Explicit approximate controllability of the Schrödinger equation with a polarizability term.

Morgan MORANCEY

CMLA, ENS Cachan

Sept. 2011

Control of dispersive equations, Benasque.

Outline

1. Model studied and strategy
   - Bilinear controlled Schrödinger equation and polarizability
   - LaSalle invariance principle

2. Previous results
   - Semiglobal weak stabilization in the dipolar approximation
   - Finite dimension approximation of the polarizability system

3. Explicit approximate controllability with polarizability
   - Study of the averaged system
   - The averaging strategy in infinite dimension

Model studied

- Model of a quantum particle in a potential V

$$
\begin{cases}
i\partial_t \psi = \big(-\Delta + V(x)\big)\psi + u(t)Q_1(x)\psi & , \quad x \in D, \\
\psi_{|\partial D} = 0, \\
\psi(0,\cdot) = \psi^0,
\end{cases}
\tag{1.1}
$$

where

- ψ is the wave function,
- $D \subset \mathbb{R}^m$ is a bounded regular domain,
- $V \in C^\infty(\overline{D}, \mathbb{R})$ is the potential,
- the control u is the real amplitude of the electric field,
- $Q_1 \in C^\infty(\overline{D}, \mathbb{R})$ is the dipolar moment,

Model studied

- Model of a quantum particle in a potential V with a polarizability term.

$$\begin{cases} i\partial_t\psi = \big(-\Delta + V(x)\big)\psi + u(t)Q_1(x)\psi + u(t)^2 Q_2(x)\psi, & x \in D, \\ \psi_{|\partial D} = 0, \\ \psi(0,\cdot) = \psi^0, \end{cases} \tag{1.1}$$

where

- ψ is the wave function,
- $D \subset \mathbb{R}^m$ is a bounded regular domain,
- $V \in C^\infty(\overline{D}, \mathbb{R})$ is the potential,
- the control u is the real amplitude of the electric field,
- $Q_1 \in C^\infty(\overline{D}, \mathbb{R})$ is the dipolar moment,
- $Q_2 \in C^\infty(\overline{D}, \mathbb{R})$ is the polarizability moment.

Notations

- $\mathcal{S} := \left\{\psi \in L^2(D,\mathbb{C}); ||\psi||_{L^2} = 1\right\}.$
- $< f, g > := \int_D f(x)\overline{g(x)}\mathrm{d}x$, for $f, g \in L^2$.
- Let $(\lambda_k)_{k\in\mathbb{N}^*}$ be the non decreasing sequence of eigenvalues of the operator $\big(-\Delta + V\big)$ with domain $H^2 \cap H^1_0$.
- Let $(\varphi_k)_{k\in\mathbb{N}^*}$ be the associated sequence of eigenvectors in $\mathcal{S}$.
- $\mathcal{C} := \{c\varphi_1; c \in C, |c| = 1\}.$

Goal : Find a control u such that $\psi \to \varphi_1$.

LaSalle invariance principle in infinite dimensions. I

1. Lyapunov function. $\mathcal{L} : H \to \mathbb{R}$ non negative, $\mathcal{L}(x) = 0 \iff x = \tilde{x}$ and $\mathcal{L}(x) \underset{x\to\infty}{\longrightarrow} +\infty$.
2. Non increasing along trajectories

$$t \mapsto \mathcal{L}(x(t)) \text{ non increasing },$$

so

$$\mathcal{L}(x(t)) \underset{t\to+\infty}{\to} \alpha.$$

3. Invariant set. We assume x solution of the PDE and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(x(t)) \equiv 0, \forall t \geq 0 \Longrightarrow x(t) \equiv \tilde{x}, \forall t \geq 0.$$

4. Let $(t_n)_{n\in\mathbb{N}} \nearrow +\infty$. $\mathcal{L}(x(t_n)) \leq \mathcal{L}(x(t_0))$ so $(x(t_n))_{n\in\mathbb{N}}$ is bounded.

$$x(t_n) \underset{n\to+\infty}{\rightharpoonup} x_\infty.$$

LaSalle invariance principle in infinite dimensions. II

5. Continuity with respect to the initial condition. Let $x_\infty(\cdot)$ initiated from x_∞.

$$x_n(t) := x(t+t_n) \underset{n\to\infty}{\rightharpoonup} x_\infty(t), \forall t \geq 0.$$

6. Conclusion. Continuity of the Lyapunov function for the weak topology.

$$\mathcal{L}(x_n(t)) \underset{n\to\infty}{\to} \mathcal{L}(x_\infty(t)), \quad \forall t \geq 0,$$
$$\mathcal{L}(x(t_n+t)) \underset{n\to\infty}{\to} \alpha, \quad \forall t \geq 0.$$

So (invariant set)

$$x_\infty(t) = \tilde{x}, \quad \forall t \geq 0,$$

hence

$$x(t) \underset{t\to\infty}{\rightharpoonup} \tilde{x}.$$

- System studied in [Beauchard and Nersesyan, 2010].

$$
\begin{cases}
i\partial_t \psi = \big(-\Delta + V(x) \big)\psi + u(t)Q(x)\psi, \\
\psi_{|\partial D} = 0, \\
\psi(0,\cdot) = \psi^0.
\end{cases}
\tag{2.1}
$$

- Lyapunov function

$$
\mathcal{L}(\psi) := \gamma ||(-\Delta + V)P\psi||^2_{L^2} + \big(1 - |\langle \psi, \varphi_1 \rangle|^2\big),
$$

with P the orthogonal projection on Span $\{\varphi_k, k \geq 2\}$ and $\gamma > 0$.

- Hypotheses
  - $\langle Q\varphi_1, \varphi_k \rangle \neq 0$, for all $k \geq 2$.
  - $\lambda_1 - \lambda_j \neq \lambda_p - \lambda_q$, for all $\{1, j\} \neq \{p, q\}$ and $j \neq 1$.

Semiglobal weak stabilization using feedback law

- We consider the feedback law

$$u(\psi) := -\mathrm{Im}\left[\langle \gamma(-\Delta + V)P(Q\psi), (-\Delta + V)P\psi\rangle - \langle Q\psi, \varphi_1\rangle\langle\varphi_1, \psi\rangle\right]. \tag{2.2}$$

Theorem

Under the previous hypotheses, there exists $J \subset \mathbb{R}_+^*$ finite or countable such that for any $\psi^0 \in \mathcal{S} \cap H_0^1 \cap H^2$ not belonging to $\mathcal{C}$, there exists $\gamma^* := \gamma^*(||\psi^0||_{L^2}) > 0$ such that the solution of the system (2.1) with control u defined in (2.2) with $\gamma \in (0, \gamma^*)\backslash J$ and initial condition ψ^0 satisfies (up to a global phase)

$$\psi(t) \underset{t\to\infty}{\rightharpoonup} \varphi_1, \quad \text{in } H_w^2.$$

Beyond the dipolar approximation I

$$\begin{cases} i\dfrac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \big(H_0 + u(t)H_1 + u(t)^2 H_2\big)\psi(t), \\ \psi(0,\cdot) = \psi^0. \end{cases} \tag{2.3}$$

with $\psi(\cdot) \in \mathbb{C}^n$, H_0, H_1 and H_2 are $n \times n$ Hermitian matrices. $\lambda_1, \cdots, \lambda_n$ eigenvalues of H_0 and $\varphi_1, \cdots, \varphi_n$ the associated eigenvectors.

- Studied in [Grigoriu et al., 2009].
- Improved in [Coron et al., 2009].

Strategy : Use of a time periodic feedback

$$u(t,\psi) := \alpha(\psi) + \beta(\psi) \sin\left(\frac{t}{\varepsilon}\right).$$

Beyond the dipolar approximation II

$$i\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \Big(H_0 + \alpha(\psi)H_1 + \beta(\psi)\sin\left(\frac{t}{\varepsilon}\right)H_1 + \alpha^2(\psi)H_2 + 2\alpha(\psi)\beta(\psi)\sin\left(\frac{t}{\varepsilon}\right)H_2 + \beta^2(\psi)\sin^2\left(\frac{t}{\varepsilon}\right)H_2\Big)\psi(t). \quad (2.4)$$

- Use of the averaged system. Let f be T periodic and $f_{av}(x) = \frac{1}{T}\int_0^T f(t,x)\mathrm{d}t$.

$$\dot{x}(t) = f(t, x(t)) \Longrightarrow \dot{x}_{av}(t) = f_{av}(x_{av}(t)).$$

  This leads to

$$i\frac{\mathrm{d}}{\mathrm{d}t}\psi_{av}(t) = \Big(H_0 + \alpha(\psi_{av})H_1 + \left(\alpha^2(\psi_{av}) + \frac{1}{2}\beta^2(\psi_{av})\right)H_2\Big)\psi_{av}(t). \quad (2.5)$$

- Stabilization of the averaged system.
- Approximation by averaging.

Application of the LaSalle invariance principle I

- Lyapunov function.

$$\mathcal{L}(\psi_{av}(t)) := ||\psi_{av}(t) - \varphi_1||^2.$$

- Choice of the feedbacks.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(\psi_{av}(t)) = 2\alpha I_1(\psi_{av}(t)) + (2\alpha^2 + \beta^2) I_2(\psi_{av}(t)),$$

where $I_j(\psi_{av}(t)) = \mathrm{Im}(\langle H_j \psi_{av}(t), \varphi_1\rangle$.
Let $k \in \left(0, \frac{1}{||H_2||}\right)$. The choice of feedbacks

$$\alpha(\psi_{av}(t)) := -k I_1(\psi_{av}(t)),$$
$$\beta(\psi_{av}(t)) := (I_2(\psi_{av}(t)))^-,$$

leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(\psi_{av}(t)) = -2\Big(k I_1(\psi_{av}(t))^2(1 - k I_2(\psi_{av}(t))) + \frac{1}{2}(I_2(\psi_{av}(t))^-)^3\Big) \leq 0.$$

Application of the LaSalle invariance principle II

- Invariant set. Assume $\lambda_j \neq \lambda_l$ for $j \neq l$ and for any $j \in \{2, \cdots, n\}$, $\langle H_1\varphi_j, \varphi_1\rangle \neq 0$ or $\langle H_2\varphi_j, \varphi_1\rangle \neq 0$. Then

$$\psi_{av}(\cdot) \text{ solution of (2.5) with } \mathcal{L}(\psi_{av}(\cdot)) \text{ constant } \text{ implies } \psi_{av}(\cdot) \equiv \pm\varphi_1.$$

Under the previous hypothesis, the averaged system is globally asymptotically stable on $\mathbb{S}^{2n-1}\backslash\{-\varphi_1\}$.

Approximation by averaging

Lemma of approximation

Let $T > 0$. There exists C and $\varepsilon_0 > 0$ such that, for every $\tau \in \mathbb{R}$ and for every $\varepsilon \in (0, \varepsilon_0)$, if $\psi : [\tau, \tau + T] \to \mathbb{S}^{2n-1}$ is a solution of (2.4), and ψ_{av} is the solution of (2.5) such that $\psi_{av}(\tau) = \psi(\tau)$, then

$$||\psi(t) - \psi_{av}(t)|| < C\varepsilon, \quad \forall t \in [\tau, \tau + T].$$

- Combining this with the convergence of ψ_{av} we obtain

Main result

Assume that the coupling assumption and the non degeneracy of the spectrum hold. Let $\mathcal{V}$ be a neighborhood of $-\varphi_1$ and $\delta > 0$. There exists a time $T > 0$ and $\varepsilon_0 > 0$ such that every solution of (2.4) with $\varepsilon \in (0, \varepsilon_0)$ that satisfies $\psi(\tau) \in \mathbb{S}^{2n-1} \backslash \mathcal{V}$ for some $\tau > 0$ also satisfies

$$||\psi(t) - \varphi_1|| < \delta, \quad \forall t \geq \tau + T.$$

1. Model studied and strategy
   - Bilinear controlled Schrödinger equation and polarizability
   - LaSalle invariance principle

2. Previous results
   - Semiglobal weak stabilization in the dipolar approximation
   - Finite dimension approximation of the polarizability system

3. Explicit approximate controllability with polarizability
   - Study of the averaged system
   - The averaging strategy in infinite dimension

Averaged system

$$\begin{cases} i\partial_t \psi = \big(-\Delta + V(x)\big)\psi + u(t)Q_1(x)\psi + u(t)^2 Q_2(x)\psi, \\ \psi_{|\partial D} = 0, \end{cases}$$

with feedback control $u(t, \psi(t)) := \alpha(\psi(t)) + \beta(\psi(t)) \sin(t/\varepsilon)$ leads to the averaged system

$$\begin{cases} i\partial_t \psi_{av} = \big(-\Delta + V(x)\big)\psi_{av} + \alpha(\psi_{av})Q_1\psi_{av} \\ \qquad\qquad + \left(\alpha(\psi_{av})^2 + \frac{1}{2}\beta(\psi_{av})^2\right) Q_2\psi_{av}, \\ \psi_{av_{|\partial D}} = 0. \end{cases} \tag{3.1}$$

Study of the averaged system and choice of the feedbacks I

- Lyapunov function.

$$\mathcal{L}(z) := \gamma ||(-\Delta + V)Pz||^2_{L^2} + \left(1 - |\langle z, \varphi_1 \rangle|^2\right).$$

- Feedback laws

$$\alpha(z) := -kI_1(z), \quad \beta(z) := g(I_2(z)),$$

  with

$$I_j(z) := \mathrm{Im}\left(\gamma \langle (-\Delta + V)P(Q_j z), (-\Delta + V)Pz \rangle - \langle Q_j z, \varphi_1 \rangle \langle \varphi_1, z \rangle \right),$$

  $k > 0$ small enough and $g \in C^2(\mathbb{R}, \mathbb{R}^+)$ satisfying $g(x) = 0$ if and only if $x \geq 0$, g' bounded.

- Then, $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(\psi_{av}(t)) \leq 0$.
- Under the following assumptions
  - **(H1)** $\langle Q_1 \varphi_1, \varphi_k \rangle = 0 \Longrightarrow \langle Q_2 \varphi_1, \varphi_k \rangle \neq 0$,
  - **(H2)** Card $\{k \geq 2; \langle Q_1 \varphi_1, \varphi_k \rangle = 0\} < \infty$,

Study of the averaged system and choice of the feedbacks II

- **(H3)** $\lambda_1 - \lambda_k \neq \lambda_p - \lambda_q$ for $\{1, k\} \neq \{p, q\}$ and $k \neq 1$,
- **(H4)** $\lambda_p \neq \lambda_q$ for $p \neq q$,

the invariant set is included in $\mathcal{C}$.

- Continuity with respect to the initial condition and continuity of the feedback law for the weak H^2 topology.

Assume that hypotheses **(H1)-(H4)** hold. If $\psi^0 \in X_0 := \left\{z \in \mathcal{S} \cap H_0^1 \cap H^2; \Delta z \in H_0^1 \cap H^2\right\}$ with $0 < \mathcal{L}(\psi^0) < 1$, the solution of (3.1) satisfies (up to a global phase)

$$\psi_{av}(t) \underset{t \to +\infty}{\rightharpoonup} \varphi_1, \quad \text{in } H^2.$$

Approximation by averaging I

For an initial condition $\psi^0 \in X_0$, we consider the control

$$u^\varepsilon(t) := \alpha(\psi_{av}(t)) + \beta(\psi_{av}(t)) \sin(t/\varepsilon),$$

with ψ_{av} the solution of (3.1) satisfying $\psi_{av}(0, \cdot) = \psi^0$.

Let $L > 0$, $\psi^0 \in X_0$ with $0 < \mathcal{L}(\psi^0) < 1$. Let ψ_{av} be the solution of the closed loop system (3.1) with initial condition ψ^0. For any $\delta > 0$, there exists $\varepsilon_0 > 0$ such that if ψ_ε is the solution of (1.1) with initial condition ψ^0 and control u^ε with $\varepsilon \in (0, \varepsilon_0)$, then

$$||\psi_\varepsilon(t) - \psi_{av}(t)||_{H^2} \leq \delta, \quad \forall t \in [0, L].$$

Explicit approximate controllability

Main result

Assume that hypotheses **(H1)-(H4)** hold. For any $s < 2$, for any $\psi^0 \in X_0$ with $0 < \mathcal{L}(\psi^0) < 1$, there exist a strictly increasing time sequence $(T_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+^*$ tending to $+\infty$ and a decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in R_+^* such that if ψ_ε is the solution of (1.1) associated to the control u^ε with $\varepsilon \in (0, \varepsilon_n)$ and initial condition ψ^0,

$$\mathrm{dist}_{H^s}(\psi_\varepsilon(t), \mathcal{C}) \leq \frac{1}{2^n}, \quad \forall t \in [T_n, T_{n+1}].$$

Open Problems

- Convergence in the H^2 norm.
- Card $\{j \geq 2; \langle Q_1\varphi_1, \varphi_j\rangle = 0\} = \infty$.
- Approximation property on infinite time interval $[s, +\infty)$.
- Semi global exact controllability using [Beauchard and Laurent, 2010] in the 1D case with $V = 0$.

Open Problems

- Convergence in the H^2 norm.
- Card $\{j \geq 2; \langle Q_1\varphi_1, \varphi_j\rangle = 0\} = \infty$.
- Approximation property on infinite time interval $[s, +\infty)$.
- Semi global exact controllability using [Beauchard and Laurent, 2010] in the 1D case with $V = 0$.

Thank you for you attention.

Beauchard, K. and Laurent, C. (2010).
Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control.
J. Math. Pures Appl. (9), 94(5):520–554.

Beauchard, K. and Nersesyan, V. (2010).
Semi-global weak stabilization of bilinear Schrödinger equations.
C. R. Math. Acad. Sci. Paris, 348(19-20):1073–1078.

Coron, J.-M., Grigoriu, A., Lefter, C., and Turinici, G. (2009).
Quantum control design by Lyapunov trajectory tracking for dipole and polarizability coupling.
New. J. Phys., 11(10).

Grigoriu, A., Lefter, C., and Turinici, G. (2009).
Lyapunov control of Schrödinger equation:beyond the dipole approximations.
In *Proc of the 28th IASTED International Conference on Modelling, Identification and Control*, pages 119–123, Innsbruck, Austria.