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EXPLICIT ESTIMATES FOR SUMMATORY FUNCTIONS LINKED TO THE MÖBIUS μ-FUNCTION

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Abstract: Let $M(x)$ be the summatory function of the Möbius function and $R(x)$ be the remainder term for the number of squarefree integers up to x. In this paper, we prove the explicit bounds $|M(x)| < x/4345$ for $x \geqslant 2160535$ and $|R(x)| \leqslant 0.02767\sqrt{x}$ for $x \geqslant 438653$. These bounds are considerably better than preceding bounds of the same type and can be used to improve Schoenfeld type estimates.
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1. Introduction

The Möbius function μ is defined by

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1, \\ (-1)^r, & \text{if } n = p_1 p_2 ... p_r \text{ (where the } p_i \text{ are distinct primes)}, \\ 0, & \text{otherwise.} \end{cases}$$

We also define the summatory functions

$$M(x) = \sum_{n \leqslant x} \mu(n),$$

and

$$Q(x) = \sum_{n=1}^{x} |\mu(n)|$$

i.e. $Q(x)$ is the number of squarefree integers up to x.

Finally, we denote by R the remainder term $R(x) = Q(x) - \frac{6}{\pi^2} x$.

[Möbius 1832] introduced the μ-function to obtain inversion formulas for arithmetic functions (in modern terms, μ is the arithmetic convolution inverse of the constant function 1).

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The aim of this paper is to improve substantially on the following explicit upper bounds which were the best to date:

$$|R(x)| \leqslant 0.1333\sqrt{x} \quad \text{for } x \geqslant 1\,664,$$

([Cohen and Dress 1988]) and

$$|M(x)| \leqslant \frac{x}{2\,360} \quad \text{for } x \geqslant 617\,973$$

([Dress and El Marraki 1993]). This latter upper bound was itself an improvement on the result $|M(x)| \leqslant \frac{x}{1\,036}$ for $x \geqslant 120\,727$ due to ([Pereira 1989]).

The results are given simultaneously, since they are obtained using convolution formulas where each improvement of an effective estimate of one of the functions gives an improvement for the other one.

It is reasonable to ask what is the point of obtaining bounds of the form $|M(x)| \leqslant \epsilon x$ for "small" ϵ since one knows by the Prime Number Theorem that $M(x) = o(x)$. The answer to this question is in the practical use of the results.

For example, such bounds are used as the starting point for convolutions which give bounds of the form $C(\alpha)\frac{x}{(\log x)^\alpha}$, and the size of the constant $C(\alpha)$ is strongly dependent on the size of the constant ϵ. (Note that, contrary to the case of the remainder term for the Ψ or Θ functions, there does not exist effective upper bounds for $|M(x)|$ better than bounds of the form $\frac{x}{(\log x)^\alpha}$.) The first bounds of this form have been given by [Schoenfeld 1969]. Using the inequality $|M(x)| < x/4345$ given in the present paper, [El Marraki 1995] has given several bounds of this form, improving by a factor of 15 the corresponding bounds given in [Schoenfeld 1969]. For example, he shows that

$$|M(x)| < \frac{0.002969x}{(\log x)^{1/2}} \quad \text{for } x \geqslant 142194 \;.$$

However, notwithstanding these improvements, the bound $|M(x)| < x/4345$ is still the best available up to $x = 2 \cdot 10^{72}$, and so bounds of the form $|M(x)| \leqslant \epsilon x$ are interesting in themselves.

Note also that bounds of the form $|M(x)| \leqslant \epsilon x$ and those deduced from it, are also essential in the study of the discrepancy of the Farey sequence.

To bound $|R(x)|$, we use the method introduced by [Cohen et Dress 1988].

To bound $|M(x)|$ we use a method initiated by [von Sterneck 1898], which itself was inspired by Tchebychev's method for $\psi(x)$ [Tchebychev 1854]. The method that we use here contains improvements introduced by [Mac Leod 1969] and [Costa Pereira 1989] for constructing a function which in some sense is close to the constant function 1 (see Section 5 below), and a method introduced in [Dress 1977] for fine estimates of the complementary part of the bound.

2. Estimates of $Q(x)$ and $R(x)$: The basic formula

Let n be a positive integer which we shall choose later. For $j = 1, 2, ..., n$, we set $X_j := \sqrt{\frac{x}{j}}$. We have:

$$Q(x) = \sum_{a^2 \leqslant x} \mu(a)\left[\frac{x}{a^2}\right] = \sum_{a \leqslant X_n} \mu(a)\left[\frac{x}{a^2}\right] + \sum_{X_n < x \leqslant X_1} \mu(a)\left[\frac{x}{a^2}\right].$$

In the second sum, we group terms giving the same value of $\left[\frac{x}{a^2}\right]$. Since $\left[\frac{x}{a^2}\right] = j$ if and only if $\left[X_{j+1}\right] + 1 \leqslant a \leqslant \left[X_j\right]$, we get:

$$\begin{aligned} Q(x) &= \sum_{a \leqslant X_n} \mu(a)\left[\frac{x}{a^2}\right] + \sum_{j=1}^{n-1}\Bigg(\sum_{a=[X_{j+1}]+1}^{[X_j]} \mu(a)\Bigg) j \\ &= \sum_{a \leqslant X_n} \mu(a)\left[\frac{x}{a^2}\right] + \sum_{j=1}^{n-1} j\big(M(X_j) - M(X_{j+1})\big), \end{aligned}$$

hence, finally:

$$Q(x) = \sum_{a \leqslant X_n} \mu(a)\left[\frac{x}{a^2}\right] + \sum_{j=1}^{n-1} M(X_j) - (n-1)M(X_n). \tag{2.1}$$

This formula can be used in several ways, with quite different optimal values of n:

– numerical computation of $Q(x)$; if we can store in the computer's main memory all the values $M(y)$ up to $\sqrt{x}$, we must the minimize the number of terms in formula (2.1), which gives an n of the order of $x^{1/3}$, and the running time is also $O(x^{1/3})$;

– obtaining upper bounds for $|R(x+y) - R(x)|$; depending on the size of y with respect to x, the optimal value of n will either be $x^{1/3}$ or $\frac{x}{y}$; this is how the following bounds were obtained:

$$|R(x+y) - R(x)| < 0.7343\,\frac{y}{x^{1/3}} + 1.4327\,x^{1/3} \tag{2.2}$$

et

$$|R(x+y) - R(x)| < 1.6749\,\sqrt{y} + 0.6212\,\frac{x}{y}, \tag{2.3}$$

valid for all x and $y \geqslant 1$, given in [Cohen and Dress 1988]. Contrary to the other bounds, the bounds for $|R(x+y) - R(x)|$ depend very little on the quality of the effective bounds for $|M(x)|$ and cannot be substantially improved.

– bounds for $|R(x)|$, using a bound for $|M(x)|$ of the form $\epsilon(x)x$, where ϵ is either a constant or a slowly varying function which tends to 0 when $x \to \infty$.

We then choose n close to $\frac{1}{4\pi\sqrt{3\epsilon(x)}} \# \frac{0.046}{\sqrt{\epsilon(x)}}$, and we then get a bound close to $1.7\sqrt{\epsilon(x)}\sqrt{x}$. It is this type of bounds that we study in detail below (under the Riemann hypothesis, we would obtain in this way a bound of the form $O(x^{1/4}\log x)$).

To obtain bounds for $|R(x)|$ in terms of $\sqrt{x}$, we transform the first term in the fundamental formula (2.1):

$$\sum_{a\leqslant X_n} \mu(a)\Big[\frac{x}{a^2}\Big] = x \sum_{a\leqslant X_n} \frac{\mu(a)}{a^2} - \frac{1}{2}M(X_n) - \sum_{a\leqslant X_n} \mu(a)\left(\left\{\frac{x}{a^2}\right\} - \frac{1}{2}\right),$$

and we also have

$$\begin{aligned}\sum_{a\leqslant X_n} \frac{\mu(a)}{a^2} &= \frac{6}{\pi^2} - \sum_{a>X_n} \frac{\mu(a)}{a^2} = \frac{6}{\pi^2} - \int_{X_n}^{\infty} \frac{dM(t)}{t^2}\\ &= \frac{6}{\pi^2} + \frac{M(X_n)}{X_n^2} - 2\int_{X_n}^{\infty} \frac{dM(t)}{t^3}.\end{aligned}$$

Grouping all the terms, we obtain

$$\begin{aligned}R(x) &= Q(x) - \frac{6}{\pi^2}\,x\\ &= -2x\int_{X_n}^{\infty} \frac{dM(t)}{t^3} + \sum_{j=1}^{n-1} M(X_j) + \frac{1}{2}M(X_n) - \sum_{a\leqslant X_n} \mu(a)\left(\left\{\frac{x}{a^2}\right\} - \frac{1}{2}\right).\end{aligned}$$

We will use this formula in the form $|R(x)| \leqslant T_1 + T_2 + T_3$, where

$$\begin{aligned}T_1 &:= 2x\int_{X_n}^{\infty} \frac{|M(t)|}{t^3}\,dt,\\ T_2 &:= \sum_{j=1}^{n-1} |M(X_j)| + \frac{1}{2}|M(X_n)|,\\ T_3 &:= \sum_{a\leqslant X_n} \left|\mu(a)\left(\left\{\frac{x}{a^2}\right\} - \frac{1}{2}\right)\right|.\end{aligned}$$

Initially, we can apply these formulas by using the bounds $|M(x)| \leqslant \frac{x}{2\,360}$ for $x \geqslant 617\,973$ (see introduction).

As mentioned in the introduction, there will be an interaction between all the bounds. We will first give the basic lemma, not with the bound $\frac{x}{2\,360}$ but with the parametric bounds $|M(x)| < \epsilon\, x$ for $x \geqslant x_1 = x_1(\epsilon)$ and $|R(x)| < b\sqrt{x}$ for $x \geqslant x_2 = x_2(b)$.

Lemma 1. *For* $X_n = \sqrt{\frac{x}{n}} \geqslant \max(x_1, x_2)$, *we have*

$$T_1 \leqslant 2\epsilon \sqrt{n}\sqrt{x}$$
$$T_2 \leqslant 2\epsilon \sqrt{n}\sqrt{x} - 1.46\,\epsilon\sqrt{x}$$
$$T_3 \leqslant \sqrt{\Big(\frac{6\sqrt{x}}{\pi^2\sqrt{n}} + b\frac{x^{1/4}}{n^{1/4}}\Big)\Big(\frac{\sqrt{x}}{18\sqrt{n-0.02}} + \frac{(12x)^{1/3}}{6} - \frac{2}{3}(n-\frac{1}{2})\Big)}$$

The proofs of the first two bounds are immediate:

$$T_1 = 2x\int_{X_n}^{\infty} \frac{|M(t)|}{t^3}dt \leqslant 2\epsilon x \int_{X_n}^{\infty}\frac{dt}{t^2} = 2\epsilon\,\frac{x}{X_n},$$

and

$$T_2 = \sum_{j=1}^{n-1} |M(X_j)| + \frac{1}{2}\left|M(X_n)\right| \leqslant \epsilon\Big(X_1 + X_2 + ... + X_{n-1} + \frac{1}{2}X_n\Big)$$
$$= \epsilon\sqrt{x}\Big(1 + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n-1}} + \frac{1}{2\sqrt{n}}\Big)$$

and the result follows from the elementary inequality $\Big(1+\frac{1}{\sqrt{2}}+...+\frac{1}{\sqrt{n-1}}+\frac{1}{2\sqrt{n}}\Big) < 2\sqrt{n} - 1.46$.

The proof of the third bound is much more tricky. We first use Schwartz's inequality:

$$T_3 \leqslant \sqrt{\sum_{a\leqslant X_n} \mu^2(a)}\sqrt{\sum_{a\leqslant X_n} |\mu(a)|\big(\{\frac{x}{a^2}\} - \frac{1}{2}\big)^2}.$$

Let $h(a)$ be the characteristic function of integers which are not divisible by 4 or by 9. We have $|\mu(a)| \leqslant h(a)$, so if we set:

$$S(z) := \sum_{a\leqslant z} h(a)\Big(\{\frac{x}{a^2}\} - \frac{1}{2}\Big)^2,$$

we have

$$T_3 \leqslant \sqrt{Q(X_n)}\sqrt{S(X_n)}.$$

We trivially have

$$Q(X_n) \leqslant \frac{6\sqrt{x}}{\pi^2\sqrt{n}} + b\,\frac{x^{1/4}}{n^{1/4}}.$$

To bound $S(X_n)$, we set $S_k := \sum_{X_{k+1}+1\leqslant a\leqslant X_k} h(a)\big(\{\frac{x}{a^2}\}\big)^2$ and $H(z) := \sum_{a\leqslant z} h(z) = [z] - \left[\frac{z}{4}\right] - \left[\frac{z}{9}\right] + \left[\frac{z}{36}\right]$. Thus, we have $S(X_n) < S_n + S_{n+1} + ... + S_N + \frac{1}{4}H(X_N)$.

We first note that $H(z) = \frac{2}{3}z + \rho(z)$ with $|\rho(z)| \leqslant \frac{4}{3}$.

An easy but detailed and delicate study of a few functions gives the following inequality:

$$S_k = \left(\left(\frac{16k}{9} + \frac{4}{3} + \frac{1}{6k}\right)\sqrt{k} - \left(\frac{16k}{9} + \frac{4}{9} + \frac{1}{6(k+1)}\right)\sqrt{k+1}\right)\sqrt{x} + \frac{1}{4}\left(\rho\left(\sqrt{\frac{x}{k}}\right) - \rho\left(\sqrt{\frac{x}{k+1}}\right)\right) + r_k \quad \text{with} \quad |r_k| \leqslant \frac{2}{3}.$$

For $k \geqslant 4$, the main term can be bounded by

$$\frac{1}{18}\left(\frac{1}{\sqrt{k-0.02}} - \frac{1}{\sqrt{k+0.98}}\right)\sqrt{x}.$$

Finally, we obtain

$$S(X_n) \leqslant \frac{\sqrt{x}}{18\sqrt{n-0.02}} + \frac{1}{9}\left(\sqrt{\frac{x}{N}} + 6N\right) - \frac{2}{3}\left(n - \frac{1}{2}\right).$$

The best bound is obtained by choosing $N = \left[\left(\frac{x}{144}\right)^{1/3}\right]$, which gives

$$S(X_n) \leqslant \frac{\sqrt{x}}{18\sqrt{n-0.02}} + \frac{(12\,x)^{1/3}}{6} - \frac{2}{3}\left(n - \frac{1}{2}\right)$$

as claimed.

Note that the bounds for $Q(X_n)$ and $S(X_n)$ are valid for $n \geqslant 4$ and $\sqrt{\frac{x}{n}} \geqslant x_2$.

3. Estimates for small x

The bound for T_3 given in the above lemma is useful only when x is rather large. Hence it is important to have estimates for "small" values of x.

Proposition 2. *We have*

$$\left|R(x)\right| \leqslant 0.02\sqrt{x} \quad \text{for } x \in \left[2\,050\,244\,,\ 1.6\,10^{15}\right].$$

We first show that the bounds (2.2) and (2.3) for $|R(x+y) - R(x)|$ imply that, if we know that $|R(x)| \leqslant 0.001\sqrt{x}$ and $|R(1.0001\,x)| \leqslant 0.001\sqrt{1.0001\,x}$ then $|R(z)| \leqslant 0.02\sqrt{z}$ for all z in the interval $[x, 1.0001\,x]$. We first use (2.2) and (2.3) together to give a single increasing bound $f_x(y)$ for $|R(x+y) - R(x)|$, where

$$f_x(y) = 0.7343\,\frac{y}{x^{1/3}} + 1.4327\,x^{1/3} \text{ if } y < 1.3007\,x^{2/3}$$
$$f_x(y) = 1.6749\,\sqrt{y} + 0.6212\,\frac{x}{y} \text{ if } y \geqslant 1.3007\,x^{2/3}.$$

We have $f_x(0.00005\,y) = 0.36715\,10^{-4}\,x^{2/3} + 1.4327\,x^{1/3}$ if $x < (26014)^3 = 1.76044\,10^{13}$, and $f_x(0.00005\,y) = 1.18433\,10^{-2}\sqrt{x} + 12414$ if $x \geqslant 1.76044\,10^{13}$. Hence our claim is valid as soon as $f_x(0.00005\,y) \leqslant 0.019\sqrt{x}$, which is true when $x \geqslant 5.92\,10^{11}$.

The explicit computations of the necessary values of $R(x)$ are done using formula (2.1) with $x = \left[\frac{x^{1/3}}{2}\right]$.

Thus we have proved that the bound of Proposition 2 is valid on the interval

$$[5.92\,10^{11}\,,\ 1.6\,10^{15}].$$

It is a simple (but long) matter to check explicitly that it is also valid on the interval $[2\,050\,244, 5.92\,10^{11}]$, thus proving the proposition.

4. Estimates for $Q(x)$ and $R(x)$: Numerical results

Theorem 3. *We have*

$$\left|R(x)\right| \leqslant 0.036438\sqrt{x} \qquad \text{for } x \geqslant 82\,005$$

Form $82\,005$ to $2\,050\,244$, we check directly. From $2\,050\,244$ to $1.6\,10^{15}$, the result follows from the proposition above. Above $1.6\,10^{15}$ we use the lemma of Section 2 with $\epsilon = \frac{1}{2\,360}$, $x_1 = 617\,973$, $b = 0.1333$, $x_2 = 1\,664$. One easily checks that the optimal value of n is $n = 109$. Thus, for $x \geqslant 1.6\,10^{15}$ we obtain

$$Q(X_n) \leqslant \sqrt{\frac{x}{109}}\left(\frac{6}{\pi^2} + b\frac{109^{1/4}}{(1.6\,10^{15})^{1/4}}\right)$$
$$S(X_n) \leqslant \sqrt{\frac{x}{109}}\left(\frac{\sqrt{109}}{18\sqrt{107.98}} + \frac{12^{1/3}\,109^{1/2}}{6(1.6\,10^{15})^{1/6}}\right)$$

hence

$$T_3 \leqslant \sqrt{Q(X_n)S(X_n)} = 0.019361\sqrt{x}.$$

Since

$$T_1 + T_2 \leqslant 0.847458\,10^{-3}\left(2\sqrt{109} - 0.73\right)\sqrt{x} = 0.017077\sqrt{x},$$

Theorem 3 follows.

5. Estimates for $M(x)$: description of the method

Let E be the function $E(x) = \sum_{n\leqslant x}\mu(n)\left[\frac{x}{n}\right]$, which is constant and equal to 1, and which trivially satisfies the condition

$$\sum_{n\leqslant x}\mu(n)\left(1 - E\left(\frac{x}{n}\right)\right) = M(x) - \sum_{n\leqslant x}\mu(n).$$

We consider an approximation $F(x) = \sum_{i=1}^{m} c_i \left[\frac{x}{a_i}\right]$ of this function. We have

$$\sum_{n\leqslant x} \mu(n)\Big(1 - F\big(\frac{x}{n}\big)\Big) = M(x) - \sum_{a_i\leqslant x} c_i \ .$$

If $\sum_{i=1}^{m} \frac{c_i}{a_i} = 0$ and if F is a "good" approximation of E, we can thus obtain good bounds for $|M(x)|$. We refer to [El Marraki 1991] for a detailed description of this method.

Here, we use the same method with a more efficient function F and better bounds for $R(x)$ (which are used in the final estimates).

We use the following function

$$F(x) = \sum_{i=1}^{m} c_i \left[\frac{x}{a_i}\right] + c_{m+1}\left[\frac{x}{a_{m+1}}\right] + c_{m+2}\left[\frac{x}{a_{m+2}}\right]$$

with the following four conditions:

(i) $F(x) = 1$ for $1 \leqslant x \leqslant k_0$,

(ii) the function $G(x) := |1 - F(x)|$ is close to 1 for $k_0 \leqslant x \leqslant 2\,k_0$,

(iii) $I(F) := \frac{6}{\pi^2} \int \frac{|1-F(u)|}{u^2} du$ is "small",

(iv) the function F is periodic, i.e. $\sum_{i=1}^{m+2} \frac{c_i}{a_i} = 0$ (this last condition is absolutely necessary).

We build the function F in the following way. For condition (i), we choose an integer k_0, we set $m_1 = Q(k_0 - 1)$, and we define the first m_1 terms of F by setting (a_i) to be the increasing sequence of squarefree integers less than or equal to k_0 and $c_i = \mu(a_i)$. We then add corrective terms of the form $c_i\left[\frac{x}{a_i}\right]$ so that condition (ii) is satisfied. We then compute the integral $I(f)$, and we add other terms so that condition (iii) is satisfied. Finally, we add two terms with not necessary integral a_i so that the periodicity condition (iv) is satisfied.

We have written a program which, given k_0, automatically computes a number of possible functions, which only differ by the number of corrective terms. Heuristic reasons and experimental observation show that efficient values of k_0 are obtained when both $M(k_0)$ and $m(k_0) = \sum_{n\leqslant k_0} \frac{\mu(n)}{n}$ are small. The upper bound $|M(x)| \leqslant \frac{x}{2\,360}$ (see Introduction) was obtained using a function where $k_0 = 26\,442$, for which $M(k_0) = -1$ and $m(k_0) = 1.59\,10^{-5}$.

Here we will use a function where $k_0 = 100\,882$, for which $M(k_0) = 1$ and $m(k_0) = 4.34\,10^{-7}$. This function F, given by our program, is built with $m_1 = 61\,334$ and $m = 63\,951$ as follows:

$$F(x) = \sum_{i=1}^{61\,334} c_i \left[\frac{x}{a_i}\right] + \sum_{i=61\,335}^{63\,951} c_i \left[\frac{x}{a_i}\right] + \left[\frac{x}{1\,436\,009.0}\right] - \left[\frac{x}{47\,666\,734\,237\,381.39}\right]$$

We obtain an upper bound $\max(G)$ of G by splitting $G = |1 - F|$ into an affine part (which vanishes) and a periodic part:

$$\begin{aligned}|1-F(x)| &= \Big|1+\frac{1}{2}\sum_{i=1}^{m+2} c_i - x\sum_{i=1}^{m+2}\frac{c_i}{a_i} + \sum_{i=1}^{m+2} c_i\Big(\Big\{\frac{x}{a_i}\Big\}-\frac{1}{2}\Big)\Big| \\ &\leqslant \Big|1+\frac{1}{2}\sum_{i=1}^{m+2} c_i\Big| + 0 + \Big|\sum_{i=1}^{m_1} c_i\Big(\Big\{\frac{x}{a_i}\Big\}-\frac{1}{2}\Big)\Big| + \Big|\sum_{i=m_1+1}^{m+2} c_i\Big(\Big\{\frac{x}{a_i}\Big\}-\frac{1}{2}\Big)\Big|.\end{aligned}$$

The term $\big|1+\frac{1}{2}\sum_{i=1}^{m+2} c_i\big|$ is equal to 2.

We bound the term $\big|\sum_{i=1}^{m_1} c_i\big(\{\frac{x}{a_i}\}-\frac{1}{2}\big)\big|$ by using a technique due to Costa Pereira and explained in detail in Lemma 5.1 of [El Marraki 1991]. The result is that this term is bounded by $14\,891$. Finally, the term $\big|\sum_{i=m_1+1}^{m+2} c_i\big(\{\frac{x}{a_i}\}-\frac{1}{2}\big)\big|$ can be trivially bounded by $\frac{1}{2}\sum_{i=m_1+1}^{m+2}|c_i|$, which gives 7634.5.

The properties of the function F can be summarized in the following table:

k_0	m_1	m	$\max(G)$
100 882	61 334	63 951	22 527.5

6. Estimates for $M(x)$: Numerical results

To bound $|M(x)|$ we use the same method as in [Dress et El Marraki 1993]. It can be summarized in the following lemma:

Lemma 4. *If $x \geqslant \tau$ and $\tau \geqslant \max(a_i, (N+1)x(b))$, we have*

$$\big|M(x)\big| \leqslant \Big(\frac{6}{\pi^2}u + \frac{b\,v}{\sqrt{\tau}} + \frac{w}{\tau}\Big)x,$$

where, if we denote by N the upper bound for direct computations,

$$u(N) = \sum_{n=1}^{N}\frac{1}{n}\big(G(n)-G(n-1)\big) \quad - \quad \frac{1}{N+1}\,G(N),$$

$$v(N) = \sum_{n=1}^{N}\frac{1}{\sqrt{n}}\big|G(n)-G(n-1)\big| \quad - \quad \frac{1}{\sqrt{N+1}}\,G(N)$$

$$u = u(N) + \frac{1}{N+1}\,\max(G),\;\; v = v(N) + \frac{1}{\sqrt{N+1}}\,\max(G),\;\; w = \Big|\sum_{i=1}^{m+2} c_i\Big|.$$

Recall also the following two results.

$$\big|M(x)\big| \leqslant 0.570591\sqrt{x} \qquad \text{for } 33 \leqslant x \leqslant 10^{12} \tag{6.1}$$

[Dress 1993]

$$\big|M(x)\big| \leqslant \frac{x}{4\,257} \qquad \text{for } 10^{12} \leqslant x \leqslant 2.590\,10^{15} \tag{6.2}$$

[Dress and El Marraki 1993].

Theorem 5. *We have*

$$\left|M(x)\right| \leqslant \frac{x}{4\,257} \qquad \text{for } x \geqslant 2\,159\,561$$

To prove this, we use the function F given above (for which $w = 8$), the value $b = 0.036438$ given in Theorem 3, and $\tau = 2.590\,10^{15}$. We split the interval $[2\,159\,561\,;\,+\infty[$ in four consecutive sub-intervals as follows.

(i) For $x \in [2\,159\,561\,;5\,900\,070]$: we make a direct check on the computer.
(ii) For $x \in [5\,900\,070\,;\,10^{12}]$: the bound (6.1) implies $|M(x)| \leqslant \frac{x}{4257}$.
(iii) For $x \in [10^{12}\,;\,2.590\,10^{15}]$: this is (6.2).
(iv) Finally, for $x \in [2.590\,10^{15}, +\infty]$: the bound follows from the use of the function $F(x)$ as explained in Lemma 4.

The first table below gives the basic results for using Lemma 4 up to $N = 1.62\,10^9$. The total running time on a Sparcstation 10 was approximately 6 months. We decided to stop there since we would only very slightly improve the final result by going any further.

$\frac{N}{10^6}$	$u(N)$	$v(N)$
60	0.00033225264406	6993.26247
120	0.00033278045122	9798.92803
180	0.00033297524343	11924.63733
240	0.00033307188379	13701.60869
300	0.00033313444950	15258.91664
360	0.00033317548479	16659.86951
420	0.00033320547881	17943.01439
480	0.00033322821023	19132.56364
540	0.00033324602713	20245.63763
600	0.00033326069524	21294.91286
660	0.00033327290186	22290.10075
720	0.00033328301510	23238.79402
780	0.00033329184416	24146.22679
840	0.00033329909164	25017.15026
900	0.00033330533451	25855.13303
960	0.00033325085140	26664.09029
1020	0.00033332552935	27446.77662
1080	0.00033332997683	28205.29763
1140	0.00033333387044	28941.39330
1200	0.00033333739947	29656.91071
1260	0.00033334068787	30353.50586
1320	0.00033334359056	30560.02903
1380	0.00033334623636	31222.56618
1440	0.00033334870142	31869.60823
1500	0.00033335094814	32502.25952
1560	0.00033335300301	33121.36884
1620	0.00033335488681	33727.85805

The second table gives the results obtained after applying Lemma 4, with $b = 0.036438$ and $\tau = 2.590\,10^{15}$. The term $w(=8)$, which only changes the 10-th significant figure, has been completely neglected. The bound is given through the value $u = u(N) + \frac{1}{N+1}\max(G)$, $v = v(N) + \frac{1}{\sqrt{N+1}}\max(G)$, $M = \frac{6}{\pi^2}u + \frac{bv}{\sqrt{\tau}}$ and $\frac{1}{M}$.

$\frac{N}{10^6}$	$\frac{6}{\pi^2}u$	$\frac{bv}{\sqrt{\tau}}$	$M = \frac{6}{\pi^2}u + \frac{bv}{\sqrt{\tau}}$	$\frac{1}{M}$
60	0.00043023667963	0.00000500916169	0.00043524584133	2297.55
180	0.00027850843979	0.00000853907793	0.00028704751773	3483.74
300	0.00024817171955	0.00001092610491	0.00025909782447	3859.55
420	0.00023517196904	0.00001284773765	0.00024801970670	4031.94
540	0.00022795054659	0.00001449629117	0.00024244683777	4124.62
660	0.00022335574718	0.00001596003228	0.00023931577946	4178.58
780	0.00022017493691	0.00001728894260	0.00023746387950	4211.17
900	0.00021784209912	0.00001851245600	0.00023635455513	4230.93
1020	0.00021606416987	0.00001965201822	0.00023571618809	4242.39
1140	0.00021465591992	0.00002072211581	0.00023537803574	4248.48
1200	0.00021405740402	0.00002123440431	0.00023529180832	4250.04
1260	0.00021351594765	0.00002173314560	0.00023524909325	4250.81
1320	0.00021302366185	0.00002188100289	0.00023490466474	4257.05
1380	0.00021257418079	0.00002235536058	0.00023492954137	4256.60

Note that $I(F) := \frac{6}{\pi^2}\int \frac{|1-F(u)|}{u^2}du\# \frac{1}{4\,930}$.

It is trivial to control the accuracy of the computations: we deal with sums $\sum_n \frac{1}{n}\big(G(n) - G(n-1)\big)$ and $\sum_n \frac{1}{\sqrt{n}}\big|G(n) - G(n-1)\big|$, where $G(n) - G(n-1)$ is exact since it is integral. If we sum up to $N = 1.62\,10^9$ using double precision floating point real numbers in C, which have a relative accuracy of at least 10^{-15}, the accuracy of the final result is at least 10^{-6}, which is more than enough.

This finishes the proof of Theorem 5.

7. Final Numerical Estimates

Theoreme 3 bis. *We have*

$$\big|R(x)\big| \leqslant 0.02767\sqrt{x} \quad \text{for } x \geqslant 438\,653$$

We follow again the proof of Theorem 3, but now with $\epsilon = \frac{1}{4\,257}$, $x(\epsilon) = 2\,159\,561$, $b = 0.036438$, $x_1(b) = 82\,005$. The optimal value of n for minimizing the sum $T_1 + T_2 + T_3$ is now $n = 197$. Thus, we have for $x \geqslant 1.6\,10^{15}$:

$$Q(X_n) \leqslant \sqrt{\frac{x}{197}}\left(\frac{6}{\pi^2} + b\frac{197^{1/4}}{(1.6\,10^{15})^{1/4}}\right)$$
$$S(X_n) \leqslant \sqrt{\frac{x}{197}}\left(\frac{\sqrt{197}}{18\sqrt{196.98}} + \frac{12^{1/3}\,197^{1/2}}{6(1.6\,10^{15})^{1/6}}\right)$$

hence

$$T_3 \leqslant \sqrt{Q(X_n)S(X_n)} = 0.01479\,\sqrt{x}.$$

Since

$$T_1 + T_2 \leqslant 0.472478\,10^{-3}\,\big(2\sqrt{197} - 0.73\big)\sqrt{x} = 0.012845\,\sqrt{x},$$

we obtain the theorem.

Theorem 5 bis. *We have*

$$\big|M(x)\big| \leqslant \frac{x}{4\,345} \qquad \textit{for } x \geqslant 2\,160\,535$$

We follow the proof of Theorem 5. The following table gives the results of applying Lemma 4 with $b = 0.02767$ and $\tau = 2.401\,10^{15}$. As in the preceding table, the bound is given through the value $u = u(N) + \frac{1}{N+1}\max(G)$, $v = v(N) + \frac{1}{\sqrt{N+1}}\max(G)$, $M = \frac{6}{\pi^2}u + \frac{bv}{\sqrt{\tau}}$ and $\frac{1}{M}$.

$\frac{N}{10^6}$	$\frac{6}{\pi^2}u$	$\frac{bv}{\sqrt{\tau}}$	$M = \frac{6}{\pi^2}u + \frac{bv}{\sqrt{\tau}}$	$\frac{1}{M}$
60	0.00043023667963	0.00000395069480	0.00043418737443	2303.15
180	0.00027850843979	0.00000673471787	0.00028524315766	3505.78
300	0.00024817171955	0.00000861735126	0.00025678907082	3894.25
420	0.00023517196904	0.00001013293110	0.00024530490015	4076.56
540	0.00022795054659	0.00001143313505	0.00023938368164	4177.39
660	0.00022335574718	0.00001258757859	0.00023594332577	4238.31
780	0.00022017493691	0.00001363568192	0.00023381061883	4276.97
900	0.00021784209912	0.00001460065937	0.00023244275849	4302.13
1020	0.00021606416987	0.00001549942503	0.00023156359490	4318.47
1140	0.00021465591992	0.00001634340437	0.00023099932430	4329.02
1260	0.00021351594765	0.00001714079730	0.00023065674496	4335.45
1380	0.00021257418079	0.00001763153440	0.00023020571519	4343.94
1440	0.00021216218064	0.00001799690788	0.00023015908852	4344.82
1500	0.00021178312766	0.00001835415541	0.00023013728308	4345.23
1560	0.00021143322102	0.00001870375628	0.00023013697731	4345.24
1620	0.00021110922195	0.00001904623100	0.00023015545295	4344.89

This proves Theorem 5 bis.

It is not really useful to iterate once more the process, since the improvements would be marginal (around 0.9% for $|R(x)|$ and 0.07% for $|M(x)|$).

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