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Explicit Representation of Cost-Efficient Strategies

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1 Outline

- Part 1: Optimal strategies for law-invariant preferences
  - · You only care about the **distribution** of final wealth
  - · e.g. "Tail=1 and Head=0" $\longleftrightarrow$ "Tail=0 and Head=1"
- Part 2: Optimal strategies with additional state-dependent constraints
  - · You also care about the states of the world where wealth is received
  - · e.g. Money has "more value" in a crisis

2 Main Results (1)

- Part 1: Optimal strategies for law-invariant preferences
  - · Are always simple (“two fund theorem”)
  - · Are increasing in the “market asset”
  - · Perform badly in crisis situations and add to systemic risk

3 Main Results (2)

- Part 2: Optimal strategies with additional state-dependent constraints
    - Conditionally increasing in “market asset” (“three fund theorem”)
    - Able to cope with crisis regimes

4 Notations and Assumptions

4.1 Black-Scholes Market

• There is a bank account earning a constant risk free rate $r > 0$. Its current value B_0 is known and for its future (fixed) value B_T we have that $B_T = B_0 \cdot e^{rT}$.

• There is a risky asset. Its current price S_0 is known and for its future (random) value S_T we have that $S_T = S_0 \cdot e^{R(t)}$ with stochastic return distributed (under $\mathbb{P}$) as

$$R(t) \sim N(\mu t, \sigma^2 t).$$

• Consider a contract with payoff Y_T at maturity time T: if Y_T depends on the final asset value S_T only, then Y_T is **path-independent.** Otherwise it is path-dependent.

• *Examples of path-independent payoffs*

- $Y_T = Max[S_T - D, 0]$ (Plain vanilla call)

- $Y_T = \alpha_1 \frac{S_T}{S_0} + \alpha_2 e^{rT}$ (Buy and hold)

• *Examples of path-dependent payoffs*

- $Y_T = Max[\frac{1}{n}\sum_{i=1}^{n} S_{t_i} - D, 0]$ with $0 \leq t_i \leq T$.(Asian call option)

- $Y_T = \sum_{i=0}^{n-1} \alpha_i \frac{S_T}{S_{t_i}}$ (Dollar cost averaging)

- Very complicated $Y_T \longleftrightarrow$ 'Clickfunds', 'CPPI strategies',...

4.2 Pricing

- Let $C(\cdot)$ be the cost functional. The market is arbitrage-free.

- Cost at $t = 0$ of a random payoff Y_T must be equal to

$$C(Y_T) = e^{-rT} E_{\mathbb{Q}} [Y_T] \, ,$$

where the expectation is taken with respect to a, so-called risk-neutral, probability measure $\mathbb{Q}$.

• Important observation:

$$\boxed{C(Y_T) = E_{\mathbb{Q}}\left[e^{-rT} Y_T\right],}$$

is equivalent to

$$\boxed{C(Y_T) = E_{\mathbb{P}}\left[\xi_T Y_T\right]},$$

where ξ_T is the stochastic discount factor (deflator, state-price density, pricing kernel, Radon-Nikodym derivative).

• In our setting it shows that

$$\xi_T = a.S_T^{-b},$$

for some (known) coefficients $a, b > 0$.

5 Law-Invariant preferences

- In this case investors only care about the distribution of final wealth.

- e.g. expected utility: E[u(Y_T)]=E[u(Z_T)] if $Y_T \sim Z_T$ (under $\mathbb{P}$)

- Most decision theories exhibit law invariance: Expected Utility theory, Mean-Variance optimization, Distortion Risk Measures (Yaari), Target Probability Maximization, Cumulative Prospect Theory (Kahneman & Tversky),...

- How making payoffs better?

6 A First Way to Improve Payoffs

• A given payoff Y_T can be dominated by a new payoff H_T^* defined as

$$\boxed{H_T^* = E_{\mathbb{P}}\left[Y_T | S_T\right],}$$

• Indeed:

- Clearly, H_T^* is less risky than Y_T (it has less spread)
- Surprisingly, H_T^* has the same cost as Y_T.

• Note that by construction H_T^* is a plain function of the market asset S_T and is path-independent.

7 Example - Asian Option

- Consider the initial path-dependent payoff Y_2 :

$$\boxed{Y_2 = (S_1 S_2)^{\frac{1}{2}}}$$

  - Y_2 is path-dependent.
  - Y_2 is lognormally distributed ($\mathbb{P}$ and $\mathbb{Q}$).

- Consider now the payoff $H_2^* = E_{\mathbb{P}}[Y_2|S_2]$:

$$\boxed{H_2^* = (S_2)^{\frac{3}{4}} e^{\frac{1}{16}\sigma^2}}$$

  - H_2^* is path-independent.
  - H_2^* is lognormally distributed ($\mathbb{P}$ and $\mathbb{Q}$).

H_2^* less risky than Y_2?

- Under the physical measure $\mathbb{P}$ we find that

  - $Y_2 \stackrel{d}{=} \mathsf{LN}(\frac{3}{2}\mu, \frac{5}{4}\sigma^2)$
  - $H_2^* \stackrel{d}{=} \mathsf{LN}(\frac{3}{2}\mu + \frac{1}{16}\boldsymbol{\sigma}^2, \frac{5}{4}\sigma^2 - \frac{1}{8}\boldsymbol{\sigma}^2)$
  - It is clear that H_2^* is 'less risky' than Y_2.

H_2^* same cost than Y_2?

- For the costs we find that

$$C(Y_2) = C(H_2^*) = e^{-\frac{1}{2}r - \frac{1}{8}\sigma^2}$$

8 A Second Way to Improve Payoffs

• Let Y_T be distributed with F and denote by F_{S_T} the distribution for S_T. Then, Y_T can be dominated by a new payoff Z_T^* defined as

$$\boxed{Z_T^* = F^{-1}(F_{S_T}(S_T)),}$$

• Indeed:

- Z_T^* is distributed with F as well.
- Z_T^* has lower cost than Y_T. (sketch proof on next slide)

• Hence, all decision makers (with increasing preferences) prefer Z_T^* above Y_T.

• Note that Z_T^* is an increasing function of the market asset S_T and thus also path-independent.

9 Sketch Proof (1)

• Remark that we want the optimal strategy Y_T to have a distribution F. Hence it writes as

$$Y_T = F^{-1}(U),$$

for some standard uniform random variable U that we need choose in a clever way...

10 Sketch Proof (2)

- We also know the distribution for ξ_T. Hence

$$\begin{aligned}\underset{Y_T\sim F}{\text{Min}}\left(E_{\mathbb{P}}\left[\xi_T Y_T\right]\right) &= \underset{Y_T\sim F}{\text{Min}}\left(\frac{E_{\mathbb{P}}\left[\xi_T Y_T\right]-E_{\mathbb{P}}\left[\xi_T\right]E_{\mathbb{P}}\left[Y_T\right]}{\text{Stdev}_{\mathbb{P}}[\xi_T]\text{Stdev}_{\mathbb{P}}[Y_T]}\right)\\ &= \underset{Y_T\sim F}{\text{Min}}\left(\text{Corr}_{\mathbb{P}}[\xi_T, Y_T]\right)\end{aligned}$$

Clearly, minimal correlation occur when $Y_T = F^{-1}(U)$ and thus also U is decreasing in ξ_T. Hence, U must write as $1 - F_{\xi_T}(\xi_T)$ and the optimal choice for Y_T becomes

$$\begin{aligned}Z_T^* &= F^{-1}(1 - F_{\xi_T}(\xi_T))\\ &= F^{-1}(F_{S_T}(S_T)).\end{aligned}$$

11 Example - Geometric Asian Option

•Now we have

$$\boxed{Z_2^* = (S_2)^{\frac{5}{8}} e^{(\frac{3}{2} - 2\frac{5}{8})\mu}},$$

• we find that

∘ $Z_2^* \stackrel{d}{=} Y_2 \stackrel{d}{=} \mathsf{LN}(\frac{3}{2}\mu, \frac{5}{4}\sigma^2)$

∘ $\frac{C(Z_2^*)}{C(Y_2)} < 1.$

12 Drawback cost-efficient payoffs

• They are increasing in "the market" (S_T). When market are bullish they perform nicely but in **bear markets they provide poor performance**.

• In practice, investments strategies set-up by investors are inspired by law-invariant frameworks. they are thus to a great extent correlated with the market as well ("we are all long with the market"):

∘ During a crisis, values of most investment funds decline

∘ Systemic risk

13 State-dependent Preferences (1)

- Problem considered so far:

  - $\underset{Y_T \sim F}{\text{Min}} \left(E_{\mathbb{P}}\left[\xi_T Y_T\right]\right)$

  - Payoff that solves this problem is cost-efficient

- New problem

  - $\underset{Y_T \sim F, \mathbb{S}}{\text{Min}} \left(E_{\mathbb{P}}\left[\xi_T Y_T\right]\right)$

  - Here $\mathbb{S}$ is a set of constraints on interaction between S_T and Y_T

  - Payoff that solves this problem is **constrained cost-efficient**

14 State-dependent Preferences (2)

- Examples:

  - We want $\Pr(Y_T > 100 | S_T < 80) \geq 0.8$

  - We want Y_T to be independent of S_T when $S_T < 80$.

- Approach amounts to imposing constraints on the dependence (copula) between S_T and Y_T.

- Technically, we need to find bounds on copulas...(we have paper in JAP, extending earlier works of respectively Nelsen, Tankov and Rüschendorf).

- Cost-efficient strategies are the ones that yield maximal dependence with S_T while satisfying the constraints $\mathbb{S}$.

  - No constraints: $Z_T^* = f(S_T)$ with f increasing (two-fund theorem).

  - Constraints $\mathbb{S}$: $Z_T^* = f(S_t, S_T)$ with f increasing in S_T given S_t.

    - Z_T^* is still rather simple (three-fund theorem)

References

[1] Dybvig, P, H., 1988. Inefficient Dynamic Portfolio Strategies or How To Throw Away a Million Dollars in the Stock Market., The Review of Financial Studies.

[2] Vanduffel, S., Chernih, A., Maj, M., Schoutens, W., 2008, "A note on the suboptimality of path-dependent payoffs in Lévy markets", Applied Mathematical Finance.

[3] Bernard, C., Jiang, X., Vanduffel, S., 2012. "Note on Improved Frechet bounds and model-free pricing of multi-asset options", Journal of Applied Probability.

[4] Bernard, C., Vanduffel, S., 2012, "Financial Bounds for Insurance Claims", Journal of Risk and Insurance.

[5] Bernard, C., Boyle, P., Vanduffel, S., 2011, "Explicit Characterisation of cost-efficient strategies", Working paper.