Can. J. Math., Vol. XXXIX, No. 4, 1987, pp. 983-1000

EXPLICIT FORMULAS FOR THE ASSOCIATED JACOBI POLYNOMIALS AND SOME APPLICATIONS

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1. Introduction and notation. In this paper we determine closed-form expressions for the associated Jacobi polynomials, i.e., the polynomials satisfying the recurrence relation for Jacobi polynomials with n replaced by $n + c$, for arbitrary real $c \geqq 0$. One expression allows us to give in closed form the $[n - 1/n]$ Padé approximant for what is essentially Gauss' continued fraction, thus completing the theory of explicit representations of main diagonal and off-diagonal Padé approximants to the ratio of two Gaussian hypergeometric functions and their confluent forms, an effort begun in [2] and [**19**]. (We actually give only the $[n - 1/n]$ Padé element, although other cases are easily constructed, see [**19**] for details.)

We also determine the weight function for the polynomials in certain cases where there are no discrete point masses. Concerning a weight function for these polynomials, so many writers have obtained so many partial results that our formula should be considered an epitome rather than a real discovery, see the discussion in Section 3. (Nevai [**12**] has, for all practical purposes, solved the problem when c is an integer.) Finally we construct a generating function for the polynomials using a fairly deep result of [**9**]. Even special cases of this formula seem to be new.

We employ the notation of [**8**] for our special functions, except in a few cases, where ad hoc notation is defined when it is first used. ${}_pF_q\left(\begin{smallmatrix}a_p\\ b_q\end{smallmatrix}; x\right)$ denotes the generalized hypergeometric function with p numerator parameters, $a_1, a_2, \ldots, a_p$, q denominator parameters, $b_1, b_2, \ldots, b_q$, and argument x. When $p = 2$ and $q = 1$, we drop the subscripts (Gauss' function). ${}_pF_q$ is meromorphic in its parameters. It is our general policy not to give conditions on the parameters to make the function defined because obvious limiting processes can always be invoked to obtain formulas valid in these cases. For instance, formula (28) contains $\Gamma(\gamma + c)$, formula (33) the term $1/\beta$. We do not require, in the first instance, that $\gamma + c \neq 0, -1, -2$, nor in the second $\beta \neq 0$. This convention keeps our formulas from being buried under an avalanche of irrelevant restrictions.

Although we generally assume all parameters to be real, many of our formulas, particularly the purely algebraic ones, will be valid for the parameters complex.

Received November 7, 1985. This research was supported by the National Science Foundation under Grant No. DMS-8419086.

2. Background and basic formulas. Let $\{p_n(x)\}$ be a system of orthogonal polynomials satisfying the recurrence

$$p_{n+1} = (A(n)x + B(n))p_n - C(n)p_{n-1},$$
$$(1) \qquad n = 0, 1, 2, \dots,$$
$$p_{-1} = 0;\ p_1 = 1.$$

If $A(t)$, $B(t)$, $C(t)$ are defined for all $t > 0$, then for any fixed real $c \geqq 0$ we may define the *associated polynomials* $p_n(x, c)$ by means of the recurrence

$$p_{n+1} = (A(n + c)x + B(n + c))p_n - C(n + c)p_{n-1},$$
$$(2) \qquad n = 0, 1, 2, \dots,$$
$$p_{-1} = 0;\ p_1 = 1.$$

In general, these polynomials will also be orthogonal with respect to some positive measure on the real line but, at least in the cases of the classical orthogonal polynomials, one should expect the measure to be much more exotic than the measure for the original polynomials $p_n(x)$.

For any given system of orthogonal polynomials, it is usually of great interest to determine explicit formulas for the associated polynomials, i.e., some reasonably simple expression for the coefficients $\mu_{nk}(c)$ in the expression

$$(3) \qquad p_n(x; c) = \sum_{k=0}^{n} \mu_{nk}(c)x^k, \quad n = 0, 1, 2, \dots.$$

There are several reasons for this. First, the rational function

$$(4) \qquad p_{n-1}(x; c + 1)/p_n(x; c),$$

is the $[n - 1/n]$ Padé approximant for a significant function (the Hilbert transform of the measure). Askey and Wimp [**1**] did this for the associated Laguerre polynomials $L_n^{\alpha}(x; c)$, finding that

$$(5) \qquad \mu_{nk}(c) = \frac{(c + \alpha + 1)_n(-n)_k}{n!(c + 1)_k(c + \alpha + 1)_k} \times {}_3F_2\left(\begin{matrix} k - n, c, c + \alpha \\ c + \alpha + k + 1, c + k + 1 \end{matrix}; 1\right)$$

and this enabled them to construct a closed-form expression for the $[n - 1/n]$ Padé approximant to the function

$$(6) \qquad \Psi(a + 1, b; x)/\Psi(a, b; x).$$

(These Padé approximants are, of course, just truncated continued fractions, the continued fraction for (6) having been known for a long time, see [**16**, p. 350].) In [**19**] the present author discovered the associated polynomials necessary to construct the $[n - 1/n]$ Padé approximant for

(7) $\quad \Phi'(a;\, b;\, x)/\Phi(a;\, b;\, x).$

The work in the present paper will result in a formula for the $[n - 1/n]$ approximant to Gauss' continued fraction.

Second and third reasons for the advantages of having an explicit formula available for $p_n(x;\, c)$ are that often it helps to analyze the measure for the polynomials (see how Askey and Wimp, [**2**], used it to do this) and to allow interesting limiting cases to be constructed.

To motivate the derivation of our main formula, we introduce the moments

$$(8) \quad m_k(c) := \int_{-\infty}^{\infty} t^k d\mu(t;\, c),$$

where $d\mu$ is the measure for $p_n(x;\, c)$. If $\mu_{nk}(c)$ is known for any c, and if $m_k(c)$ is also known, $\mu_{nk}(c + 1)$ may be obtained as follows. The formula [**8**, v. 2, p. 162 (6)] implies

$$(9) \quad p_n(x;\, c + 1) = \frac{1}{A(c)m_0(c)} \int_{-\infty}^{\infty} \frac{p_{n+1}(x;\, c) - p_{n+1}(t;\, c)}{x - t} d\mu(t;\, c).$$

The polynomials defined by the expression above are the traditional "associated" polynomials, usually written $q_{n+1}(x)$. But they satisfy the same recurrence as $p_n(x;\, c)$ except that $q_0 = 0$, $q_1 = 1$. Thus, obviously,

$$q_n(x) = p_{n-1}(x;\, c + 1).$$

Putting the formula for p_{n+1} into the integrand and expanding $(x^k - t^k)/(x - t)$ gives

$$(10) \quad \mu_{nk}(c + 1) = \frac{1}{A(c)m_0(c)} \sum_{r=k}^{n} \mu_{n+1,r+1}(c)m_{r-k}(c).$$

It is not feasible to iterate this equation to determine $\mu_{nk}(c + j)$ from $\mu_{nk}(c)$ since the necessary moments $m_k(c + s)$, $s = 0, 1, 2, \dots, j - 1$, will not be known. But, as the reader shall see, the formula can provide useful insights. (It is also possible to derive a recurrence which expresses $\mu_{nk}(c + 2)$ directly in terms of other coefficients,

$$(11) \quad \mu_{nk}(c + 2) = C(c + 1)^{-1} \times \{B(c)\mu_{n+1,k}(c + 1) + A(c)\mu_{n+1,k-1}(c + 1) - \mu_{n+2,k}(c)\}.$$

Though this can be iterated, it does not appear to be very useful.)

Now the *associated Jacobi polynomials* $P_n^{(\alpha,\beta)}(x;\, c)$ are the polynomials satisfying the recurrence

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$$\begin{aligned}
&2(n + c + 1)(n + c + \gamma)(2n + 2c + \gamma - 1)p_{n+1} \\
&= (2n + 2c + \gamma)[\,(2n + 2c + \gamma - 1)(2n + 2c + \gamma + 1)x \\
(12)\quad &+ (\gamma - 1)(\gamma - 2\beta - 1)\,]p_n - 2(n + c + \gamma - \beta - 1) \\
&\times (n + c + \beta)(2n + 2c + \gamma + 1)p_{n-1}, \\
&n = 0, 1, 2, \dots, \gamma = \alpha + \beta + 1, p_{-1} = 0; p_0 = 1.
\end{aligned}$$

Rather than work with these polynomials, we shall use instead the *shifted* polynomials

$$(13)\quad R_n^{(\alpha,\beta)}(x; c) = P_n^{(\alpha,\beta)}(2x - 1; c).$$

Note for $c = 1$ the polynomials are a constant multiple of the shifted "associated" polynomials $q_n^{(\alpha,\beta)}(2x - 1)$,

$$\begin{aligned}
(14)\quad q_n^{(\alpha,\beta)}(x) := \int_{-1}^{1} &(t - x)^{-1}(1 - t)^{\alpha}(1 + t)^{\beta}[P_n^{(\alpha,\beta)}(t) \\
&- P_n^{(\alpha,\beta)}(x)\,]dt
\end{aligned}$$

i.e.,

$$(15)\quad q_{n+1}^{(\alpha,\beta)}(2x - 1) = (\gamma + 1)2^{\gamma-1}B(\alpha + 1, \beta + 1)R_n^{(\alpha,\beta)}(x).$$

Using (13) we can write our expansions in powers of x rather than $(x + 1)/2$, a space saver. For these polynomials (see [**8**, v. 2, p. 170 (16)]),

$$(16)\quad \mu_{nk}(0) = \frac{(-1)^n(-n)_k(n + \gamma)_k(\beta + 1)_n}{n!k!(\beta + 1)_k}$$

and, of course, the recurrence satisfied is the above with x replaced by $2x - 1$.

Also

$$(17)\quad m_k(0) = \int_0^1 (1 - x)^{\alpha}x^{\beta+k}dx = B(\alpha + 1, \beta + k + 1).$$

Applying the formula (10) gives

$$\begin{aligned}
(18)\quad \mu_{nk}(1) &= \frac{(-1)^n(\gamma + 2)_n(\beta + 2)_n(-n)_k(n + \gamma + 2)_k}{(\gamma + 1)_n n!(2)_k(\beta + 2)_k} \\
&\times {}_4F_3\left(\begin{matrix} k - n, n + \gamma + k + 2, \beta + 1, 1 \\ \beta + k + 2, k + 2, \gamma + 1 \end{matrix}; 1\right), \\
&\gamma = \alpha + \beta + 1.
\end{aligned}$$

A comparison of this with the Askey-Wimp formula (5) and formula (2.12) of [**19**] which is essentially the case of the associated Bessel polynomials strongly implies that a similar formula should hold for polynomial (13). In fact, an abundance of experimentation led us to the expression below, which turns out to be

THEOREM 1.

$$(19)\quad R_n^{(\alpha,\beta)}(x;c) = \frac{(-1)^n(\gamma+2c)_n(\beta+c+1)_n}{(\gamma+c)_n n!} \times \sum_{k=0}^{n} \frac{(-n)_k(n+\gamma+2c)_k x^k}{(c+1)_k(\beta+c+1)_k} \times {}_4F_3\left(\begin{matrix} k-n,\ n+\gamma+k+2c,\ \beta+c,\ c \\ \beta+k+c+1,\ k+c+1,\ \gamma+2c-1 \end{matrix};1\right).$$

Proof. The proof is not a simple one, and depends on some substantial results, including a transformation theorem for ${}_4F_3$'s of unit argument.

First we construct an explicit Wronskian-type formula for R_n.

Let

$$(20)\quad u_n := (-1)^n\frac{\Gamma(n+\beta+c+1)}{\Gamma(n+c+1)}F\left(\begin{matrix}-n-c,\ n+\gamma+c\\ \beta+1\end{matrix};x\right).$$

Then the usual power series arguments (Sister Celine's technique, [**14**, Chapter 14]) show u_n satisfies the same recurrence as R_n. The function

$$(21)\quad v_n := \frac{(-1)^n\Gamma(n+\gamma+c-\beta)}{\Gamma(n+\gamma+c)} \times F\left(\begin{matrix}-n-\beta-c,\ n+\gamma+c-\beta\\ 1-\beta\end{matrix};x\right)$$

can be shown to satisfy the recurrence also by the substitutions $n \to -n$, $\beta \to -\beta$, $c \to \beta - c - \gamma$, $\gamma \to \gamma - 2\beta$ and the identification $v_n = p_{-n}$. Note that

$$(22)\quad v_n^* := x^{-\beta}v_n$$

satisfies the same second order linear differential equation as u_n, namely

$$(23)\quad x(1-x)y'' + [(\beta+1)-(\gamma+1)x]y' + (n+c)(n+\gamma+c)y = 0,$$

(see the tabulation [**8**, v. 1, p. 105 (17) and p. 56 (1)]). Elementary arguments show the Wronskian of any two solutions y_1, y_2 of this equation has the form

$$(24)\quad W(y_1, y_2) = Kx^{-\beta-1}(1-x)^{\beta-\gamma},$$

a result which will be of use later.

By the theory of linear difference equations, an appropriate linear combination of u_n, v_n must give R_n. That combination is easily seen to be

$$(25)\quad \begin{cases} R_n = \dfrac{u_{-1}v_n - v_{-1}u_n}{\Delta}, & n = -1, 0, 1, \dots, \\ \Delta := u_{-1}v_0 - v_{-1}u_0. \end{cases}$$

Surprisingly, Δ turns out to be simple.

LEMMA 1.

$$(26)\quad F\left(\begin{matrix} a+1, b-1 \\ c \end{matrix}; x\right) = \frac{1}{a(c-b)}\Bigg\{x(1-x)(a+1-b) \times \frac{d}{dx}F\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) + a[-x(a+1-b) + (c-b)]F\left(\begin{matrix} a, b \\ c \end{matrix}; x\right)\Bigg\}.$$

Proof. This follows by application of the differential-difference relations [**8**, v. 1, p. 102 (21), (23)] and the differential equation (23).

By using this lemma, Δ may be expressed in terms of $W(u_n, v_n^*)$ and the constant K in the formula (24) may be found by expanding out in a few powers of x. The final result is

$$(27)\quad \Delta = \frac{(1-x)^{\beta+1-\gamma}\beta\Gamma(\beta+c)(\gamma+2c-1)\Gamma(\gamma+c-\beta-1)}{\Gamma(c+1)\Gamma(\gamma+c)}.$$

We now use a Kummer transformation [**8**, v. 1, p. 105 (2)] on v_{-1} and v_n in the numerator to cancel out the factor of $(1-x)^{\beta+1-\gamma}$ arising in the denominator. We thus arrive at the representation

$$(28)\quad R_n = \frac{(-1)^n\Gamma(c+1)\Gamma(\gamma+c)}{\beta\Gamma(\beta+c)(\gamma+2c-1)\Gamma(\gamma+c-\beta-1)} \times \Bigg\{\frac{\Gamma(\gamma+c-\beta-1)\Gamma(n+\beta+c+1)}{\Gamma(\gamma+c-1)\Gamma(n+c+1)}F\left(\begin{matrix} c, 2-\gamma-c \\ 1-\beta \end{matrix}; x\right) \times F\left(\begin{matrix} -n-c, n+\gamma+c \\ \beta+1 \end{matrix}; x\right) - \frac{\Gamma(\beta+c)}{\Gamma(c)}\frac{\Gamma(n+\gamma+c-\beta)}{\Gamma(n+c+\gamma)} \times F\left(\begin{matrix} 1-c, \gamma+c-1 \\ \beta+1 \end{matrix}; x\right)F\left(\begin{matrix} n+c+1, 1-n-\gamma-c \\ 1-\beta \end{matrix}; x\right)\Bigg\}.$$

(An irony of working with such formulas is that it is far from obvious that the expression on the right is a polynomial in x.)

On each of the the last two F's above we again use a Kummer transformation and then take Cauchy products of power series by means of the formula [**8**, v. 1, p. 187 (14)]. The resulting series involves gamma functions and the two hypergeometric functions

$$(29)\quad \begin{cases} {}_4F_3\left(\begin{matrix} c, 2-\gamma-c, -k, -\beta-k \\ 1-\beta, n+c+1-k, 1-n-\gamma-c-k \end{matrix}; 1\right) \\ {}_4F_3\left(\begin{matrix} \beta+c, \beta+2-\gamma-c, -k, \beta-k \\ \beta+1, n+c+1+\beta-k, \beta+1-n-\gamma-c-k \end{matrix}; 1\right) \end{cases}$$

where k is the summation variable.

We will submit each of these two quantities to a transformation which is an analog of a well-known transformation of a ${}_3F_2$ with unit argument, see [**3**, p. 15 (2)].

LEMMA 2.

$$(30)\quad {}_{p+1}F_p\left(\begin{matrix} a_{p+1} \\ b_p \end{matrix}; 1\right) = \sum_{j=2}^{p+1} \Omega_j \; {}_{p+1}F_p\left(\begin{matrix} a_j, 1+a_j-b_p \\ (1+a_j-a_{p+1})^* \end{matrix}; 1\right)$$

where

$$(31)\quad \Omega_j = \frac{\Gamma(1-a_1)}{\Gamma(1+a_j-a_1)} \prod_{r=1}^{p} \frac{\Gamma(b_r)}{\Gamma(b_r-a_j)} \prod_{\substack{r=2 \\ r\neq j}}^{p+1} \frac{\Gamma(a_r-a_j)}{\Gamma(a_r)}.$$

(The * indicates the parameter corresponding to 1 is to be deleted.)

Proof. The proof follows Bailey's proof for the case $p = 2$ exactly. The function

$$(32)\quad e^{\pm\pi is}\Gamma(-s)\frac{\Gamma(a_1+s)\ldots\Gamma(a_{p+1}+s)}{\Gamma(b_1+s)\ldots\Gamma(b_p+s)}$$

is integrated along an appropriate contour. Both sides of the resulting equality are multiplied by $e^{\pm\pi ia}$ and the two equations subtracted. This gives the lemma.

In this lemma we let $p = 2$. The result we apply to the previous series and the two ${}_4F_3$'s (29), choosing first

$$(a_1, a_2, a_3, a_4) = (-k, -\beta-k, c, 2-\gamma-c)$$

and then

$$(a_1, a_2, a_3, a_4) = (\beta-k, \beta+c, -k, \beta+2-\gamma-c).$$

We are left with six ${}_4F_3$'s. Three of these are identically zero, two cancel, and the remaining one is that displayed in the theorem. This completes the proof.

Despite the eye appeal of (19), the respresentation (28) is more useful. It reveals, for instance, that

$$(33)\quad R_n^{(\alpha,\beta)}(0;\,c) = \frac{(-1)^n}{\beta(\gamma + 2c - 1)}\left\{\frac{\Gamma(c+1)(\gamma+c-1)}{\Gamma(\beta+c)}\right.$$

$$\left.\times\frac{\Gamma(n+\beta+c+1)}{\Gamma(n+c+1)} - \frac{c\Gamma(\gamma+c)\Gamma(n+\gamma+c-\beta)}{\Gamma(\gamma+c-\beta-1)\Gamma(n+\gamma+c)}\right\}$$

$$(34)\quad = \frac{(-1)^n}{\beta(\gamma+c-1)}\left\{\frac{\Gamma(c+1)(\gamma+c-1)n^{\beta}}{\Gamma(\beta+c)}\right.$$

$$\left. - \frac{c\Gamma(\gamma+c)n^{-\beta}}{\Gamma(\gamma+c-\beta-1)}\right\}\left[1 + O\left(\frac{1}{n}\right)\right]$$

and (33) serves to sum a terminating Saalschutzian ${}_4F_3$ with slightly specialized parameters:

$$(35)\quad {}_4F_3\left(\begin{matrix}-n,\, n+a,\, b,\, c\\ a-1,\, b+1,\, c+1\end{matrix};\,1\right)$$

$$= \frac{n!bc}{(b-c)(a-1)_{n+1}}\left[\frac{(a-c-1)_{n+1}}{(c)_{n+1}} - \frac{(a-b-1)_{n+1}}{(b)_{n+1}}\right].$$

This formula seems new and cannot be obtained by using the known transformation formulas for a Saalschutzian ${}_4F_3$ (see [**3**, p. 56 (1)]).

Another interesting but rather specialized Saalschutzian ${}_4F_3$ can be summed by means of the following trick. It is easily verified that

$$\Gamma(n+c+1)R_n^{(-1/2,-1/2)}(x;\,c)\Big/\left(\frac{1}{2}\right)_{n+c}$$

satisfies

$$(36)\quad p_{n+1} = 2(2x-1)p_n - p_{n-1}.$$

Thus

$$(37)\quad R_n^{(-1/2,-1/2)}(x;\,c) = \frac{\left(c+\frac{1}{2}\right)_n}{(c+1)_n}U_n(2x-1),$$

(the shifted Chebyshev polynomial). Equating kth powers of x on both sides gives

$$(38)\quad \left({}_4F_3\begin{matrix}k-n,\, n+k+2c,\, c-\frac{1}{2},\, c\\ k+c+\frac{1}{2},\, k+c+1,\, 2c-1\end{matrix};\,1\right)$$

$$= \frac{\Gamma(n+k+2)\Gamma(c+k+1)\Gamma\left(c+k+\frac{1}{2}\right)2^{2c-2}}{(n+c)\Gamma\left(k+\frac{3}{2}\right)\Gamma(n+2c+k)\Gamma(k+1)}.$$

We suspect this formula also holds for k arbitrary complex.

Letting $x \to x/\gamma$, $\gamma \to \infty$ in (19) yields the associated Laguerre polynomials. Letting $x \to x\beta$, $P_n \to \beta^{-n}R_n$, $\beta \to \infty$, gives the polynomials discussed by the present author, [**19**], essentially the associated Bessel polynomials. If then one takes $x \to ix$, $c \to i\kappa + c/2$, $\gamma \to 1 - 2i\kappa$, one obtains polynomials shown in [**19**] to be orthogonal on a discrete point set, namely, the zeros of

$$(39)\quad {}_1F_1\left(\frac{c}{2} + i\kappa;\, c;\, \frac{-i}{x}\right)$$

which are real. A special case of this, $\kappa = 0$, gives the Lommel polynomials, and the discrete point set is then the zeros of

$$J_{(c-1)/2}\left(\frac{1}{2x}\right),$$

see [**19**]. These facts will also turn out to be limiting cases of our work in Section 3.

The following special cases are easily demonstrated from (19) and [**8**, v. 1, p. 188 (3)]:

$$(40)\quad \begin{cases} R_n^{(\alpha,-c)}(x;\,c) = \dfrac{(\alpha+c+1)_n n!}{(\alpha+1)_n(c+1)_n} R_n^{(\alpha,-c)}(x), \\ R_n^{(-c-\beta,\beta)}(x;\,c) = R_n^{(c+\beta,-\beta)}(x); \\ R_n^{(-c,\beta)}(x;\,c) = \dfrac{(\beta+c+1)_n n!}{(\beta+1)_n(c+1)_n} R_n^{(-c,\beta)}(x). \end{cases}$$

These, along with $c = 0$, are the only cases where R_n so simplifies.

A fact not obvious from (19) but which follows directly from the recurrence is

$$(41)\quad R_n^{(\alpha,\beta)}(1-x;\,c) = (-1)^n R_n^{(\beta,\alpha)}(x;\,c).$$

(This provides a check in the first and third formulas above.)

A number of algebraic relationships are known for special cases of the Jacobi polynomials, viz.,

$$(42)\quad R_n^{(\alpha,-1/2)}(x^2) = \frac{(2n)!(\alpha+1)_n}{n!(\alpha+1)_{2n}} R_{2n}^{(\alpha,\alpha)}\left(\frac{x+1}{2}\right),$$

see [**10**, p. 437 (5)].

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Formula (28) allows us to generalize these formulas. The relation (42) generalizes in a highly unexpected way. Using [**8**, v. 1, p. 111 (10), (14), and (41)] and a little algebra shows:

$$(43)\quad R_n^{(\alpha,-1/2)}(x^2;\, c) = \frac{2^{2n+1}\left(c+\frac{1}{2}\right)_n(\alpha+c)_{n+1}c}{x(\alpha+2c)_{2n+1}\left(2c+\alpha-\frac{1}{2}\right)} R_{2n+1}^{(\alpha,\alpha)}\left(\frac{1+x}{2};\, 2c-1\right).$$

(A limit must be taken when $c \to 0$.)

Other identities are possible, e.g.,

$$(44)\quad R_n^{(\alpha,1/2)}(x^2;\, c) = \frac{(2c+\alpha+1)_{2n+1}}{x2^{2n+1}\left(2c+\alpha+\frac{1}{2}\right)(c+1)_n\left(\alpha+c+\frac{3}{2}\right)_n} \times R_{2n+1}^{(-\alpha,\alpha)}\left(\frac{1+x}{2};\, 2c+\alpha\right).$$

The number of such formulas seems to be large.

The representation (28) also provides a description of the asymptotic behavior of R_n since the asymptotics of u_n, v_n are known (see [**11**, v. 1, p. 237 (8)]). The required tool, which is due to Watson, can most conveniently be written

$$(45)\quad F\left(\begin{matrix} a+n,\, b-n \\ c \end{matrix};\, \sin^2\theta\right) \sim \frac{\Gamma(c)n^{1/2-c}}{\sqrt{\pi}} \frac{(\cos\theta)^{c-a-b-1/2}}{(\sin\theta)^{c-1/2}} \cos\left[2n\theta+(a-b)\theta-\frac{\pi}{2}\left(c-\frac{1}{2}\right)\right],$$

$$n \to \infty,\ \theta \in (0, \pi).$$

After a little algebra, one finds

$$R_n^{(\alpha,\beta)}(\sin^2\theta;\, c) \sim \frac{(-1)^n\Gamma(c+1)\Gamma(\gamma+c)}{\sqrt{\pi n}\beta\Gamma(\beta+c)(\gamma+2c-1)\Gamma(\gamma+c-\beta-1)} \times \left\{\frac{\Gamma(\gamma+c-\beta-1)\Gamma(\beta+1)}{\Gamma(\gamma+c-1)(\cos\theta)^{\alpha+1/2}(\sin\theta)^{\beta+1/2}}\right.$$

$$(46)\quad \times F\left(\begin{matrix} c,\, 2-\gamma-c \\ 1-\beta \end{matrix};\, \sin^2\theta\right)\cos\left[2n\theta+(\gamma+2c)\theta-\frac{\pi}{2}\left(\beta+\frac{1}{2}\right)\right]$$

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$$- \frac{\Gamma(\beta + c)\Gamma(1 - \beta)}{\Gamma(c)(\cos\theta)^{1/2-\alpha}(\sin\theta)^{1/2-\beta}} F\left(\begin{matrix} 1 - c, \gamma + c - 1 \\ \beta - 1 \end{matrix}; \sin^2\theta\right)$$

$$\times \cos\left[2n\theta + (\gamma + 2c)\theta - \frac{\pi}{2}\left(\frac{1}{2} - \beta\right)\right]\Bigg\},$$

$n \to \infty$, $\theta \in (0, \pi)$.

The representation (28) is also useful for deriving a generating function for R_n, as we shall see later.

Another interesting consequence of (28) depends on the following observation: if y and w each satisfy a second order linear homogeneous differential equation, then their product satisfies a fourth order equation. The computations to derive the equation, however, are truly formidable, see [**17**, p. 146] for a suggestion of the details involved. Computer algebra is the only feasible approach. In this case we find, using MACSYMA, that R_n satisfies the equation

$$(47) \quad A_0(x)y^{IV} + A_1(x)y''' + A_2(x)y'' + A_3(x)y' + A_4(x)y = 0,$$

where

$$(48) \quad \begin{cases} A_0 = (x - 1)^2x^2; \\ A_1 = 5(x - 1)x(2x - 1); \\ A_2 = -[x^2(2K + 2C + \gamma^2 - 25) \\ \qquad + x(-2K - 2C - 2\beta\gamma - 2\gamma + 2\beta + 26) + \beta^2 - 4]; \\ A_3 = -3[x(2K + 2C + \gamma^2 - 5) \\ \qquad\qquad - K - C - \gamma\beta - \gamma + \beta + 3]; \\ A_4 = ((K - C)^2 - 2K - 2C + 1 - \gamma^2) \quad \text{and} \\ K = (n + c)(n + \gamma + c) \\ C = (c - 1)(\gamma + c - 1). \end{cases}$$

3. Padé approximants and the weight function. For this discussion it is convenient, because of their simple asymptotics, to take as solutions of the recurrence the (shifted) *extended Jacobi functions*

$$(49) \quad w_n(c) := P_{n+c}^{(\alpha,\beta)}(2x - 1) = \frac{(\alpha + 1)_{n+c}}{\Gamma(n + c + 1)} \times F\left(\begin{matrix} -n - c, n + \gamma + c \\ \alpha + 1 \end{matrix}; 1 - x\right)$$

$$(50) \quad t_n(c) := \hat{Q}_{n+c}^{(\alpha,\beta)}(x) := Q_{n+c}^{(\alpha,\beta)}(2x - 1)$$

$$= \frac{\Gamma(n + c + \alpha + 1)\Gamma(n + c + \beta + 1)}{2\Gamma(2n + 2c + \gamma + 1)} \times (x - 1)^{-n-\alpha-1-c}x^{-\beta}$$

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$$\times F\left(\begin{matrix} n+c+1, n+c+\alpha+1 \\ 2n+2c+\gamma+1 \end{matrix}; \frac{1}{1-x}\right).$$

x will be the complex plane off the cut [0, 1]. $w_n(c)$ is obviously a solution and $t_n(c)$ can be shown to be one by Sister Celine's techniques.

We can write

$$(51) \quad R_n^{(\alpha,\beta)}(x; c) = \frac{t_{-1}(c)w_n(c) - w_{-1}(c)t_n(c)}{\Delta(c)}$$

$$(52) \quad \Delta(c) = t_{-1}(c)w_0(c) - w_{-1}(c)t_0(c)$$

$$= \frac{(2c+\gamma-1)\Gamma(c+\alpha)\Gamma(c+\beta)(x-1)^{-\alpha}x^{-\beta}}{2\Gamma(c+1)\Gamma(c+\gamma)}$$

by [**8**, v. 2, p. 172 (26)]. In $C - [0, 1]$, w_n dominates t_n exponentially, as results of Watson show [**11**, v. 1, p. 237 (11)]. In fact, the asymptotic theory of linear difference equations [**18**] shows that t_n/w_n has a complete asymptotic expansion of Poincaré type,

$$(53) \quad \begin{cases} \dfrac{t_n}{w_n} \sim C(x)\tau(x)^{-4n}\left[1 + \dfrac{c_1}{n} + \dfrac{c_2}{n^2} + \dots\right], \\ \tau(x) = \sqrt{x} + \sqrt{x-1}, \ n \to \infty, \ x \notin [0, 1]. \end{cases}$$

We now follow the construction given in [**8**, v. 2, 10.5]. The nth convergent to the continued fraction generated by the recurrence for R_n is F_n/G_n where

$$(54) \quad F_n = R_{n-1}^{(\alpha,\beta)}(x; c+1), \quad G_n = R_n^{(\alpha,\beta)}(x; c).$$

$$(55) \quad \frac{F_n}{G_n} = \left[\frac{t_{-1}(c+1)w_{n-1}(c+1) - w_{-1}(c+1)t_{n-1}(c+1)}{t_{-1}(c)w_n(c) - w_{-1}(c)t_n(c)}\right] \times \frac{\Delta(c)}{\Delta(c+1)}.$$

Now $w_{n-1}(c+1) = w_n(c)$, $t_{n-1}(c+1) = t_n(c)$. Dividing numerator and denominator by $w_n(c)$ shows

$$(56) \quad \frac{F_n}{G_n} = \frac{t_0(c)\Delta(c)}{t_{-1}(c)\Delta(c+1)} + O(\tau^{-4n}),$$

$$n \to \infty, \ x \notin [0, 1].$$

F_n/G_n is the $[n - 1/n]$ Padé approximant (about ∞) to the function on the right, and the above formula asserts convergence as $n \to \infty$ for all $x \notin [0, 1]$.

Redefining things a little, we arrive at the following

THEOREM 2. *The* $[n - 1/n]$ *Padé approximant about* 0 *to the function*

$$(57)\quad f(x) := F\left(\begin{matrix} a+1, b+1 \\ c+2 \end{matrix}; x\right) \Big/ F\left(\begin{matrix} a, b \\ c \end{matrix}; x\right)$$

is U_n/V_n *where*

$$(58)\quad U_n = \frac{-n(c+n+1)x^{n-1}}{(a+1)(b+1)} \sum_{k=0}^{n-1} \frac{(1-n)_k(n+c+2)_k x^{-k}}{(a+2)_k(b+2)_k} \times {}_4F_3\left(\begin{matrix} k+1-n, n+k+c+2, a+1, b+1 \\ k+a+2, k+b+2, c+2 \end{matrix}; 1\right)$$

$$(59)\quad V_n = x^n \sum_{k=0}^{n} \frac{(-n)_k(n+c+1)_k x^{-k}}{(a+1)_k(b+1)_k} \times {}_4F_3\left(\begin{matrix} k-n, n+k+c+1, a, b \\ k+a+1, k+b+1, c \end{matrix}; 1\right).$$

U_n/V_n *converges* $O[\sigma(x)^{4n}]$ *in the cut plane* $C - [1, \infty)$ *to* $f(x)$, *where*

$$(60)\quad \sigma(x) := \sqrt{x}/(1 + \sqrt{1-x}).$$

The case $a = 0$ is a result foreshadowed by Laguerre and given by Luke [**11**, v. 2, p. 168]. Confluent cases were given in [**2**] and [**19**].

Now we consider the measure for R_n. Let

$$(61)\quad g(x) := \frac{t_0(c)}{t_{-1}(c)}.$$

The F in $t_{-1}(c)$ has parameters $(c, c + \alpha, 2c + \gamma - 1)$, argument $1/(1 - x)$. Using a Kummer transformation produces an F with parameters $(c, c + \beta, 2c + \gamma - 1)$, argument $1/x$.

Consequently that function will be positive for all real $x \notin [0, 1]$ provided $c \geqq 0$, $c + \alpha > 0$, $c + \beta > 0$, for instance. (Other conditions are also possible.) Thus, the denominator of g cannot vanish on $R - [0, 1]$ and f can have no poles there, so the measure can have no point masses there (see the discussion in [**1**, Chapter 2]). This is the only case we will consider here. The same theorems used in [**6**] show the measure is continuous in $(0, 1)$ and has support in $[0, 1]$ and the formulas (34), (41) can be used to rule out mass points at 0 or 1, again, see [**6**]. The Stieltjes inversion formula gives

$$(62)\quad w(t, c) = \lim_{\epsilon \to 0} [g(t - i\epsilon) - g(t + i\epsilon)].$$

Determining the weight function is extremely tedious. We shall only sketch the derivation.

Let

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$$(63)\quad u(x) := \frac{h(c)}{xh(c-1)},$$

$$(64)\quad h(c) := F\left(\begin{matrix} c+1, c+\beta+1 \\ 2c+\gamma+1 \end{matrix}; \frac{1}{x}\right).$$

$u(x)$ is a constant multiple of $t_0(c)/t_{-1}(c)$ and hence the Stieltjes transform of $w(c, t)$.

We have, by [**8**, v. 1, p. 108 (2)]

$$(65)\quad h(c) = B_1(c)(xe^{-\pi i})^{c+1}F\left(\begin{matrix} c+1, 1-c-\gamma \\ 1-\beta \end{matrix}; x\right)$$
$$+ B_2(c)(xe^{-\pi i})^{c+\beta+1}F\left(\begin{matrix} c+\beta+1, \beta+1-\gamma-c \\ \beta+1 \end{matrix}; x\right).$$

We need to evaluate

$$(66)\quad u^+(x) - u^-(x) := r(x)$$

where u^+ denotes the value of u above the cut $[0, 1]$, u^- the value of u below the cut.

We have

$$(67)\quad r(x) = \frac{[h^+(c)h^-(c-1) - h^+(c-1)h^-(c)]}{xh^+(c-1)h^-(c-1)}.$$

The numerator is now expanded and Lemma 1 used on $h^+(c)$, $h^-(c)$. After being subjected to much algebra, the numerator reduces to a Wronskian. We have

THEOREM 3. *Let* $c \geqq 0$, $c + \alpha > 0$, $c + \beta > 0$. *Then* $\{R_n^{(\alpha,\beta)}(t; c)\}$ *is an orthogonal set on* $[0, 1]$ *with respect to the weight function*

$$(68)\quad w(t; c)$$
$$= \frac{(1-t)^\alpha t^\beta}{\left|F\left(\begin{matrix} c, 2-\gamma-c \\ 1-\beta \end{matrix}; t\right) + K(c)(te^{\pi i})^\beta F\left(\begin{matrix} \beta+c, \beta+2-\gamma-c \\ \beta+1 \end{matrix}; t\right)\right|^2},$$

$$(69)\quad K(c) = \frac{\Gamma(-\beta)\Gamma(c+\beta)\Gamma(c+\gamma-1)}{\Gamma(\beta)\Gamma(c+\gamma-\beta-1)\Gamma(c)}.$$

As a curious example of this theorem, ($\beta = 1 - c$) we find that the polynomials $R_n^{(\alpha,1-c)}(t; c)$ are orthogonal on $[0, 1]$ with respect to the weight function

$$(70)\quad w(t) = t^{c-1}|B(c+\alpha, 1-c) - e^{\pi ic}B_t(1-c, -\alpha)|^{-2}.$$

Here B_t is the incomplete beta function, [**8**, v. 1, p. 87].

When $c = k = 1, 2, \ldots$, is integral, the weight function can be simply expressed in terms of $R_k(t)$ and shifted "associated" polynomials

$$q_k^{(\alpha,\beta)}(2t - 1) = (\text{const.})R_{k-1}^{(\alpha,\beta)}(t;\ 1).$$

A convenient alternate form of (68) to use in this case is

$$(71)\quad w(t, c) = \frac{(\text{const.})(1 - t)^\alpha t^\beta}{[(t - 1)^\alpha t^\beta \hat{Q}_{c-1}(t)]^+ [(t - 1)^\alpha t^\beta \hat{Q}_{c-1}(t)]^-}.$$

Now by [**8**, v. 2, p. 171 (25)], we have

$$(72)\quad (t - 1)^\alpha t^\beta \hat{Q}_{k-1}(t) = \frac{B(\alpha + 1, \beta + 1)}{2}\left\{t^{-1}F\left(\begin{matrix}1, \beta + 1\\ \gamma + 1\end{matrix}; \frac{1}{t}\right)\right.$$
$$\left.\times R_{k-1}(t) - (\gamma + 1)R_{k-2}(t;\ 1)\right\},$$
$$k = 1, 2, \ldots .$$

(We omit superscripts for convenience.) In a number of interesting cases, the F can be summed. ($\alpha = \beta = 0$; $\gamma = 0$; $\alpha = 0$, $\beta = -1/2$, etc.)

Special cases of the formulas in this section have been considered by many writers. Nevai [**12**] has determined a formula for $w(t, c)$ for integer c for a class of polynomials generalizing the Bernstein-Szëgo polynomials. The case (72) can be determined from his work. Our formula only produces an explicit evaluation of an integral in his master formula, p. 414. See also [**4**, **6**, **13**, **15**]. Pollaczek [**13**] gave the formula for $w(t, c)$ for general c in the case $\alpha = \beta$ (the associated ultra-spherical polynomials).

If β is an integer in (68), limits must be taken, but the techniques for doing this are well-established, see [**11**] for many examples. Logarithms arise in these cases.

4. A generating function. We start with an intriguing expansion which has been derived independently by several authors.

THEOREM 4. *Let t, x, a, b, c, be complex*, $x \notin [1, \infty)$,

$$|t| < |\sqrt{x} + \sqrt{x - 1}|^{-2}.$$

Then

$$(73)\quad \sum_{n=0}^{\infty} \frac{(c + a)_n (b)_n t^n}{n!(a + b + 1)_n} F\left(\begin{matrix}-n - a, n + b\\ c\end{matrix}; x\right)$$
$$= \left(\frac{2}{t}\right)^b (Z_2 - 1)^{-a-c}(Z_2 + 1)^{c+a-b}$$
$$\times F\left(\begin{matrix}-a, b\\ c\end{matrix}; \frac{1 - Z_1}{2}\right) F\left(\begin{matrix}a + c, a + 1\\ a + b + 1\end{matrix}; \frac{2}{1 - Z_2}\right),$$

where

$$(74)\quad Z_1 = \frac{1-\phi}{t},\ Z_2 = \frac{1+\phi}{t},\ \phi = \sqrt{(1-t)^2 + 4xt}.$$

Flensted-Jensen and Koornwinder first stated this in full generality in 1975, [**9**]. In 1977 Cohen, [7], gave it for the case where a is an integer (although his proof doesn't really require this hypothesis). Bellandi Fo and de Oliveira, unaware of either earlier paper, published it in 1982, [**5**]. The proofs are radically different. Cohen's, which is very elegant, relies only on an elementary (and clever) series argument.

Note all the functions above are multivalent, and one must be careful to select the correct branches. The branch of ϕ is taken which is positive when t is small positive.

We now multiply both sides of (28) by

$$(-1)^n(c+\gamma)_n(c+1)_n t^n/n!(\gamma+2c+1)_n$$

and sum from $n = 0$ to ∞, using first the above result with

$$(a, b, c) = (c, \gamma + c, \beta + 1),$$

next with

$$(a, b, c) = (\gamma + c - 1, c + 1, 1 - \beta),$$

then replace t by $-t$. Everything fits together very neatly and after a Kummer transformation on the third, fourth and fifth F we arrive at

THEOREM 5. *Subject to the conditions of the previous theorem*

$$(75)\quad \sum_{n=0}^{\infty} \frac{(c+\gamma)_n(c+1)_n t^n R_n^{(\alpha,\beta)}(x;c)}{n!(\gamma+2c+1)_n}$$

$$= \frac{1}{\beta(\gamma+2c-1)}\left(\frac{2}{1+t+\rho}\right)^{\gamma+c}$$

$$\times \left\{(\beta+c)(\gamma+c-1)F\left(\begin{matrix}c, 2-\gamma-c\\ 1-\beta\end{matrix}; x\right)\right.$$

$$\times F\left(\begin{matrix}-c, \gamma+c\\ \beta+1\end{matrix}; \frac{1+t-\rho}{2t}\right)F\left(\begin{matrix}\gamma+c-\beta, \gamma+c\\ \gamma+2c+1\end{matrix}; \frac{2t}{1+t+\rho}\right)$$

$$- c(\gamma+c-\beta-1)\left(\frac{1+t+\rho}{2}\right)^{\beta}F\left(\begin{matrix}c+\beta, 2-\gamma-c+\beta\\ \beta+1\end{matrix}; x\right)$$

$$\times F\left(\begin{matrix}\gamma+c-\beta, -c-\beta\\ 1-\beta\end{matrix}; \frac{1+t-\rho}{2t}\right)$$

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$$\times F\left(\begin{matrix}\gamma + c + \beta, \gamma + c \\ \gamma + 2c + 1\end{matrix}; \frac{2t}{1 + t + \rho}\right)\Bigg\},$$

$$\rho = \sqrt{(1 + t)^2 - 4xt}.$$

There is no doubt one of the most exotic generating functions known which utilizes only fairly ordinary functions. Yet, in a sense, it is the generating function for R_n. True, it does not reduce when $c = 0$ to the standard generating function for $R_n^{(\alpha,\beta)}(x)$,

$$(76) \quad \sum_{n=0}^{\infty} t^n R_n^{(\alpha,\beta)}(x) = \frac{2^{\gamma-1}}{\rho}(1 - t + \rho)^{-\alpha}(1 + t + \rho)^{-\beta}$$

yielding instead

$$(77) \quad \sum_{n=0}^{\infty} \frac{t^n}{(n + \gamma)} R_n^{(\alpha,\beta)}(x)$$
$$= \frac{1}{\gamma}\left(\frac{2}{1 + t + \rho}\right)^{\gamma} F\left(\begin{matrix}\gamma - \beta, \gamma \\ \gamma + 1\end{matrix}; \frac{2t}{1 + t + \rho}\right),$$

but the generating function which does have this property, and which can be obtained from (75) by multiplying by $t^{\gamma+2c}$ and differentiating with respect to t, is horribly complicated. The simplest generalization does not always yield the simplest specialization.

Letting $x \to x/\gamma$, $\gamma \to \infty$, gives a generating function for $L_n^{\beta}(x; c)$ found by Askey and Wimp **[2]**.

When $\alpha = \beta$ the formula produces a different generating function from the one given by Bustoz and Ismail for the associated ultraspherical polynomials, **[6]**. Theirs, which involves the Appell function F_1, does reduce, when $c = 0$, to a standard generating function for $C_n^{(\lambda)}(x)$. $\alpha = \beta$ produces no simplification in our formula.

Acknowledgement. The author thanks Alberto Izaguirre of the University of Pennsylvania for assisting him in the computer algebra computations of Section 2.

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