Theta} = \Delta\omega, \end{cases} \tag{11}$$

where the operator $\wedge : \mathbb{R}^3 \mapsto \mathbb{R}^{3\times 3}$ is the skew-symmetric operator defined in [3], [8] as $\widehat{a}\, b \triangleq a \times b$.

Equation (11) can be expressed as

$$J\Delta\ddot{\Theta} + \Psi'(0)\Delta\dot{\Theta} - (\boldsymbol{\kappa} - \mu)\frac{\widehat{\rho}^{\,2}}{\|\rho\|}\Delta\Theta = 0. \tag{12}$$

Now note that $\widehat{\rho}^{\,2}$ is a rank 2, symmetric, negative-semidefinite matrix. Thus, it follows that one can simultaneously diagonalize both J and the matrix given by $\widehat{\rho}^{\,2}$. Thus, there exists a non-singular matrix H such that $J = HH^{\mathrm{T}}$ and

$$-(\boldsymbol{\kappa} - \mu)\frac{\widehat{\rho}^{\,2}}{\|\rho\|} = H\Omega H^{\mathrm{T}},$$

where Ω is a diagonal matrix [9]. Clearly, the diagonal elements of Ω are the eigenvalues of $-(\boldsymbol{\kappa}/\|\rho\| - mg)$.

Denote $\Omega = \mathrm{diag}(\lambda_1, \lambda_2, 0)$, where λ_1 and λ_2 are positive numbers. Define $x \triangleq H^{\mathrm{T}}\Delta\Theta$ and denote $D = H^{-1}\Psi'(0)\, H^{-\mathrm{T}}$. Note that since $\Psi'(0)$ is symmetric and positive definite, $D^{\mathrm{T}} = D$ and D is positive definite. Now, one can express (12) as

$$\ddot{x} + D\dot{x} + \Omega x = 0. \tag{13}$$

If Ψ is chosen such that D is diagonal, then equation (13) represents a set of three decoupled second order differential equations. The coordinates $x = (x_1, x_2, x_3)$ are the modal coordinates of the linearized equations.

Note that (13) does not depend on x_3. Since $\Gamma = R^{\mathrm{T}}e_3$, to the first order, Γ is approximated by $[I - \widehat{\Delta\Theta}]\Gamma_i$. Then, denoting $\Delta\Gamma \triangleq \Gamma - \Gamma_i = \widehat{\Gamma}_i\Delta\Theta$, it can be shown that $\Delta\Gamma$ depends on (x_1, x_2) only and hence, is independent of the value of x_3. Thus, the linearization of (10) is independent of x_3. Next, we state a result whose proof is similar to Proposition 1 in [8].

Proposition 1: *Consider the fully actuated 3D pendulum given by (1). Choose a controller as given in (5). Then, the inverted equilibrium of the closed-loop (10) is asymptotically stable and the convergence is locally exponential.*

Remark 1: Note that the controller presented in [4] for stabilization of the inverted equilibrium is equivalent to the controller (8) with the value of $\boldsymbol{\kappa}$ chosen exactly to be μ. Thus, the closed-loop system in [4] is not locally exponentially stable.

Remark 2: Another advantage of the modified control law given by (8) is that it modifies the gravity potential forces by domination rather than by cancellation. Thus, we only need an upper bound on the value of μ.

VIII. Experiments on Stabilization of the Inverted Equilibrium: Modified Controller

In this section, we present experimental results obtained for stabilization of the inverted equilibrium. The inverted equilibrium corresponds to $\Gamma_i = [0\ 0\ -1]^{\mathrm{T}}$ and $\omega = 0$. We use the modified controller to obtain both global asymptotic stability and local exponential stability. Note that the upper bound on μ is given by $\mu_{\max} = 0.6149$. Thus, the control law is given as

$$u = -P\omega + k\left[\frac{1 - \Gamma_i^{\mathrm{T}}\Gamma}{1 - \frac{1}{4}(\Gamma_i^{\mathrm{T}}\Gamma - 1)^2} + \boldsymbol{\kappa}\right](\Gamma_i \times \Gamma), \tag{14}$$

where $\boldsymbol{\kappa} > \mu_{\max} = 0.6149$. For the experiments we choose $\boldsymbol{\kappa} = 10$. This gain was selected by trial and error to achieve reasonably fast local responses without large control torques.

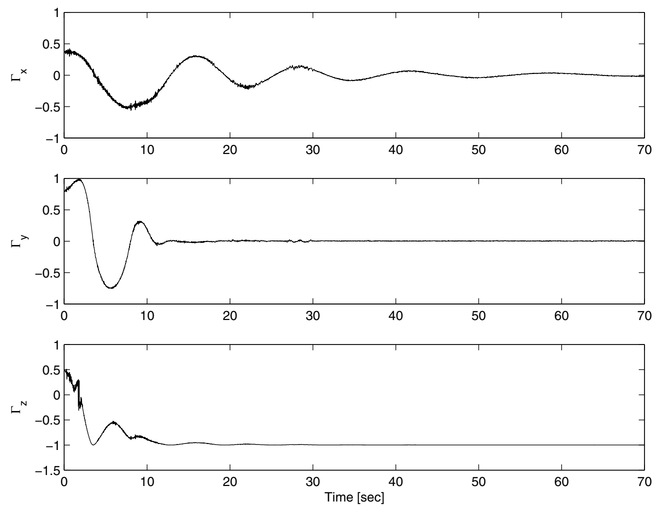

Fig. 7. Plot of the Γ vector for $P = \mathrm{diag}(5, 5, 10)$ and $k = 15$

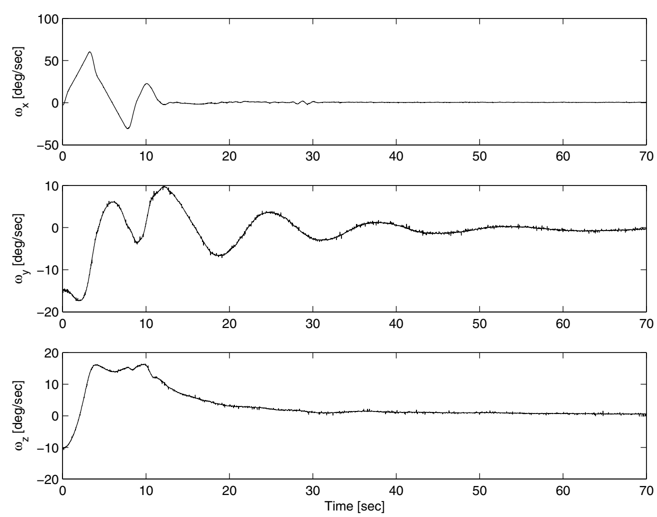

Fig. 8. Plot of the angular velocities for $P = \mathrm{diag}(5, 5, 10)$ and $k = 15$

The experiments were performed for two values of damping gain P. However, we present results corresponding to the damping gain $P = \mathrm{diag}(5, 5, 10)$ and $k = 15$. The gains were chosen by trial and error to obtain reasonable convergence rate while avoiding excessively high control torques. The plots for the reduced attitude vector Γ and angular velocities of the closed-loop system for the controller with $P = \mathrm{diag}(5, 5, 10)$ and $k = 15$ are presented in figures 7 and 8.

IX. Swing-up Problem of the 3D pendulum

In this section, we present results obtained for swinging up the TACT using the controller (14). The control torque is saturated due to the limited torque the thrusters can produce. Although the controller may demand large torque, the saturated controller still stabilizes the inverted equilibrium and swings up the TACT successfully.

Experiments were done for four different sets of gains. In this paper, however, we present results for the gains $P = \mathrm{diag}(5, 5, 10)$ and $k = 30$. Figures 9 and 10 depict plots for the reduced attitude vector Γ and angular velocities for the swing up problem.

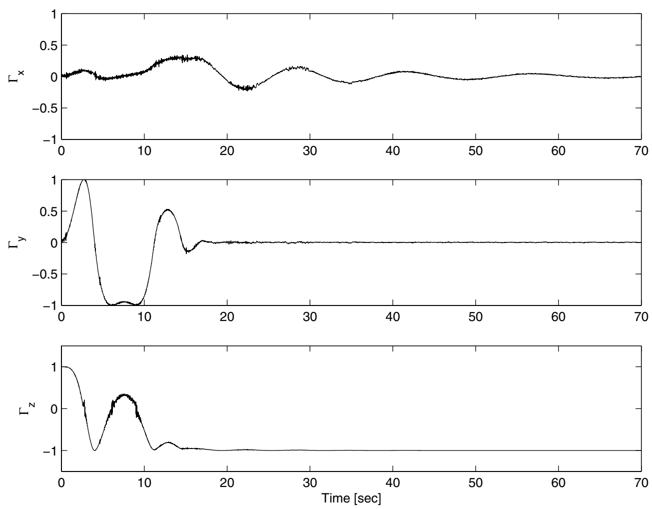

Fig. 9. Plot of the Γ vector for swing-up problem of 3D pendulum for $P = \mathrm{diag}(5, 5, 10)$ and $k = 30$

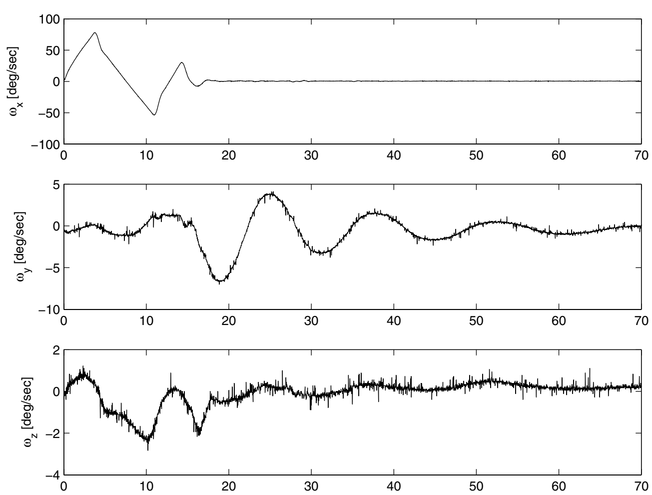

Fig. 10. Plot of the angular velocities for swing-up problem of 3D pendulum for $P = \mathrm{diag}(5, 5, 10)$ and $k = 30$

X. Conclusions

In this paper, we present experimental results that verify the theory of stabilization of the hanging equilibrium and the inverted equilibrium for the 3D rigid pendulum, using controllers designed in [4]. These experiments were performed on the TACT, an experimental testbed at the University of Michigan [5]. We discuss implementation issues and present a method to estimate the parameter required for the controller.

Experimental results for the stabilization of the hanging equilibrium indicate slow convergence due to lack of exponential stability. Thus, a modification for the controller is presented for improved local performance. It is shown that the modified controller provides local exponential stabilization of the inverted equilibrium of a 3D pendulum, with an almost global domain of attraction. Experimental results are presented for stabilization of the inverted equilibrium using the modified controller. Finally, we use this controller to successfully swing up the TACT from rest at the hanging equilibrium to the inverted equilibrium. Thus, experimental results illustrate the closed loop properties of the 3D pendulum system.

One important assumption of this paper is full control actuation. Controllers can be developed to stabilize an equilibrium in the case of underactuation; see for example [10]. As part of our future work, this will be studied by experiments on a TACT implementation.

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