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EXPLICIT ESTIMATION OF ERROR CONSTANTS APPEARING IN NON-CONFORMING LINEAR TRIANGULAR FINITE ELEMENT METHOD[1]

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Abstract. The non-conforming linear (P_1) triangular FEM can be viewed as a kind of the discontinuous Galerkin method, and is attractive in both the theoretical and practical purposes. Since various error constants must be quantitatively evaluated for its accurate a priori and a posteriori error estimates, we derive their theoretical upper bounds and some computational results. In particular, the Babuška-Aziz maximum angle condition is required just as in the case of the conforming P_1 triangle. Some applications and numerical results are also included to see the validity and effectiveness of our analysis.

Keywords: FEM; non-conforming linear triangle; a priori error estimate; a posteriori error estimate; error constant; Raviart-Thomas element

MSC 2010: 65N15, 65N30

1. Introduction

As a well-known alternative to the conforming linear (P_1) triangular finite element for approximation of the first-order Sobolev space (H^1), the non-conforming P_1 element is considered a classical discontinuous Galerkin finite element [4] and has various interesting properties from both the theoretical and practical standpoints [11], [28]. In particular, its a priori error analysis was performed in a fairly early stage of

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mathematical analysis of FEM (Finite Element Method), and recently a posteriori error analysis has been rapidly developing as well. For accurate error estimation of such an FEM, various error constants must be evaluated quantitatively [2], [6], [8], [11].

Based on our preceding works on the constant (P_0) and the conforming P_1 triangles [17], [18], we here give some results for error constants required for the analysis of the non-conforming P_1 triangle. More specifically, we first summarize a priori error estimation of the present non-conforming FEM, where several error constants appear. In this process, we use the lowest-order Raviart-Thomas triangular $H(\mathrm{div})$ element to deal with the inter-element discontinuity of the approximate functions [9], [19]. Then we introduce some constants related to a reference triangle, some of which are popular in the P_0 and the conforming P_1 cases. We give some theoretical results for the upper bounds of such constants. Finally, we include some numerical results to support the validity of such upper bounds. Our results can be effectively used in the quantitative a priori and a posteriori error estimates for the non-conforming P_1 triangular FEM.

2. A priori error estimation

We here summarize a priori error estimation of the non-conforming P_1 triangular FEM. Let Ω be a bounded convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$, and let us consider a weak formulation of the Dirichlet boundary value problem for the Poisson equation: Given $f \in L_2(\Omega)$, find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall\, v \in H_0^1(\Omega). \tag{2.1}$$

Here, $L_2(\Omega)$ and $H_0^1(\Omega)$ are the usual Hilbertian Sobolev spaces associated to Ω, ∇ is the gradient operator, and $(\cdot,\cdot)$ stands for the inner products for both $L_2(\Omega)$ and $L_2(\Omega)^2$. It is well known that the solution exists uniquely in $H_0^1(\Omega)$ and also belongs to $H^2(\Omega)$.

Let us consider a regular family of triangulations $\{\mathcal{T}^h\}_{h>0}$ of Ω, to which we associate the non-conforming P_1 finite element spaces $\{V^h\}_{h>0}$. Each V^h is constructed over a certain $\mathcal{T}^h$, and the functions in V^h are linear in each $K \in \mathcal{T}^h$ with continuity only at midpoints of edges, and also vanish at the midpoints on the $\partial\Omega$ to approximate the homogeneous Dirichlet condition [11], [28]. Then the finite element solution $u_h \in V^h$ is determined, for a given $f \in L_2(\Omega)$, by

$$(\nabla_h u_h, \nabla_h v_h) = (f, v_h) \quad \forall\, v_h \in V^h, \tag{2.2}$$

where ∇_h is the "non-conforming" or discrete gradient defined as an $L_2(\Omega)^2$-valued operator by the element-wise relations $(\nabla_h v)|_K = \nabla(v|_K)$ for all $v \in V^h + H^1(\Omega)$ and for all $K \in \mathcal{T}^h$. Equation (2.2) is formally of the same form as in the conforming case, so that, for the error analysis, it is natural to consider an appropriate interpolation operator Π_h, e.g., the Crouzeix-Raviart interpolation, from $H_0^1(\Omega)$ (or its intersection with some other spaces) to V^h. However, the situation is not so simple. That is, using the Green formula, we have

$$(\nabla_h u_h, \nabla_h v_h) = (\nabla_h u, \nabla_h v_h) - \sum_{K \in \mathcal{T}^h} \int_{\partial K} v_h \frac{\partial u}{\partial n}\Big|_{\partial K} \mathrm{d}\gamma \quad \forall\, v_h \in V^h, \tag{2.3}$$

where $\frac{\partial u}{\partial n}|_K$ denotes the trace of the derivative of u in the outward normal direction of ∂K, and $\mathrm{d}\gamma$ is the infinitesimal element of ∂K. Conventional efforts of error analysis have been focused on the estimation of the second term on the right-hand side of (2.3), which is absent in the conforming case. To cope with such difficulty, we introduce the lowest-order Raviart-Thomas triangular $H(\mathrm{div})$ finite element space W^h associated to each $\mathcal{T}^h$ (see [9], [19]). Then, noticing that the normal component of for all $q_h \in W^h$ is constant and continuous along each inter-element edge, we can derive

$$(q_h, \nabla_h v_h) + (\mathrm{div}\, q_h, v_h) = 0. \tag{2.4}$$

From (2.2) and (2.4),

$$(\nabla_h u_h, \nabla_h v_h) = (q_h, \nabla_h v_h) + (\mathrm{div}\, q_h + f, v_h). \tag{2.5}$$

By puting $-(\nabla u, \nabla_h v_h)$ on both sides of (2.6), we have, for any $q_h \in W^h$, $v_h \in V^h$,

$$(\nabla_h u_h - \nabla u, \nabla_h v_h) = (q_h - \nabla u, \nabla_h v_h) + (\mathrm{div}\, q_h + f, v_h). \tag{2.6}$$

Then by Lemma 6 of [15], a refinement of Strang's second lemma [11], we have[2]

$$\begin{aligned} \|\nabla u - \nabla_h u_h\|^2 = & \inf_{v_h \in V^h} \|\nabla u - \nabla_h v_h\|^2 \\ & + \Big[\sup_{w_h \in V^h \setminus \{0\}} \frac{(q_h - \nabla u, \nabla_h w_h) + (\mathrm{div}\, q_h + f, w_h)}{\|\nabla_h w_h\|} \Big]^2, \end{aligned} \tag{2.7}$$

[2] The proof restricted to (2.2) is simple. Let P_h be the projection that projects V to V^h, with respect to $(\nabla_h \cdot, \nabla_h \cdot)$. Then $\|\nabla_h u_h - \nabla u\|^2 = \|\nabla_h P_h u - \nabla u\|^2 + \|\nabla_h (u_h - P_h u)\|^2$. Noticing that $\|\nabla_h (u_h - P_h u)\|^2 = (\nabla_h (u_h - P_h u), \nabla_h u_h - \nabla u)$ and applying (2.6), we can easily get (2.7).

where $\|\cdot\|$ stands for the norms of both $L_2(\Omega)$ and $L_2(\Omega)^2$. Using the Fortin operator $\Pi_h^F\colon\ H(\mathrm{div};\Omega)\cap H^{1/2+\delta}(\Omega)^2 \to W^h$ ($\delta>0$) (cf. [9]) and the orthogonal projection $Q_h\colon\ L_2(\Omega)\to X^h :=$ space of step functions over $\mathcal{T}^h$, we obtain the a priori error estimate

$$\begin{aligned}(2.8)\quad \|\nabla u-\nabla_h u_h\|^2 \leqslant \inf_{v_h\in V^h}\|\nabla u-\nabla_h v_h\|^2 \\ +\Big[\|\nabla u-\Pi_h^F\nabla u\| + \sup_{w_h\in V^h\setminus\{0\}}\frac{(f-Q_hf, w_h-Q_hw_h)}{\|\nabla_h w_h\|}\Big]^2,\end{aligned}$$

where q_h in (2.7) is taken as $\Pi_h^F\nabla u$.

We can obtain a more concrete error estimate in terms of the mesh parameter $h_*>0$ (see the definition of h_* in (2.47); h will be used with a different meaning later) by deriving estimates such as for all $v\in H_0^1(\Omega)\cap H^2(\Omega)$ and for all $g\in H^1(\Omega)+V^h$,

$$\begin{aligned}(2.9)\qquad \|v-\Pi_hv\| \leqslant \gamma_0h_*^2|v|_2,\quad \|\nabla v-\nabla_h\Pi_hv\|\leqslant\gamma_1h_*|v|_2,\\ \|\nabla v-\Pi_h^F\nabla v\|\leqslant\gamma_2h_*|v|_2,\quad \|g-Q_hg\|\leqslant\gamma_3h_*\|\nabla_hg\|,\end{aligned}$$

where $|\cdot|_k$ denotes the standard semi-norm of $H^k(\Omega)$, $k\in\mathbb{N}$, and γ_i's are positive error constants dependent only on $\{\mathcal{T}^h\}_{h>0}$.

Then we obtain, for the solution $u\in H_0^1(\Omega)\cap H^2(\Omega)$,

$$\|\nabla u-\nabla_hu_h\|\leqslant\begin{cases} h_*\sqrt{\gamma_1^2|u|_2^2+(\gamma_2|u|_2+\gamma_3\|f\|)^2} & \text{for } f\in L_2(\Omega),\\ h_*\sqrt{\gamma_1^2|u|_2^2+(\gamma_2|u|_2+\gamma_3^2h_*|f|_1)^2} & \text{for } f\in H^1(\Omega),\end{cases}$$

where the term $|u|_2$ can be bounded as $|u|_2\leqslant\|f\|$ for the considered Ω.

We can also use Nitsche's trick to evaluate the a priori L_2 error of u_h (see [11], [20]). That is, let us define $\psi\in H_0^1(\Omega)(\cap H^2(\Omega))$ for $e^h := u-u_h$ by

$$(\nabla\psi,\nabla v)=(e^h,v)\quad\forall\, v\in H_0^1(\Omega),$$

Then for all $v_h\in V^h$ and for all $q_h,\tilde q_h\in W^h$, by noticing

$$\begin{cases}\|e^h\|^2=(e^h,e^h)=(\mathrm{div}\,\tilde q_h+e^h,e^h)+(\tilde q_h,\nabla_he^h)\\ (-\nabla_hv_h,\nabla_he^h)+(\nabla_hv_h,\nabla u)+(-v_h,f)=0\\ (-\nabla\psi,\nabla u-q_h)+(\psi,\mathrm{div}\,q_h+f)=0\\ (\nabla_hv_h,-q_h)+(v_h,\mathrm{div}\,q_h)=0\end{cases}$$

we have

$$\begin{aligned}\|e^h\|^2=(\tilde q_h-\nabla_hv_h,\nabla_he^h)+(\nabla_hv_h-\nabla\psi,\nabla u-q_h)\\ +(\psi-v_h,\mathrm{div}\,q_h+f)+(\mathrm{div}\,\tilde q_h+e^h,e^h).\end{aligned}$$

Substituting $v_h = \Pi_h \psi, q_h = \Pi_h^F \nabla u$ and $\tilde{q}_h = \Pi_h^F \nabla \psi$ above, we find

$$\begin{aligned}\|e^h\|^2 = &(\Pi_h^F \nabla\psi - \nabla\psi + \nabla\psi - \nabla_h \Pi_h \psi, \nabla_h e^h) + (\nabla \Pi_h \psi - \nabla\psi, \nabla u - \Pi_h^F \nabla u) \\ &+ (\psi - \Pi_h\psi, f - Q_h f) + (e^h - Q_h e^h, e^h - Q_h e^h),\end{aligned}$$

since $\operatorname{div} q_h = \operatorname{div} \Pi_h^F \nabla u = -Q_h f$ and $\operatorname{div} \tilde{q}_h = \operatorname{div} \Pi_h^F \nabla \psi = -Q_h e^h$. Then we have, by (2.9) as well as the relations $|u|_2 \leqslant \|f\|$ and $|\psi|_2 \leqslant \|e^h\|$,

$$\|e^h\|^2 \leqslant [(\gamma_1 + \gamma_2) h_* \|\nabla_h e^h\| + (\gamma_0 + \gamma_1\gamma_2) h_*^2 \|f\|] \|e^h\| + \gamma_3^2 h_*^2 \|\nabla_h e^h\|^2,$$

where the term $\gamma_0 h_*^2 \|f\| \cdot \|e^h\|$ can be replaced by $\gamma_0 \gamma_3 h_*^3 |f|_1 \|e^h\|$ if $f \in H^1(\Omega)$. This may be considered a quadratic inequality for e^h, and solving it gives an expected order estimate $\|u - u_h\| = \|e^h\| = O(h_*^2)$:

$$\begin{aligned}&\|e^h\| \leqslant \frac{h_*}{2}\Big(A_1 + \sqrt{A_1^2 + 4A_2}\Big), \\ &A_1 := (\gamma_1 + \gamma_2)\|\nabla_h e^h\| + (\gamma_0 + \gamma_1\gamma_2) h_* \|f\|, \\ &A_2 := \gamma_3^2 h_* \|\nabla_h e^h\|^2.\end{aligned}$$

Relation to Raviart-Thomas mixed FEM. We have already introduced the Raviart-Thomas space W^h for auxiliary purposes. But it is well known that the present non-conforming FEM is closely related to the Raviart-Thomas mixed FEM [3], [25]. Here we will summarize the implementation of such a mixed FEM by slightly modifying the original nonconforming P_1 scheme described by (2.2). The original idea in [3], [25] is based on the enrichment by the conforming cubic bubble functions with the L_2 projection into W^h, but we here adopt non-conforming quadratic bubble ones to make the modification procedure a little simpler.[3]

First, we replace f in (2.2) by $Q_h f$. Then u_h is modified to $u_h^* \in V^h$ defined by

$$(\nabla_h u_h^*, \nabla_h v_h) = (Q_h f, v_h) \quad \forall\, v_h \in V^h. \tag{2.10}$$

Secondly, we introduce the space V_B^h of non-conforming quadratic bubble functions by defining its basis function φ_K associated to each $K \in \mathcal{T}^h$ such that φ_K vanishes outside K and its value at $x \in K$ is given by

$$\varphi_K(x) = \frac{1}{2}|x - x^G|^2 - \frac{1}{12}\sum_{i=1}^{3} |x^{(i)} - x^G|^2, \tag{2.11}$$

[3] In 2015, Hu and Ma showed the same result about the relation between the enriched FEM and Raviart-Thomas FEM, along with the extension to the general dimensional space [14].

where $|\cdot|$ is the Euclidean norm of $\mathbb{R}^2$, x^G the barycenter of K, and $x^{(i)}$, $i = 1, 2, 3$ the ith vertex of K. It is easy to see that the line integration of φ_K for each e of K vanishes

$$\int_e \varphi_K \, \mathrm{d}\gamma = 0. \tag{2.12}$$

Now the enriched non-conforming finite element space $\widetilde{V}^h$ is defined by the linear sum

$$\widetilde{V}^h = V^h + V_B^h. \tag{2.13}$$

By (2.12) and the Green formula, we find the following orthogonality relation for $(\nabla_h \cdot, \nabla_h \cdot)$:

$$(\nabla_h v_h, \nabla_h \beta_h) = 0 \quad \forall\, v_h \in V^h, \ \forall\, \beta_h \in V_B^h. \tag{2.14}$$

Then the modified finite element solution $\widetilde{u}_h \in \widetilde{V}^h$ is defined by

$$(\nabla_h \widetilde{u}_h, \nabla_h \tilde{v}_h) = (Q_h f, \tilde{v}_h) \quad \forall\, \tilde{v}_h \in \widetilde{V}^h. \tag{2.15}$$

Thanks to (2.14), the present $\widetilde{u}_h$ can be obtained as the sum

$$\widetilde{u}_h = u_h^* + \alpha_h, \tag{2.16}$$

where $u_h^* \in V^h$ is the solution of (2.10), and $\alpha_h \in V_B^h$ is determined by

$$(\nabla_h \alpha_h, \nabla_h \beta_h) = (Q_h f, \beta_h) \quad \forall\, \beta_h \in V_B^h, \tag{2.17}$$

i.e., completely independently of u_h^*. Moreover, α_h can be decided by element-by-element comupations. More specifically, denoting $\alpha_h|_K$ as $\alpha_K \varphi_K|K$, (2.17) leads to

$$\alpha_K (\nabla \varphi_K, \nabla \varphi_K)_K = (Q_h f, \varphi_K)_K \quad \forall\, K \in \mathcal{T}^h, \tag{2.18}$$

where $(\cdot, \cdot)$ denotes the inner products of both $L_2(K)$ and $L_2(K)^2$.

Define $\{p_h, \overline{u}_h\} \in L_2(\Omega)^2 \times X^h$ by

$$p_h = \nabla_h \widetilde{u}_h, \quad \overline{u}_h = Q_h \widetilde{u}_h. \tag{2.19}$$

By appying the Green formula to (2.15), we can show that $p_h \in W^h$, and also that the present pair $\{p_h, \overline{u}_h\}$ satisfies the determination equations of the lowest-order Raviart-Thomas mixed FEM:

$$\begin{cases} (p_h, q_h) + (\overline{u}_h, \operatorname{div} q_h) = 0 & \forall\, q_h \in W^h, \\ (\operatorname{div} p_h, \overline{v}_h) = -(Q_h f, \overline{v}_h) & \forall\, \overline{v}_h \in X^h. \end{cases} \tag{2.20}$$

By the uniqueness of the solutions, $\{p_h, \overline{u}_h\}$ is nothing but the unique solution of (2.20).

In conclusion, denoting the constant value of $Q_h f|K$ by $\overline{f}_K$ $(= \int_K f\,\mathrm{d}x/\mathrm{meas}(K))$, we have for all $K \in \mathcal{T}^h$ and for all $x \in K$ that

$$\begin{cases} \alpha_K = -\dfrac{1}{2}\overline{f}_K, \\ \widetilde{u}_h(x) = u_h^*(x) + \alpha_K \psi_K(x) = u_h^*(x) - \dfrac{1}{4}\overline{f}_K\left(|x - x^G|^2 - \dfrac{1}{6}\displaystyle\sum_{i=1}^{3}|x^{(i)} - x^G|^2\right), \end{cases}$$

and

$$\begin{cases} p_h(x) = \nabla_h u_h^*(x) - \dfrac{1}{2}\overline{f}_K(x - x^G), \\ \overline{u}_h(x) = u_h^*(x^G) - \dfrac{1}{16}\overline{f}_K\left(|x^G|^2 - \dfrac{1}{3}\displaystyle\sum_{i=1}^{3}|x^{(i)}|^2\right), \end{cases} \tag{2.21}$$

which coincide with those in [25] and are easy to compute by post-processing.

A posteriori error estimation. The consideration in the preceding section suggests the a posteriori error estimation based on the hypercircle method [12], [19].

Taking into account of the fact that $p_h \in W^h$, obtained in the preceding section, belongs to $H(\mathrm{div};\Omega)$ with $\mathrm{div}\, p_h = -Q_h f$, we find that, for all $v \in H_0^1(\Omega)$,

$$\begin{gathered} \|\nabla v - p_h\|^2 = \|\nabla(v - u^h)\|^2 + \|\nabla u^h - p_h\|^2, \\ \left\|\nabla u^h - \frac{1}{2}(\nabla v + p_h)\right\| = \frac{1}{2}\|\nabla v - p_h\|, \end{gathered} \tag{2.22}$$

where $u^h \in H_0^1(\Omega)$ is the solution of (2.1) with f replaced by $Q_h f$:

$$(\nabla u^h, \nabla v) = (Q_h f, v) \quad \forall\, v \in H_0^1(\Omega). \tag{2.23}$$

The relation (2.22) implies that the three points ∇u^h, ∇v and p_h in $L_2(\Omega)^2$ make a hypercircle, the first having a right inscribed angle. Noting that $(f - Q_f, v) = (f - Q_h f, v - Q_h v)$ for all $v \in H_0^1(\Omega) \subset L_2(\Omega)$, we have by (2.8) that

$$|u - u^h|_1 = \|\nabla(u - h^h)\| \leqslant \gamma_3 h_* \|f - Q_h f\| \quad (\leqslant \gamma_3^2 h_*^2 |f|_1 \text{ if } f \in H^1(\Omega)). \tag{2.24}$$

Taking an appropriate $v \in H_0^1(\Omega)$, we obtain a posteriori error estimates related $p_h = \nabla_h \widetilde{u}_h$:

$$\|\nabla u - p_h\| \leqslant \|\nabla(u - u^h)\| + \|\nabla u^h - p_h\| \leqslant \|\nabla(u - u^h)\| + \|\nabla v - p_h\|, \tag{2.25}$$

$$\left\|\nabla u - \frac{1}{2}(\nabla v + p_h)\right\| \leqslant \|\nabla(u - u^h)\| + \frac{1}{2}\|\nabla v - p_h\|. \tag{2.26}$$

A typical example of v is the conforming P_1 finite element solution $u_h^C \in V_C^h$, where V_c^h is the conforming P_1 space over T^h. Another example is a function $v_C^h \in V_C^h$ obtained by appropriate post-processing of u_h or u_h^*, such as nodal averaging or smoothing. A cheap method of constructing a nice v_C^h may be also an interesting subject. Again, we need the constant γ_3 to evaluate the term $\|\nabla(u-u^h)\|$ above. If we use $\nabla_h u_h$ based on the original $u_h \in V^h$ in (2.2), instead of the modified one $\widetilde{u}_h \in V^h$, we must evaluate some additional terms. Fortunately, such evaluation can be done explicitly by using γ_3 and some positive constants related to $\{\psi_K\}_{K\in\mathcal{T}^h}$.

Error constants. To analyze the error constants in (2.8), let us consider their element-wise counterparts. Let h, α, and θ be positive constants such that

$$h > 0, \quad 0 < \alpha \leqslant 1, \quad \frac{\pi}{3} \leqslant \cos^{-1}\frac{\alpha}{2} \leqslant \theta < \pi. \tag{2.27}$$

Then we define the triangle $T_{\alpha,\theta,h}$ by $\triangle OAB$ with three vertices $O(0,0)$, $A(h,0)$, and $B(\alpha h\cos\theta, \alpha h\sin\theta)$. From (2.27), AB is shown to be the edge of maximum length, i.e., $\overline{AB} \geqslant h \geqslant \alpha h$ so that $h = \overline{OA}$ here denotes the medium edge length, unlike the usual usage as the largest one [11]. A point on the closure $\overline{T}_{\alpha,\theta,h}$ is denoted by $x = \{x_1, x_2\}$, and the three edges e_i's, $i = 1,2,3$, are defined by $\{e_1, e_2, e_3\} = \{OA, OB, AB\}$.

By an appropriate congruent transformation in $\mathbb{R}^2$, we can configure any triangle as $T_{\alpha,\theta,h}$. Following the usage in [5], we will use abbreviated notation $T_{\alpha,\theta} = T_{\alpha,\theta,1}$, $T_\alpha = T_{\alpha,\pi/2}$ and $T = T_1$ (Fig. 1). We will also use the notation $\|\cdot\|_{T_{\alpha,\theta,h}}$ and $|\cdot|_{k,T_{\alpha,\theta,h}}$ as the norms of $L_2(T_{\alpha,\theta,h})$ and semi-norms of $H^k(T_{\alpha,\theta,h})$, where the subscript $T_{\alpha,\theta,h}$ will be usually omitted.

Let us define the following closed linear spaces for functions over $T_{\alpha,\theta,h}$:

$$V^0_{\alpha,\theta,h} = \left\{ v \in H^1(T_{\alpha,\theta,h}) \;\middle|\; \int_{T_{\alpha,\theta,h}} v(x)\,\mathrm{d}x = 0 \right\}, \tag{2.28}$$

$$V^i_{\alpha,\theta,h} = \left\{ v \in H^1(T_{\alpha,\theta,h}) \;\middle|\; \int_{e_i} v(s)\,\mathrm{d}s = 0 \right\}, \quad i = 1,2,3, \tag{2.29}$$

$$V^{\{1,2\}}_{\alpha,\theta,h} = \left\{ v \in H^1(T_{\alpha,\theta,h}) \;\middle|\; \int_{e_1} v(s)\,\mathrm{d}s = \int_{e_2} v(s)\,\mathrm{d}s = 0 \right\}, \tag{2.30}$$

$$V^{\{1,2,3\}}_{\alpha,\theta,h} = \left\{ v \in H^1(T_{\alpha,\theta,h}) \;\middle|\; \int_{e_i} v(s)\,\mathrm{d}s = 0, \quad i = 1,2,3 \right\}, \tag{2.31}$$

$$V^4_{\alpha,\theta,h} = \left\{ v \in H^2(T_{\alpha,\theta,h}) \;\middle|\; \int_{e_i} v(s)\,\mathrm{d}s = 0, \quad i = 1,2,3 \right\}. \tag{2.32}$$

We will again use abbreviations like $V^0_{\alpha,\theta} = V^0_{\alpha,\theta,1}$, $V^0_\alpha = V^0_{\alpha,\pi/2}$, $V^0 = V^0_1$, etc.

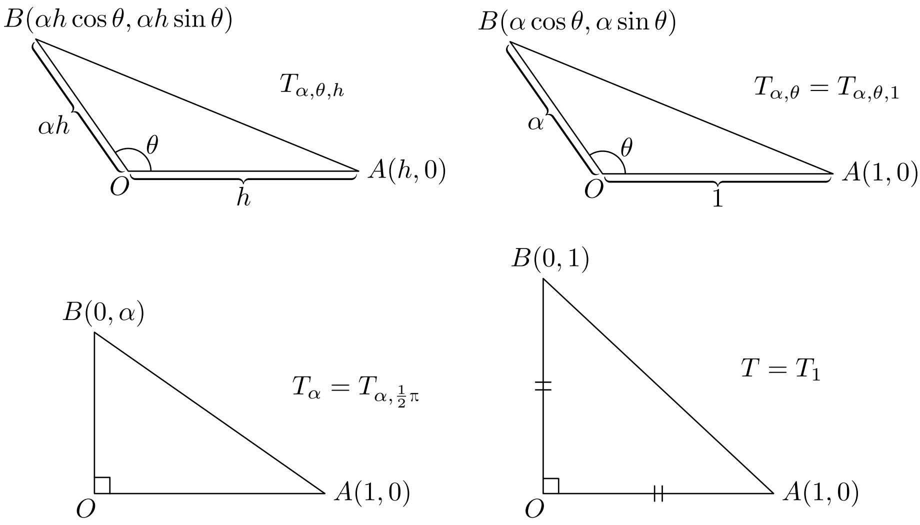

Figure 1. Notation for triangles: $T_{\alpha,\theta} = T_{\alpha,\theta,1}$, $T_\alpha = T_{\alpha,\frac{1}{2}\pi}$, $T = T_1$.

Let us consider the P_0 interpolation operator $\Pi^0_{\alpha,\theta,h}$ and non-conforming P_1 one $\Pi^{1,N}_{\alpha,\theta,h}$ for functions on $T_{\alpha,\theta,h}$ [8], [11]: $\Pi^0_{\alpha,\theta,h}v$ for all $v \in H^1(T_{\alpha,\theta,h})$ is a constant function such that

$$(\Pi^0_{\alpha,\theta,h}v)(x) = \frac{\int_{T_{\alpha,\theta,h}} v(y)\,\mathrm{d}y}{|T_{\alpha,\theta,h}|} \quad \forall\, x \in T_{\alpha,\theta,h}, \tag{2.33}$$

while $\Pi^{1,N}_{\alpha,\theta,h}v$ for all $v \in H^1(T_{\alpha,\theta,h})$ is a linear function such that

$$\int_{e_i} (\Pi^{1,N}_{\alpha,\theta,h}v)(s)\,\mathrm{d}s = \int_{e_i} v(s)\,\mathrm{d}s \quad \text{for } i = 1,2,3. \tag{2.34}$$

To analyze these interpolation operators, let us estimate the positive constants defined by

$$C_J(\alpha,\theta,h) = \sup_{v \in V^J_{\alpha,\theta,h}\setminus\{0\}} \frac{\|v\|}{|v|_1}, \quad J = 0,1,2,3,\{1,2\},\{1,2,3\}, \tag{2.35}$$

$$C_4(\alpha,\theta,h) = \sup_{v \in V^4_{\alpha,\theta,h}\setminus\{0\}} \frac{|v|_1}{|v|_2}, \quad C_5(\alpha,\theta,h) = \sup_{v \in V^4_{\alpha,\theta,h}\setminus\{0\}} \frac{\|v\|}{|v|_2}. \tag{2.36}$$

We will again use abbreviated notation $C_J(\alpha,\theta) = C_J(\alpha,\theta,1)$, $C_J(\alpha) = C_J(\alpha,\frac{1}{2}\pi)$, $C_J = C(1)$ and also $C_{J,\alpha,\theta} := C_J(\alpha,\theta)$ for every possible subscript J.

By a simple scale change, we find that $C_J(\alpha,\theta,h) = hC(\alpha,\theta)$, $J \neq 5$, and $C_5(\alpha,\theta,h) = h^2 C_5(\alpha,\theta)$. Now, by noticing $v - \Pi^0_{\alpha,\theta,h}v \in V^0_{\alpha,\theta,h}$ for $v \in H^1(T_{\alpha,\theta,h})$

and $v - \Pi^{1,N}_{\alpha,\theta,h} v \in V^4_{\alpha,\theta,h}$ for $v \in H^2(T_{\alpha,\theta,h})$, we can easily obtain the popular interpolation error estimates on $T_{\alpha,\theta,h}$ (see [8], [11]):

$$\|v - \Pi^0_{\alpha,\theta,h} v\| \leqslant C_0(\alpha,\theta) h |v|_1 \quad \forall\, v \in H^1(T_{\alpha,\theta,h}), \tag{2.37}$$

$$|v - \Pi^{1,N}_{\alpha,\theta,h} v|_1 \leqslant C_4(\alpha,\theta) h |v|_2 \quad \forall\, v \in H^2(T_{\alpha,\theta,h}), \tag{2.38}$$

$$\|v - \Pi^{1,N}_{\alpha,\theta,h} v\| \leqslant C_5(\alpha,\theta) h^2 |v|_2 \quad \forall\, v \in H^2(T_{\alpha,\theta,h}). \tag{2.39}$$

We can show that the following relations hold for the constants $C_{J,\alpha,\theta}$ ($:= C_J(\alpha,\theta)$):

$$C_{4,\alpha,\theta} \leqslant C_{0,\alpha,\theta}, \quad C_{5,\alpha,\theta} \leqslant C_{0,\alpha,\theta} C_{\{1,2,3\},\alpha,\theta} \leqslant C_{0,\alpha,\theta} C_{\{1,2\},\alpha,\theta}. \tag{2.40}$$

An estimation rougher than the latter of (2.40) is $C_{5,\alpha,\theta} \leqslant C_{0,\alpha,\theta} \min_{i=1,2,3} C_{i,\alpha,\theta}$. To show the former of (2.40), we first derive $\int_{T_{\alpha,\theta}} \partial v/\partial x_i \,\mathrm{d}x = 0$ for all $v \in V^4_{\alpha,\theta}$, $i = 1,2$, by considering the definition in (2.32) and applying the Gauss formula. Then we can easily obtain the desired result by noticing the definition of $C_0(\alpha,\theta)$. To derive the latter of (2.40), we should evaluate $\|v\|/|v|_1$ and $|v|_1/|v|_2$ for all $v \in V^4_{\alpha,\theta}$, $i = 1,2$. The former quotient can be evaluated by using $C_{\{1,2,3\}}(\alpha,\theta)$, while the latter can be done by $C_4(\alpha,\theta)$ together with the former of (2.40). Clearly, $C_{\{1,2,3\}}(\alpha,\theta) \leqslant C_{\{1,2\}}(\alpha,\theta)$, hence we have the latter of (2.40).

Thus we can give quantitative interpolation estimates from (2.37) throught (2.39), if we succeed in evaluating or bounding the constants $C_J(\alpha,\theta)$'s explicitly for all possible J. Among them, $C_0(\alpha,\theta)$ and $C_{\{1,2\}}(\alpha,\theta)$ are important as may be seen from (2.40). Notice that each of such constants can be characterized by minimization of a kind of Rayleigh quotient [5], [26], [27]. Then it is equivalent to finding the minimum eigenvalue of a certain eigenvalue problem expressed by a weak formulation for a partial differential equation with some auxiliary conditions.

Moreover, we already derived some results for $C_i(\alpha,\theta)$ for $i = 0,1,2$ (see [17], [18]).[4] In particular, $C_0 = 1/\pi$, and C_1 ($= C_2$) is equal to the maximum positive solution of the equation $1/\mu + \tan(1/\mu) = 0$ for μ. The constants $C_J(\alpha,\theta)$'s for $J = 0,1,2,3,4,5,\{1,2\},\{1,2,3\}$ are bounded uniformly for $\{\alpha,\theta\}$. More specifically, their explicit upper bounds are given in terms of α,θ and their values at $\{\alpha,\theta\} = \{1,\frac{1}{2}\pi\}$. Furthermore, $C_J(\alpha)$'s except for $J = 4$ are monotonically increasing in α. The asymptotic behavior of the constants $C_J(\alpha)$'s for $\alpha \downarrow 0$ can be also analyzed as in [18]. As a result, the interpolation by the non-conforming P_1 triangle is robust to the distortion of $T_{\alpha,\theta}$. This fact does not necessarily imply the robustness of the final error estimates for $u - u_h$, since the analysis of the Fortin interpolation has not been performed yet.

[4] K. Kobayashi also developed upper bounds for the error constants, see e.g. [21], [22].

R e m a r k 2.1. Instead of $\Pi^{1,N}_{\alpha,\theta,h}$, it is also possible to consider an interpolation operator using the function values at midpoints of edges. Such an operator is definable for continuous functions over $\overline{T}_{\alpha,\theta,h}$, but not so for functions in $H^1(T_{\alpha,\theta,h})$. Moreover, its analysis would be different from that for $\Pi^{1,N}_{\alpha,\theta,h}$.

Determination of $C_{\{1,2\}}$. From the preceding observations, we can give explicit upper bounds of various interpolation constants associated to the non-conforming P_1 triangle, provided that the value of $C_{\{1,2\}}$ is determined. This becomes indeed possible by adopting essentially the same idea and techniques to determine C_0 and C_1 $(= C_2)$:

Theorem 2.1. $C_{\{1,2\}} = C_{\{1,2\}}(1, \frac{1}{2}\pi, 1)$ *is equal to the maximum positive solution of the transcendental equation for* μ*:*

$$\frac{1}{2\mu} + \tan\frac{1}{2\mu} = 0. \tag{2.41}$$

The above implies that $C_{\{1,2\}} = \frac{1}{2}C_1$ $(= \frac{1}{2}C_2)$, *and hence is bounded as, with numerical verification,*

$$0.24641 < C_{\{1,2\}} < 0.24647. \tag{2.42}$$

R e m a r k 2.2. Thus $\frac{1}{4}$ is a simple but nice upper bound. Numerically, we have $C_{\{1,2\}} = 0.2464562258\ldots$

P r o o f. By using the technique for determination of C_0 and $C_1 = C_2$ in [17] and [19], we obtain the following equation for μ,

$$1 + \frac{1}{2\mu}\sin\frac{1}{\mu} - \cos\frac{1}{\mu} = 0, \tag{2.43}$$

whose maximum positive solution is the desired $C_{\{1,2\}}$. By the double-angle formulas, the above is transformed into

$$\Bigl(2\sin\frac{1}{2\mu} + \frac{1}{\mu}\cos\frac{1}{2\mu}\Bigr)\sin\frac{1}{2\mu} = 0. \tag{2.44}$$

It is now easy to derive (see 2.41), and also to draw other conclusions by using the result in [17], [19]. $\square$

Analysis of Fortin's interpolation. This section is devoted to the analysis of Fortin's interpolation operator $\Pi^F_{\alpha,\theta}$ (see [9]) for each $T_{\alpha,\theta}$. Given $q \in H(\mathrm{div}; T_{\alpha,\theta}) \cap H^{1/2+\delta}(T_{\alpha,\theta})^2$ ($\delta > 0$), Fortin's interpolation $q_h = \{\alpha_1 + \alpha_3 x_1, \alpha_2 + \alpha_3 x_2\}$ (α_i being constants) satisfies,

$$\int_{e_i} (q_h - q) \cdot \vec{n}\, \mathrm{d}s = 0, \quad i = 1, 2, 3.$$

To consider the error estimation for Fortin's interpolation, we quote a result about the error estimation for the Lagrange interpolation function. Define a constant C_F by

$$C_F := \sup_{q \in W(T_{\alpha,\theta})} \frac{\|q\|}{|q|_1}.$$

Here $W(T_{\alpha,\theta})$ is defined by

$$W(T_{\alpha,\theta}) := \left\{ q \in H(T_{\alpha,\theta})^2 \;\Big|\; \int_{e_i} q \cdot \vec{\tau}\, \mathrm{d}s = 0,\ i = 1, 2, 3 \right\},$$

where $\vec{\tau}$ denotes the unit tangent vector along edges. Such a constant has been used to bound the Lagrange interpolation error constant (Theorem 2 of [24]), which has an explicit upper bound $C_F \leqslant C_6(\alpha, \theta)$.

$$C_6(\alpha, \theta) := \frac{\sqrt{c_1^2 + c_2^2 + 2c_1c_2\cos^2\theta + (c_1 + c_2)\sqrt{c_1^2 + c_2^2 + 2c_1c_2\cos 2\theta}}}{\sqrt{2}\sin\theta}, \tag{2.45}$$

where c_i stands for $C_i(\alpha, \theta)$, $i = 1, 2$, for the purpose of abbreviation.

The following theorem gives the error constant for Fortin's interpolation, where the technique in the proof is following the one used in Theorem 5.1 of [10].[5]

Theorem 2.2. *For $q = \{q_1, q_2\} \in (H^1(T_{\alpha,\theta}))^2$ we have*

$$\|q - \Pi^F_{\alpha,\theta} q\| \leqslant C_6(\alpha, \theta)|q|_1. \tag{2.46}$$

Proof. Let $\widehat{w}$ be the rotation of $w := q - \Pi^F_{\alpha,\theta} q$ by $\frac{1}{2}\pi$, then it is easy to verify that $\int_{e_i} \widehat{w} \cdot \vec{\tau}\, \mathrm{d}s = 0$, $i = 1, 2, 3$. Hence,

$$(\|w\| =)\|\widehat{w}\| \leqslant C_6(\alpha, \theta)|\widehat{w}|_1 \ (= C_6(\alpha, \theta)|w|_1).$$

Rewrite the vector w as $w = (w_1, w_2)$ and decompose $|w|_1^2$ as

$$|w|_1^2 = \left\| w_{1,x} - \frac{\mathrm{div}\, w}{2} \right\|^2 + \|w_{1,y}\|^2 + \|w_{2,x}\|^2 + \left\| w_{2,y} - \frac{\mathrm{div}\, w}{2} \right\|^2 + \frac{\|\mathrm{div}\, w\|^2}{2}.$$

[5] The result below is an improvement of the error estimation of [23], which involves another constant C_7 along with the term $\|\mathrm{div}\, q\|$, which however can be removed.

Also, noticing that for $q_h = (q_{h_1}, q_{h_2}) := \Pi^F_{\alpha,\theta} q$,

$$q_{h_1,x} - \frac{\operatorname{div} q_h}{2} = q_{h_2,y} - \frac{\operatorname{div} q_h}{2} = q_{h_1,y} = q_{h_2,x} = 0$$

and the orthogonal decomposition of $\operatorname{div} w$,

$$\|\operatorname{div} w\|^2 + \|\operatorname{div} q_h\|^2 = \|\operatorname{div} q\|^2,$$

we have $|w|_1^2 \leqslant |q|_1^2$, which leads to the conclusion. $\square$

R e m a r k 2.3. Because of the factor $\sin\theta$ in (2.45), the maximum angle condition applies to estimate (2.46) (see [1], [5], [19]). On the other hand, the estimates for $\Pi^0_{\alpha,\theta,h}$ and $\Pi^{1,N}_{\alpha,\theta,h}$ are free from such conditions as may be seen from (2.40) and the comments there.

Global interpolation operators. So far, we have introduced and analyzed local interpolation operators $\Pi^0_{\alpha,\theta,h}, \Pi^{1,N}_{\alpha,\theta,h}$ and $\Pi^F_{\alpha,\theta,h}$. For each $K \in \mathcal{T}^h$, we can find an appropriate $T_{\alpha,\theta,h}$ congruent to K under a mapping $\Phi_K\colon K \to T_{\alpha,\theta,h}$. Then it is natural to define the P_1 non-conforming interpolation operator $\Pi_h\colon H^1_0(\Omega) \to V^h$ by $\Pi_h u|_K = \Pi^{1,N}_{\alpha,\theta,h}(v|_K \circ \Phi_K^{-1}) \circ \Phi_K$ for all $v \in H^1_0(\Omega)$ and for all $K \in \mathcal{T}^h$. Similarly, the orthogonal projection operator $Q_h\colon L_2(\Omega) \to X^h$ is related to $\Pi^0_{\alpha,\theta,h}$, while the global Fortin operator Π^F_h is defined through $\Pi^F_{\alpha,\theta,h}$, Φ_K and the Piola transformation for 2D contravariant vector fields [3].

For each $K \in \mathcal{T}^h$, define $\{\alpha_K, \theta_K, h_K\}$ as $\{\alpha, \theta, h\}$ of the associated $T_{\alpha,\theta,h}$. Then, our analysis shows that the estimates in (2.9) can be concretely given, for all $v \in H^1_0(\Omega) \cap H^2(\Omega)$ and for all $g \in H^1(\Omega) + V^h$, by

$$\|v - \Pi_h v\| \leqslant C^h_5 h_*^2 |v|_2 \leqslant C^h_0 C^h_{\{1,2\}} h_*^2 |v|_2,\ \|\nabla v - \nabla \Pi_h v\| \leqslant C^h_4 h_* |v|_2 \leqslant C^h_0 h_* |v|_2$$
$$\|\nabla v - \Pi^F_h \nabla v\| \leqslant C^h_6 h_* |v|_2,\ \|g - Q_h g\| \leqslant C^h_0 h_* \|\nabla_h g\|,$$

where

$$h_* = \max_{K\in T^h} h_K, \quad C^h_J := \max_{K\in\mathcal{T}^h} C_J(\alpha_K, \theta_K), \quad J = 0,4,5,6,7,\{1,2\}. \tag{2.47}$$

R e m a r k 2.4. Relations such as (2.16), (2.19), and (2.21) may suggest the possibility of finding interpolations for ∇u in W^h rather than the one by the Fortin operator, which are free from the maximum angle condition [5]. However, $\nabla_h(\Pi_h u + \alpha_h)$, for example, is not shown to belong to W^h, because we cannot prove the inter-element continuity of normal components unlike $\nabla_h \hat{u}_h$. Our numerical results show that the maximum angle condition is probably essential for the non-conforming P_1 triangle. See also [1] for related topics.

Numerical results. First, we performed numerical computations to see the actual dependence of various constants on α and θ by adopting the conforming P_1 element and a kind of discrete Kirchhoff plate bending element [16], the latter of which is used to deal directly with the 4th order partial differential eigenvalue problems related to $C_4(\alpha,\theta)$ and $C_5(\alpha,\theta)$. That is, we obtained some numerical results for $C_4(\alpha)$ and $C_5(\alpha)$, $\theta = \frac{1}{2}\pi$ together with their upper bounds. We used the uniform triangulation of the entire domain T_α: T_α is subdivided into small triangles, all being congruent to $T_{\alpha,\pi/2,h}$ with e.g. $h = \frac{1}{20}$.

The left-hand side of Fig. 2 shows the graphs of approximate $C_4(\alpha)$ and $C_0(\alpha)$ versus $\alpha \in\,]0,1]$, while the right-hand side shows similar graphs for $C_5(\alpha)$ and $C_0(\alpha)C_{\{1,2\}}(\alpha)$. In both cases, the theoretical upper bounds based on (2.40) give fairly good approximations to the considered constants $C_4(\alpha)$ and $C_5(\alpha)$. Asymptotic behavior of the constants for $\alpha \downarrow 0+$ observed in the figures can be analyzed as in [18].

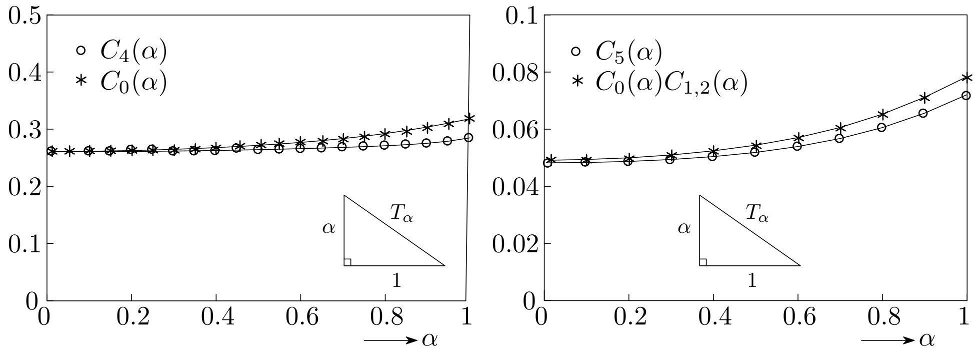

Figure 2. Numerical results for $C_4(\alpha)$ & $C_0(\alpha)$ (left), and for $C_5(\alpha)$ & $C_0(\alpha)C_{\{1,2\}}(\alpha)$ (right); $0 < \alpha \leqslant 1$.

We also tested numerically the validity of our a priori error estimate for $\|\nabla u - \nabla_h u_h\|$. That is, we chose Ω as the unit square $\{x = \{x_1, x_2\} \mid 0 < x_1, x_2 < 1\}$ and f as $f(x_1,x_2) = \sin \pi x_1 \sin \pi x_2$, and consider the $N \times N$ Friedrichs-Keller type uniform triangulations ($N \in \mathbb{N}$). In such situation, $u(x_1,x_2) = \frac{1}{2}\pi^{-2} \sin \pi x_1 \sin \pi x_2$, and all the triangles are congruent to a right isosceles triangle $T_{1,\pi/2,1/N}$, i.e., $h_* = h = 1/N$. Moreover, we can use the following values or upper bounds for the necessary constants:

$$C_0^h = C_0 = \frac{1}{\pi}, \quad C_{\{1,2\}}^h = C_{\{1,2\}} < \frac{1}{4}, \quad C_6^h = C_1 = C_2 < \frac{1}{2}.$$

Moreover, under a current boundary condition and domain shape, we have $|u|_2 = \|\Delta u\| = \|f\|$, see e.g. Theorem 4.3.1.4 of [13]. Then, since $f \in H^1(\Omega)$, the a priori error estimation is given as

$$\|\nabla u - \nabla_h u_h\| \leqslant h_* \sqrt{\frac{1}{\pi^2}\|f\|^2 + \left(\frac{1}{2}\|f\| + \frac{h_*}{\pi^2}\|\nabla f\|\right)^2}.$$

Figure 3 shows the comparison of the actual $\|\nabla u - \nabla_h u_h\|$ and its a priori estimate based on our analysis. The difference is still large, but anyway our analysis appears to give correct upper bounds and order of errors. Probably, a posteriori estimation mentioned previously would give more realistic results.[6]

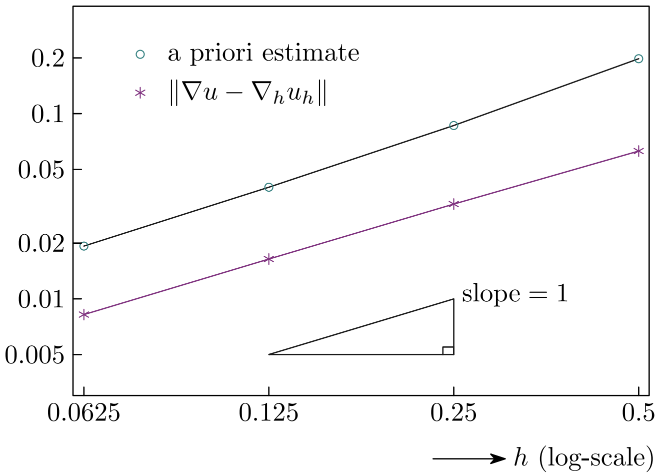

Figure 3. $\|\nabla u - \nabla_h u_h\|$ and its a priori estimates versus h.

Concluding remarks. We have obtained some theoretical and numerical results for several error constants associated to the non-conforming P_1 triangle. These results are hoped to be effectively used in quantitative error estimates, which are necessary for adaptive mesh refinements [7] and numerical verifications. Especially for numerical verification of partial differential equations by Nakao's method [26], accurate bounding of various error constants is essential. Moreover, we are planning to extend our analysis to its 3D counterpart, i.e., the non-conforming P_1 tetrahedron with face DOF's.

[6] Another kind of a priori error estimation is given in [10], which gives larger (worse) estimation compared to our proposed estimation, if the two estimations are applied to the example used in [10].

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