Explicit refinements of Böcherer's conjecture for Siegel modular forms of squarefree level

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Abstract

We formulate an explicit refinement of Böcherer's conjecture for Siegel modular forms of degree 2 and squarefree level, relating weighted averages of Fourier coefficients with special values of L-functions. To achieve this, we compute the relevant local integrals that appear in the refined global Gan-Gross-Prasad conjecture for Bessel periods as proposed by Yifeng Liu. We note several consequences of our conjecture to arithmetic and analytic properties of L-functions and Fourier coefficients of Siegel modular forms.

We also compute and write down the relevant ε-factors for all dihedral twists of non-supercuspidal representations of GSp_4 over a non-archimedean local field. By comparing this with the conditions for a local Bessel model to exist, we verify the local Gan-Gross-Prasad conjecture in these cases for the generic L-packets and also demonstrate its severe failure for the non-generic L-packets.

Contents

1 Introduction

1.1 Böcherer's conjecture

Let f be a Siegel cusp form[1] of degree 2 and weight k for the group $\mathrm{Sp}_4(\mathbb{Z})$. Then f has a Fourier expansion

$$f(Z) = \sum_S a(f,S) e^{2\pi i \mathrm{Tr}(SZ)},$$

where the Fourier coefficients $a(f,S)$ are indexed by matrices S of the form

$$S = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}, \qquad a,b,c \in \mathbb{Z}, \qquad a > 0, \qquad \mathrm{disc}(S) := b^2 - 4ac < 0. \tag{1}$$

For two matrices S_1, S_2 as above, we write $S_1 \sim S_2$ if there exists $A \in \mathrm{SL}_2(\mathbb{Z})$ such that $S_1 = {}^tAS_2A$. Using the defining relation for Siegel cusp forms, we see that

$$a(f,S_1) = a(f,S_2) \qquad \text{if } S_1 \sim S_2, \tag{2}$$

thus showing that $a(f,S)$ only depends on the $\mathrm{SL}_2(\mathbb{Z})$-equivalence class of the matrix S, or equivalently, only on the proper equivalence class of the associated binary quadratic form.

Let $d < 0$ be a fundamental discriminant[2]. Put $K = \mathbb{Q}(\sqrt{d})$ and let Cl_K denote the ideal class group of K. It is well-known that the $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of binary quadratic forms of discriminant d are in natural bijective correspondence with the elements of Cl_K. In view of the comments above, it follows that the notation $a(f,c)$ makes sense for $c \in \mathrm{Cl}_K$. We define

$$R(f,K) = \sum_{c \in \mathrm{Cl}_K} a(f,c) = \sum_{\substack{S/\sim \\ \mathrm{disc}(S)=d}} a(F,S). \tag{3}$$

For odd k, it is easy to see that $R(f,K)$ equals 0. If k is even, Böcherer [2] made a remarkable conjecture that relates the quantity $R(f,K)$ to the central value of a certain L-function.

1.1 Conjecture. (Böcherer [2]) *Let k be even, $f \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ be a non-zero Hecke eigenform and π_f be the associated automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$. Then there exists a constant c_f depending only on f such that for any imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$ with $d < 0$ a fundamental discriminant, we have*

$$|R(f,K)|^2 = c_f \cdot w(K)^2 \cdot |d|^{k-1} \cdot L_{\mathrm{f}}(1/2, \pi_f \otimes \chi_d).$$

Above, $\chi_d = \left(\frac{d}{\cdot}\right)$ is the Kronecker symbol (i.e., the quadratic Hecke character associated via class field theory to the field $\mathbb{Q}(\sqrt{d})$), $w(K)$ denotes the number of distinct roots of unity inside K, and $L_{\mathrm{f}}(s, \pi_f \otimes \chi_d)$ denotes the (finite part of the) associated degree 4 Langlands L-function.[3]

As far as we know, Böcherer did not speculate what the constant c_f is exactly (except for the case when f is a Saito-Kurokawa lift). For many applications, both arithmetic and analytic, it would be of interest to have a conjectural formula where the constant c_f is known precisely, and one of the original motivations of this paper was to do exactly that.

[1] For definitions and background on Siegel cusp forms, see [18].

[2] Recall that an integer n is a fundamental discriminant if *either* n is a squarefree integer congruent to 1 modulo 4 *or* $n = 4m$ where m is a squarefree integer congruent to 2 or 3 modulo 4.

[3] All global L-functions in this paper are normalized to have the functional equation taking $s \mapsto 1-s$. We denote the completed L-functions by $L(s, \ )$ and reserve $L_{\mathrm{f}}(s, \ )$ for the finite part of the L-function, i.e., without the gamma factor $L_\infty(s, \ )$. Thus $L(s, \ ) = L_{\mathrm{f}}(s, \ )L_\infty(s, \ )$.

This paper deals with refinements of Böcherer's conjecture for Siegel newforms with squarefree level. A special case of the results of this paper is the following *precise formula* for c_f, valid for all eigenforms f of full level and even weight k that are not Saito-Kurokawa lifts:

$$c_f = \frac{2^{4k-5}\cdot\pi^{2k+2}}{(2k-2)!}\cdot\frac{L_{\mathrm{f}}(1/2,\pi_f)}{L_{\mathrm{f}}(1,\pi_f,\mathrm{Ad})}\cdot\langle f,f\rangle. \tag{4}$$

Above, $\langle f,f\rangle$ is the Petersson norm of f normalized as in (82), $L_{\mathrm{f}}(s,\pi_f,\mathrm{Ad})$ is the finite part of the adjoint (degree 10) L-function for π_f, and $L_{\mathrm{f}}(s,\pi_f)$ is the finite part of the spinor (degree 4) L-function for π_f.

Conjecture 1.1 can be extended naturally as follows. For any character Λ of the finite group Cl_K, we make the definition

$$R(f,K,\Lambda) = \sum_{c\in\mathrm{Cl}_K} a(f,c)\Lambda^{-1}(c). \tag{5}$$

Let $\mathcal{AI}(\Lambda^{-1})$ be the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ given by the automorphic induction of Λ^{-1} from $\mathbb{A}_K^\times$; it is generated by (the adelization of) the theta-series of weight 1 and character χ_d given by

$$\theta_{\Lambda^{-1}}(z) = \sum_{0\neq\mathfrak{a}\subset O_K}\Lambda^{-1}(\mathfrak{a})e^{2\pi i N(\mathfrak{a})z}.$$

We make the following refined version of Conjecture 1.1 (a further generalization of which to the case of squarefree levels is stated as Theorem 1.13 later in this introduction).

1.2 Conjecture. *Let $f\in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ be a non-zero Hecke eigenform of even weight k and π_f the associated automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$. Suppose that f is not a Saito-Kurokawa lift. Then for any imaginary quadratic field $K=\mathbb{Q}(\sqrt{d})$ with $d<0$ a fundamental discriminant and any character Λ of Cl_K, we have*

$$\begin{aligned}\frac{|R(f,K,\Lambda)|^2}{\langle f,f\rangle} &= (2^{2k-5}\cdot\pi)\ w(K)^2\ |d|^{k-1}\frac{L(1/2,\pi_f\times\mathcal{AI}(\Lambda^{-1}))}{L(1,\pi_f,\mathrm{Ad})}\\ &= \frac{2^{4k-5}\cdot\pi^{2k+2}}{(2k-2)!}\ w(K)^2\ |d|^{k-1}\frac{L_{\mathrm{f}}(1/2,\pi_f\times\mathcal{AI}(\Lambda^{-1}))}{L_{\mathrm{f}}(1,\pi_f,\mathrm{Ad})}\end{aligned}$$

1.3 Remark. *We note that $L_{\mathrm{f}}(1,\pi_f,\mathrm{Ad})$ is not zero by Theorem 5.2.1 of [25].*

1.4 Remark. *The values of the relevant archimedean L-factors used above are as follows,*

$$\begin{aligned}L_\infty(1/2,\pi_f\times\mathcal{AI}(\Lambda^{-1})) &= 2^4(2\pi)^{-2k}(\Gamma(k-1))^2,\\ L_\infty(1,\pi_f,\mathrm{Ad}) &= 2^5(2\pi)^{-4k-1}(\Gamma(k-1))^2\,\Gamma(2k-1).\end{aligned}$$

1.5 Remark. *If $\Lambda=1$, then $R(f,K,1)=R(f,K)$ and*

$$L(1/2,\pi_f\otimes\mathcal{AI}(1)) = L(1/2,\pi_f)L(1/2,\pi_f\otimes\chi_d).$$

Hence, Conjecture 1.1 is a special case of Conjecture 1.2 with the particular value of c_f given in (4).

1.6 Remark. *For Saito-Kurokawa lifts, a similar equality is actually a theorem; see Section 4.4.*

1.7 Remark. *Note that Conjecture 1.2 implies that whenever $R(f,K,\Lambda)\neq 0$, one also has $L(1/2,\pi_f\times\mathcal{AI}(\Lambda^{-1}))\neq 0$. This is a special case of the Gan-Gross-Prasad conjectures [12].*

As we explain in the next section, Conjecture 1.2 follows from the refined Gan-Gross-Prasad (henceforth GGP) conjectures as stated by Yifeng Liu [21]. These refined conjectures involve certain local factors which are essentially integrals of local matrix coefficients. The primary goal of the present paper is to compute these local factors exactly in several cases. The calculation of the local factor at infinity (for all even k) is what allows us to make Conjecture 1.2. We also calculate these local factors for certain Iwahori-spherical representations. As a consequence, we are able to generalize Conjecture 1.2 to all newforms of squarefree level. For the definition of the local factors and an overview of our results, we refer the reader to Section 1.3; for the resulting explicit refinement of Böcherer's conjecture, see Section 1.5 and in particular Theorem 1.13. The details and proofs of these appear in Sections 3 and 4 respectively. We also prove some consequences of these refined conjectures in Section 4.5 (some of these consequences are summarized later in this introduction in Section 1.6).

A secondary goal of this paper is to compute some local ε-factors related to what is known as the local GGP conjecture (which is now a theorem in the case of interest to us), and simultaneously write down conditions for a local Bessel model to exist. Our calculations will provide an explicit verification of the local GGP conjecture in our case for all non-supercuspidal representations. For further details of this result and its connection to the global GGP conjecture and Liu's conjecture, we refer the reader to Sections 1.4 and 2.

1.2 Bessel periods and the refined Gan-Gross-Prasad conjecture

We now explain how Conjectures 1.1 and 1.2 fit into the framework of the GGP conjectures made in [12]. Let F be a number field, $H \subseteq G$ be reductive groups over F, and π (resp. σ) be an irreducible, unitary, cuspidal, automorphic representation of $G(\mathbb{A}_F)$ (resp. $H(\mathbb{A}_F)$).

Then if ϕ_π and ϕ_σ are automorphic forms in the space of π and σ respectively, one can form the global period

$$P(\phi_\pi, \phi_\sigma) = \int\limits_{H(F)Z(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \phi_\pi(h)\phi_\sigma(h)dh.$$

The basic philosophy is that, in certain cases, the quantity $|P(\phi_\pi, \phi_\sigma)|^2$ should be essentially equal to the value $L(1/2, \pi \otimes \sigma)$. The earliest prototype of this is due to Waldspurger [44], who dealt with the case $H = \mathrm{SO}(2)$, $G = \mathrm{SO}(2,1) \simeq \mathrm{PGL}(2)$. The case $H = \mathrm{SO}(2,1)$, $G = \mathrm{SO}(2,2)$ is the famous triple product formula (note that $\mathrm{SL}(2) \times \mathrm{SL}(2)$ is a double cover for $\mathrm{SO}(2,2)$), developed and refined by various people over the years, including Harris-Kudla [14], Watson [46], Ichino [15], and Nelson-Pitale-Saha [23]. Gross and Prasad made a series of conjectures in this vein for orthogonal groups; this was extended to other classical groups by Gan-Gross-Prasad [12]. In their original form, the Gross-Prasad and GGP conjectures do not predict an exact formula for $|P(\phi_\pi, \phi_\sigma)|^2$ but instead deal with the conditions for its non-vanishing in terms of the non-vanishing of $L(1/2, \pi \otimes \sigma)$. However, a recent work of Yifeng Liu [21], extending that of Ichino-Ikeda [16] and Prasad-Takloo-Bighash [29], makes a refined GGP conjecture by giving a precise conjectural formula for $|P(\phi_\pi, \phi_\sigma)|^2$ for a wide family of automorphic representations.

The case of Liu's conjecture that interests us in this paper is $G = \mathrm{PGSp}(4) \simeq \mathrm{SO}(3,2)$, and $H = T_S N$ equal to the *Bessel subgroup*, which is an enlargement of $T_S \simeq \mathrm{SO}(2)$ with a unipotent subgroup N.

Let S be a symmetric 2×2 matrix that is anisotropic over F; in particular this implies that $d = \mathrm{disc}(S)$ is not a square in F. Define a subgroup T_S of GL_2 by

$$T_S(F) = \{g \in \mathrm{GL}_2(F) : \ {}^t g S g = \det(g) S\}. \tag{6}$$

It is easy to verify that $T_S(F) \simeq K^\times$ where $K = F(\sqrt{d})$. We consider T_S as a subgroup of GSp_4 via

$$T_S \ni g \longmapsto \begin{bmatrix} g & 0 \\ 0 & \det(g)\,{}^t g^{-1} \end{bmatrix} \in \mathrm{GSp}_4. \tag{7}$$

Let us denote by N the subgroup of GSp_4 defined by

$$N = \{n(X) = \begin{bmatrix} 1_2 & X \\ 0 & 1_2 \end{bmatrix} \mid {}^t X = X\}.$$

Fix a non-trivial additive character ψ of $F\backslash\mathbb{A}_F$. We define the character θ_S on $N(\mathbb{A})$ by $\theta(n(X)) = \psi(\mathrm{Tr}(SX))$. Let Λ be a character of $T_S(F)\backslash T_S(\mathbb{A}_F) \simeq K^\times\backslash\mathbb{A}_K^\times$ such that $\Lambda|_{\mathbb{A}_F^\times} = \omega_\pi$. Let (π, V_π) be a cuspidal, automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ with trivial central character. For an automorphic form $\phi \in V_\pi$, we define the global Bessel period $B(\phi,\Lambda)$ on $\mathrm{GSp}_4(\mathbb{A})$ by

$$B(\phi,\Lambda) = \int\limits_{\mathbb{A}_F^\times T_S(F)\backslash T_S(\mathbb{A}_F)} \int\limits_{N(F)\backslash N(\mathbb{A}_F)} \phi(tn)\Lambda^{-1}(t)\theta_S^{-1}(n)\,dn\,dt. \tag{8}$$

The GGP conjecture in this case can be stated as follows.

1.8 Conjecture. (Special case of global GGP) *Let π, Λ be as above. Suppose that for almost all places v of F, the local representation π_v is generic. Suppose also that there exists an automorphic form ϕ in the space of π such that $B(\phi,\Lambda) \neq 0$. Then $L(1/2, \pi \times \mathcal{AI}(\Lambda^{-1})) \neq 0$.*

1.9 Remark. *The global GGP [12] actually predicts more in this case, namely that $L(1/2, \pi \times \mathcal{AI}(\Lambda^{-1})) \neq 0$ if and only if there exists a representation π' in the Vogan packet of π and an automorphic form ϕ' in the space of π' such that $B(\phi',\Lambda) \neq 0$.*

1.10 Remark. *In the special case $F = \mathbb{Q}$, if we take ϕ to be the adelization of a weight k Siegel cusp form, ψ the standard additive character, $d < 0$ a fundamental discriminant, and Λ corresponding to a character of Cl_K, then up to a constant, one has the relation (see [10])*

$$B(\phi,\Lambda) = e^{-2\pi \mathrm{Tr}(S)} R(f, K, \Lambda).$$

We now state Liu's refinement of Conjecture 1.8 in our case. For simplicity, we restrict to the case $F = \mathbb{Q}$ and write $\mathbb{A}$ for $\mathbb{A}_\mathbb{Q}$. Let the matrix S, the group $T = T_S$, the field $K = \mathbb{Q}(\sqrt{d})$, and the character χ_d be as above. Let $\pi = \otimes_v \pi_v$ be an irreducible, unitary, cuspidal, automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ with trivial central character and Λ be a unitary Hecke character of $K^\times\backslash\mathbb{A}_K^\times$ such that $\Lambda|_{\mathbb{A}^\times} = 1$. Fix the local Haar measures dn_v on $N(\mathbb{Q}_v)$ (resp. dt_v on $T(\mathbb{Q}_v)$) to be the standard additive measure at all places (resp. such that the maximal compact subgroup has volume one at almost all places). Fix the global measures dn and dt to be the *Tamagawa measures*. Note that $dn = \prod_v dn_v$. Define the constant C_T by $dt = C_T \prod_v dt_v$. For each automorphic form ϕ in the space of π, define the global Bessel period $B(\phi,\Lambda)$ via (8).

For each place v, fix a $\mathrm{GSp}_4(\mathbb{Q}_v)-$invariant Hermitian inner product $\langle\,,\rangle_v$ on π_v. For $\phi_v \in V_{\pi_v}$, define

$$J_v(\phi_v) = \frac{L(1,\pi_v,\mathrm{Ad})L(1,\chi_{d,v}) \int_{\mathbb{Q}_v^\times\backslash T(\mathbb{Q}_v)} \int_{N(\mathbb{Q}_v)} \frac{\langle \pi_v(t_v n_v)\phi_v,\phi_v\rangle}{\langle\phi_v,\phi_v\rangle} \Lambda_v^{-1}(t_v)\theta_S^{-1}(n_v)\,dt_v\,dn_v}{\zeta_{\mathbb{Q}_v}(2)\zeta_{\mathbb{Q}_v}(4)L(1/2,\pi_v \otimes \mathcal{AI}(\Lambda_v^{-1}))}.$$

Strictly speaking, the integral above may not converge absolutely, in which case one defines it via regularization (see [21, p. 6]). It can be shown that $J_v(\phi_v) = 1$ for almost all places. We now state the refined conjecture as phrased by Liu.

1.11 Conjecture. (Yifeng Liu [21]) *Let π, Λ be as above. Suppose that for almost all places v of $\mathbb{Q}$ the local representation π_v is generic. Let $\phi = \otimes_v \phi_v$ be an automorphic form in the space of π. Then*

$$\frac{|B(\phi,\Lambda)|^2}{\langle\phi,\phi\rangle} = \frac{C_T}{2S_\pi}\frac{\xi(2)\xi(4)L(1/2,\pi\times\mathcal{AI}(\Lambda^{-1}))}{L(1,\pi,\mathrm{Ad})L(1,\chi_d)} \prod_v J_v(\phi_v), \tag{9}$$

where ξ denotes the completed Riemann zeta function and S_π denotes a certain integral power of 2, related to the Arthur parameter of π. In particular,

$$S_\pi = \begin{cases} 4 & \text{if } L(s,\pi) = L(s,\pi_1)L(s,\pi_2) \text{ for irreducible cuspidal representations } \pi_i \text{ of } \mathrm{GL}_2(\mathbb{A}), \\ 2 & \text{if } L(s,\pi) = L(s,\Pi) \text{ for an irreducible cuspidal representation } \Pi \text{ of } \mathrm{GL}_4(\mathbb{A}). \end{cases}$$

Liu gave a proof of the above conjecture in the special case of Yoshida lifts. The proof in the case of the non-endoscopic analog of Yoshida lifts has been very recently given by Corbett [7]. To the best of our knowledge, Conjecture 1.11 remains open for all automorphic forms ϕ that are not lifts of some sort.

In view of Remark 1.10, it is clear that a precise computation of the factors $J_v(\phi_v)$ above will enable us to write down explicit refinements ofBöcherer's conjecture.

1.3 Local results I: computing local integrals

We now briefly describe our main local result. Fix the following purely local data (where for convenience we have dropped all subscripts).

i) A non-archimedean local field F of characteristic 0. We let $\mathfrak{o}$ denote the ring of integers of F and q the cardinality of the residue class field.

ii) A symmetric invertible matrix $S = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$. Put $d = b^2 - 4ac$. Denote $K = F(\sqrt{d})$ if d is a non-square in F and $K = F \times F$ if d is a square in F. Assume that $a, b \in \mathfrak{o}$, $c \in \mathfrak{o}^\times$, and d generates the discriminant ideal[4] of K/F. Let $\chi_{K/F}$ denote the quadratic character on $F^\times$ associated to the extension K/F.

iii) Haar measures dn and dt on $N(F)$ and $F^\times\backslash T_S(F)$ respectively, normalized so that the subgroup $N(\mathfrak{o})$ and the maximal compact subgroup of $F^\times\backslash T_S(F)$ have volume 1.

iv) An irreducible, admissible, unitary representation (π, V_π) of $\mathrm{GSp}_4(F)$ with trivial central character[5] and invariant inner product $\langle\, , \,\rangle$, and a choice of vector $\phi \in V_\pi$.

v) A character Λ on $K^\times$, which we also think of as a character on $T_S(F)$, that satisfies $\Lambda|_{F^\times} = 1$.

vi) An additive character ψ on F of conductor $\mathfrak{o}$. Define a character θ_S on $N(F)$ by $\theta(n(X)) = \psi(\mathrm{Tr}(SX))$.

Given all the above data, we define the local quantity

$$J(\phi) := \frac{L(1,\pi,\mathrm{Ad})L(1,\chi_{K/F}) \int_{F^\times\backslash T_S(F)} \int_{N(F)} \frac{\langle \pi(tn)\phi,\phi\rangle}{\langle\phi,\phi\rangle} \Lambda^{-1}(t)\theta_S^{-1}(n)\, dn\, dt}{\zeta_F(2)\zeta_F(4)L(1/2,\pi\times\mathcal{AI}(\Lambda^{-1}))}, \tag{10}$$

where $\zeta_F(s) = (1-q^{-s})^{-1}$. Note that the above quantity is exactly what appears in Liu's refined GGP conjecture (9) above. If all the data above is *unramified* (see Section 3.1), then one can show that $J(\phi) = 1$. Note also that $J(\phi)$ does not change if ϕ is multiplied by a constant.

Define the congruence subgroup P_1 of $\mathrm{GSp}_4(\mathfrak{o})$ as follows

$$P_1 = \left\{ A \in \mathrm{GSp}_4(\mathfrak{o}) : A \in \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \mathfrak{o} \end{bmatrix} \right\}. \tag{11}$$

Assume that π contains a non-zero vector fixed by P_1. If π is spherical (unramified), we assume further that it is generic. Using the standard classification (see [38] or [27] or the tables later in this paper) this implies that π is an Iwahori spherical representation of type I, IIa, IIIa, Vb, Vc, VIa or VIb.

For each π as above, we choose a certain orthogonal basis $\mathcal{B}$ of the P_1-fixed subspace of π. Then for each vector ϕ in $\mathcal{B}$, we *exactly compute* the local integrals $J(\phi)$, under the assumptions that F has odd residual characteristic, K/F is an unramified field extension and Λ is unramified. For the complete results, see Theorem 3.11, which is the technical heart of this paper. Its proof relies on computations of various local integrals and matrix coefficients, which are performed in Section 3.

[4] When $K = F \times F$, the discriminant ideal is just $\mathfrak{o}^\times$.

[5] Our results immediately extend to the slightly more general case of unramified central character by considering a suitable unramified twist of π.

1.4 Local results II: computation of local ε-factors

As mentioned earlier, we also compute the local factors $\varepsilon(1/2, \pi \otimes \mathcal{AI}(\Lambda^{-1}), \psi)$ for all irreducible, admissible, non-supercuspidal representations of $\mathrm{GSp}(4, F)$. Combined with the existence conditions for local Bessel functionals from [33], we verify the following theorem for all non-supercuspidal π.

1.12 Theorem. (Prasad - Takloo-Bighash, Waldspurger) *Let F be a local field and $\{\pi\}$ be an irreducible, admissible, generic L-packet of $\mathrm{GSp}(4, F)$. Let $S \in \mathrm{Sym}_2(F)$ have non-zero determinant and let Λ be a character of the associated quadratic extension $K^\times$ such that $\Lambda\big|_{F^\times} = \omega_\pi$. Then the following are equivalent.*

i) *There exists a representation $\pi' \in \{\pi\}$ admitting a (Λ, θ)-Bessel functional.*

ii) *$\varepsilon(1/2, \pi \otimes \mathcal{AI}(\Lambda^{-1}), \psi) = 1$.*

The above theorem, which is a special case of the local GGP conjectures, has been proved by Prasad and Takloo-Bighash [29] (in the case of odd residual characteristic) and Waldspurger [45] (in much more generality, for tempered L-packets for orthogonal groups). The relation of the above Theorem with global GGP (Conjecture 1.8) is clear. Indeed, if $L(1/2, \pi \otimes \mathcal{AI}(\Lambda^{-1})) \neq 0$, then the global ε-factor $\varepsilon(1/2, \pi \otimes \mathcal{AI}(\Lambda^{-1}), \psi)$ must equal 1, which is certainly true if all the local ε-factors are equal to 1. On the other hand, $B(\phi, \Lambda) \neq 0$ implies that the representation π has a global Bessel model, in which case all the local components of π must admit Bessel functionals.

In Section 2 of this paper, we list all the non-supercuspidal representations π of $\mathrm{GSp}(4, F)$, and write down the precise conditions for a Bessel functional to exist. Then we compute and write down the values $\varepsilon(1/2, \pi \otimes \mathcal{AI}(\Lambda^{-1}), \psi)$ for each such π. In the process, we not only verify the above theorem for the generic L-packets, but also show its severe failure for the non-generic L-packets. We hope that the explicit table for ε-factors we provide in Section 2 will be of use to those interested in the local theory.

1.5 A refined conjecture in the classical language

We now use our results described in Section 1.3 to translate Conjecture 1.11 to the classical language of Fourier coefficients for Siegel modular forms of squarefree level.

For any congruence subgroup Γ of $\mathrm{Sp}_4(\mathbb{Z})$, and any integer k, let $S_k(\Gamma)$ denote the space of Siegel cusp forms of weight k with respect to Γ. For any positive integer N, let the congruence subgroup $\Gamma_0(N)$ be defined as follows:

$$\Gamma_0(N) = \mathrm{Sp}_4(\mathbb{Z}) \cap \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{bmatrix}. \tag{12}$$

Now, suppose that N is squarefree. By a newform of weight k for $\Gamma_0(N)$, we mean an element $f \in S_k(\Gamma_0(N))$ with the following properties.

i) f lies in the orthogonal complement of the space of oldforms $S_k(\Gamma_0(N))^{\mathrm{old}}$, as defined in [38].

ii) f is an eigenform for the local Hecke algebras for all primes p not dividing N and an eigenfunction of the $U(p)$ operator (see [36]) at all primes dividing N.

iii) The adelization of f generates an irreducible cuspidal representation π_f of $\mathrm{GSp}_4(\mathbb{A})$.

It is known that any newform f defined as above is of one of two types:

i) **CAP**: A CAP newform f is one for which the associated automorphic representation $\pi_f = \otimes_v \pi_{f,v}$ is nearly equivalent to a constituent of a global induced representation of a proper parabolic subgroup of $\mathrm{GSp}_4(\mathbb{A})$. These newforms f fail the Ramanujan conjecture at all primes. Furthermore, if $k \geq 3$, the CAP newforms are exactly those that are obtained via Saito-Kurokawa lifting from classical newforms of weight $2k-2$ and level N [39]. For $k = 1, 2$, one can potentially also have CAP newforms that are not of Saito-Kurokawa type; these are associated to the Borel or Klingen parabolics. Finally, if f is CAP, the representations $\pi_{f,v}$ are non-generic for all places v.

ii) **Non-CAP**: These are those newforms f that are not CAP. Their degree 4 L-functions are always entire. Weissauer proved [47] that the non-CAP newforms always satisfy the Ramanujan conjecture at all primes $p \nmid N$ whenever $k \geq 3$. Another key property of the non-CAP newforms is that the local representations $\pi_{f,p}$ are generic for all $p \nmid N$. In terms of the standard classification, the representations $\pi_{f,p}$ are type I if $p \nmid N$, and one of types IIa, IIIa, Vb/c, VIa, VIb if $p|N$. (Conjecturally, they are not of type Vb/c, since these are non-tempered representations.)

Given any newform f in $S_k(\Gamma_0(N))$, any fundamental discriminant $d < 0$, and any character Λ of the ideal class group of $K = \mathbb{Q}(\sqrt{d})$, we can define the quantity $R(f, K, \Lambda)$ exactly as in (5). Then our local results lead to the following theorem.

1.13 Theorem. *Let N be squarefree and f be a non-CAP newform in $S_k(\Gamma_0(N))$. For $d < 0$ a fundamental discriminant, and Λ a character of Cl_K, define the quantity $R(f, K, \Lambda)$ as in (5). Let $\pi_f = \otimes_v \pi_{f,v}$ be the automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ associated to f. Suppose that N is odd, k is even, and $\left(\frac{d}{p}\right) = -1$ for all primes p dividing N. Then, assuming the truth of Conjecture 1.11, we have*

$$\frac{|R(f,K,\Lambda)|^2}{\langle f, f\rangle} = (2^{2k-s}\cdot\pi)\ w(K)^2\ |d|^{k-1}\frac{L(1/2,\pi_f\times\mathcal{AI}(\Lambda^{-1}))}{L(1,\pi_f,\mathrm{Ad})}\prod_{p|N}J_p, \tag{13}$$

where $s = 6$ if f is a weak Yoshida lift[6] *in the sense of [35] and $s = 5$ otherwise. The quantities J_p for $p|N$ are given as follows:*

$$J_p = \begin{cases} (1+p^{-2})(1+p^{-1}) & \text{if } \pi_{f,p} \text{ is of type IIIa},\\ 2(1+p^{-2})(1+p^{-1}) & \text{if } \pi_{f,p} \text{ is of type VIb},\\ 0 & \text{otherwise.}\end{cases}$$

1.14 Remark. *The assumptions N odd and $\left(\frac{d}{p}\right) = -1$ arise since our local computations are done only for unramified field extensions of odd residue characteristic. The assumption $k \in 2\mathbb{Z}$ is related to Saito-Kurokawa lifts. It would be of interest to remove all these assumptions and get a more general result.*

1.6 Some consequences of the refined conjecture

The formula (13) is the promised explicit refinement of Böcherer's conjecture for Siegel newforms of squarefree level. Our main global result, Theorem 4.8, is a mild extension of this formula that includes the case of oldforms. Historically, exact formulas relating periods and L-functions (such as Watson's triple product formula [46] or Kohnen's refinement of Waldspurger's formula [19]) have had various applications, ranging from quantum unique ergodicity in level aspect [23] to non-vanishing of central values modulo primes [6]. So one can expect Theorem 4.8 to have many interesting consequences as well. Towards the end of this paper (see Section 4.5) we work out some of these consequences. We note a few of them below.

Averages of L-functions. By combining Theorem 4.8 with some results proved in [20] and [8], we obtain some quantitative asymptotic formulas for averages of L-functions. For example, if $\mathcal{B}_k^T$ is an orthogonal Hecke basis for the space of Siegel cusp forms of full level and even weight k that are not Saito-Kurokawa lifts, then we show that Conjecture 1.11 implies:

$$\sum_{f\in\mathcal{B}_k^T}\frac{L_{\mathrm{f}}(1/2,\pi_f\times\mathcal{AI}(\Lambda^{-1}))}{L_{\mathrm{f}}(1,\pi_f,\mathrm{Ad})\ L_{\mathrm{f}}(1,\chi_d)} = \frac{k^3}{2l\pi^7}+O(k^{7/3}), \tag{14}$$

where $l = 1$ if $\Lambda^2 = 1$ and $l = 2$ otherwise. In fact, we prove a version of (14) for forms of squarefree level; see Theorem 4.11 and Corollary 4.12.

[6] In this case, one can show that a weak Yoshida lift is automatically a Yoshida lift; see Corollary 4.17 of [35].

Bounds for Fourier coefficients. Theorem 4.8 combined with the GRH allows us to predict the following best possible upper bound for the size of $R(f,K,\Lambda)$ for all non-CAP newforms f in $S_k(\Gamma_0(N))$,

$$|R(f,K,\Lambda)| \ll_\epsilon \langle f,f\rangle^{1/2}(2\pi e)^k\ k^{-k+\frac{3}{4}}|d|^{\frac{k-1}{2}}(Nkd)^\epsilon. \tag{15}$$

Arithmeticity of L-values. The refined Böcherer conjecture implies that for all f, Λ as in Theorem 1.13 such that f has algebraic integer Fourier coefficients, the quantity

$$\pi^{2k+2}\cdot\langle f,f\rangle\cdot\frac{L_{\mathrm f}(1/2,\pi_f\times\mathcal{AI}(\Lambda^{-1}))}{L_{\mathrm f}(1,\pi_f,\mathrm{Ad})} \tag{16}$$

is, up to an exactly specified rational number, an *algebraic integer* (see Proposition 4.14). We find this interesting because even the algebraicity of (16) appears to not have been conjectured previously.

The above results are of course conditional on Conjecture 1.11. However, there has been promising progress towards a proof of Conjecture 1.11 by methods of the relative trace formula, pioneered by Furusawa, Martin, and Shalika [11]. Thus it is not unlikely that Theorem 1.13 will be unconditionally proven in the not-too-distant future. Moreover, there are important cases where Conjecture 1.11 is already proven. This includes in particular the *Yoshida lifts*, which are described in more detail later. Applying our formula to Yoshida lifts, we note the following unconditional *integrality* result about families of L-values of classical modular forms, which may be of independent interest:

1.15 Proposition. *Let p be an odd prime and let g_1, g_2 be distinct classical primitive forms of level p and weights ℓ, 2 respectively, where $\ell\equiv 2 \pmod 4$. Assume that the Atkin-Lehner eigenvalues of g_1 and g_2 at p are equal. Then there exists a positive real number Ω (depending only on g_1, g_2) such that for any imaginary quadratic field K with the property that p is inert in K, and any character Λ of Cl_K,*

$$\Omega\times\mathrm{disc}(K)^{\ell/2}\times L_{\mathrm f}(1/2,\pi_{g_1}\times\mathcal{AI}(\Lambda^{-1}))\ L_{\mathrm f}(1/2,\pi_{g_2}\times\mathcal{AI}(\Lambda^{-1}))$$

is a totally-positive algebraic integer in the field generated by Λ-values and the coefficients of g_1, g_2.

1.7 Notations

Throughout this paper, the letter G will denote the algebraic group GSp(4) defined as follows:

$$\mathrm{GSp}(4,R)=\{g\in\mathrm{GL}(4,R):\ {}^tgJg=\lambda(g)J,\ \lambda(g)\in R^\times\},\qquad J=\begin{bmatrix}&&1&\\&&&1\\-1&&&\\&-1&&\end{bmatrix}.$$

The subgroup for which $\lambda(g)=1$ is denoted $\mathrm{Sp}(4,R)$. We denote by N the unipotent radical of the Siegel parabolic subgroup of $\mathrm{GSp}(4,R)$. Explicitly,

$$N=\{\begin{bmatrix}1&&x&y\\&1&y&z\\&&1&\\&&&1\end{bmatrix}:\ x,y,z\in R\}. \tag{17}$$

The symbol $\mathbb{A}$ will denote the ring of adeles over $\mathbb{Q}$. For an additive group R, $\mathrm{Sym}(2,R)$ denotes the group of size 2 symmetric matrices over R. All (local and global) L-functions and ε-factors are defined via the local Langlands correspondence. The global L-functions denoted by $L(s,\ )$ include the archimedean factors. The finite part of an L-function (without the archimedean factors) is denoted by $L_{\mathrm f}(s,\ )$.

2 Local representations, local Bessel models, and ε-factors

The following notations will be used throughout Sections 2 and 3. Let F be a non-archimedean local field of characteristic zero, $\mathfrak{o}$ the ring of integers of F, and $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$. Let $\varpi\in\mathfrak{o}$ be a fixed generator of $\mathfrak{p}$, and $q=\#\mathfrak{o}/\mathfrak{p}$ be the cardinality of the residue class field. The normalized valuation on F is denoted by v, and the normalized absolute value by $|\cdot|$. The character of $F^\times$ obtained by restricting $|\cdot|$ to $F^\times$ is denoted by ν. We fix a non-trivial character ψ of F with conductor $\mathfrak{o}$ once and for all.

2.1 Non-supercuspidal representations of $\mathrm{GSp}(4,F)$

Table (18) lists all the irreducible, admissible, non-supercuspidal representations of $G(F)$.

Type		representation	ρ	g	P_1	K
I		$\chi_1\times\chi_2\rtimes\sigma$ (irreducible)	$\chi_1\chi_2\sigma\oplus\chi_1\sigma\oplus\chi_2\sigma\oplus\sigma$	•	4	1
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$	$\chi^2\sigma\oplus\sigma\oplus\chi\sigma\,\mathrm{st}(2)$	•	1	0
	b	$\chi 1_{\mathrm{GL}(2)}\rtimes\sigma$	$\chi^2\sigma\oplus\sigma\oplus\nu^{1/2}\chi\sigma\oplus\nu^{-1/2}\chi\sigma$		3	1
III	a	$\chi\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$\chi\sigma\,\mathrm{st}(2)\oplus\sigma\,\mathrm{st}(2)$	•	2	0
	b	$\chi\rtimes\sigma 1_{\mathrm{GSp}(2)}$	$\nu^{1/2}\chi\sigma\oplus\nu^{-1/2}\chi\sigma\oplus\nu^{1/2}\sigma\oplus\nu^{-1/2}\sigma$		2	1
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$\sigma\,\mathrm{st}(4)$	•	0	0
	b	$L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	$\nu\sigma\,\mathrm{st}(2)\oplus\nu^{-1}\sigma\,\mathrm{st}(2)$		2	0
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma)$	$\nu^{3/2}\sigma\oplus\sigma\,\mathrm{st}(2)\oplus\nu^{-3/2}\sigma$		1	0
	d	$\sigma 1_{\mathrm{GSp}(4)}$	$\nu^{3/2}\sigma\oplus\nu^{1/2}\sigma\oplus\nu^{-1/2}\sigma\oplus\nu^{-3/2}\sigma$		1	1
V	a	$\delta^{(*)}([\xi,\nu\xi],\nu^{-1/2}\sigma)$	$\sigma\,\mathrm{st}(2)\oplus\xi\sigma\,\mathrm{st}(2)$	•	0	0
	a*	(supercuspidal)	$\sigma\,\mathrm{st}(2)\oplus\xi\sigma\,\mathrm{st}(2)$		0	0
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$\nu^{1/2}\sigma\oplus\nu^{-1/2}\sigma\oplus\xi\sigma\,\mathrm{st}(2)$		1	0
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\xi\nu^{-1/2}\sigma)$	$\sigma\,\mathrm{st}(2)\oplus\nu^{1/2}\xi\sigma\oplus\nu^{-1/2}\xi\sigma$		1	0
	d	$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$	$\nu^{1/2}\sigma\oplus\nu^{-1/2}\sigma\oplus\nu^{1/2}\xi\sigma\oplus\nu^{-1/2}\xi\sigma$		2	1
VI	a	$\tau(S,\nu^{-1/2}\sigma)$	$\sigma\,\mathrm{st}(2)\oplus\sigma\,\mathrm{st}(2)$	•	1	0
	b	$\tau(T,\nu^{-1/2}\sigma)$	$\sigma\,\mathrm{st}(2)\oplus\sigma\,\mathrm{st}(2)$		1	0
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$\nu^{1/2}\sigma\oplus\sigma\,\mathrm{st}(2)\oplus\nu^{-1/2}\sigma$		0	0
	d	$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$	$\nu^{1/2}\sigma\oplus\nu^{1/2}\sigma\oplus\nu^{-1/2}\sigma\oplus\nu^{-1/2}\sigma$		2	1
VII		$\chi\rtimes\pi$	$\chi\varphi_\pi\oplus\varphi_\pi$	•	0	0
VIII	a	$\tau(S,\pi)$	$\varphi_\pi\oplus\varphi_\pi$	•	0	0
	b	$\tau(T,\pi)$	$\varphi_\pi\oplus\varphi_\pi$		0	0
IX	a	$\delta(\nu\xi,\nu^{-1/2}\pi)$	$\mathrm{st}(2)\otimes\varphi_\pi$	•	0	0
	b	$L(\nu\xi,\nu^{-1/2}\pi)$	$\nu^{1/2}\varphi_\pi\oplus\nu^{-1/2}\varphi_\pi$		0	0
X		$\pi\rtimes\sigma$	$\sigma\omega_\pi\oplus\sigma\oplus\sigma\varphi_\pi$	•	0	0
XI	a	$\delta^{(*)}(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$\sigma\,\mathrm{st}(2)\oplus\sigma\varphi_\pi$	•	0	0
	a*	(supercuspidal)	$\sigma\,\mathrm{st}(2)\oplus\sigma\varphi_\pi$		0	0
	b	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$\nu^{1/2}\sigma\oplus\nu^{-1/2}\sigma\oplus\sigma\varphi_\pi$		0	0

(18)

For further explanation of the notation, we refer to Sect. 2.2 of [32] (also see the comments in Section 3.3 of this paper). Also listed in the table are two types of supercuspidal representations, denoted Va* and XIa*. The reason that these supercuspidal representations are included in the list is that they share an L-packet with a non-supercuspidal representation. Namely, $\{\mathrm{Va},\mathrm{Va}^*\}$ constitute an L-packet, and $\{\mathrm{XIa},\mathrm{XIa}^*\}$ also constitute an L-packet. These are the only L-packets for $\mathrm{GSp}(4)$ that contain both a supercuspidal and a non-supercuspidal representation. Other two-element L-packets consisting of non-supercuspidal representations are $\{\mathrm{VIa},\mathrm{VIb}\}$ and $\{\mathrm{VIIIa},\mathrm{VIIIb}\}$. The supercuspidal representations Va* and XIa* can be defined in terms of the theta correspondence between $G(F)$ and $\mathrm{GO}(X)$, where X is a 4-dimensional orthogonal space; see [33].

The column ρ lists the relevant L-parameters. Our calculations of ε-factors later in this section are based on these. These L-parameters are listed in Table A.7 of [32] as explicit homomorphisms of the Weil-Deligne group W_F' into the dual group $G(\mathbb{C})$. For the current purposes it is sufficient and more appropriate to have the L-parameters listed as abstract 4-dimensional representations of W_F', neglecting the embedding of the image into the dual group. Let $\mathrm{sp}(n)$ be the n-dimensional indecomposable representation of W_F' defined in §5 of [34]. In matrix form, $\mathrm{sp}(n)=(\sigma,N)$, where

$$\sigma(w)=\begin{bmatrix}1&&&\\&\nu(w)&&\\&&\ddots&\\&&&\nu(w)^{n-1}\end{bmatrix},\qquad N=\begin{bmatrix}0&&&\\1&0&&\\&\ddots&\ddots&\\&&1&0\end{bmatrix}.$$

Here, ν is the normalized absolute value of $F^\times$, identified with a character of the Weil group W_F. Let us introduce the symbol $\mathrm{st}(n):=\nu^{-(n-1)/2}\mathrm{sp}(n)$. Then the semisimple part of $\mathrm{st}(n)$ has determinant 1. Note that $\mathrm{st}(2)=\nu^{-1/2}\mathrm{sp}(2)$ is the L-parameter of $\mathrm{St}_{\mathrm{GL}(2)}$, and $\mathrm{st}(4)=\nu^{-3/2}\mathrm{sp}(4)$ is the L-parameter of $\mathrm{St}_{\mathrm{GSp}(4)}$. In groups VII - XI, π is a supercuspidal representation of $\mathrm{GL}(2,F)$, and φ_π is its L-parameter.

The "g" column indicates which representations are generic. The last two columns P_1 and K give the dimensions of the subspace of vectors fixed by the Siegel congruence subgroups P_1 (see (48)) and the maximal compact subgroup $K=G(\mathfrak{o})$ respectively, assuming that the inducing characters are unramified. This data will only be useful in Section 3. The representations IVb, IVc are never unitary and hence are not relevant for our global applications. The representations that contain a K-fixed vector are known as spherical representations; we see from the Table that these are of types I, IIb, IIIb, IVd, Vd, or VId.

2.2 Bessel models for GSp(4)

For $a,b,c\in F$ let

$$S=\begin{bmatrix}a&b/2\\b/2&c\end{bmatrix}. \tag{19}$$

Assume that $d=b^2-4ac$ is not zero and put $D=d/4=-\det(S)$. If $D\notin F^{\times2}$, then let $\Delta=\sqrt{D}$ be a square root of D in an algebraic closure $\bar F$, and $K=F(\Delta)$. If $D\in F^{\times2}$, then let $\sqrt{D}$ be a square root of D in $F^\times$, $K=F\times F$, and $\Delta=(-\sqrt{D},\sqrt{D})\in K$. In both cases K is a two-dimensional F-algebra containing F, $K=F+F\Delta$, and $\Delta^2=D$. Let

$$A=A_S=\{\begin{bmatrix}x+yb/2&yc\\-ya&x-yb/2\end{bmatrix}:\ x,y\in F\}. \tag{20}$$

Then A is a two-dimensional F-algebra naturally containing F. One can verify that

$$A=\{g\in \mathrm{M}_2(F):\ {}^tgSg=\det(g)S\}. \tag{21}$$

We write $T=T_S=A^\times$. We define an isomorphism of F-algebras,

$$A\xrightarrow{\sim}K,\qquad \begin{bmatrix}x+yb/2&yc\\-ya&x-yb/2\end{bmatrix}\longmapsto x+y\Delta. \tag{22}$$

The restriction of this isomorphism to T is an isomorphism $T\xrightarrow{\sim}K^\times$, and we identify characters of T and characters of $K^\times$ via this isomorphism. We embed T into $G(F)$ via the map

$$t\longmapsto\begin{bmatrix}t&\\&\det(t)t'\end{bmatrix},\qquad t\in T,\quad t'={}^tt^{-1}. \tag{23}$$

For S as in (19), we define a character $\theta=\theta_{a,b,c}=\theta_S$ of N by

$$\theta(\begin{bmatrix}1&&x&y\\&1&y&z\\&&1&\\&&&1\end{bmatrix})=\psi(ax+by+cz)=\psi(\mathrm{tr}(S\begin{bmatrix}x&y\\y&z\end{bmatrix})) \tag{24}$$

for $x,y,z\in F$. Every character of N is of this form for uniquely determined a,b,c in F, or, alternatively, for a uniquely determined symmetric 2×2 matrix S. It is easily verified that $\theta(tnt^{-1})=\theta(n)$ for $n\in N$ and $t\in T$. We refer to $T\ltimes N$ as the *Bessel subgroup* defined by the character θ (or the matrix S). Given a character Λ of T (identified with a character of $K^\times$ as explained above), we can define a character $\Lambda\otimes\theta$ of $T\ltimes N$ by

$$(\Lambda\otimes\theta)(tn)=\Lambda(t)\theta(n)\qquad\text{for } n\in N \text{ and } t\in T.$$

Every character of $T\ltimes N$ whose restriction to N coincides with θ is of this form for an appropriate Λ.

Let (π,V) be an admissible representation of $G(F)$. Let S and θ be as above, and let Λ be a character of the associated group T. A non-zero element β of $\mathrm{Hom}_{T\ltimes N}(V,\mathbb{C}_{\Lambda\otimes\theta})$ is called a (Λ,θ)-*Bessel functional* for π. We note that if π admits a central character ω_π and a (Λ,θ)-Bessel functional, then $\Lambda|_{F^\times}=\omega_\pi$.

It is possible to determine the Bessel functionals admitted by the non-supercuspidal representations of $G(F)$. The following was proved in [33].

2.1 Theorem. *The table below shows the possible Bessel functionals for all irreducible, admissible, non-supercuspidal representations of $G(F)$. The column "$K\leftrightarrow\xi$" indicates that the field K is the quadratic extension of F corresponding to the non-trivial, quadratic character ξ of $F^\times$; this is only relevant for representations in groups V and IX. The pairs of characters (χ_1,χ_2) in the "$K=F\times F$" column for types IIIb and IVc refer to the characters of $T=\{\mathrm{diag}(a,b,b,a)\,:\; a,b\in F^\times\}$ given by $\mathrm{diag}(a,b,b,a)\mapsto\chi_1(a)\chi_2(b)$. In representations of group IX, the symbol κ denotes a non-Galois-invariant character of $K^\times$, where K is the quadratic extension corresponding to ξ. The Galois conjugate of κ is denoted by κ'. Finally, the symbol N in the table stands for the norm map $\mathrm{N}_{K/F}$. In the split case, the character $\sigma\circ\mathrm{N}$ is the same as (σ,σ).*

		representation	(Λ,θ)-Bessel functional exists exactly for ...		
			$K=F\times F$	K/F a field extension	
				$K\leftrightarrow\xi$	$K\not\leftrightarrow\xi$
I		$\chi_1\times\chi_2\rtimes\sigma$ (irreducible)	all Λ	all Λ	
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$	all Λ	$\Lambda\neq(\chi\sigma)\circ\mathrm{N}$	
	b	$\chi 1_{\mathrm{GL}(2)}\rtimes\sigma$	$\Lambda=(\chi\sigma)\circ\mathrm{N}$	$\Lambda=(\chi\sigma)\circ\mathrm{N}$	
III	a	$\chi\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	all Λ	all Λ	
	b	$\chi\rtimes\sigma 1_{\mathrm{GSp}(2)}$	$\Lambda\in\{(\chi\sigma,\sigma),(\sigma,\chi\sigma)\}$	—	
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	all Λ	$\Lambda\neq\sigma\circ\mathrm{N}$	
	b	$L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	$\Lambda=\sigma\circ\mathrm{N}$	$\Lambda=\sigma\circ\mathrm{N}$	
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma)$	$\Lambda=(\nu^{\pm1}\sigma,\nu^{\mp1}\sigma)$	—	
	d	$\sigma 1_{\mathrm{GSp}(4)}$	—	—	
V	a	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	all Λ	$\Lambda\neq\sigma\circ\mathrm{N}$	$\sigma\circ\mathrm{N}\neq\Lambda\neq(\xi\sigma)\circ\mathrm{N}$
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$\Lambda=\sigma\circ\mathrm{N}$	—	$\Lambda=\sigma\circ\mathrm{N}$
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\xi\nu^{-1/2}\sigma)$	$\Lambda=(\xi\sigma)\circ\mathrm{N}$	—	$\Lambda=(\xi\sigma)\circ\mathrm{N}$
	d	$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$	—	$\Lambda=\sigma\circ\mathrm{N}$	—
VI	a	$\tau(S,\nu^{-1/2}\sigma)$	all Λ	$\Lambda\neq\sigma\circ\mathrm{N}$	
	b	$\tau(T,\nu^{-1/2}\sigma)$	—	$\Lambda=\sigma\circ\mathrm{N}$	
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$\Lambda=\sigma\circ\mathrm{N}$	—	
	d	$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$	$\Lambda=\sigma\circ\mathrm{N}$	—	
VII		$\chi\rtimes\pi$	all Λ	all Λ	
VIII	a	$\tau(S,\pi)$	all Λ	$\mathrm{Hom}_T(\pi,\mathbb{C}_\Lambda)\neq0$	
	b	$\tau(T,\pi)$	—	$\mathrm{Hom}_T(\pi,\mathbb{C}_\Lambda)=0$	
IX	a	$\delta(\nu\xi,\nu^{-1/2}\mathcal{AI}_{K/F}(\kappa))$	all Λ	$\kappa\neq\Lambda\neq\kappa'$	all Λ
	b	$L(\nu\xi,\nu^{-1/2}\mathcal{AI}_{K/F}(\kappa))$	—	$\Lambda=\kappa$ or $\Lambda=\kappa'$	—
X		$\pi\rtimes\sigma$	all Λ	$\mathrm{Hom}_T(\sigma\pi,\mathbb{C}_\Lambda)\neq0$	
XI	a	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	all Λ	$\Lambda\neq\sigma\circ\mathrm{N}$ and $\mathrm{Hom}_T(\sigma\pi,\mathbb{C}_\Lambda)\neq0$	
	b	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$\Lambda=\sigma\circ\mathrm{N}$	$\Lambda=\sigma\circ\mathrm{N}$ and $\mathrm{Hom}_T(\pi,\mathbb{C}_1)\neq0$	
Va*		$\delta^*([\xi,\nu\xi],\nu^{-1/2}\sigma)$	—	$\Lambda=\sigma\circ\mathrm{N}$	—
XIa*		$\delta^*(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	—	$\Lambda=\sigma\circ\mathrm{N}$ and $\mathrm{Hom}_T(\pi^{\mathrm{JL}},\mathbb{C}_1)\neq0$	

2.3 Automorphic induction

Let K/F be a quadratic field extension, or $K=F\times F$. Let Λ be a character of $K^\times$. In §1 and §4 of [17], an irreducible, admissible representation $\pi(\Lambda)$ has been attached to this data. Here, we write $\mathcal{AI}_{K/F}(\Lambda)$ (or $\mathcal{AI}(\Lambda)$ if K and F are clear) for this representation and say that it is obtained via *automorphic induction*. The name is justified by the fact that, in case K is a field, the L-parameter of $\mathcal{AI}_{K/F}(\Lambda)$ is $\mathrm{ind}_{W_K}^{W_F}(\Lambda)$, where W_F is the Weil group of F, the index-2 subgroup W_K is the Weil group of K, and Λ is identified with a character of W_K.

We will list a number of properties of the representations $\mathcal{AI}_{K/F}(\Lambda)$. We denote by $\chi_{K/F}$ the quadratic character of $F^\times$ that is trivial precisely on elements that are norms from $K^\times$. Hence, if $K = F \times F$, then $\chi_{K/F}$ is the trivial character of $F^\times$.

- If $\Lambda = \chi \circ \mathrm{N}_{K/F}$ for a character χ of $F^\times$, then $\mathcal{AI}_{K/F}(\Lambda) = \chi \times (\chi\chi_{K/F})$ is an irreducible principal series representation of $\mathrm{GL}(2, F)$.
- Assume that K is a field and Λ is not of the above form. Then $\mathcal{AI}_{K/F}(\Lambda)$ is an irreducible, admissible, supercuspidal representation of $\mathrm{GL}(2, F)$.
- The central character of $\mathcal{AI}_{K/F}(\Lambda)$ is $(\Lambda\big|_{F^\times})\chi_{K/F}$.
- If K is a field, then the ε-factor of $\mathcal{AI}_{K/F}(\Lambda)$ is
$$\varepsilon(s, \mathcal{AI}_{K/F}(\Lambda), \psi) = \varepsilon(s, \chi_{K/F}, \psi)\varepsilon(s, \Lambda, \psi \circ \mathrm{tr}_{K/F}). \tag{25}$$
If $K = F \times F$, then $\Lambda = (\Lambda_1, \Lambda_2)$ with characters Λ_i of $F^\times$, and
$$\varepsilon(s, \mathcal{AI}_{K/F}(\Lambda), \psi) = \varepsilon(s, \Lambda_1, \psi)\varepsilon(s, \Lambda_2, \psi).$$
- Characterization in case K is a field (see [13]): For π an irreducible, admissible representation of $\mathrm{GL}(2, F)$,
$$\pi \cong \mathcal{AI}_{K/F}(\Lambda) \text{ for some } \Lambda \qquad \Longleftrightarrow \qquad \pi \cong \pi \otimes \chi_{K/F}. \tag{26}$$

2.4 Some properties of ε-factors

Let $\mathrm{Irr}(n, F)$ be the set of equivalence classes of irreducible, admissible representations of $\mathrm{GL}(n, F)$. Via the local Langlands correspondence, an ε-factor $\varepsilon(s, \pi, \psi)$ is attached to each $\pi \in \mathrm{Irr}(n, F)$ (and our fixed choice of additive character ψ). Recall that $\varepsilon(s, \pi, \psi)\varepsilon(1 - s, \pi^\vee, \psi) = \omega_\pi(-1)$, and in particular
$$\varepsilon(1/2, \pi, \psi)\varepsilon(1/2, \pi^\vee, \psi) = \omega_\pi(-1), \tag{27}$$
where $\pi^\vee$ is the congragredient of π, and ω_π is the central character of π.

Let K/F be a cyclic extension of prime degree ℓ. *Base change* and *automorphic induction* are maps
$$\mathcal{BC}_{K/F} : \mathrm{Irr}(n, F) \longrightarrow \mathrm{Irr}(n, K), \tag{28}$$
$$\mathcal{AI}_{K/F} : \mathrm{Irr}(n, K) \longrightarrow \mathrm{Irr}(n\ell, F). \tag{29}$$
Let $\chi_{K/F}$ be the character of $F^\times$ corresponding to the extension K/F via local class field theory. Let $\psi_K = \psi \circ \mathrm{tr}_{K/F}$. By Proposition 3.1 of [28], we have the adjointness formula
$$\varepsilon(s, \pi \otimes \mathcal{AI}_{K/F}(\Lambda), \psi)\varepsilon(s, 1_{K^\times}, \psi_K)^{mn} = \varepsilon(s, \mathcal{BC}_{K/F}(\pi) \otimes \Lambda, \psi_K) \prod_{j=0}^{\ell-1} \varepsilon(s, \chi_{K/F}^j, \psi)^{nm} \tag{30}$$
for $\pi \in \mathrm{Irr}(m, F)$ and $\Lambda \in \mathrm{Irr}(n, K)$. If we set $m = \ell = 2$ and $n = 1$, evaluate at $1/2$ and observe (27), we obtain
$$\varepsilon(1/2, \pi \otimes \mathcal{AI}_{K/F}(\Lambda), \psi) = \varepsilon(1/2, \mathcal{BC}_{K/F}(\pi) \otimes \Lambda, \psi_K)\, \chi_{K/F}(-1). \tag{31}$$
We remark that, for $m = n = 1$, $\ell = 2$ and π the trivial representation of $F^\times$, formula (30) reduces to (25).

Let S be as in (19) with $\det(S) \neq 0$. Let K be the quadratic extension associated to S, let A be as in (20), and let $T = A^\times$. Then $T \cong K^\times$ via the map in (22). Let Λ be a character of T, identified with a character of $K^\times$. Let (π, V) be an irreducible, admissible representation of $\mathrm{GL}(2, F)$. By the arguments in Proposition 9 of [43], the space $\mathrm{Hom}_T(\pi, \mathbb{C}_\Lambda)$ is at most one-dimensional. Evidently, a necessary condition for $\mathrm{Hom}_T(\pi, \mathbb{C}_\Lambda)$ to be non-zero is $\Lambda\big|_{F^\times} = \omega_\pi$. Assuming that $\Lambda\big|_{F^\times} = \omega_\pi$, it was proved in Lemma 1.5 of [42] that $\varepsilon(1/2, \mathcal{BC}_{K/F}(\pi) \otimes \Lambda^{-1}, \psi_K)$ is independent of ψ, and is an element of $\{\pm 1\}$. The main result of [42] and [37] is the following ε-factor criterion for $\mathrm{Hom}_T(\pi, \mathbb{C}_\Lambda)$ to be non-zero.

2.2 Theorem. (Saito, Tunnell) *Let (π, V) be an infinite-dimensional, irreducible, admissible representation of $\mathrm{GL}(2,F)$. Let S be as in (19) with $\det(S) \neq 0$. Let K be the quadratic extension associated to S, let A be as in (20), and let $T = A^\times$. Let Λ be a character of T, identified with a character of $K^\times$, such that $\Lambda\big|_{F^\times} = \omega_\pi$. Then*

$$\mathrm{Hom}_T(\pi, \mathbb{C}_\Lambda) \neq 0 \qquad \Longleftrightarrow \qquad \varepsilon(1/2, \mathcal{BC}_{K/F}(\pi) \otimes \Lambda^{-1}, \psi_K) = \omega_\pi(-1). \tag{32}$$

In view of (31), we may rewrite (32) as

$$\mathrm{Hom}_T(\pi, \mathbb{C}_\Lambda) \neq 0 \qquad \Longleftrightarrow \qquad \varepsilon(1/2, \pi \otimes \mathcal{AI}_{K/F}(\Lambda^{-1}), \psi) = \omega_\pi(-1)\chi_{K/F}(-1). \tag{33}$$

For Λ such that $\Lambda\big|_{F^\times} = \sigma^2$, it is known that

$$\dim(\mathrm{Hom}_T(\sigma \mathrm{St}_{\mathrm{GL}(2)}, \mathbb{C}_\Lambda)) = \begin{cases} 0 & \text{if } K \text{ is a field and } \Lambda = \sigma \circ \mathrm{N}_{K/F}, \\ 1 & \text{otherwise;} \end{cases} \tag{34}$$

see Proposition 1.7 and Theorem 2.4 of [42]. In combination with (33), we obtain

$$\varepsilon(1/2, \sigma \mathrm{St}_{\mathrm{GL}(2)} \otimes \mathcal{AI}(\Lambda^{-1}), \psi) = \chi_{K/F}(-1) \qquad \Longleftrightarrow \qquad K = F \times F \text{ or } \Lambda \neq \sigma \circ \mathrm{N}_{K/F}. \tag{35}$$

2.5 Calculation of ε-factors

Many of the calculations in the proof of the following theorem have also been done in [29].

2.3 Theorem. *Let (π, V) be an irreducible, admissible representation of $G(F)$. Let S be as in (19). Let Λ be a character of the associated quadratic extension $K^\times$ such that $\Lambda\big|_{F^\times} = \omega_\pi$.*

i) The value $\varepsilon(1/2, \pi \otimes \mathcal{AI}(\Lambda^{-1}), \psi)$ is independent of ψ, and can only be ± 1.

ii) If π is non-supercuspidal, or is in an L-packet with a non-supercuspidal representation, then the value $\varepsilon(1/2, \pi \otimes \mathcal{AI}(\Lambda^{-1}), \psi)$ is given in the table below. The column "$K \leftrightarrow \xi$" indicates that the field K is the quadratic extension of F corresponding to the non-trivial, quadratic character ξ of $F^\times$; this is only relevant for representations in groups V and IX. In representations of group IX, the symbol κ denotes a non-Galois-invariant character of $K^\times$, where K is the quadratic extension corresponding to ξ. The Galois conjugate of κ is denoted by κ'. Finally, the symbol N in the table stands for the norm map $\mathrm{N}_{K/F}$.

Proof. Let $\varphi_\pi : W_F' \to \mathrm{GL}(V)$ be the L-parameter of π, where V is a 4-dimensional space. There exists on V a non-degenerate symplectic form $\langle \cdot, \cdot \rangle$ such that

$$\langle \varphi_\pi(g) v_1, \varphi_\pi(g) v_2 \rangle = \lambda(\varphi_\pi(g)) \langle v_1, v_2 \rangle$$

for all $v_1, v_2 \in V$ and $g \in W_F'$. Here, λ is the multiplier homomorphism, and the character $\lambda \circ \varphi_\pi$ of W_F corresponds to the character ω_π of $F^\times$. Let $\tau = \mathcal{AI}(\Lambda^{-1})$, and let $\varphi_\tau : W_F' \to \mathrm{GL}(W)$ be the L-parameter of τ, where W is a 2-dimensional space. By Lemma 2.4.1 of [32], there exists on W a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ such that

$$\langle \varphi_\pi(g) w_1, \varphi_\pi(g) w_2 \rangle = \chi_{K/F}(g) \det(\varphi_\tau(g)) \langle w_1, w_2 \rangle$$

for all $w_1, w_2 \in W$ and $g \in W_F'$. Our hypothesis $\Lambda\big|_{F^\times} = \omega_\pi$ is equivalent to

$$\lambda(\varphi_\pi(g)) = \chi_{K/F}(g) \det(\varphi_\tau(g))^{-1}$$

for all $g \in W_F'$. It follows that the space $V \otimes W$ is endowed with a symplectic form such that $\varphi_\pi(g) \otimes \varphi_\tau(g)$ is contained in $\mathrm{Sp}(V \otimes W)$ for all $g \in W_F'$. By i) of the Proposition in §11 of [34], the value

$\varepsilon(1/2, \pi \otimes \mathcal{AI}(\Lambda^{-1}), \psi)$ is independent of ψ, and by iv) of the Proposition in §12 of [34], it can only take the values ± 1.

<table>
<tr><th colspan="3" rowspan="3">representation π</th><th colspan="3">$\varepsilon(1/2, \pi \otimes \mathcal{AI}_{K/F}(\Lambda^{-1}), \psi) = 1$ exactly for ...</th></tr>
<tr><th rowspan="2">$K = F \times F$</th><th colspan="2">K/F a field extension</th></tr>
<tr><th>$K \leftrightarrow \xi$</th><th>$K \not\leftrightarrow \xi$</th></tr>
<tr><td>I</td><td></td><td>$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)</td><td>all Λ</td><td colspan="2">all Λ</td></tr>
<tr><td rowspan="2">II</td><td>a</td><td>$\chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$</td><td>all Λ</td><td colspan="2">$\Lambda \neq (\chi\sigma) \circ \mathrm{N}$</td></tr>
<tr><td>b</td><td>$\chi 1_{\mathrm{GL}(2)} \rtimes \sigma$</td><td>all Λ</td><td colspan="2">all Λ</td></tr>
<tr><td rowspan="2">III</td><td>a</td><td>$\chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(2)}$</td><td>all Λ</td><td colspan="2">all Λ</td></tr>
<tr><td>b</td><td>$\chi \rtimes \sigma 1_{\mathrm{GSp}(2)}$</td><td>all Λ</td><td colspan="2">all Λ</td></tr>
<tr><td rowspan="4">IV</td><td>a</td><td>$\sigma \mathrm{St}_{\mathrm{GSp}(4)}$</td><td>all Λ</td><td colspan="2">$\Lambda \neq \sigma \circ \mathrm{N}$</td></tr>
<tr><td>b</td><td>$L(\nu^2, \nu^{-1}\sigma \mathrm{St}_{\mathrm{GSp}(2)})$</td><td>all Λ</td><td colspan="2">all Λ</td></tr>
<tr><td>c</td><td>$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$</td><td>all Λ</td><td colspan="2">$\Lambda \neq \sigma \circ \mathrm{N}$</td></tr>
<tr><td>d</td><td>$\sigma 1_{\mathrm{GSp}(4)}$</td><td>all Λ</td><td colspan="2">all Λ</td></tr>
<tr><td rowspan="4">V</td><td>a/a*</td><td>$\delta^{(*)}([\xi, \nu\xi], \nu^{-1/2}\sigma)$</td><td>all Λ</td><td>all Λ</td><td>$\sigma \circ \mathrm{N} \neq \Lambda \neq (\xi\sigma) \circ \mathrm{N}$</td></tr>
<tr><td>b</td><td>$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$</td><td>all Λ</td><td>$\Lambda \neq \sigma \circ \mathrm{N}$</td><td>$\Lambda \neq (\xi\sigma) \circ \mathrm{N}$</td></tr>
<tr><td>c</td><td>$L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$</td><td>all Λ</td><td>$\Lambda \neq \sigma \circ \mathrm{N}$</td><td>$\Lambda \neq \sigma \circ \mathrm{N}$</td></tr>
<tr><td>d</td><td>$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$</td><td>all Λ</td><td>all Λ</td><td>all Λ</td></tr>
<tr><td rowspan="4">VI</td><td>a</td><td>$\tau(S, \nu^{-1/2}\sigma)$</td><td rowspan="2">all Λ</td><td colspan="2" rowspan="2">all Λ</td></tr>
<tr><td>b</td><td>$\tau(T, \nu^{*-1/2}\sigma)$</td></tr>
<tr><td>c</td><td>$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$</td><td>all Λ</td><td colspan="2">$\Lambda \neq \sigma \circ \mathrm{N}$</td></tr>
<tr><td>d</td><td>$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$</td><td>all Λ</td><td colspan="2">all Λ</td></tr>
<tr><td>VII</td><td></td><td>$\chi \rtimes \pi$</td><td>all Λ</td><td colspan="2">all Λ</td></tr>
<tr><td rowspan="2">VIII</td><td>a</td><td>$\tau(S, \pi)$</td><td rowspan="2">all Λ</td><td colspan="2" rowspan="2">all Λ</td></tr>
<tr><td>b</td><td>$\tau(T, \pi)$</td></tr>
<tr><td rowspan="2">IX</td><td>a</td><td>$\delta(\nu\xi, \nu^{-1/2}\mathcal{AI}_{K/F}(\kappa))$</td><td>all Λ</td><td>$\kappa \neq \Lambda \neq \kappa'$</td><td>all Λ</td></tr>
<tr><td>b</td><td>$L(\nu\xi, \nu^{-1/2}\mathcal{AI}_{K/F}(\kappa))$</td><td>all Λ</td><td>all Λ</td><td>all Λ</td></tr>
<tr><td>X</td><td></td><td>$\pi \rtimes \sigma$</td><td>all Λ</td><td colspan="2">$\mathrm{Hom}_T(\sigma\pi, \mathbb{C}_\Lambda) \neq 0$</td></tr>
<tr><td rowspan="2">XI</td><td>a/a*</td><td>$\delta^{(*)}(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$</td><td>all Λ</td><td colspan="2">$(\Lambda \neq \sigma \circ \mathrm{N}$ and $\mathrm{Hom}_T(\sigma\pi, \mathbb{C}_\Lambda) \neq 0)$
or $(\Lambda = \sigma \circ \mathrm{N}$ and $\mathrm{Hom}_T(\pi^{\mathrm{JL}}, \mathbb{C}_1) \neq 0)$</td></tr>
<tr><td>b</td><td>$L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$</td><td>all Λ</td><td colspan="2">$\mathrm{Hom}_T(\sigma\pi, \mathbb{C}_\Lambda) \neq 0$</td></tr>
</table>

From now on we abbreviate $\varepsilon(1/2, \pi, \psi)$ by $\varepsilon(\pi)$ for representations π of $\mathrm{GL}(n, F)$ or the Weil-Deligne group W_F'. As above, let $\tau = \mathcal{AI}(\Lambda^{-1})$, and let $\varphi_\tau : W_F' \to \mathrm{GL}(W)$ be the L-parameter of τ. The central character of τ is $\omega_\tau = (\Lambda\big|_{F^\times})^{-1}\chi_{K/F}$. By (26), we have $\chi_{K/F} \otimes \tau = \tau$. It follows that the contragredient of τ is given by

$$\tau^\vee = (\Lambda\big|_{F^\times}) \times \tau = \omega_\pi \times \tau. \tag{36}$$

We give full details of the proof only for Types I and II, and omit the details in the remaining cases in the interest of brevity.

Type I: Let $\pi = \chi_1 \times \chi_2 \rtimes \sigma$, an irreducible principle series representation. Its central character is $\chi_1\chi_2\sigma^2$. The L-parameter of π is the direct sum of the four one-dimensional representations σ, $\chi_1\sigma$, $\chi_2\sigma$

and $\chi_1\chi_2\sigma$. Hence,

$$\begin{aligned}\varepsilon(\pi\otimes\tau) &= \varepsilon(\sigma\times\tau)\varepsilon(\chi_1\chi_2\sigma\times\tau)\varepsilon(\chi_1\sigma\times\tau)\varepsilon(\chi_2\sigma\times\tau)\\ &= \varepsilon(\sigma\times\tau)\varepsilon(\sigma^{-1}\times\tau^\vee)\varepsilon(\chi_1\sigma\times\tau)\varepsilon((\chi_1\sigma)^{-1}\times\tau^\vee)\\ &= \omega_{\sigma\times\tau}(-1)\omega_{\chi_1\sigma\times\tau}(-1)\\ &= (\sigma^2\omega_\tau)(-1)(\chi_1^2\sigma^2\omega_\tau)(-1)\\ &= 1.\end{aligned}$$

Type IIa: Let $\pi=\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$. Its central character is $\chi^2\sigma^2$. The L-parameter of π is the direct sum of the one-dimensional σ and $\chi^2\sigma$, and the two-dimensional $\chi\sigma\mathrm{st}(2)$. Hence

$$\begin{aligned}\varepsilon(\pi\otimes\tau) &= \varepsilon(\sigma\times\tau)\varepsilon(\chi^2\sigma\times\tau)\varepsilon(\chi\sigma\mathrm{St}_{\mathrm{GL}(2)}\times\tau)\\ &= \varepsilon(\sigma\times\tau)\varepsilon(\sigma^{-1}\times\tau^\vee)\varepsilon(\chi\sigma\mathrm{St}_{\mathrm{GL}(2)}\times\tau)\\ &= \omega_\tau(-1)\varepsilon(\chi\sigma\mathrm{St}_{\mathrm{GL}(2)}\times\tau)\\ &= \chi_{K/F}(-1)\varepsilon(\chi\sigma\mathrm{St}_{\mathrm{GL}(2)}\times\tau).\end{aligned}$$

By (35), $\varepsilon(\pi\otimes\tau)=1$ if and only if $K=F\times F$ or $\Lambda\neq(\chi\sigma)\circ\mathrm{N}_{K/F}$. (This calculation has been done in Sect. 3.1 of [29].)

Type IIb: Let $\pi=\chi 1_{\mathrm{GL}(2)}\rtimes\sigma$. Its central character is $\chi^2\sigma^2$. The L-parameter of π is the direct sum of σ, $\chi^2\sigma$, $\nu^{1/2}\chi\sigma$ and $\nu^{-1/2}\chi\sigma$. Hence

$$\begin{aligned}\varepsilon(\pi\otimes\tau) &= \varepsilon(\sigma\times\tau)\varepsilon(\chi^2\sigma\times\tau)\varepsilon(\nu^{1/2}\chi\sigma\times\tau)\varepsilon(\nu^{-1/2}\chi\sigma\times\tau)\\ &= \varepsilon(\sigma\times\tau)\varepsilon(\sigma^{-1}\times\tau^\vee)\varepsilon(\nu^{1/2}\chi\sigma\times\tau)\varepsilon(\nu^{-1/2}\chi^{-1}\sigma^{-1}\times\tau^\vee)\\ &= \omega_\tau(-1)\omega_\tau(-1)\\ &= 1.\end{aligned}$$

As mentioned above, we leave the calculations for the remaining cases, which are essentially similar, to the reader. ■

Comparing the previous table with the table of Bessel models, we obtain the following corollary. The same result was stated as Theorem 2 of [29], for odd residual characteristic, but including supercuspidal L-packets as well.

2.4 Corollary. *Let π be an irreducible, admissible, non-supercuspidal, generic representation of $G(F)$. Let S be as in (19). Let Λ be a character of the associated quadratic extension $K^\times$ such that $\Lambda\big|_{F^\times}=\omega_\pi$. Then the following are equivalent.*

i) There exists a representation in the L-packet of π admitting a (Λ,θ)-Bessel functional.

ii) $\varepsilon(1/2,\pi\otimes\mathcal{AI}(\Lambda^{-1}),\psi)=1$.

Further comparisons of the two tables show that *the statement of the corollary is wrong for non-supercuspidal representations that do not share an L-packet with a generic representation.* For example, representation VId satisfies the second condition but not the first.

3 The local integral in the non-archimedean case

3.1 Notations and basic facts

We continue to use the notations of Section 2. *We also enforce the assumptions stated at the beginning of Section 1.3.* In particular,

- For the matrix S, assume $c \in \mathfrak{o}^\times$ and that $d = b^2 - 4ac$ generates the discriminant ideal of K/F.[7]
- (π, V) is a unitary, irreducible, admissible representation of $G(F)$ with trivial central character and invariant hermitian inner product $\langle \cdot, \cdot \rangle$.
- All the Haar measures are normalized appropriately, so the maximal compact subgroups have volume 1.

For any vector $\phi \in V$, define Φ_ϕ to be the function on $G(F)$ defined by

$$\Phi_\phi(g) = \langle \pi(g)\phi, \phi \rangle / \langle \phi, \phi \rangle. \tag{37}$$

For each $t \in T = T_S \simeq K^\times$, consider the integrals

$$J_0^{(k)}(\phi, t) := \int\limits_{N(\mathfrak{p}^{-k})} \Phi_\phi(tn)\theta_S^{-1}(n)\, dn, \tag{38}$$

where k is a positive integer. It was proved by Liu [21] that for each t the values $J_0^{(k)}(\phi, t)$ stabilize as $k \to \infty$, and hence the definition

$$J_0(\phi, t) = \lim_{k \to \infty} J_0^{(k)}(\phi, t) \tag{39}$$

is meaningful. Define the quantity

$$J_0(\phi) = \int\limits_{F^\times \backslash T} J_0(\phi, t)\Lambda^{-1}(t)\, dt \tag{40}$$

and the normalized quantity

$$J(\phi) = \frac{L(1, \pi, \mathrm{Ad})L(1, \chi_{K/F})}{\zeta_F(2)\zeta_F(4)L(1/2, \pi \times \mathcal{AI}(\Lambda^{-1}))}\, J_0(\phi). \tag{41}$$

3.1 Proposition. (Liu [21]) *Suppose that Λ is an unramified character, π is an unramified type I representation, and ϕ is the (unique up to multiples) $G(\mathfrak{o})$-fixed vector in π. Then $J(\phi) = 1$.*

The goal for the rest of this section is to compute $J(\phi)$ in some additional cases.

3.2 Basic structure theory

In this subsection we recall some of the structure theory of $G(F)$ concerning Weyl groups, parahoric subgroups, and the Iwahori-Hecke algebra. Many of the stated facts can be verified directly. For others, we refer to [4] and the references therein.

Weyl groups

Let D be the diagonal torus of $\mathrm{Sp}(4)$. Let $\tilde{N}$ be the normalizer of D in $\mathrm{Sp}(4)$. Let

$$\tilde{W} = \tilde{N}(F)/D(\mathfrak{o}) \qquad \text{and} \qquad W = \tilde{N}(F)/D(F). \tag{42}$$

The Weyl group W has 8 elements and is generated by the images of

$$s_1 = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}, \quad s_2 = \begin{bmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{bmatrix}.$$

[7] As explained in [27, Section 2.3], there is no loss of generality in making this assumption.

There is an exact sequence

$$1 \longrightarrow D(F)/D(\mathfrak{o}) \longrightarrow \tilde{W} \longrightarrow W \longrightarrow 1, \tag{43}$$

where $D(F)/D(\mathfrak{o}) \cong \mathbb{Z}^2$ via the map $\mathrm{diag}(a,b,a^{-1},b^{-1}) \mapsto (v(a),v(b))$. The *Atkin-Lehner element* of $G(F)$ is defined by

$$\eta = \begin{bmatrix} &&&-1\\ &&1&\\ &\varpi&&\\ -\varpi&&& \end{bmatrix} = s_2s_1s_2 \begin{bmatrix} \varpi&&&\\ &-\varpi&&\\ &&1&\\ &&&-1 \end{bmatrix} = \begin{bmatrix} -1&&&\\ &1&&\\ &&-\varpi&\\ &&&\varpi \end{bmatrix} s_2s_1s_2. \tag{44}$$

Set

$$s_0 = \eta s_2 \eta^{-1} = \begin{bmatrix} 1&&&\\ &&&\varpi^{-1}\\ &&1&\\ &-\varpi&& \end{bmatrix}. \tag{45}$$

Then $\tilde{W}$ is generated by the images of s_0, s_1, s_2. (We will not distinguish in notation between the matrices s_0, s_1, s_2 and their images in the Weyl groups.) There is a length function on the Coxeter group $\tilde{W}$, which we denote by ℓ.

Parahoric subgroups

The Iwahori subgroup is defined as

$$I = G(\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o}&\mathfrak{p}&\mathfrak{o}&\mathfrak{o}\\ \mathfrak{o}&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\ \mathfrak{p}&\mathfrak{p}&\mathfrak{o}&\mathfrak{o}\\ \mathfrak{p}&\mathfrak{p}&\mathfrak{p}&\mathfrak{o} \end{bmatrix}. \tag{46}$$

If $J \subsetneq \{0,1,2\}$, then the corresponding standard parahoric subgroup P_J is generated by I and s_j, where j runs through J. More precisely,

$$P_J = \bigsqcup_{w \in \langle s_j \colon j \in J\rangle} IwI. \tag{47}$$

The parahoric corresponding to $\{1\}$ is called the *Siegel congruence subgroup* and denoted by P_1. The parahoric corresponding to $\{2\}$ is called the *Klingen congruence subgroup* and denoted by P_2. Explicitly,

$$P_1 = G(\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o}&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\ \mathfrak{o}&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\ \mathfrak{p}&\mathfrak{p}&\mathfrak{o}&\mathfrak{o}\\ \mathfrak{p}&\mathfrak{p}&\mathfrak{o}&\mathfrak{o} \end{bmatrix}, \qquad P_2 = G(\mathfrak{o}) \cap \begin{bmatrix} \mathfrak{o}&\mathfrak{p}&\mathfrak{o}&\mathfrak{o}\\ \mathfrak{o}&\mathfrak{o}&\mathfrak{o}&\mathfrak{o}\\ \mathfrak{o}&\mathfrak{p}&\mathfrak{o}&\mathfrak{o}\\ \mathfrak{p}&\mathfrak{p}&\mathfrak{p}&\mathfrak{o} \end{bmatrix}. \tag{48}$$

Note that P_1 is normalized by η, but P_2 is not. We let $P_0 := \eta P_2 \eta^{-1}$; this is the parahoric corresponding to $\{0\}$. The parahoric corresponding to $\{1,2\}$ is $K := G(\mathfrak{o})$.

The Iwahori-Hecke algebra

We recall the structure of the Iwahori-Hecke algebra $\mathcal{I}$, which is the convolution algebra of compactly supported left and right I-invariant functions on $G(F)$. Explicitly, for T and T' in $\mathcal{I}$, their product is given by

$$(T \cdot T')(x) = \int\limits_{G(F)} T(xy^{-1})T'(y)\,d^Iy.$$

Here, d^Iy is the Haar measure on $G(F)$ which gives I volume 1. The characteristic function of I is the identity element of $\mathcal{I}$; we denote it by e. The characteristic function of ηI is an element of $\mathcal{I}$, which we denote again by η. For $j = 0,1,2$ let e_i be the characteristic function of Is_iI. Then $\mathcal{I}$ is generated by e_0, e_1, e_2 and η. For $w \in \tilde{W}$, let $q_w = \#IwI/I$. It is easy to verify that

$$q_{s_i} = q \qquad \text{for } i = 0,1,2. \tag{49}$$

It is known that

$$q_{w_1w_2} = q_{w_1}q_{w_2} \qquad \text{if} \quad \ell(w_1w_2) = \ell(w_1) + \ell(w_2). \tag{50}$$

For $w \in \tilde{W}$, let $T_w \in \mathcal{I}$ be the characteristic function of IwI. It is known that

$$T_{w_1w_2} = T_{w_1} \cdot T_{w_2} \qquad \text{if} \quad \ell(w_1w_2) = \ell(w_1) + \ell(w_2). \tag{51}$$

Action of the Iwahori-Hecke algebra on smooth representations

The Iwahori-Hecke algebra $\mathcal{I}$ acts on our representation (π, V) by

$$Tv = \int\limits_{G(F)} T(g)\pi(g)v\, d^I g, \qquad T \in \mathcal{I},\ v \in V.$$

We denote by V^I the subspace of I-invariant vectors. The action of $\mathcal{I}$ induces an endomorphism of V^I. If V is irreducible, then so is V^I.

Set

$$d_i = \frac{1}{q+1}(e + e_i), \qquad i \in \{0, 1, 2\}. \tag{52}$$

Then $d_i^2 = d_i$, and as operator on V, the element d_i acts as a projection onto the space of fixed vectors V^{P_i}. We refer to the operator d_1 as *Siegelization*, and to d_2 as *Klingenization*.

Our goal

As mentioned earlier, our goal is to compute the quantities $J(\phi)$ for suitable vectors $\phi \in \pi$ in certain cases where ϕ is not the spherical vector in a type I representation. Specifically, we will cover the cases where π has a P_1-fixed vector, and we will take ϕ to be such a vector. If π is spherical (i.e., has a K-fixed vector), we will assume it is generic; no such assumption will be made for the remaining representations.

Let us look more closely at the relevant representations π. Looking at the last two columns of Table (18), we see that such a π must be one of types I, IIa, IIIa, Vb/c, VIa or VIb. (Note that the representations IVb, IVc are not unitary). For each of these representations π, we will choose ϕ to be any member of a suitable orthogonal basis for the space of P_1-fixed vectors. We will evaluate $J(\phi)$ exactly in each of these cases under certain additional assumptions which are listed later.

3.3 Calculation of matrix coefficients

We consider the endomorphisms of the space V^{P_1} induced by the elements

$$d_1 e_1 e_0 e_1 \qquad \text{and} \qquad d_1 e_0 e_1 e_0 \tag{53}$$

of the Iwahori-Hecke algebra $\mathcal{I}$. (Since the element $e_0 e_1 e_0$ commutes with e_1, we could have omitted the projection d_1 on the second operator, but we will carry it along for symmetry.) In this section we will calculate the matrix coefficients

$$\lambda(\phi) := \frac{\langle d_1 e_1 e_0 e_1 \phi, \phi\rangle}{\langle \phi, \phi\rangle} \qquad \text{and} \qquad \mu(\phi) := \frac{\langle d_1 e_0 e_1 e_0 \phi, \phi\rangle}{\langle \phi, \phi\rangle} \tag{54}$$

for certain π and certain $\phi \in V^{P_1}$. The results will be needed as input for the calculation of the quantity $J_0(\phi)$ in the subsequent sections. Evidently, the matrix coefficients (54) depend neither on the normalization of the inner product, nor on the normalization of the vector ϕ.

All irreducible, admissible (π, V) for which V^{P_1} is not zero can be realized as subrepresentations of a full induced representation $\chi_1 \times \chi_2 \rtimes \sigma$ with unramified χ_1, χ_2, σ. Since we are working with a version of GSp(4) which is different from the one in [32], it is necessary to clarify the notation. Let

$$G' = \{g \in \mathrm{GL}_4 :\ {}^t g J' g = \lambda J',\ \lambda \in \mathrm{GL}_1\}, \qquad J' = \begin{bmatrix} &&&1\\ &&1&\\ &-1&&\\ -1&&& \end{bmatrix}.$$

Then G' is the *symmetric* version of GSp(4); this is the one used in [32]. There is an isomorphism between $G(F)$ and $G'(F)$ given by switching the first two rows and the first two columns. For example, the Iwahori subgroup I defined in (46) corresponds to the subgroup I' of $G'(\mathfrak{o})$ consisting of matrices which are upper triangular mod $\mathfrak{p}$. If (π, V) is a representation of $G(F)$, let (π', V) be the representation of $G'(F)$ obtained by composing π with the isomorphism $G(F) \cong G'(F)$. This establishes a one-one correspondence between representations of $G(F)$ and representations of $G'(F)$. When we write $\chi_1 \times \chi_2 \rtimes \sigma$, we mean the representation of $G(F)$ that corresponds to the representation of $G'(F)$ which in [32] was denoted by the same symbol.

In the following we will rely, sometimes without mentioning it, on Table A.15 of [32], which lists the dimensions of the spaces of fixed vectors under all parahoric subgroups for all Iwahori-spherical representations of $G(F)$.

Type I

Let $\pi = \chi_1 \times \chi_2 \rtimes \sigma$ with unramified χ_1, χ_2, σ. Let V be the standard model of π, and let V^I be the subspace of I-invariant vectors. Then V^I has the basis f_w, $w \in W$, where f_w is the unique I-invariant element of V with $f_w(w) = 1$ and $f_w(w') = 0$ for $w' \in W$, $w' \neq w$. It is convenient to order the basis as follows:

$$f_e, \quad f_1, \quad f_2, \quad f_{21}, \quad f_{121}, \quad f_{12}, \quad f_{1212}, \quad f_{212}, \tag{55}$$

where we have abbreviated $f_1 = f_{s_1}$ and so on. A basis for the four-dimensional space V^{P_1} is given by

$$f_e + f_1, \qquad f_2 + f_{21}, \qquad f_{121} + f_{12}, \qquad f_{1212} + f_{212}. \tag{56}$$

Having fixed the basis (55), the operators e_0, e_1, e_2 and η on V^I become 8×8–matrices. They depend on the Satake parameters

$$\alpha = \chi_1(\varpi), \qquad \beta = \chi_2(\varpi), \qquad \gamma = \sigma(\varpi) \tag{57}$$

and are given[8] in Lemma 2.1.1 of [38]. Calculating with the elements (53) thus comes down to simple linear algebra.

Now assume that χ_1, χ_2, σ are unitary characters, or equivalently, that α, β, γ have absolute value 1. In this case $\chi_1 \times \chi_2 \rtimes \sigma$ is an irreducible representation of type I. It is unitary and tempered, with hermitian inner product on V given by

$$\langle f, f' \rangle = \int\limits_K f(g)\, \overline{f'(g)}\, dg. \tag{58}$$

The vectors (55) are orthogonal, since they are supported on disjoint cosets. Let $\phi_1, \ldots, \phi_4$ be the vectors in (56), in this order. Then $\phi_1, \ldots, \phi_4$ is an orthogonal basis of V^{P_1}. The vector $\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4$ is the K-fixed vector. If we write $d_1 e_1 e_0 e_1 \phi_i = \sum_{j=1}^4 c_{ij} \phi_j$, $d_1 e_0 e_1 e_0 \phi_i = \sum_{j=1}^4 c'_{ij} \phi_j$, then the quantities defined in (54) are given by $\lambda(\phi_i) = c_{ii}$ and $\mu(\phi_i) = c'_{ii}$. Working out the linear algebra, we find that

$$\lambda(\phi_1) = (q-1)q^2, \qquad \lambda(\phi_2) = \frac{q-1}{q+1} q^2, \qquad \lambda(\phi_3) = \frac{q-1}{q+1} q^2, \qquad \lambda(\phi_4) = 0, \tag{59}$$

$$\mu(\phi_1) = (q-1)q^2, \qquad \mu(\phi_2) = (q-1)q, \qquad \mu(\phi_3) = 0, \qquad \mu(\phi_4) = 0. \tag{60}$$

Type IIIa

Let $\pi = \chi \rtimes \sigma \mathrm{St}_{\mathrm{GSp}(2)}$ with $\chi \notin \{1, \nu^{\pm 2}\}$. Then π is a representation of type IIIa. We assume $\chi\sigma^2 = 1$, so that π has trivial central character. We realize π as a subrepresentation of $\chi \times \nu \rtimes \nu^{-1/2}\sigma$, and set

$$\alpha = \chi(\varpi), \qquad \beta = q^{-1}, \qquad \gamma = q^{1/2}\delta,$$

[8] The matrix for e_1 in Lemma 2.1.1 of [38] contains a typo: The two lower right 2×2-blocks need to be conjugated by $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.

where $\delta = \sigma(\varpi)$. Let V be the standard space of the full induced representation $\chi \times \nu \rtimes \nu^{-1/2}\sigma$, and let U be the subspace of V realizing π. To determine the two-dimensional space U^{P_1}, we observe that $\dim U^{P_2} = 1$, so that the Klingenization map $d_2 : U^{P_1} \to U^{P_2}$ has a non-trivial kernel. The condition $d_2\phi = 0$ characterizes a unique vector in V^{P_1}, which must thus lie in U^{P_1}. A second, linearly independent vector can then be obtained by applying η. Thus we find that U^{P_1} is spanned by

$$\phi_1 = {}^t[q, q, -1, -1, 0, 0, 0, 0] \qquad \text{and} \qquad \phi_2 = {}^t[0, 0, 0, 0, -q, -q, 1, 1]. \tag{61}$$

These vectors are eigenvectors for the endomorphism $T_{1,0} = e_2e_1e_2\eta$:

$$T_{1,0}\phi_1 = \alpha\delta q\phi_1, \qquad T_{1,0}\phi_2 = \delta q\phi_2. \tag{62}$$

Now assume that χ and σ are unitary. In this case π is unitary and tempered. Since the invariant inner product $\langle\cdot,\cdot\rangle$ is not a priori given by formula (58), the following lemma is not obvious:

3.2 Lemma. *The vectors ϕ_1 and ϕ_2 are orthogonal.*

Proof. First note that

$$\langle\pi(\eta)v, w\rangle = \langle v, \pi(\eta)w\rangle \qquad \text{for all } v, w \in U; \tag{63}$$

this is immediate, since η^2 acts as the identity on V. By Proposition 2.1.2 of [38], we have

$$\langle e_iv, w\rangle = \langle v, e_iw\rangle \qquad \text{for all } v, w \in U^I \text{ and } i \in \{1, 2\}. \tag{64}$$

We will now calculate $\langle T_{1,0}\phi_1, \phi_2\rangle$ in two different ways. On the one hand, by (62),

$$\langle T_{1,0}\phi_1, \phi_2\rangle = \alpha\delta q\langle\phi_1, \phi_2\rangle. \tag{65}$$

On the other hand, by (63) and (64), $\langle T_{1,0}\phi_1, \phi_2\rangle = \langle\phi_1, \eta e_2e_1e_2\phi_2\rangle$. A calculation using the explicit form (61) shows that $\eta e_2e_1e_2\phi_2 = \delta^{-1}q\phi_2$. Hence

$$\langle T_{1,0}\phi_1, \phi_2\rangle = \delta q\langle\phi_1, \phi_2\rangle. \tag{66}$$

Our assertion follows from (65) and (66), since $\alpha \neq 1$ for representations of type IIIa. ∎

In view of Lemma 3.2, we can now calculate the quantities (54) for the vectors ϕ_1 and ϕ_2 similar to the type I case. The result is

$$\lambda(\phi_i) = -\frac{q^2}{q+1}, \qquad \mu(\phi_i) = 0, \tag{67}$$

for $i = 1, 2$.

Type VIb

Consider the representation $\pi = \tau(T, \nu^{-1/2}\sigma)$ of type VIb. We will assume $\sigma^2 = 1$, so that π has trivial central character. We may realize π as a subrepresentation of $1_{F^\times} \times \nu \rtimes \nu^{-1/2}\sigma$. Hence we set

$$\alpha = 1, \qquad \beta = q^{-1}, \qquad \gamma = q^{1/2}\delta,$$

where $\delta = \sigma(\varpi) \in \{\pm1\}$. Let V be the standard space of the full induced representation $1_{F^\times} \times \nu \rtimes \nu^{-1/2}\sigma$, and let U be the subspace of V realizing π. We know $\dim U^{P_1} = 1$ and $\dim U^{P_2} = 0$. Hence U^{P_1} must be spanned by a vector ϕ_1 with $d_2\phi_1 = 0$. This condition characterizes a unique element of V^{P_1} (up to scalars), namely

$$\phi_1 = {}^t[q^2, q^2, -q, -q, -q, -q, 1, 1]. \tag{68}$$

It follows that ϕ_1 spans U^{P_1}. A calculation then shows that

$$\lambda(\phi_1) = -q^2 \qquad \text{and} \qquad \mu(\phi_1) = q. \tag{69}$$

Types IIa, Vb/c and VIa

Let (π, V) be an Iwahori-spherical representation of type IIa, Vb, Vc or VIa. Then V^{P_1} is one-dimensional by Table (18) in Section 2.1. Assume in addition that π is unitary and has trivial central character. If ϕ_1 is a vector spanning V^{P_1}, then calculations similar to the above show that

$$\lambda(\phi_1) = \frac{q-1}{q+1}q^2 \qquad \text{and} \qquad \mu(\phi_1) = -q. \tag{70}$$

We mention these representations just for completeness; they are not relevant for our global applications, since, by Theorem 2.1 they do not admit a special, non-split Bessel model.

3.4 Preliminary calculation of J_0

Let I be the Iwahori subgroup, defined as in (46). Let Φ_0 be any function $G(F) \to \mathbb{C}$ that is left and right I-invariant. Assume that

- The residual characteristic of F is odd
- $d := b^2 - 4ac$ is in $\mathfrak{o}^\times$ but not in $\mathfrak{o}^{\times 2}$. Then $K = F(\sqrt{d})$ is the unramified quadratic extension of F. Since $1 + \mathfrak{p} \subset \mathfrak{o}^{\times 2}$, this implies $a \in \mathfrak{o}^\times$.

Our goal in this section is to calculate the integrals

$$J_0^{(k)}(\Phi_0) := \int\limits_{N(\mathfrak{p}^{-k})} \Phi_0(n)\theta_S^{-1}(n)\,dn, \tag{71}$$

where k is a positive integer. Our main result is the following.

3.3 Proposition. *Let $\Phi_0 : G(F) \to \mathbb{C}$ be left and right I-invariant. Then the integrals (71) stabilize as $k \to \infty$, so that the definition*

$$J_0(\Phi_0) := \lim_{k\to\infty} \int\limits_{N(\mathfrak{p}^{-k})} \Phi_0(n)\theta_S^{-1}(n)\,dn \tag{72}$$

is meaningful. Moreover, we have

$$\begin{aligned} J_0(\Phi_0) = \Phi_0(1) &- \Phi_0(\begin{bmatrix} 1 &&& \\ & \varpi^{-1} && \\ && 1 & \\ &&& \varpi \end{bmatrix} s_1s_2s_1) \\ &- q\Phi_0(\begin{bmatrix} \varpi^{-1} &&& \\ & 1 && \\ && \varpi & \\ &&& 1 \end{bmatrix} s_2) + q\Phi_0(\begin{bmatrix} \varpi^{-1} &&& \\ & \varpi^{-1} && \\ && \varpi & \\ &&& \varpi \end{bmatrix} s_2s_1s_2s_1). \end{aligned}$$

3.4 Remark. *The quantities $J_0(\Phi_0)$ defined above are related to the quantities $J_0(\phi, t)$ defined in (39). Indeed, let $\phi \in V^I$, and define Φ_ϕ as in (37). Then Φ_ϕ is bi-I-invariant, and we see that $J_0(\Phi_\phi) = J_0(\phi, 1)$.*

We begin with some preliminary calculations. For any $A \in \Gamma^0(\mathfrak{p}) = \mathrm{GL}(2, \mathfrak{o}) \cap \left[\begin{smallmatrix} \mathfrak{o} & \mathfrak{p} \\ \mathfrak{o} & \mathfrak{o} \end{smallmatrix}\right]$, a simple calculation gives

$$J_0^{(k)}(\Phi_0) = \int\limits_{\mathrm{Sym}(2,\mathfrak{p}^{-k})} \Phi_0(\left[\begin{smallmatrix} 1 & X \\ & 1 \end{smallmatrix}\right])\psi^{-1}(\mathrm{tr}(S'X))\,dX, \qquad \text{where } S' = {}^tASA.$$

Choosing $A = \left[\begin{smallmatrix} 1 & \\ x & 1 \end{smallmatrix}\right]$ with $x = -c^{-1}b/2$ produces a diagonal matrix S'. This shows that in the following we may (and will) assume $b = 0$. For m, n, r non-negative integers, let

$$J_0^{(m,n),z} = \int\limits_{\varpi^{-m}\mathfrak{o}^\times} \int\limits_{\mathfrak{p}^{-n}} \Phi_0(\left[\begin{smallmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right])\psi^{-1}(ax + cz)\, dy\, dx,$$

$$J_0^{x,(n,r)} = \int\limits_{\mathfrak{p}^{-n}} \int\limits_{\varpi^{-r}\mathfrak{o}^\times} \Phi_0(\left[\begin{smallmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right])\psi^{-1}(ax + cz)\, dz\, dy.$$

3.5 Lemma. *If $m > 1$, then $J_0^{(m,n),z} = 0$. If $r > 1$, then $J_0^{x,(n,r)} = 0$.*

Proof. Take $w \in \mathfrak{p}$ and write $1 + w = u^2$ with $u \in \mathfrak{o}^\times$. Then

$$J_0^{(m,n),z} = \int\limits_{\varpi^{-m}\mathfrak{o}^\times} \int\limits_{\mathfrak{p}^{-n}} \Phi_0(\left[\begin{smallmatrix} u^{-1} & & & \\ & 1 & & \\ & & u & \\ & & & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} u & & & \\ & 1 & & \\ & & u^{-1} & \\ & & & 1 \end{smallmatrix}\right])\psi^{-1}(ax + cz)\, dy\, dx$$

$$= \int\limits_{\varpi^{-m}\mathfrak{o}^\times} \int\limits_{\mathfrak{p}^{-n}} \Phi_0(\left[\begin{smallmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right])\psi^{-1}(ax + cz)\psi^{-1}(axw)\, dy\, dx.$$

Applying $\int_{\mathfrak{p}} \ldots dw$ to both sides shows that $J_0^{(m,n),z} = 0$ if $x \notin \mathfrak{p}^{-1}$, i.e., if $m > 1$. A similar argument shows that $J_0^{x,(n,r)} = 0$ if $r > 1$. ∎

3.6 Lemma. *For any $k \geq 1$, we have $J_0^{(k)}(\Phi_0) = J_0^{(1)}(\Phi_0)$.*

Proof. Assume that $k > 1$. By Lemma 3.5,

$$J_0^{(k)}(\Phi_0) = \int\limits_{\mathfrak{p}^{-1}} \int\limits_{\mathfrak{p}^{-k}} \int\limits_{\mathfrak{p}^{-1}} \Phi_0(\left[\begin{smallmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right])\psi^{-1}(ax + cz)\, dz\, dy\, dx.$$

Using the left invariant properties of Φ_0, and a simple calculation using the identity

$$\left[\begin{smallmatrix} 1 & y \\ & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & \\ y^{-1} & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} y & \\ & y^{-1} \end{smallmatrix}\right]\left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ y^{-1} & 1 \end{smallmatrix}\right], \tag{73}$$

one can check that

$$\int\limits_{\mathfrak{p}^{-1}} \int\limits_{\varpi^{-\ell}\mathfrak{o}^\times} \int\limits_{\mathfrak{p}^{-1}} \Phi_0(\left[\begin{smallmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right])\psi^{-1}(ax + cz)\, dz\, dy\, dx = 0 \qquad \text{for } \ell \geq 2. \tag{74}$$

This implies our assertion. ∎

Proof of Proposition 3.3. By Lemma 3.6, the integrals $J_0^{(k)}(\Phi_0)$ stabilize, and in fact

$$J_0(\Phi_0) = \int\limits_{\mathfrak{p}^{-1}} \int\limits_{\mathfrak{p}^{-1}} \int\limits_{\mathfrak{p}^{-1}} \Phi_0(\left[\begin{smallmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right])\psi^{-1}(ax + cz)\, dz\, dy\, dx.$$

Write $J_0(\Phi_0) = J_1 + J_2$, where J_1 is the part where $x \in \mathfrak{o}$, and J_2 is the part where $x \in \varpi^{-1}\mathfrak{o}^\times$. Evidently,

$$J_1 = \int\limits_{\mathfrak{p}^{-1}} \int\limits_{\mathfrak{p}^{-1}} \Phi_0(\left[\begin{smallmatrix} 1 & & & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{smallmatrix}\right])\psi^{-1}(cz)\, dz\, dy.$$

Let J_{11} be the part where $y \in \mathfrak{o}$, and let J_{12} be the part where $y \notin \mathfrak{o}$. Using (73), we have

$$J_{11} = \Phi_0(1) - \Phi_0(\begin{bmatrix} 1 &&& \\ & \varpi^{-1} && \\ && 1 & \\ &&& \varpi \end{bmatrix} s_1 s_2 s_1).$$

By a conjugation of the argument of Φ_0 and the use of bi-I-invariance we see that, for any $w \in \mathfrak{o}$,

$$\begin{aligned} J_{12} &= \int\limits_{\varpi^{-1}\mathfrak{o}^\times} \int\limits_{\mathfrak{p}^{-1}} \Phi_0(\begin{bmatrix} 1 &&& y \\ & 1 & y & z \\ && 1 & \\ &&& 1 \end{bmatrix})\psi^{-1}(cz)\,dz\,dy \\ &= \int\limits_{\varpi^{-1}\mathfrak{o}^\times} \int\limits_{\mathfrak{p}^{-1}} \Phi_0(\begin{bmatrix} 1 &&& y \\ & 1 & y & z+2wy \\ && 1 & \\ &&& 1 \end{bmatrix})\psi^{-1}(cz)\,dz\,dy. \end{aligned}$$

The element w may depend on y and z. Choosing $w = -y^{-1}z/2$, we see $J_{12} = 0$. Thus

$$J_1 = J_{11} = \Phi_0(1) - \Phi_0(\begin{bmatrix} 1 &&& \\ & \varpi^{-1} && \\ && 1 & \\ &&& \varpi \end{bmatrix} s_1 s_2 s_1).$$

A similar (but slightly more involved) computation gives

$$J_2 = J_{21} + J_{22} = -q\Phi_0(\begin{bmatrix} \varpi^{-1} &&& \\ & 1 && \\ && \varpi & \\ &&& 1 \end{bmatrix} s_2) + q\Phi_0(\begin{bmatrix} \varpi^{-1} &&& \\ & \varpi^{-1} && \\ && \varpi & \\ &&& \varpi \end{bmatrix} s_2 s_1 s_2 s_1).$$

This concludes the proof. ∎

3.7 Corollary. *If Φ_0 is left and right P_1-invariant, then*

$$J_0(\Phi_0) = \Phi_0(1) - (q+1)\Phi_0(\begin{bmatrix} \varpi^{-1} &&& \\ & 1 && \\ && \varpi & \\ &&& 1 \end{bmatrix} s_2) + q\Phi_0(\begin{bmatrix} \varpi^{-1} &&& \\ & \varpi^{-1} && \\ && \varpi & \\ &&& \varpi \end{bmatrix} s_2 s_1 s_2). \tag{75}$$

3.5 The value of $J(\phi)$ for certain representations

We return to the setup of Section 3.1. Let (π, V) be an irreducible, admissible, unitary representation of $G(F)$ with trivial central character, and Λ be a character of T satisfying $\Lambda|_{F^\times} = 1$. For $\phi \in V$, let the function $\Phi_\phi(g)$ and the quantity $J_0(\phi)$ be as defined in (37), (40) respectively.

3.8 Lemma. *Suppose that the following hold.*

i) The residual characteristic of F is odd.

ii) $K = F(\sqrt{d})$ is the unramified quadratic extension of F.

iii) Λ is an unramified character.

iv) ϕ is a P_1-invariant vector.

Then $J_0(\phi) = J_0(\Phi_\phi)$. Hence, by Corollary 3.7,

$$J_0(\phi) = 1 - (q+1)\Phi_\phi(g_1) + q\Phi_\phi(g_2), \tag{76}$$

where

$$g_1 = \begin{bmatrix} \varpi^{-1} &&& \\ & 1 && \\ && \varpi & \\ &&& 1 \end{bmatrix} s_2, \qquad g_2 = \begin{bmatrix} \varpi^{-1} &&& \\ & \varpi^{-1} && \\ && \varpi & \\ &&& \varpi \end{bmatrix} s_2 s_1 s_2. \tag{77}$$

Proof. The assumptions imply that the natural map $\mathfrak{o}^\times\backslash T(\mathfrak{o}) \to F^\times\backslash T$ is an isomorphism and $\Lambda = 1$. Hence $J_0(\phi) = \int_{\mathfrak{o}^\times\backslash T(\mathfrak{o})} J_0(\phi,t)dt$. Moreover, we note that $T(\mathfrak{o}) \subset P_1$. It follows that $J_0(\phi,t) = J_0(\phi,1)$ for all $t \in T(\mathfrak{o})$, and therefore $J_0(\phi) = J_0(\phi,1)$. By Remark 3.4, it follows that $J_0(\phi) = J_0(\Phi_\phi)$. ∎

3.9 Remark. *Suppose that the first three assumptions of Lemma 3.8 hold, but ϕ is no longer assumed to be P_1-invariant, but merely I-invariant. Then with some additional work one can show*

$$\begin{aligned} J_0(\phi) = \frac{1}{1+q}\Big[1 &+ q\Phi_\phi(s_1) - q^2\Phi_\phi(\begin{bmatrix} \varpi^{-1}&&&\\&1&&\\&&\varpi&\\&&&1\end{bmatrix} s_2) - q\Phi_\phi(\begin{bmatrix} 1&&&\\&\varpi^{-1}&&\\&&1&\\&&&\varpi\end{bmatrix} s_1s_2) \\ &- q\Phi_\phi(\begin{bmatrix} \varpi^{-1}&&&\\&1&&\\&&\varpi&\\&&&1\end{bmatrix} s_2s_1) - \Phi_\phi(\begin{bmatrix} 1&&&\\&\varpi^{-1}&&\\&&1&\\&&&\varpi\end{bmatrix} s_1s_2s_1) \\ &+ q\Phi_\phi(\begin{bmatrix} \varpi^{-1}&&&\\&\varpi^{-1}&&\\&&\varpi&\\&&&\varpi\end{bmatrix} s_2s_1s_2) + q^2\Phi_\phi(\begin{bmatrix} \varpi^{-1}&&&\\&\varpi^{-1}&&\\&&\varpi&\\&&&\varpi\end{bmatrix} s_1s_2s_1s_2)\Big]. \end{aligned}$$

Using (76), we now explicitly compute $J_0(\phi)$ for the relevant vectors ϕ. The values $\Phi_\phi(g_i)$ depend on the representation π. We can convert an operator $\pi(g)$ on V^I into an element of the Iwahori-Hecke algebra $\mathcal{I}$. Recall that e, the characteristic function of I, is the identity element of $\mathcal{I}$. For $g \in G(F)$, let T_g be the characteristic function of IgI. We have the following easy lemma, whose proof we omit.

3.10 Lemma. *Let (π, V) be a smooth representation of $G(F)$. Let g be any element of $G(F)$. Then $e \circ \pi(g) = \frac{1}{\#IgI/I} T_g$ as operators on V^I.*

Applying Lemma 3.10 to the elements g_1 and g_2 in (77), we see that

$$e \circ \pi(g_i) = \frac{1}{\#Ig_iI/I}\,\mathrm{char}(Ig_iI) \tag{78}$$

as operators on V^{P_1}. Note that, as elements of $\tilde{W}$,

$$g_1 = s_1s_0s_1, \qquad g_2 = s_0s_1s_0. \tag{79}$$

From (49), (50), (51) and (78), we therefore get

$$e \circ \pi(g_1) = q^{-3}\, e_1e_0e_1, \qquad e \circ \pi(g_2) = q^{-3}\, e_0e_1e_0. \tag{80}$$

Substituting into (76), we obtain the formula

$$J_0(\phi) = 1 - (q+1)q^{-3}\lambda(\phi) + q^{-2}\mu(\phi), \tag{81}$$

with $\lambda(\phi)$ and $\mu(\phi)$ as defined in (54). (We can insert d_1 into the inner product because ϕ is assumed to be P_1-invariant.)

The quantities $\lambda(\phi)$ and $\mu(\phi)$ have been calculated in Sect. 3.3 for various vectors in a number of representations. Using these, we can now compute $J_0(\phi)$ in each case using (81). Furthermore, writing

$$M(\pi) = \frac{L(1,\pi,\mathrm{Ad})L(1,\chi_{K/F})}{\zeta_F(2)\zeta_F(4)L(1/2,\pi\times\mathcal{AI}(\Lambda^{-1}))},$$

we have $J(\phi) = M(\pi)J_0(\phi)$. We can write down $M(\pi)$ in each case using the relevant data from [38] and [1]. As a result, we can compute $J(\phi)$ in each case as well. The results are summarized in the following Theorem.

3.11 Theorem. *Suppose that π is an irreducible, admissible, unitary, Iwahori-spherical representation of $\mathrm{GSp}(4,F)$ with trivial central character. We assume that π is one of the following types: I, IIa, IIIa, Vb/c, VIa or VIb. For type I, let the vectors $\phi_1,\ldots,\phi_4$ be the vectors in (56), in this order. For type IIIa, let ϕ_1 and ϕ_2 be as given in (61); in all other cases let ϕ_1 be the essentially unique P_1-invariant vector.*

Suppose that the residual characteristic of F is odd, $K=F(\sqrt{d})$ is the unramified quadratic extension of F, and Λ is an unramified character. Then the quantities $J_0(\phi)$, $J(\phi)$ are as given by the last two columns of the table below.

type	$M(\pi)$	ϕ	$J_0(\phi)$	$J(\phi)$
I	$L(1,\pi,\mathrm{Std})(1-q^{-4})$	ϕ_1	q^{-1}	$q^{-1}L(1,\pi,\mathrm{Std})(1-q^{-4})$
		ϕ_2	1	$L(1,\pi,\mathrm{Std})(1-q^{-4})$
		ϕ_3	q^{-1}	$q^{-1}L(1,\pi,\mathrm{Std})(1-q^{-4})$
		ϕ_4	1	$L(1,\pi,\mathrm{Std})(1-q^{-4})$
IIIa	$1+q^{-2}$	ϕ_1	$1+q^{-1}$	$(1+q^{-2})(1+q^{-1})$
		ϕ_2	$1+q^{-1}$	$(1+q^{-2})(1+q^{-1})$
VIb	$1+q^{-2}$	ϕ_1	$2(1+q^{-1})$	$2(1+q^{-2})(1+q^{-1})$
IIa, Vb/c, VIa		ϕ_1	0	0

3.12 Remark. *For π of type I, with $\pi=\chi_1\times\chi_2\rtimes\sigma$, we performed our calculations in Section 3.3 under the assumption π tempered, while here we merely assume π unitary. We exploit here the fact that the values of $J_0(\phi)$ for spherical generic representations vary analytically with the Satake parameters [21, Theorem 2.2]. Note also that for type I, using the notations introduced in Section 3.3, we have*

$$L(1,\pi,\mathrm{Std})=\big((1-q^{-1})(1-\alpha q^{-1})(1-\alpha^{-1}q^{-1})(1-\beta q^{-1})(1-\beta^{-1}q^{-1})\big)^{-1}.$$

3.13 Remark. *Note that for representations of type IIa, Vb/c and VIa we obtain $J_0(\phi_1)=0$ for the essentially unique P_1-invariant vector ϕ_1. This is consistent with the fact that these representations do not admit a non-split Bessel functional with trivial character Λ on the torus $T(F)$; see Theorem 2.1. Hence, such representations will not be relevant for our global applications to Siegel modular forms.*

3.14 Remark. *Note that whenever $J(\phi)\neq 0$, we have $\sum_i J(\phi_i)=2+O(q^{-1})$.*

4 Global results

In this final section, we return to a global setup. We begin with some basic results on the correspondence between Siegel modular forms and automorphic representations on $G(\mathbb{A})$. Then we move on to the relation between Bessel periods and Fourier coefficients and the classical reformulation of Conjecture 1.11. Finally, we work out several consequences of our explicit refinement of Böcherer's conjecture. We remind the reader that all our measures on adelic groups are equal to the respective Tamagawa measures.

4.1 Siegel modular forms and representations

We use the definitions and some basic properties of Siegel modular forms of degree 2 without proof; we refer the reader to [38] or [35] for details. Let $\Gamma_0(N)\subseteq \mathrm{Sp}_4(\mathbb{Z})$ denote the Siegel congruence subgroup of squarefree level N, defined in (12). If $p|N$ is prime, set $P_{1,p}(N)=P_1$, the local analog of $\Gamma_0(N)$ defined in (11); if $p\nmid N$ is prime, set $P_{1,p}(N)=\mathrm{GSp}_4(\mathbb{Z}_p)$. We let $S_k(\Gamma_0(N))$ denote the space of Siegel cusp forms of weight k with respect to the group $\Gamma_0(N)$. For any $f \in S_k(\Gamma_0(N))$, define the Petersson inner product

$$\langle f, f\rangle = \frac{1}{[\mathrm{Sp}_4(\mathbb{Z}) : \Gamma_0(N)]} \int\limits_{\Gamma_0(N)\backslash\mathbb{H}_2} |f(Z)|^2 (\det Y)^{k-3}\, dX\, dY. \tag{82}$$

The space $S_k(\Gamma_0(N))$ has a natural orthogonal (with respect to the Petersson inner product) decomposition

$$S_k(\Gamma_0(N)) = S_k(\Gamma_0(N))^{\mathrm{old}} \oplus S_k(\Gamma_0(N))^{\mathrm{new}}$$

into the space of oldforms and newforms in the sense of [38].

For any $\phi \in L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))$, define $\langle \phi, \phi\rangle = \int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} |\phi(g)|^2 dg$.

4.1 Proposition. *Suppose that N is a squarefree integer and k a positive integer. Let π be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character. Suppose that*[9] *$\pi_\infty \simeq L(k,k)$ and π_p has a non-zero $P_{1,p}(N)$-fixed vector for all primes $p|N$. Let $\phi \in V_\pi \subset L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))$ be a non-zero function such that*[10] *$\phi(gk_Nk_\infty) = \det(J(k_\infty, iI_2))^{-k}\phi(g)$ for all $k_\infty \in K_\infty$, $k_N \in \prod_{p<\infty} P_{1,p}(N)$. Define the function f on $\mathbb{H}_2$ via $f(g_\infty i) = \det(J(g_\infty, iI_2))^k \phi(g_\infty)$. Then f is a non-zero element of $S_k(\Gamma_0(N))$ and is an eigenfunction for the local Hecke algebras at all $p \nmid N$. The function f belongs to $S_k(\Gamma_0(N))^{\mathrm{old}}$ if and only if π_p is spherical for some $p|N$. Furthermore*

$$\frac{\langle f, f\rangle}{\mathrm{vol}(\mathrm{Sp}_4(\mathbb{Z})\backslash\mathbb{H}_2)} = \frac{\langle \phi, \phi\rangle}{\mathrm{vol}(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))}.$$

Proof. This is routine; see Section 3.2 of [38]. ∎

Given any $f \in S_k(\Gamma_0(N))$, we can define its adelization as in [38] or [35]. A non-zero $f \in S_k(\Gamma_0(N))$ arises via Proposition 4.1 if and only if its adelization generates an irreducible automorphic representation, in which case the adelization precisely equals the function ϕ in the proposition. It is easy to see that the set of such f spans the space $S_k(\Gamma_0(N))$, a fact that follows immediately from the decomposition of the cuspidal automorphic spectrum into a direct sum of irreducible representations.

Let π, ϕ, f be as in Proposition 4.1. Then using the results of [38] (or see table (18) earlier) we see that the local representations π_p at finite primes have the following properties.

- If $p \nmid N$, then π_p is spherical, i.e., one of types I, IIb, IIIb, IVd, Vd, VId. Of these, types IIb, IIIb, IVd, Vd, VId are non-tempered (as well as non-generic). Type I is generic; moreover it is tempered provided the inducing characters are unitary.
- If $p|N$, then π_p is either spherical (in which case it is one of those from the above list) or non-spherical, in which case it is one of types IIa, IIIa, Vb/c, VIa or VIb. Of these, Vb/c are non-tempered and non-generic, VIb is tempered and non-generic while the rest are tempered and generic.

A representation π of $G(\mathbb{A})$ is said to be CAP if it is nearly equivalent to a constituent of a global induced representation of a proper parabolic subgroup of $G(\mathbb{A})$. Otherwise we say that it is non-CAP. A non-CAP representation is type I at all primes where π_p is spherical, and hence is generic almost everywhere. A CAP representation *is not generic at any place*, and hence cannot have a type I component anywhere. Further, a non-CAP representation is *expected* to satisfy the generalized Ramanujan conjecture, which postulates that it must be tempered at all places and hence conjecturally can never equal one of the representations IIb, IIIb, IVd, Vb/c, Vd or VId.

If $k \geq 3$, and π, f are as in Proposition 4.1, then the following additional facts are known to be true:

[9] See the paper [24] for the definition of $L(k,\ell)$; this was also called $\mathcal{E}(k,\ell)$ previously by us.

[10] Our conditions on π imply that such a function always exists.

- If π is CAP then π must be P-CAP (CAP associated to the Siegel parabolic) and therefore π_p is type IIb whenever it is spherical (see Corollary 4.5 of [26]).
- If π is non-CAP, then it is *tempered* type I whenever it is spherical. This is due to Weissauer [47].

We let $S_k(\Gamma_0(N))^{\rm CAP}$, (resp. $S_k(\Gamma_0(N))^{\rm T}$) denote the subspace spanned by all the f as in Proposition 4.1 for which the associated π are CAP (resp. non-CAP). The letter T is chosen because the non-CAP forms are (conjecturally) precisely the ones that are tempered everywhere. Recall that if $k \geq 3$, then it is known that the space $S_k(\Gamma_0(N))^{\rm CAP}$ is spanned precisely by the Saito-Kurokawa lifts, and that the representations attached to the space $S_k(\Gamma_0(N))^{\rm T}$ are tempered at all unramified places.

It is clear that we have orthogonal (with respect to Petersson inner product) direct sum decompositions

$$S_k(\Gamma_0(N)) = S_k(\Gamma_0(N))^{\rm CAP} \oplus S_k(\Gamma_0(N))^{\rm T},$$

$$S_k(\Gamma_0(N))^{\rm old} = S_k(\Gamma_0(N))^{\rm old,CAP} \oplus S_k(\Gamma_0(N))^{\rm old,T},$$
$$S_k(\Gamma_0(N))^{\rm new} = S_k(\Gamma_0(N))^{\rm new,CAP} \oplus S_k(\Gamma_0(N))^{\rm new,T},$$
$$S_k(\Gamma_0(N))^{\rm CAP} = S_k(\Gamma_0(N))^{\rm new,CAP} \oplus S_k(\Gamma_0(N))^{\rm old,CAP},$$
$$S_k(\Gamma_0(N))^{\rm T} = S_k(\Gamma_0(N))^{\rm new,T} \oplus S_k(\Gamma_0(N))^{\rm old,T}.$$

Recall that Conjecture 1.11 only applies to those π that are generic almost everywhere, i.e., only to non-CAP π. However the CAP case is actually much easier, and Qiu [30] has recently proved a theorem that asserts that an analog of Conjecture 1.11 holds for all CAP representations.

4.2 Newforms and orthogonal Hecke bases

Let N be a squarefree integer. We say that a non-zero f in $S_k(\Gamma_0(N))$ is a **newform** if

i) $f \in S_k(\Gamma_0(N))^{\rm new}$.

ii) For each prime $p|N$, f is an eigenfunction of the $U(p)$ operator (see [36, Section 2.3] for the definition).

iii) The adelization ϕ of f generates an irreducible representation π (in other words, π, f are in the situation of Proposition 4.1).

The conditions imply that the automorphic form ϕ corresponds to a *factorizable* vector $\phi = \otimes_v \phi_v$ in π. Indeed, for any $p|N$, the local representation π_p has a unique $P_{1,p}(N)$-invariant vector, except if π_p is of type IIIa, in which case the space is two dimensional but has (up to multiples) exactly two vectors that are eigenfunctions of the local analog of $U(p)$, which is the $T_{1,0}$ operator considered earlier.

Using the result from [22], it can be shown that if $f \in S_k(\Gamma_0(N))$ is an eigenfunction of the local Hecke algebras at all places, then it is a newform. However the converse is not true (newforms are not necessarily an eigenfunction of the local Hecke algebra at all primes, in fact they fail to be so precisely at the primes p where the corresponding π_p is a type IIIa representation).

4.2 Lemma. *The space $S_k(\Gamma_0(N))^{\rm new}$ has an orthogonal basis consisting of newforms.*

Proof. This is immediate by decomposing the space $S_k(\Gamma_0(N))^{\rm new}$ into mutually orthogonal subspaces corresponding to irreducible automorphic representations, and then using the fact that the two local vectors in the Type IIIa representation are orthogonal. ∎

Next, for any four positive, mutually coprime, squarefree integers a, b, c, d, we will define a linear map $\delta_{a,b,c,d}$. This map will take $S_k(\Gamma_0(e))^{\rm T}$ to $S_k(\Gamma_0(abcde))^{\rm T}$ for each positive squarefree integer e coprime to $abcd$. It is defined as follows.

Let $f \in S_k(\Gamma_0(e))^{\mathrm{T}}$ be such that its adelization ϕ generates an irreducible representation π. (It suffices to define the map on such f because these span the full space $S_k(\Gamma_0(e))^{\mathrm{T}}$.) The automorphic form ϕ corresponding to f factors as $\phi = \phi_S \otimes_{p \nmid e} \phi_p$ where S denotes the set of places dividing e. Let p be any prime dividing $abcd$. Then the local vector ϕ_p is a spherical vector in a type I representation. Using the notation of Section 3.3, we have an orthogonal decomposition $\phi_p = \phi_{1,p} + \phi_{2,p} + \phi_{3,p} + \phi_{4,p}$. Define $\delta_{a,b,c,d}(\phi) = \phi_S \otimes_{p \nmid e} \phi'_p$ where $\phi'_p = \phi_p$ if $p \nmid abcd$, $\phi'_p = \phi_{1,p}$ if $p|a$, $\phi'_p = \phi_{2,p}$ if $p|b$, $\phi'_p = \phi_{3,p}$ if $p|c$, $\phi'_p = \phi_{4,p}$ if $p|d$. Using Proposition 4.1, we see that this takes f to an element $\delta_{a,b,c,d}(f)$ of $S_k(\Gamma_0(abcde))^{\mathrm{T}}$. Note that $\delta_{1,1,1,1}$ is just the identity map.

4.3 Proposition. *For any five coprime squarefree positive integers a, b, c, d, e, the map $\delta_{a,b,c,d}$ is an injective linear map that takes $S_k(\Gamma_0(e))^{\mathrm{T}}$ to $S_k(\Gamma_0(abcde))^{\mathrm{T}}$. Furthermore, for any positive squarefree integer N, we have an orthogonal direct sum decomposition*

$$S_k(\Gamma_0(N))^{\mathrm{T}} = \bigoplus_{\substack{a,b,c,d,e \\ abcde=N}} \delta_{a,b,c,d}\left(S_k(\Gamma_0(e))^{\mathrm{new},\mathrm{T}}\right)$$

Proof. By construction, for each a, b, c, d, e, the space $\delta_{a,b,c,d}\left(S_k(\Gamma_0(e))^{\mathrm{new},\mathrm{T}}\right)$ is the span of all those f in $S_k(\Gamma_0(N))^{\mathrm{T}}$ that generate an irreducible representation $\pi_f = \otimes'_v \pi_v$ with the property that π_p is spherical if and only if $p \nmid e$, and moreover, $\phi_p = \phi_{1,p}$ for all primes dividing a, $\ldots$, $\ldots$, $\phi_p = \phi_{4,p}$ for all primes dividing d. Clearly, as a, b, c, d, e vary, this takes care of all representations $\pi = \pi_f$ that come from Proposition 4.1. The orthogonality and injectivity properties come from the corresponding properties of the local maps which were proved earlier. ∎

4.4 Corollary. *Let $N > 0$ be squarefree. For all positive integers $e|N$, let $\mathcal{B}_{k,e}^{\mathrm{new},\mathrm{T}}$ be an orthogonal basis of $S_k(\Gamma_0(e))^{\mathrm{new},\mathrm{T}}$ consisting of newforms (such a basis is known to exist by Lemma 4.2). Then the set $\{\delta_{a,b,c,d}(f)\}_{\substack{a,b,c,d,e \\ abcde=N \\ f \in \mathcal{B}_{k,e}^{\mathrm{new},\mathrm{T}}}}$ gives an orthogonal basis for $S_k(\Gamma_0(N))^{\mathrm{T}}$.*

Proof. This follows immediately from the above proposition. ∎

Thus, we have constructed a nice orthogonal Hecke basis for $S_k(\Gamma_0(N))^{\mathrm{T}}$. The adelization ϕ of any element of our basis generates an irreducible representation. Moreover, ϕ is a factorizable vector and its local components ϕ_p are *precisely* the local vectors for which we have computed $J(\phi_p)$ previously.

4.3 Bessel periods and Fourier coefficients

Let $f \in S_k(\Gamma_0(N))$ have the Fourier expansion as in (1),

$$f(Z) = \sum_T a(f, T) e^{2\pi i \,\mathrm{Tr}(TZ)}.$$

Let ϕ_f be the function associated to f via strong approximation, as in [38] or [35]. Let $d < 0$ be a fundamental discriminant and put

$$S = S(d) = \begin{cases} \begin{bmatrix} \frac{-d}{4} & 0 \\ 0 & 1 \end{bmatrix} & \text{if } d \equiv 0 \pmod 4, \\ \begin{bmatrix} \frac{1-d}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} & \text{if } d \equiv 1 \pmod 4. \end{cases}$$

Given S as above, let the group T_S be defined as before. So $T_S \simeq K^\times$ where $K = \mathbb{Q}(\sqrt{d})$. Define

$$\mathrm{Cl}_K = T(\mathbb{A})/T(\mathbb{Q})T(\mathbb{R}) \prod_{p<\infty} (T(\mathbb{Q}_p) \cap \mathrm{GL}_2(\mathbb{Z}_p)).$$

Then Cl_K can be naturally identified with the ideal class group of K. Let t_c, $c \in \mathrm{Cl}_K$, be coset representatives of Cl_K with $t_c \in \prod_{p<\infty} T(\mathbb{Q}_p)$. We can write $t_c = \gamma_c m_c \kappa_c$ with $\gamma_c \in \mathrm{GL}(2,\mathbb{Q})$, $m_c \in \mathrm{GL}(2,\mathbb{R})^+$, and $\kappa_c \in \prod_{p<\infty} \mathrm{GL}(2,\mathbb{Z}_p)$. The matrices $S_c = \det(\gamma_c)^{-1}\ ({}^t\gamma_c) S \gamma_c$ have discriminant d, and form a set of representatives of the $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of primitive semi-integral positive definite matrices of discriminant d.

Let $\psi : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$ be the character such that $\psi(x) = e^{2\pi i x}$ if $x \in \mathbb{R}$ and $\psi(x) = 1$ for $x \in \mathbb{Z}_p$. One obtains a character θ_S of $N(\mathbb{Q})\backslash N(\mathbb{A})$ by $\theta_S(\left[\begin{smallmatrix}1 & X\\ & 1\end{smallmatrix}\right]) = \psi(\mathrm{Tr}(SX))$. Let Λ be a character of Cl_K. Then, as before, we can define the Bessel period

$$B(\phi_f, \Lambda) = \int\limits_{\mathbb{A}^\times T(\mathbb{Q})\backslash T(\mathbb{A})} \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \phi_f(tu)\Lambda^{-1}(t)\theta_S^{-1}(n)\,dn\,dt. \tag{83}$$

Then we have the following result (see e.g. Prop. 4.3 of [36]).

4.5 Proposition. *We have*

$$B(\phi_f, \Lambda) = r e^{-2\pi \mathrm{Tr}(S)} \sum_{c \in \mathrm{Cl}_K} \Lambda(c)^{-1} a(f, S_c)$$

where r is an absolute constant.

The constant r depends only on the normalization of measures and can be easily calculated in our case (as we are using the global Tamagawa measure). However, we will use a trick that will get rid of this constant later, so we do not calculate it here.

Let π, f, ϕ be as in Propostion 4.1 and Λ be a character of Cl_K as above. We will use the notation

$$R(f, K, \Lambda) = \sum_{c \in \mathrm{Cl}_K} \Lambda(c)^{-1} a(f, c). \tag{84}$$

Then, if Conjecture 1.11 holds for the representation π, we can conclude the following: *Assuming $f \in S_k(\Gamma_0(N))^T$, we have*

$$\frac{|R(f,K,\Lambda)|^2}{\langle f, f\rangle} = \frac{C e^{4\pi \mathrm{Tr}(S))}}{2 S_\pi} \frac{L(1/2, \pi \times \mathcal{AI}(\Lambda^{-1}))}{L(1,\pi,\mathrm{Ad}) L(1,\chi_d)} \cdot J_\infty \cdot \prod_{p|N} J(\phi_p), \tag{85}$$

where C is an absolute constant. We will now compute the quantity CJ_∞.

4.4 The Saito-Kurokawa trick

A modified version of (85) is in fact known to (unconditionally) hold for CAP representations. This is a theorem due to Qiu [30]. We quote Qiu's result in a very special case.

4.6 Theorem. *Let f be a full-level Saito-Kurokawa form of weight k that is an eigenfunction for all Hecke operators. Let π_0 be the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ that π_f is associated to, i.e., $L(s,\pi_f) = L(s,\pi_0)\xi(s+1/2)\xi(s-1/2)$. Then for $d < 0$ a fundamental discriminant, $K = \mathbb{Q}(\sqrt{d})$, we have*

$$\frac{|R(f,K,1)|^2}{\langle f, f\rangle} = \frac{C e^{4\pi \mathrm{Tr}(S))}}{\xi(2)} \frac{L(1/2, \pi_0 \otimes \chi_d) L(0,\chi_d)}{L(3/2,\pi_0) L(1,\pi_0,\mathrm{Ad})} \cdot J_\infty, \tag{86}$$

where C is the same constant as in (85).

4.7 Remark. *Crucially for us, the quantities C and J_∞ appearing in the above Theorem are the same as those that appear in (85). This can be checked by comparing the definition of the local integrals J_v of Liu and Qiu, which are the same up to regularization.*

Let g be the classical cusp form of half-integral weight $k-1/2$ that gives rise to the lift f. We have the following classical results:

$$|R(f,K,1)|^2 = |d|\left(\frac{w(K)L(1,\chi_d)}{2}\right)^2 |c(g,d)|^2, \tag{87}$$

which can be derived by combining the equation on page 74 of [9] with the class number formula. Also, from [5], Corollary 6.3,[11]

$$\frac{|c(g,d)|^2}{\langle f,f\rangle} = \frac{3\pi}{k-1}|d|^{k-3/2}2^{2k+1}\frac{L_{\mathrm{f}}(1/2,\pi_0\otimes\chi_d)}{L_{\mathrm{f}}(3/2,\pi_0)L(1,\pi_0,\mathrm{Ad})} \tag{88}$$

Hence (86) gives us

$$CJ_\infty = 2^{2k-3}L(1,\chi_d)e^{-4\pi\mathrm{Tr}(S)}|d|^{k-1}w(K)^2\pi. \tag{89}$$

The above analysis is valid whenever there exists a Saito-Kurokawa lift of weight k, i.e., for all even $k\geq 10$. We claim that the same formula for CJ_∞ holds for all even integers $k\geq 2$. This can be seen by looking at Saito-Kurokawa lifts with level (which exist for all $k\geq 2$) and performing a similar analysis. In particular, we use the main Theorem of [5] (the only thing we need is that as a function of k, the main formula of [5] does not depend on whether k is greater or less than 10). Moreover, an analogue of Qiu's theorem remains valid in this setup. *We conclude that (89) holds for all even integers $k\geq 2$.*

4.5 The main result and some consequences

Throughout this subsection, N denotes a positive squarefree odd integer and $k\geq 2$ an *even* integer. We begin by stating the main theorem of this paper.

4.8 Theorem. *Let $f\in S_k(\Gamma_0(N))^{\mathrm{T}}$. Assume that $f=\delta_{a,b,c,d}(g)$ where $abcde=N$ and g is a newform in $S_k(\Gamma_0(e))^{\mathrm{T}}$; note that f itself is a newform if and only if $a=b=c=d=1$ (whence $f=g$). Let π be the representation attached to g (or equivalently, to f). Let $d<0$ be a fundamental discriminant such that $\left(\frac{d}{p}\right)=-1$ for all $p|N$ and let Λ be an ideal class character of $K=\mathbb{Q}(\sqrt{d})$. Then, assuming the truth of Conjecture 1.11 for π, we have*

$$\begin{aligned}\frac{|R(f,K,\Lambda)|^2}{\langle f,f\rangle} &= (2^{2k-s}\cdot\pi)w(K)^2\ |d|^{k-1}\frac{L(1/2,\pi\times\mathcal{AI}(\Lambda^{-1}))}{L(1,\pi,\mathrm{Ad})}\prod_{p|N}J_p\\ &= \frac{2^{4k-s}\cdot\pi^{2k+2}}{(2k-2)!}\ w(K)^2\ |d|^{k-1}\frac{L_{\mathrm{f}}(1/2,\pi\times\mathcal{AI}(\Lambda^{-1}))}{L_{\mathrm{f}}(1,\pi,\mathrm{Ad})}\prod_{p|N}J_p,\end{aligned}$$

where $s=6$ if f is a weak Yoshida lift in the sense of [35] and $s=5$ otherwise. The quantities J_p are given as follows:

$$J_p=\begin{cases}L(1,\pi_p,\mathrm{Std})(1-p^{-4}) & \text{if } p|bd,\ \text{(cannot occur if } f \text{ is a newform)},\\ L(1,\pi_p,\mathrm{Std})(1-p^{-4})p^{-1} & \text{if } p|ac,\ \text{(cannot occur if } f \text{ is a newform)},\\ (1+p^{-2})(1+p^{-1}) & \text{if } p|e \text{ and } \pi_p \text{ is of type IIIa},\\ 2(1+p^{-2})(1+p^{-1}) & \text{if } p|e \text{ and } \pi_p \text{ is of type VIb},\\ 0 & \text{otherwise.}\end{cases}$$

Proof. This follows by combining (85) with Theorem 3.11 and the computation of CJ_∞ from the previous section. Note that $S_\pi=4$ if f is a weak Yoshida lift and $S_\pi=2$ otherwise. ∎

[11] Note that the formula stated in [5] differs from ours by a factor of $6=[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(4)]$. This is because the Kohnen–Zagier formula is quoted incorrectly in Lemma 6.1 of [5], owing to a different normalization of the Petersson inner product.

4.9 Remark. *If $f \in S_k(\Gamma_0(N))^{\mathrm{CAP}}$, a similar formula can be derived using the result of Qiu [30]. However, whenever $k \geq 3$ and $f \in S_k(\Gamma_0(N))^{\mathrm{CAP}}$, then f is automatically a Saito-Kurokawa lift, in which case the relevant formula has already been written down by Brown [5].*

For each f as in Theorem 4.8, let us use the notation

$$J_{f,N} = \prod_{p|N} J_p. \tag{90}$$

We have the following bound, which follows immediately from the formulas for J_p,

$$J_{f,N} \ll_\epsilon N^\epsilon. \tag{91}$$

We now work out several consequences of Theorem 4.8.

Yoshida lifts

We recall the representation theoretic construction of scalar valued Yoshida lifts described in [36]. Let N_1, N_2 be two positive, squarefree integers such that $M = \gcd(N_1, N_2) > 1$. Let g_1 be a classical newform of weight $2k-2$ and level N_1 and g_2 be a classical newform of weight 2 and level N_2, such that g_1 and g_2 are not multiples of each other. Assume that for all primes p dividing M the Atkin-Lehner eigenvalues of g_1 and g_2 coincide. Put $N = \mathrm{lcm}(N_1, N_2)$. For any $p|M$, let δ_p be the common Atkin-Lehner eigenvalue of g_1 and g_2. Then for any divisor M_1 of M with an *odd* number of prime factors, there exists a non-zero holomorphic Siegel cusp form $f = f_{g_1,g_2;M_1}$ such that f is a newform in $S_k(\Gamma_0(N))^{\mathrm{T}}$. Furthermore, the automorphic representation π generated by the adelization of f has the following properties.

i) $L(s, \pi_f) = L(s, \pi_{g_1}) L(s, \pi_{g_2})$.

ii) If $p|N$, $p \nmid M$, then π_p is of type IIa.

iii) If $p|M$, $p \nmid M_1$, then π_p is of type VIa. The associated character σ_p is trivial if $\delta_p = -1$ and unramified quadratic if $\delta_p = 1$.

iv) If $p|M_1$, then π_p is of type VIb. The associated character σ_p is trivial if $\delta_p = -1$ and unramified quadratic if $\delta_p = 1$.

Let $f = f_{g_1,g_2;M_1}$ be a Yoshida lift, as above. Then $S_\pi = 4$. It is of interest to see what Theorem 4.8 gives us in this case. Indeed, in this case Conjecture 1.11 has been proved already, so we get an *unconditional statement*.

4.10 Proposition. *Let N_1, N_2 be two positive, squarefree odd integers such that $N = \mathrm{lcm}(N_1, N_2)$ and $M = \gcd(N_1, N_2) > 1$. Let g_1 be a classical newform of weight $2k-2$ (k even) and level N_1 and g_2 be a classical newform of weight 2 and level N_2, such that g_1 and g_2 are not multiples of each other. Assume that for all primes p dividing M the Atkin-Lehner eigenvalues of g_1 and g_2 coincide. Let M_1 be any divisor of M with an* odd *number of prime factors, and let $f = f_{g_1,g_2;M_1}$ be a corresponding Yoshida lift. Then, for any $d < 0$ a fundamental discriminant such that $\left(\frac{d}{p}\right) = -1$ for all $p|N$, and any ideal class character Λ of $K = \mathbb{Q}(\sqrt{d})$, we have*

$$\frac{|R(f,K,\Lambda)|^2}{\langle f,f\rangle} = \begin{cases} \dfrac{2^{4k-6}\pi^{2k+2}w(K)^2|d|^{k-1}}{(2k-2)!} \times \prod\limits_{p|N}\left(2(1+p^{-2})(1+p^{-1})\right) & \\ \times \dfrac{L_{\mathrm{f}}(1/2, \pi_{g_1} \times \mathcal{AI}(\Lambda^{-1})) L_{\mathrm{f}}(1/2, \pi_{g_2} \times \mathcal{AI}(\Lambda^{-1}))}{L_{\mathrm{f}}(1, \pi_{g_1}, \mathrm{Ad}) L_{\mathrm{f}}(1, \pi_{g_2}, \mathrm{Ad}) L_{\mathrm{f}}(1, \pi_{g_1} \times \pi_{g_2})} & \text{if } N_1 = N_2 = M = M_1, \\ 0 & \text{otherwise.} \end{cases} \tag{92}$$

In the special case $\Lambda = 1$, the above result can probably also be derived from the formulas in [3].

Averages of L-functions

For each $e|N$, let $\mathcal{B}_{k,e}^{\mathrm{new,T}}$ be an orthogonal basis of $S_k(\Gamma_0(e))^{\mathrm{new,T}}$ consisting of newforms. Put

$$\mathcal{B}_{k,N}^{\mathrm{T}} = \bigcup_{abcde=N} \{\delta_{a,b,c,d}(g) : g \in \mathcal{B}_{k,e}^{\mathrm{new,T}}\}.$$

Then by Corollary 4.4, $\mathcal{B}_{k,N}^{\mathrm{T}}$ is an orthogonal basis for $S_k(\Gamma_0(N))^{\mathrm{T}}$. Each $f \in \mathcal{B}_{k,N}^{\mathrm{T}}$ is a Hecke eigenform at places not dividing N, and gives rise to an irreducible cuspidal automorphic representation π_f.

4.11 Theorem. *Let $k \geq 6$ be even and N squarefree and odd. Let $\mathcal{B}_{k,N}^{\mathrm{T}}$ be as above. Suppose that Conjecture 1.11 holds. Fix a fundamental discriminant $d < 0$ and an ideal class character Λ of $K = \mathbb{Q}(\sqrt{d})$. Suppose that $\left(\frac{d}{p}\right) = -1$ for all $p|N$. Put $l = 1$ if $\Lambda^2 = 1$ and $l = 2$ otherwise. Then*

$$\sum_{f \in \mathcal{B}_{k,N}^{\mathrm{T}}} \frac{L_{\mathrm{f}}(1/2, \pi_f \times \mathcal{AI}(\Lambda^{-1}))}{L_{\mathrm{f}}(1, \pi_f, \mathrm{Ad})} \cdot J_{f,N} \cdot u_f = \frac{k^3\ L_{\mathrm{f}}(1, \chi_d)\ [\mathrm{Sp}_4(\mathbb{Z}) : \Gamma_0(N)]}{2\ l\ \pi^7} \Big(1 + O(k^{-2/3} N^{-1})\Big), \quad (93)$$

where $J_{f,N}$ is as defined in (90) and $u_f = 1/2$ if f is a weak Yoshida lift, and $u_f = 1$ otherwise. The implied constant in O depends only on d.

Proof. Let

$$t_{k,d} = \frac{2^{4k-5} \cdot \pi^{2k+2}}{(2k-2)!}\ w(K)^2\ |d|^{k-1}.$$

Assuming Conjecture 1.11, our main theorem above says

$$\frac{|R(f, K, \Lambda)|^2}{\langle f, f \rangle} = t_{k,d} \cdot u_f \cdot \frac{L_{\mathrm{f}}(1/2, \pi_f \times \mathcal{AI}(\Lambda^{-1}))}{L_{\mathrm{f}}(1, \pi_f, \mathrm{Ad})} J_{f,N}.$$

On the other hand, by Theorem 7.3 and Corollary 9.4 of [8], we know that

$$\sum_{f \in \mathcal{B}_{k,N}^{\mathrm{T}}} \frac{2\pi^7 l}{k^3 t_{k,d}\ [\mathrm{Sp}_4(\mathbb{Z}) : \Gamma_0(N)] L_{\mathrm{f}}(1, \chi_d)} \frac{|R(f, K, \Lambda)|^2}{\langle f, f \rangle} = 1 + O_d(N^{-1} k^{-2/3}).$$

This completes the proof. ∎

If we put $N = 1$ in (93), we get the result (14) stated in the introduction. Furthermore, we can use the above theorem and sieve for various divisors of N to get a version of (93) that only sums over newforms, i.e., elements of $\mathcal{B}_{k,N}^{\mathrm{new,T}}$. We state and prove such a result in the simplest case of prime modulus.

4.12 Corollary. *Let p be an odd prime and let $\mathcal{B}_{k,p}^{\mathrm{new,T}}$ be an orthogonal basis of $S_k(\Gamma_0(p))$ consisting of newforms. Let $\Phi_{k,p}^{\mathrm{new,T}}$ be the set of distinct irreducible subspaces[12] of $L^2(Z(\mathbb{Q})G(\mathbb{Q})\backslash G(\mathbb{A}))$ generated by the adelizations of elements of $\mathcal{B}_{k,p}^{\mathrm{new,T}}$.*

Suppose that Conjecture 1.11 holds. Fix a fundamental discriminant $d < 0$ such that $\left(\frac{d}{p}\right) = -1$, and an ideal class character Λ of $K = \mathbb{Q}(\sqrt{d})$. Put $l = 1$ if $\Lambda^2 = 1$ and $l = 2$ otherwise. Then

$$\sum_{\substack{\pi \in \Phi_{k,p}^{\mathrm{new,T}} \\ \pi_p \in \{\mathrm{IIIa,\ VIb}\}}} \frac{L_{\mathrm{f}}(1/2, \pi \times \mathcal{AI}(\Lambda^{-1}))}{L_{\mathrm{f}}(1, \pi, \mathrm{Ad})}\ u_\pi = \frac{k^3 p^3 L_{\mathrm{f}}(1, \chi_d)}{2\ l\ \pi^7} + O_d(k^{7/3} p^2 + k^3),$$

where $u_\pi = 1/2$ if π is a weak endoscopic lift, and $u_\pi = 1$ otherwise.

[12]The "multiplicity one" conjecture predicts that two such subspaces are distinct if and only if the corresponding automorphic representations are non-isomorphic; however we do not assume the truth of this conjecture here.

Proof. This follows from Theorem 4.11 for $N = 1$ and $N = p$, combined with the observation that for any $\pi \in \Phi_{k,p}^{\mathrm{new,T}}$ with π_p of Type IIIa or VIb, we have $\sum_{\substack{f\in\mathcal{B}_{k,p}^{\mathrm{new,T}}\\ \pi_f=\pi}} J_{f,p} = 2 + O(p^{-1})$, a fact that follows from Remark 3.14. ∎

Size of Fourier coefficients

Another interesting consequence of Theorem 4.8 is that it allows us to *predict* the best possible upper bound for $|R(f, K, \Lambda)|$.

4.13 Proposition. *Let f, π, d, K, Λ be as in Theorem 4.8. Assume the truth of both Conjecture 1.11 and the Generalized Riemann Hypothesis for π. Then*

$$|R(f, K, \Lambda)| \ll_\epsilon \langle f, f\rangle^{1/2}(2\pi e)^k\ k^{-k+\frac{3}{4}}|d|^{\frac{k-1}{2}}(Nkd)^\epsilon.$$

Proof. This follows from Theorem 4.8, the bound (91), and Stirling's formula. Note that the GRH for π implies that the quotient of the various finite parts of L-functions appearing in 4.8 is bounded by $(Nkd)^\epsilon$. ∎

Unfortunately, the above proposition does not seem to predict the strongest expected bound on an *individual* Fourier coefficient $|a(F, S)|$. That would require knowledge of the distribution of the size and arguments of the complex numbers $R(f, K, \Lambda)$ as Λ varies over characters of Cl_K, which appears to be a difficult problem. We note here that a conjecture of Resnikoff and Saldana [31] predicts that as d varies, we have $|a(F, S)| \ll_F d^{k/2-3/4+\epsilon}$ for any S with $\mathrm{disc}(S) = d$. The bound of Proposition 4.13 and the conjectured bound of Resnikoff and Saldana do not appear to imply each other in any direction without assuming equidistribution of the arguments of the Fourier coefficients.

Integrality of L-values

If $L(s, \mathcal{M})$ is an L-series associated to an arithmetic object $\mathcal{M}$, it is of interest to study its values at certain critical points $s = m$. For these critical points, conjectures due to Deligne predict that $L(m, \mathcal{M})$ is the product of a suitable transcendental number Ω and an algebraic number. One can go even further and try to predict what primes divide the numerator and denominator of the algebraic number above (once the period is suitably normalized). This goes beyond Deligne's conjecture and is related to the Bloch-Kato conjectures. Our main theorem implies the following result in the spirit of the above conjectures.

4.14 Proposition. *Let k be even, N squarefree and odd, and $f \in S_k(\Gamma_0(N))^{\mathrm{T}}$ be a newform such that all its Fourier coefficients are algebraic integers.*[13] *Let π be the representation attached to f, and assume that π_p is of type* IIIa *or* VIb *for all $p|N$. Assume the truth of Conjecture 1.11 for π. Then, for all fundamental discriminants $d < 0$ such that $\left(\frac{d}{p}\right) = -1$ for all $p|N$ and all ideal class characters Λ of $K = \mathbb{Q}(\sqrt{d})$, we have*

$$\langle f, f\rangle \times \frac{L_{\mathrm{f}}(1/2, \pi_f \times \mathcal{AI}(\Lambda^{-1}))}{L_{\mathrm{f}}(1, \pi_f, \mathrm{Ad})} \times \frac{3^2\ 2^{4k-3}\pi^{2k+2}\ |d|^{k-1}\sigma_0(N)\sigma_1(N)\sigma_2(N)}{(2k-2)!\ N^3}$$

is an algebraic integer, where $\sigma_i(N) = \sum_{1\le d|N} d^i$.

Proof. This follows from Theorem 4.8 as our assumptions imply that $R(f, K, \Lambda)$ is an algebraic integer. ∎

[13] Using essentially the same method as Prop. 3.18 of [35], it can be shown that the space $S_k(\Gamma_0(N))^{\mathrm{new,T}}$ has an orthogonal basis consisting of newforms whose Fourier coefficients are algebraic integers.

4.15 Remark. *The above proposition shows that we should expect* $\pi^{2k+2} \cdot \langle f, f\rangle \cdot \frac{L_{\mathrm{f}}(1/2,\pi_f \times \mathcal{AI}(\Lambda^{-1}))}{L_{\mathrm{f}}(1,\pi_f,\mathrm{Ad})}$ *to be an algebraic number for all newforms* f *in* $S_k(\Gamma_0(N))^{\mathrm{T}}$ *with algebraic coefficients. To the best of our knowledge, even this weaker assertion is not proven at the present, except in the very special case of a Yoshida lift, where it follows by combining the results of [41] and [35].*

Proof of Proposition 1.15. We first claim that there exists a Yoshida lift f of g_1, g_2 whose Fourier coefficients are algebraic *integers* in the CM-field generated by g_1, g_2. Indeed, by Theorem 6 of [35], it follows that such a lift exists whose Fourier coefficients are algebraic *numbers* in the field generated by g_1, g_2. Now, the main result of [40] assures us that one can always multiply any Siegel modular form (of any level) with algebraic Fourier coefficients with a large enough integer such that the Fourier coefficients are algebraic integers. This proves the claim.

Given such a lift f, we now apply Proposition 4.10. Note that $|R(f, K, \Lambda)|^2$ is a totally positive algebraic integer in the field generated by g_1, g_2 and the values of Λ (indeed the latter field is a CM-field, so the absolute-value squared of any element of this field is totally positive). We can absorb all the remaining terms appearing in Proposition 4.10 that depend only on f, into the constant Ω. This immediately yields the proof.

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