# EXPLICIT SOLUTIONS OF OPTIMIZATION MODELS AND DIFFERENTIAL GAMES WITH NONSMOOTH (ASYMMETRIC) REFERENCE-PRICE EFFECTS

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Models in marketing with asymmetric reference effects lead to nonsmooth optimization problems and differential games which cannot be solved using standard methods. In this study, we introduce a new method for calculating explicitly optimal strategies, open-loop equilibria, and closed-loop equilibria of such nonsmooth problems. Application of this method to the case of asymmetric reference-price effects with loss-aversive consumers leads to the following conclusions: (1) When the planning horizon is infinite, after an introductory stage the optimal price stabilizes at a steady-state price, which is slightly below the optimal price in the absence of reference-price effects. (2) The optimal strategy is the same as in the symmetric case, but with the loss parameter determined by the initial reference-price. (3) Competition does not change the qualitative behavior of the optimal strategy. (4) Adopting an appropriate constant-price strategy results in a minute decline in profits.

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## 1. INTRODUCTION

The study of optimal dynamic strategies in economics and in marketing has a long history. A unique feature of various models in marketing is an inherent asymmetry in the effect of a perception of *gains* versus that of *losses*. This asymmetry in consumers' behavior, which is predicted by prospect theory (Kahneman and Tversky 1979) and confirmed in numerous experimental studies, implies that the corresponding optimization problem is nonsmooth and therefore cannot be solved using standard optimization methods. Although there is a large body of literature on optimal control and differential games in economics, very little is known about obtaining explicit solutions when the model is nonsmooth. As a result, in many such cases "an explicit solution … did not seem possible" (Kopalle and Winer 1996), and calculations of optimal strategies were limited to numerical simulations using dynamic programming (e.g., Greenleaf 1995, Kopalle et al. 1996, Kopalle and Winer 1996).

In this study, we present a new method for obtaining explicit solutions to various nonsmooth optimization problems which arise in quantitative marketing research. Although we mainly focus on asymmetric reference-price effects, our method can be applied to other nonsmooth models in marketing, such as promotion (Lattin and Bucklin 1989) and product quality (Kopalle and Winer 1996). The purpose of this paper is thus twofold: (1) To present a new method for obtaining explicit solutions to optimization problems with nonsmooth references, and (2) to analyze the effect of asymmetric reference-price effects on optimal strategies.

Reference price is the price consumers have in mind and to which they compare the shelf price of a specific product. The main concept behind reference-price effects is that differences between the reference price $r$ and the shelf price $p$ affect the demand for that product. Specifically, when $r > p$, consumers are likely to sense a *gain*, or a "deal" that will lead to increased demand, and when $r < p$, consumers are likely to sense a *loss* that will have a negative effect on demand. The asymmetry in reference price effects comes from the fact that consumers are typically loss-aversive, i.e., the effect of losses on their demand is greater than that of gains (e.g., Weiner 1986, Lattin and Bucklin 1989, Rajendran and Tellis 1994, Mazumdar and Papatla 2000). Because reference price is formed through past exposures of consumers to the product price, the mathematical model for the optimal strategy leads to a nonsmooth optimal control problem for a monopoly and to a nonsmooth differential game in the case of competition.

In this study, we introduce a two-stage method for solving this nonsmooth optimization problem, as follows. In the first stage, we consider the *symmetric* case when the effect of *gains* is equal to that of *losses*. In this case, the model is smooth, and we can use standard techniques to calculate the optimal strategy. Then in the second stage, we use the solution obtained in the first stage to construct the solution for the loss-aversive (nonsmooth) case. In addition, we prove that this solution is the global maximizer of the nonsmooth optimization problem.

The mathematical model of symmetric reference-price effects bears many similarities to the model of symmetric sticky-price effects, i.e., when market price does not adjust instantaneously to changes in quantities supplied. Indeed, our solutions of open-loop and closed-loop equilibria in the presence of *symmetric* reference-price effects (i.e., the first stage) are similar to the sticky-price solutions of Fershtman and Kamien (Fershtman and Kamien 1987, 1990). The methodological contribution of this study is thus in the extension of these techniques to nonsmooth models. For clarity of presentation, we first present the two-stage method for the case of a monopoly over an infinite planning horizon (§3). We then show how this method can be extended to a finite planning horizon (§4), and to the case of oligopolistic competition for both open-loop and closed-loop equilibria (§5).

The calculation of explicit solutions provides a full characterization of optimal strategies during the product lifecycle. When the planning horizon is infinite, after an introductory stage, the optimal price reaches a steady-state value, which is slightly below the one in the absence of reference-price effects. The optimal pricing strategy at the introductory stage can be "penetration" or "skimming" (i.e., below or above the steady-state level, respectively), depending on whether the initial reference-price is below or above the steady-state level, respectively. When the planning horizon is finite, there is a third stage toward the end of the planning horizon where the optimal price declines.

The availability of explicit solutions results in a considerable simplification in the analysis of the optimal strategies, because one can analyze the *solution* rather than the *equation*. Therefore, we are able to gain new insight into the characteristics of the optimization problem and solution, which is hard, or even impossible, to obtain directly from the model equations and/or from numerical simulations. Thus, for example, *we show that when consumers are loss aversive, asymmetry has no effect on the optimal strategy or on profits*, in the sense that they are equivalent to those in the symmetric case. Furthermore, competition does not lead to a qualitative change in the optimal strategy (e.g., to a cyclic strategy). In addition, we show that the loss of potential profits under a fixed-price strategy (also known as EDLP (Every Day Low Price)), is quite small (§6). This observation has considerable implications for managers and provides strong support for the constant wholesale price strategy used by Procter & Gamble or General Mills (Triplett 1994).

In §7 we briefly discuss the validity of the results of this study for general demand functions. In particular, we consider the case where the asymmetric loss-aversion function is smooth, rather than piecewise-linear. We conclude by showing how our theoretical results can be applied to a real-life situation (§8).

## 2. REFERENCE PRICE AND DEMAND

### 2.1. Reference-Price Formation

A reference price can be defined as an internal price to which consumers compare the observed price. It is constructed by consumers through personal shopping experience and exposure to price information. Most studies assume that the reference price is a weighted average of the historical price exposures of the consumer. For example, Sorger (1988) and Kopalle and Winer (1996) model the reference price $r(t)$ as a continuous weighted average of past prices with an exponentially decaying weighting function. Therefore,

$$r(t) = \beta e^{-\beta t} \int_{-\infty}^{t} e^{\beta s} p(s)\, ds, \tag{1}$$

where $\beta$ is the continuous "memory parameter." An immediate consequence of Equation (1) is that reference-price formation is given by the ordinary differential equation

$$r' = \beta(p - r), \tag{2}$$

where

$$' = (d/dt).$$

In this paper, we calculate the optimal pricing strategy over a planning interval $0 \leqslant t \leqslant T$, i.e., from $t = 0$ up to the planning horizon $T$. This problem can arise when a new product is introduced into the market, or when a retailer decides to start a new pricing policy at time $t = 0$. In both cases, the initial reference price $r_0 = r(0)$ is one of the external parameters of the model, which can be used to replace definition (1) with

$$r(t) = e^{-\beta t}\left[r_0 + \beta \int_0^t e^{\beta s} p(s)\, ds\right], \quad 0 \leqslant t. \tag{3}$$

### 2.2. Effect of Reference Price on Demand

As in Greenleaf (1995) and Kopalle et al. (1996), we assume that the demand function in the absence of reference-price effects is given by

$$Q_{\text{no-ref}} = a - \delta p, \quad a, \delta > 0, \tag{4}$$

and that in the presence of reference-price effects, the demand function becomes

$$Q(t) = a - \delta p(t) - \gamma[p(t) - r(t)], \quad \gamma > 0. \tag{5}$$

In other words, we assume that in absence of reference price the demand $Q_{\text{no-ref}}$ is linearly decreasing in $p$, and that reference-price effects are linear in $p-r$ and additive such that demand decreases when $p>r$ and vice versa.

When $\gamma$ is constant, relation (5) is *symmetric* with respect to the effect of gains and losses. However, prospect theory (Kahneman and Tversky 1979) and empirical studies (Kalwani et al. 1990, Krishnamurthi et al. 1992, Lattin and Bucklin 1989, Briesch et al. 1997) suggest that the effect of losses on demand is larger than that of gains. To account for *loss aversion*, $\gamma$ in (5) is taken as

$$\gamma = \begin{cases} \gamma_{\text{gain}} & p \leqslant r \\ \gamma_{\text{loss}} & p > r \end{cases} \qquad \text{with} \quad \gamma_{\text{gain}} \leqslant \gamma_{\text{loss}}. \tag{6}$$

In §7 we consider the case of a more general demand function.

## 3. TWO-STAGE METHOD

In this section we introduce a two-stage method for calculating explicitly the optimal pricing strategy in the presence of asymmetric reference-price effects. This method allows us to overcome the inherent difficulty in this problem, namely, the nonsmoothness of the optimization problem. To maintain a clear presentation, we begin with the case of a monopolistic retailer.

The overall profit of a monopolistic firm between $t=0$ and $t=T$ is given by

$$\Pi^T[p(t)] = \int_0^T e^{-\alpha t}[p(t)-c]Q(t)\,dt, \tag{7}$$

where $\alpha$ is the discount rate, $c$ is the production cost per unit, $Q$ is given by (5), $\gamma$ is given by (6), and $r(t)$ is given by (3). The optimization problem for a monopolist can, therefore, be written as a variational problem: *Find the optimal pricing strategy* $p_{\text{optimal}}(t)$ *for* $0 \leqslant t \leqslant T$, *such that*

$$\Pi^T[p_{\text{optimal}}(t)] = \max_{p(t)} \Pi^T[p(t)]. \tag{8}$$

To solve the optimization problem (7)–(8), we can use relations (2) and (5) to rewrite Equation (7) as $\Pi^T = \int_0^T F(t,r,r')\,dt$, where the Lagrangian density $F$ is given in terms of $r$ and $r'$ by

$$F(t,r,r') = e^{-\alpha t}\left(r+\frac{r'}{\beta}-c\right)\left[a-\delta\left(r+\frac{r'}{\beta}\right)-\gamma\frac{r'}{\beta}\right]. \tag{9}$$

### 3.1. Stage I—Symmetric Effect of Gains and Losses ($\gamma_{\text{gain}} = \gamma_{\text{loss}}$)

We begin with the symmetric case $\gamma_{\text{gain}} = \gamma_{\text{loss}}$. In this case, $F$ is smooth and application of the Euler-Lagrange equation $\partial F/\partial r - ((d/dt)(\partial F/\partial r')) = 0$ to (9) yields

$$r'' - \alpha r' - \beta\frac{2\delta(\alpha+\beta)+\alpha\gamma}{2(\gamma+\delta)}r = -\beta\frac{(\alpha+\beta)(a+\delta c)+\alpha\gamma c}{2(\gamma+\delta)}. \tag{10}$$

The solution of Equation (10) is

$$r_{\text{optimal}}(t) = p^{ss}_{\text{optimal}} + M_1 e^{-m_1 t} + M_2 e^{m_2 t}, \tag{11}$$

where $r_{\text{optimal}}(t)$ is the reference price that corresponds to $p_{\text{optimal}}(t)$, and $M_1$ and $M_2$ are constants whose value will be determined shortly,

$$m_1 = \frac{\Delta-\alpha}{2}, \qquad m_2 = \frac{\Delta+\alpha}{2}, \qquad \Delta = \sqrt{\alpha^2 + 2\beta\frac{2\delta(\alpha+\beta)+\alpha\gamma}{\gamma+\delta}}, \tag{12}$$

and

$$p^{ss}_{\text{optimal}} = \frac{(\alpha+\beta)(a+\delta c)+\alpha\gamma c}{2\delta(\alpha+\beta)+\alpha\gamma}. \tag{13}$$

The boundary condition at $t=0$ is

$$r(0) = r_0. \tag{14}$$

Because there is no constraint at $t=T$, when the planning horizon is infinite ($T=\infty$), $r$ should satisfy the free-boundary condition

$$\lim_{t\to\infty} r(t) < \infty. \tag{15}$$

Because $0 < m_1 < m_2$, condition (15) implies that $M_2 = 0$. Therefore, the solution of Equation (10) with the two boundary conditions (14)–(15) is given by

$$r_{\text{optimal}}(t) = p^{ss}_{\text{optimal}} + (r_0 - p^{ss}_{\text{optimal}})e^{-m_1 t}. \tag{16}$$

By combining this result with (2), we see that the optimal pricing strategy is given by

$$p_{\text{optimal}}(t) = p^{ss}_{\text{optimal}} + (r_0 - p^{ss}_{\text{optimal}})\left(1-\frac{m_1}{\beta}\right)e^{-m_1 t}. \tag{17}$$

To the best of our knowledge, Equation (17) is the first explicit expression of an optimal pricing strategy in the presence of *symmetric* reference-price effects. The reason that such expressions have not been calculated before is probably that previous studies calculated the optimal pricing strategy using a discrete-time formulation (Greenleaf 1995, Kopalle et al. 1996). The discrete formulation is convenient for numerical simulations using dynamic programming but cumbersome for obtaining explicit solutions. The explicit expression (17) thus illustrates the advantage of using a continuous-time formulation in this problem.

In most applications, unless $F$ is convex in $(r, r')$, it is not possible to prove that the solution of the Euler-Lagrange equation is indeed a global, or even a local, maximizer. However, in our case we can prove this result rigorously.

PROPOSITION 1. *The function* $p_{\text{optimal}}(t)$, *given by* (17), *is the global maximizer of the optimization problem* (7)–(8), *i.e.,* $\Pi^\infty[p_{\text{optimal}}(t)] \geqslant \Pi^\infty[p(t)]$ *for all functions* $p(t)$.

PROOF. See Appendix A. □

### 3.2. Stage II—Asymmetric Effect of Gains and Losses ($\gamma_{\text{gain}} < \gamma_{\text{loss}}$)

When we allow for asymmetry of the effects of gains and losses on demand using relation (6), the Lagrangian density $F$ in (9) is nonsmooth. As a result, standard optimization techniques cannot be used. Nevertheless, we now show that *it is possible to calculate explicitly the optimal pricing strategy in the asymmetric case* $\gamma_{\text{gain}} < \gamma_{\text{loss}}$. To do so, we begin by considering the exponent $m_1$ and the steady-state price $p^{ss}_{\text{optimal}}$ as functions of $\gamma$.

Lemma 3.1. *Let* $p^{ss}_{\text{optimal}}(\gamma)$ *be defined by* (13). *Then* $p^{ss}_{\text{optimal}}(\gamma)$ *decreases monotonically in* $\gamma$.

Proof. This follows directly by differentiating $p^{ss}_{\text{optimal}}$ with respect to $\gamma$. □

Note that from Lemma 3.1 we have that $p^{ss}_{\text{optimal}}(\gamma_{\text{gain}}) > p^{ss}_{\text{optimal}}(\gamma_{\text{loss}})$ when $\gamma_{\text{gain}} < \gamma_{\text{loss}}$. We now show that the solution in the asymmetric case is given by an appropriately chosen solution in the symmetric case.

Proposition 2. *Let* $0 < \gamma_{\text{gain}} < \gamma_{\text{loss}}$ *and let* $T = \infty$. *The optimal pricing strategy for the asymmetric optimization problem* (7)–(8), *with* $\gamma$ *defined by* (6), *is given by Equation* (17), *where the value of* $\gamma$ *in* $m_1$ (*Equation* (12)) *and in* $p^{ss}_{\text{optimal}}$ (*Equation* (13)) *is given by*

$$\gamma = \begin{cases} \gamma_{\text{loss}} & r_0 < p^{ss}_{\text{optimal}}(\gamma_{\text{loss}}), \\ \tilde{\gamma} & p^{ss}_{\text{optimal}}(\gamma_{\text{loss}}) < r_0 < p^{ss}_{\text{optimal}}(\gamma_{\text{gain}}), \\ \gamma_{\text{gain}} & r_0 > p^{ss}_{\text{optimal}}(\gamma_{\text{gain}}), \end{cases}$$

*where* $\tilde{\gamma}$ *is the solution of* $p^{ss}_{\text{optimal}}(\tilde{\gamma}) = r_0$, *and thus* $\gamma_{\text{gain}} < \tilde{\gamma} < \gamma_{\text{loss}}$.

Proof. See Appendix C. □

The explicit solution obtained in Proposition 2 allows us to draw the following conclusions:

1. Because the optimal solution is equal to a solution of the symmetric case, we get the surprising result that, aside from the dependence of $\gamma$ on $\gamma_{\text{loss}}$ and $\gamma_{\text{gain}}$, *loss aversion has no effect on either the optimal pricing strategy or on profits*.
2. The optimal price increases monotonically when $r_0 < p^{ss}_{\text{optimal}}(\gamma_{\text{loss}})$ and decreases monotonically when $r_0 > p^{ss}_{\text{optimal}}(\gamma_{\text{gain}})$. When $p^{ss}_{\text{optimal}}(\gamma_{\text{loss}}) < r_0 < p^{ss}_{\text{optimal}}(\gamma_{\text{gain}})$, the optimal price is given by $p_{\text{optimal}}(t) \equiv r_0$ (see Figure 1).
3. From Propositions 1 and 2 it follows that *the solution calculated in Proposition* 2 *is the global maximizer of the nonsmooth optimization problem* (7)–(8).

### 3.3. Initial and Steady-State Stages

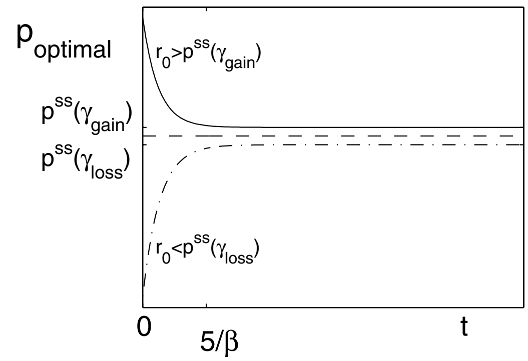
When the retailer follows the optimal pricing strategy (17), market demand is given by

$$Q_{\text{optimal}}(t) = a - \delta p^{ss}_{\text{optimal}} + D_1(r_0 - p^{ss}_{\text{optimal}})e^{-m_1 t}, \tag{18}$$

where

$$D_1 = (\delta + \gamma)\frac{m_1}{\beta} - \delta > 0. \tag{19}$$

**Figure 1.** The three cases for the optimal pricing strategy in the presence of asymmetric reference-price effects (Proposition 2).




*Note.* Initial stage of optimal pricing strategy is "skimming" when $r_0 > p^{ss}_{\text{optimal}}(\gamma_{\text{gain}})$ (solid line) and "penetration" when $r_0 < p^{ss}_{\text{optimal}}(\gamma_{\text{loss}})$ (dash-dot line). When $p^{ss}_{\text{optimal}}(\gamma_{\text{gain}}) > r_0 > p^{ss}_{\text{optimal}}(\gamma_{\text{loss}})$, the optimal strategy is $p \equiv r_0$ (dashed line).

From (17) we see that the optimal pricing strategy begins with an introductory stage that lasts several $1/m_1$ time units and is characterized by a monotonic change of the optimal price from $p_{\text{optimal}}(0)$ to $p^{ss}_{\text{optimal}}$. Because $0 \leqslant 1 - m_1/\beta \leqslant \gamma/(\gamma+\delta) \leqslant 1$, we see that $p_{\text{optimal}}(0)$ lies between $p^{ss}_{\text{optimal}}$ and $r_0$. The initial pricing strategy (penetration or skimming) depends on whether $r_0$ is higher or lower than $p^{ss}_{\text{optimal}}$ (Figure 1):

1. When $r_0 < p^{ss}_{\text{optimal}}$, the retailer faces a low initial reference price. Therefore, its goal is for consumers to grow quickly accustomed to the higher steady-state price $p^{ss}_{\text{optimal}}$, with minimal penalty in reduced demand along the way. In this case, under the optimal strategy, the penetration price $p_{\text{optimal}}(0)$ is set lower than $p^{ss}_{\text{optimal}}$. During the introductory stage the price increases monotonically, approaching $p^{ss}_{\text{optimal}}$. Throughout this stage the price is higher than the reference price, but the difference between the two prices vanishes in the steady-state stage. As a result, although the price increases, demand builds up during this stage (penetration).
2. When $r_0 > p^{ss}_{\text{optimal}}$, the retailer benefits from a high initial reference price. Therefore, its goal is to reach the lower steady-state price $p^{ss}_{\text{optimal}}$, while reaping maximal benefits from the increased demand along the way. In this case, under the optimal strategy, the initial price $p_{\text{optimal}}(0)$ is higher than $p^{ss}_{\text{optimal}}$. During the introductory stage the price decreases monotonically, approaching $p^{ss}_{\text{optimal}}$. Throughout this stage, the price remains below the reference price. The retailer, therefore, utilizes the high initial reference price to benefit from increased demand, coupled with higher initial prices (skimming).

After the introductory stage, the optimal price reaches the steady-state value $p^{ss}_{\text{optimal}}$. As a result, the reference price stabilizes at the same value and demand becomes constant. The steady-state price $p^{ss}_{\text{optimal}}$ can be written as a weighted average of $p^{\text{optimal}}_{\text{no-ref}}$ and the production cost per unit $c$:

$$p^{ss}_{\text{optimal}} = (1-\kappa_{\text{optimal}})p^{\text{optimal}}_{\text{no-ref}} + \kappa_{\text{optimal}} c,$$
$$\kappa_{\text{optimal}} = \frac{\alpha\gamma}{2\delta(\alpha+\beta)+\alpha\gamma}, \tag{20}$$

where $p^{\text{optimal}}_{\text{no-ref}}$ is the optimal price in the absence of reference-price effects (see §6.1). Because $0 \leqslant \kappa_{\text{optimal}} \leqslant 1$, the steady-state price $p^{ss}_{\text{optimal}}$ is below the optimal price in the absence of reference-price effects, i.e., $c < p^{ss}_{\text{optimal}} \leqslant p^{\text{optimal}}_{\text{no-ref}}$. In fact, because typically $\alpha \ll \beta$, we can conclude that $\kappa_{\text{optimal}} \ll 1$, and that $p^{ss}_{\text{optimal}}$ is only slightly below $p^{\text{optimal}}_{\text{no-ref}}$. Note that when either $\alpha \to 0$, or $\gamma \to 0$ or $\beta \to \infty$, $p^{ss}_{\text{optimal}}$ approaches $p^{\text{optimal}}_{\text{no-ref}}$.

## 4. FINITE PLANNING HORIZON

When the planning horizon is finite, the only difference in the model equations for the optimal pricing strategy is that condition (15) is replaced with the finite free-boundary condition (see, e.g., Kamien and Schwartz 1991):

$$F_{r'}(T, r(T), r'(T)) = 0. \tag{21}$$

The solution of Equation (10) with the two boundary conditions (14) and (21) is given by (11) with

$$\begin{aligned} M_1 = &\big(\beta[a+(\delta+\gamma)c - p^{ss}_{\text{optimal}}(2\delta+\gamma)] \\ &+ e^{m_2T}(p^{ss}_{\text{optimal}} - r_0)[2(\delta+\gamma)(m_2+\beta)-\gamma\beta]\big)\big/ \\ &\big([2(\delta+\gamma)(\beta-m_1)-\gamma\beta]e^{-m_1T} \\ &- [2(\delta+\gamma)(m_2+\beta)-\gamma\beta]e^{m_2T}\big), \\ M_2 = &\, r_0 - p^{ss}_{\text{optimal}} - M_1. \end{aligned} \tag{22}$$

Therefore, the optimal pricing strategy for a monopoly in the presence of symmetric reference-price effects over a finite planning interval is given by

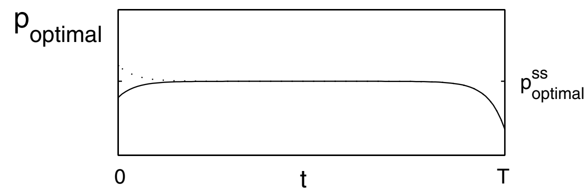
$$p_{\text{optimal}}(t) = p^{ss}_{\text{optimal}} + M_1\left(1-\frac{m_1}{\beta}\right)e^{-m_1 t} + M_2\left(1+\frac{m_2}{\beta}\right)e^{m_2 t}. \tag{23}$$

In Figure 2, we present the optimal solution (23) in a finite planning interval. As in the infinite-horizon case, the introductory stage is followed by a steady-state stage. In contrast to the infinite-horizon case, however, there is an additional third stage with decreasing price levels. Intuitively, the reason for this is that toward the end of the planning interval there is no need to make long-term reference-price considerations. Therefore the monopoly can reap maximal profits by lowering its prices to benefit from the increased demand.

## 5. OLIGOPOLISTIC COMPETITION

In this section, we extend the model to the case of oligopolistic competition of $N$ firms, possibly with different production costs. By applying the two-stage method,

**Figure 2.** The optimal strategy over a finite planning horizon (Equation (23)).




*Note.* Solid and dotted lines correspond to the cases $r_0 < p^{ss}_{\text{optimal}}$ and $r_0 > p^{ss}_{\text{optimal}}$, respectively.

we calculate the equilibrium strategies in the loss-aversive case, which turn out to have the same qualitative features as in the monopoly case. We conclude, therefore, that competition does not add a new qualitative feature to the model.

We denote the production rate at time $t$ of the $n$th firm by $q_n(t)$. The total production rate in the market is given by

$$Q(t) = \sum_{n=1}^{N} q_n(t). \tag{24}$$

As before, we assume that the total production rate is equal to the market demand rate. The total profit of the $n$th firm is given by

$$\pi_n(q_1(t), \ldots, q_N(t)) = \int_0^\infty e^{-\alpha t}[p(t)-c_n]q_n(t)\,dt, \tag{25}$$

where $c_n > 0$ is the production cost per unit of the $n$th firm, and the price-demand relation is determined from (5) and (6). Thus, whereas in the monopolistic model the decision variable can be either the price $p(t)$ or the production rate $Q(t)$, in the oligopolistic model the players' decision variables are their production levels $q_n(t)$ (Cournot competition).

### 5.1. Open-Loop Equilibrium

Let us calculate the equilibrium strategies for the case where every retailer knows the other retailers' strategies (open-loop equilibrium). The optimization problem for the $n$th retailer is

$$\max_{q_n(t)} \pi_n(q_1, \ldots, q_N), \tag{26}$$

subject to (2), (5)–(6), and (24), where $\{q_j\}_{j\neq n}$ is given.

As in the monopolistic case, since the optimization problem (26) is nonsmooth, we solve it using the two-stage method. We begin with the symmetric case.

PROPOSITION 3. *Let $\gamma_{\text{gain}} = \gamma_{\text{loss}}$ and $T = \infty$. Then, the open-loop equilibrium strategies for the optimization problem* (26), *subject to* (2), (5)–(6), *and* (24), *are given by*

$$q^n_{\text{open-loop}}(t) = \frac{1}{N}Q_{\text{open-loop}}(t) + (\bar{c}-c_n)\left[\delta + \frac{\alpha\gamma}{\alpha+\beta}\right], \quad n = 1, \ldots, N, \tag{27}$$

*where*

$$Q_{\text{open-loop}}(t) = a - \delta p^{ss}_{\text{open-loop}} + (r_0 - p^{ss}_{\text{open-loop}})\left((\delta+\gamma)\frac{m_1}{\beta} - \delta\right)e^{-m_1 t},$$

*the steady-state price is*

$$p^{ss}_{\text{open-loop}} = \frac{(\alpha+\beta)(a+\delta N\bar{c}) + \alpha\gamma N\bar{c}}{\delta(N+1)(\alpha+\beta)+\alpha N\gamma},$$

*the average production cost is given by* $\bar{c} = (\sum_{n=1}^{N} c_n)/N$, $m_1$ *is given by*

$$m_1 = \frac{\Delta-\alpha}{2} + \frac{\gamma\beta(N-1)}{2(\delta+\gamma)(N+1)},$$

*and*

$$\Delta = \sqrt{\left(\frac{\gamma\beta(N-1)}{(\delta+\gamma)(N+1)} - \alpha\right)^2 + 4\beta\frac{(\alpha+\beta)(N+1)\delta+\alpha N\gamma}{(\delta+\gamma)(N+1)}}.$$

*The corresponding price and reference-price trajectories are given by*

$$p_{\text{open-loop}}(t) = p^{ss}_{\text{open-loop}} + (r_0 - p^{ss}_{\text{open-loop}})\left(1 - \frac{m_1}{\beta}\right)e^{-m_1 t},$$

*and*

$$r_{\text{open-loop}}(t) = p^{ss}_{\text{open-loop}} + (r_0 - p^{ss}_{\text{open-loop}})e^{-m_1 t}.$$

PROOF. See Appendix D. □

From the explicit form of the solution in Proposition 3, we can draw the following conclusions:

1. The equilibrium strategies are qualitatively the same as in the monopolistic case. In particular,

(a) Because $m_1 > 0$, we see that equilibrium strategies stabilize on a constant steady-state, rather than a cyclic one. Thus, the addition of competition does not result in high-low strategies.

(b) When $r_0 > p^{ss}_{\text{open-loop}}$, then $p_{\text{open-loop}}(t)$ decreases in $t$. When, however, $r_0 < p^{ss}_{\text{open-loop}}$, then $p_{\text{open-loop}}(t)$ increases in $t$.

(c) The steady-state equilibrium price $p^{ss}_{\text{open-loop}}$ is a weighted average of the equilibrium price in the absence of reference-price effects $p^{\text{eq}}_{\text{no-ref}} = (a+\delta N\bar{c})/\delta(N+1)$ and the average production cost $\bar{c}$:

$$p^{ss}_{\text{open-loop}} = (1-\kappa_{\text{open-loop}})p^{\text{eq}}_{\text{no-ref}} + \kappa_{\text{open-loop}}\bar{c},$$
$$\kappa_{\text{open-loop}} = \frac{\alpha\gamma}{(\alpha+\beta)\delta((N+1)/N)+\alpha\gamma}.$$

Therefore, $\bar{c} \leqslant p^{ss}_{\text{open-loop}} \leqslant p^{\text{eq}}_{\text{no-ref}}$. In addition, when either $\alpha \to 0$, $\gamma \to 0$ or $\beta \to \infty$, then $p^{ss}_{\text{open-loop}} \to p^{\text{eq}}_{\text{no-ref}}$.

2. From (27) it immediately follows that firms with lower production costs per unit have higher equilibrium production paths, namely, if $c_i \geqslant c_j$ then $q^i_{\text{open-loop}}(t) \leqslant q^j_{\text{open-loop}}(t)$. In particular, when marginal costs of all firms are equal, i.e., $c_n \equiv \bar{c}$ for all $n$, then $q^n_{\text{open-loop}}(t) = (1/N)Q_{\text{open-loop}}(t)$ for all firms.

3. Clearly, when $N = 1$ the equilibrium strategy reduces to the optimal pricing strategy for a monopoly, which we calculated in §3.

To extend the results to the loss-aversive case, we consider the exponent $m_1$ and the steady-state price $p^{ss}_{\text{open-loop}}$ as functions of $\gamma$. Then, we can apply the same proof as in the monopolistic case (Appendix C), to prove the following result.

PROPOSITION 4. *Let* $0 < \gamma_{\text{gain}} < \gamma_{\text{loss}}$ *and* $T = \infty$. *Then, the open-loop equilibrium strategies for the optimization problem* (26), *subject to* (2), (5)–(6), *and* (24), *are given by* $q^n_{\text{open-loop}}(t)$ *in* (27), *where the value of* $\gamma$ *in* $p^{ss}_{\text{open-loop}}(\gamma)$ *and* $m_1(\gamma)$ *is given by*

$$\gamma = \begin{cases} \gamma_{\text{loss}} & r_0 < p^{ss}_{\text{open-loop}}(\gamma_{\text{loss}}), \\ \tilde{\gamma} & p^{ss}_{\text{open-loop}}(\gamma_{\text{loss}}) < r_0 < p^{ss}_{\text{open-loop}}(\gamma_{\text{gain}}), \\ \gamma_{\text{gain}} & r_0 > p^{ss}_{\text{open-loop}}(\gamma_{\text{gain}}), \end{cases}$$

*and* $\tilde{\gamma}$ *is the solution of* $p^{ss}_{\text{open-loop}}(\tilde{\gamma}) = r_0$.

## 5.2. Feedback (Subgame Perfect) Equilibrium

According to the definition of an open-loop equilibrium, retailers commit to their strategy and are not allowed to change it during the planning horizon. Thus, the open-loop concept is equivalent to finding the Nash equilibrium of a static game where the set of strategies for each firm is the set of all the possible production paths where each path is a time dependent function. Because of this precommitment feature, it is not clear whether the open-loop equilibrium can predict the behavior of a dynamic competition, where firms can be expected to adapt their strategies to changes in the market. A more realistic model is, therefore, the feedback equilibrium, where firms are allowed to change their strategies in response to changes in other firms strategies. This equilibrium is equivalent to a sub game perfect Nash equilibrium in a discrete game (Selten 1975) and satisfies the backward induction property.

The symmetric feedback equilibrium can be found by solving the Hamilton-Jacobi-Bellman equation. The calculations turn out to be similar to those of Fershtman and Kamien (1987) for the (smooth) sticky prices model. As in the open-loop case, this approach does not work when the asymmetry of consumers' response to gains and losses is included in the model, because then the corresponding optimization problem is nonsmooth. We shall see, however, that one can overcome this difficulty as before by applying the two-stage method: Calculate explicitly the equilibrium in the symmetric case, and then use this solution to find the equilibrium in the asymmetric case.

We recall that, in general, a firm's strategy at time $t$ depends on both the state variable $r(t)$ and on $t$. In the case of an infinite planning horizon, however, the equilibrium strategies depend only on $r(t)$ (see, e.g., Kamien and Schwartz 1991). Therefore, we can adopt the following definitions.

DEFINITION 5.1. The feedback strategy space is given by $\mathcal{Q}^{fb} = \{\hat{q}(r) \mid \hat{q}(r) \geqslant 0,\ \hat{q}(r) \text{ is continuous in } r\}$.

DEFINITION 5.2. An $N$ tuple of strategies $(\hat{q}_1(r), \ldots, \hat{q}_N(r))$, each in $\mathcal{Q}^{fb}$, is called a feedback equilibrium if for every $q_n \in \mathcal{Q}^{fb}$, $t_0$ and $r(t_0)$ and for every $n$, $n = 1, \ldots, N$,

$$\pi_n[\hat{q}_1, \ldots, \hat{q}_{n-1}, \hat{q}_n, \hat{q}_{n+1}, \ldots, \hat{q}_N;\ t_0, r(t_0)] \geqslant \pi_n[\hat{q}_1, \ldots, \hat{q}_{n-1}, q_n, \hat{q}_{n+1}, \ldots, \hat{q}_N;\ t_0, r(t_0)],$$

where $\pi_n[\hat{q}_1, \ldots, \hat{q}_{n-1}, q_n, \hat{q}_{n+1}, \ldots, \hat{q}_N;\ t_0, r(t_0)] = \int_{t_0}^{\infty} e^{-\alpha t}[p(t) - c] q_n \, dt$, and $p$ is determined from (5)–(6).

Let us denote by $V_n(t_0, r_{t_0}) = \pi_n[\hat{q}_1, \ldots, \hat{q}_N;\ t_0, r(t_0)]$ the total discounted profit of the $n$th firm under the equilibrium strategies $\{\hat{q}_n\}_{n=1}^N$ for $t_0 \leqslant t < \infty$ when the initial reference price is $r_{t_0} = r(t_0)$. Because the equilibrium feedback strategies do not depend on $t$, then $V_n = V_n(r_{t_0})$.

We now apply the two-stage method to find explicitly the feedback equilibrium strategies. To avoid technical complications we assume that the production costs are identical for all $N$ firms, i.e., $c_n = c$, $n = 1, \ldots, N$.

**5.2.1. Stage 1: The Symmetric Case.** In the symmetric case $\gamma_{\text{loss}} = \gamma_{\text{gain}}$, the optimization problem is smooth. Hence, the feedback equilibrium strategies satisfy the Hamilton-Jacobi-Bellman equations for infinite horizon planning (Starr and Ho 1969, Kamien and Schwartz 1991). Substituting $r' = \beta(p - r)$ in these equations gives

$$\alpha V_n(r) = \max_{q_n} \left\{ (p - c) q_n + \frac{dV_n(r)}{dr} \beta(p - r) \right\}. \qquad (28)$$

The solution of Equations (28) leads to the following result:

PROPOSITION 5. *The unique set of feedback equilibrium strategies is given by*

$$\hat{q}^n_{\text{feedback}}(r) = \frac{a - (\delta + \gamma)c - (\beta/N)B}{N+1} + \frac{1}{N+1}\left(\gamma - \frac{2A\beta}{N}\right) r, \quad n = 1, 2, \ldots, N, \qquad (29)$$

*where*

$$A = (\alpha + 2\beta)\frac{(\delta + \gamma)(N+1)^2}{8\beta^2 N} - \frac{\gamma}{2\beta} - \sqrt{\left((\alpha + 2\beta)\frac{(\delta + \gamma)(N+1)^2}{8\beta^2 N} - \frac{\gamma}{2\beta}\right)^2 - \frac{\gamma^2}{4\beta^2}}, \qquad (30)$$

*and*

$$B = \big(2(a - (\delta + \gamma)c)(A\beta(N-1) + N\gamma) + 2A\beta(a + (\delta + \gamma)Nc)(N+1)\big) \big/ \big((\delta + \gamma)(N+1)^2(\alpha + \beta) - 2\beta N(2A\beta + \gamma)\big). \qquad (31)$$

PROOF. See Appendix E. □

Thus, the feedback equilibrium strategies $\hat{q}^n_{\text{feedback}}$ are linear functions of $r$. Note that $A$ and $B$ are constants that depend on the demand function parameters and can thus be estimated empirically. Furthermore, the equilibrium strategies are symmetric in the sense that they are the same for all firms.

Given the equilibrium strategies, we can recover the equilibrium trajectories of $q^n(t)$, $p(t)$, and $r(t)$.

PROPOSITION 6. *The feedback equilibrium trajectories are given by*

$$q^n_{\text{feedback}}(t) = \frac{1}{N}\left[ a - \delta p^{ss}_{\text{feedback}} + \left( (\delta + \gamma)\frac{m_{\text{feedback}}}{\beta} - \delta \right) \cdot (r_0 - p^{ss}_{\text{feedback}}) e^{-m_{\text{feedback}} t} \right], \qquad (32)$$

$$p_{\text{feedback}}(t) = p^{ss}_{\text{feedback}} + \left(1 - \frac{m_{\text{feedback}}}{\beta}\right) \cdot (r_0 - p^{ss}_{\text{feedback}}) e^{-m_{\text{feedback}} t}, \qquad (33)$$

$$r_{\text{feedback}}(t) = p^{ss}_{\text{feedback}} + (r_0 - p^{ss}_{\text{feedback}}) e^{-m_{\text{feedback}} t}, \qquad (34)$$

*where*

$$p^{ss}_{\text{feedback}} = \frac{a + (\delta + \gamma)Nc + \beta B}{\delta(N+1) + \gamma N - 2A\beta}, \quad m_{\text{feedback}} = \beta\left(1 - \frac{2A\beta + \gamma}{(N+1)(\delta + \gamma)}\right) > 0. \qquad (35)$$

PROOF. The trajectory (34) was already calculated in Appendix E. From this, the expressions for $p_{\text{feedback}}(t)$ and $q^n_{\text{feedback}}(t)$ follow. □

From Proposition 6 we see that the trajectories of the feedback equilibrium strategies have the same qualitative behavior as in the case of monopoly and in the case of open-loop equilibrium, i.e., when $p^{ss}_{\text{feedback}} < r_0$, the market equilibrium price is monotonically declining towards the steady-state $p^{ss}_{\text{feedback}}$, and so on.

**5.2.2. Stage 2: The Asymmetric Case $\gamma_{\text{gain}} < \gamma_{\text{loss}}$.** As before, to calculate the solution in the asymmetric case, we consider the steady-state price $p^{ss}_{\text{feedback}}$, the constants $A$ and $B$, and the coefficient $m_{\text{feedback}}$ as functions of the parameter $\gamma$.

LEMMA 5.1. *Let $p^{ss}_{\text{feedback}}(\gamma)$ be defined by (35). Then, $p^{ss}_{\text{feedback}}(\gamma)$ decreases monotonically in $\gamma$.*

PROOF. It is easy to see that

$$\left. \frac{dp^{ss}_{\text{feedback}}(\gamma)}{d\gamma} \right|_{\gamma=0} < 0.$$

Therefore, it follows that $p^{ss}_{\text{feedback}}(\gamma)$ is monotonically decreasing for $\gamma$ sufficiently small. Although we do not have a proof for a general $\gamma$, we have verified this result numerically over a large range of parameters. □

The following proposition is an extension of Propositions 5 and 6 for the asymmetric case.

PROPOSITION 7. *Let $\gamma$ be given by* (6). *Then the feedback equilibrium strategies $\hat{q}^n_{\text{feedback}}(r)$ are given by* (29), *and the corresponding equilibrium trajectories for $q^n_{\text{feedback}}(t)$, $p_{\text{feedback}}(t)$, and $r_{\text{feedback}}(t)$ are given by* (32)–(34), *where the value of $\gamma$ in the expressions for $m(\gamma)$, $A(\gamma)$, $B(\gamma)$, and $p^{ss}_{\text{feedback}}(\gamma)$ is given by*

$$\gamma = \begin{cases} \gamma_{\text{loss}} & r_0 < p^{ss}_{\text{feedback}}(\gamma_{\text{loss}}), \\ \tilde{\gamma} & p^{ss}_{\text{feedback}}(\gamma_{\text{loss}}) < r_0 < p^{ss}_{\text{feedback}}(\gamma_{\text{gain}}), \\ \gamma_{\text{gain}} & r_0 > p^{ss}_{\text{feedback}}(\gamma_{\text{gain}}), \end{cases} \tag{36}$$

*and $\tilde{\gamma}$ is the solution of $p^{ss}_{\text{feedback}}(\tilde{\gamma}) = r_0$.*

PROOF. See Appendix F. □

## 6. COMPARISON WITH SUBOPTIMAL PRICING STRATEGIES

In this section, we compare the optimal pricing strategy in the presence of reference-price effects $p_{\text{optimal}}(t)$ with several other pricing methods that retailers can adopt. We begin with the classical demand-supply model with *no reference-price* effects. We then consider a *myopic* (greedy) model, where reference-price effects are included, but instead of global planning the retailer maximizes short-term profits. We conclude with a third strategy of an *optimal EDLP*, in which reference price is included in global planning, but the price is held constant throughout the planning horizon. For simplicity, we present the results for a monopoly and for the symmetric case $\gamma_{\text{gain}} = \gamma_{\text{loss}}$. The extensions to an oligopoly and to the loss-aversive case are straightforward.

### 6.1. No Reference-Price Model

The classical demand-supply problem for a retailer is to find the optimal price $p^{\text{optimal}}_{\text{no-ref}}$ that maximizes its profits $\Pi(p)$ in the absence of reference-price effects, namely

$$p^{\text{optimal}}_{\text{no-ref}} = \arg\max_p \Pi(p), \qquad \Pi(p) = (p-c)Q_{\text{no-ref}}(p).$$

The optimal price $p^{\text{optimal}}_{\text{no-ref}}$ can be calculated from the condition $\Pi'(p^{\text{optimal}}_{\text{no-ref}}) = 0$. For example, when $Q_{\text{no-ref}} = a - \delta p$,

$$p^{\text{optimal}}_{\text{no-ref}} = \frac{a+\delta c}{2\delta}. \tag{37}$$

Naturally, in the absence of reference-price effects (e.g., $\gamma = 0$ or $\beta \to \infty$), the optimal pricing strategy reduces to $p_{\text{optimal}}(t) \equiv p^{\text{optimal}}_{\text{no-ref}}$.

### 6.2. Myopic Price Strategy

Under the myopic strategy, the retailer takes into consideration reference-price effects. However, unlike the optimal strategy, the retailer adopts a short-sighted approach and determines the price so that it maximizes instantaneous profits:

$$\begin{aligned} &p(t) = \arg\max \pi(t), \\ &\pi(t) = (p(t)-c)[a - \delta p(t) - \gamma[p(t) - r(t)]]. \end{aligned} \tag{38}$$

The dynamics of $p$ under this myopic strategy for both finite and infinite planning horizons, is given by (Appendix B)

$$p_{\text{myopic}}(t) = p^{ss}_{\text{myopic}} + \frac{\gamma}{2(\gamma+\delta)}(r_0 - p^{ss}_{\text{myopic}})e^{-((\gamma+2\delta)/(2(\gamma+\delta)))t}, \tag{39}$$

where

$$\begin{aligned} &p^{ss}_{\text{myopic}} = (1-\kappa_{\text{myopic}})p^{\text{optimal}}_{\text{no-ref}} + \kappa_{\text{myopic}} c, \\ &\kappa_{\text{myopic}} = \frac{\gamma}{2\delta+\gamma}. \end{aligned} \tag{40}$$

We note that although the price and the reference price stabilize at the same value $p^{ss}_{\text{myopic}}$, the effect of reference price does not disappear, as is evident from the fact that $p^{ss}_{\text{myopic}} \neq p^{\text{optimal}}_{\text{no-ref}}$. Finally, we note that, as expected, as $\alpha \to \infty$ the optimal policy turns into a purely myopic one: $\lim_{\alpha\to\infty} p_{\text{optimal}}(t) = p_{\text{myopic}}(t)$.

### 6.3. Optimal EDLP Strategy

Our model does not take into account costs associated with price changes such as printing, invoice processing, and order size. Because the accumulated costs of price changes during the planning horizon increase with the number of price changes, there is a tradeoff between additional costs and profits associated with price changes. The retailer may, therefore, want to consider adopting a strategy that limits the number of price changes. Here we consider a rather extreme case—that of a pricing strategy where the price is held constant throughout the planning horizon. A priori, this constraint may seem to be too restrictive and to result in a considerable decrease in profits, compared with the optimal strategy. However, one of the surprising results of this study is that the relative loss of potential profits under this strategy, compared with the optimal one, is quite small.

Let us calculate the optimal constant price $p^{\text{optimal}}_{\text{EDLP}}$ under this strategy. In this case, the optimization problem becomes

$$\max_{p_c} \int_0^T e^{-\alpha t}(p_c - c)[a - \delta p_c - \gamma(p_c - r(t))]\,dt,$$

where $p_c$ is independent of time. Using (3), it is simple to show that in this case

$$r(t) = p_c + (r_0 - p_c)e^{-\beta t}, \tag{41}$$

and

$$\Pi^T(p_c) = (p_c - c)\left[\frac{(a-\delta p_c)}{\alpha}(1-e^{-\alpha T}) + \frac{\gamma(r_0 - p_c)}{(\alpha+\beta)}(1 - e^{-(\alpha+\beta)T})\right].$$

Therefore, the *optimal EDLP* over an infinite planning interval ($T = \infty$), is given by

$$p^{\text{optimal}}_{\text{EDLP}} = \left(1 - \kappa^{\text{optimal}}_{\text{EDLP}}\right)p^{\text{optimal}}_{\text{no-ref}} + \kappa^{\text{optimal}}_{\text{EDLP}}\frac{r_0 + c}{2}, \tag{42}$$

where

$$\kappa_{\text{EDLP}}^{\text{optimal}} = \frac{\gamma}{\delta(1+\beta/\alpha)+\gamma}. \tag{43}$$

Clearly, $0 \leqslant \kappa_{\text{EDLP}}^{\text{optimal}} \leqslant 1$ and in the absence of reference-price effects ($\gamma = 0$ or $\beta \to \infty$), $p_{\text{EDLP}}^{\text{optimal}}$ reduces to $p_{\text{no-ref}}^{optimal}$. Note that unlike $p_{\text{optimal}}^{ss}$, $p_{\text{EDLP}}^{\text{optimal}}$ does depend on $r_0$. However, the effect of $r_0$ is relatively small, since $\alpha \ll \beta$ implies that $\kappa_{\text{EDLP}}^{\text{optimal}} \ll 1$.

### 6.4. Comparison of Alternative Pricing Strategies

In this section, we compare the four pricing strategies $p_{\text{optimal}}(t)$, $p_{\text{no-ref}}^{\text{optimal}}$, $p_{\text{myopic}}(t)$, and $p_{\text{EDLP}}^{\text{optimal}}$ in terms of price dynamics and combined profits (here we use $p_{\text{no-ref}}^{\text{optimal}}$ to calculate the loss of potential profits when firms ignore reference-price effects). The price remains constant under the *no-reference price* and the *optimal EDLP* strategies. In the case of the myopic approach, after an initial transient the price stabilizes at a constant value. The optimal pricing strategy, $p_{\text{optimal}}$, has a short introductory stage and a steady-state region when the planning horizon is infinite, and an additional short final decline stage when the planning horizon is finite. In the case of the two nonconstant strategies, $p_{\text{myopic}}(t)$ and $p_{\text{optimal}}(t)$, the value of $r_0$ determines whether there is an initial decrease or increase but has no effect beyond the introductory stage. In the case of the two stationary strategies, the value of $r_0$ affects the value of $p_{\text{EDLP}}^{\text{optimal}}$, but has no effect on the value of $p_{\text{no-ref}}^{\text{optimal}}$.

Under all four strategies, price is constant except for perhaps short transients at the introductory and/or final decline stages. Therefore, one can estimate the total profits under all four strategies by $\int_0^T e^{-\alpha t}(p_c - c)(a - \delta p_c)\,dt$, where the constant price $p_c$ is equal to $p_{\text{optimal}}^{ss}, p_{\text{no-ref}}^{\text{optimal}}, p_{\text{EDLP}}^{\text{optimal}}$, and $p_{\text{myopic}}^{ss}$, for the optimal pricing strategy, the no-reference price strategy, the optimal EDLP strategy and the myopic strategy, respectively. In addition, because $\alpha \ll \beta$, in light of (20), (42)–(43) we have that

$$p_{\text{optimal}}^{ss} \approx p_{\text{EDLP}}^{\text{optimal}} \approx p_{\text{no-ref}}^{\text{optimal}}. \tag{44}$$

Therefore, although profits under $p_{\text{optimal}}(t)$ are greater than those under $p(t) \equiv p_{\text{EDLP}}^{\text{optimal}}$, which in turn are greater than those under $p(t) \equiv p_{\text{no-ref}}^{\text{optimal}}$, the differences in profits under these three strategies are small (see Figure 4). However, from (40) we see that $p_{\text{myopic}}^{ss}$ can be significantly lower than $p_{\text{no-ref}}^{\text{optimal}}$. Therefore, the loss of potential profits under the myopic strategy can be substantial. Because the penalty for adopting an optimal EDLP strategy is surprisingly small, firms should consider whether this small penalty will not be offset by the elimination of the added costs involved in price changes.

## 7. GENERAL DEMAND FUNCTIONS

In this study, we assumed that the demand rate is linear in $p$ and piecewise-linear in $(p-r)$ (see §2.2). Although these assumptions are quite common in the marketing literature, it is natural to ask whether the results of this study would remain valid for more general demand functions. The short discussion below suggests that this is indeed the case.

Let us first consider the assumption that the demand rate in the absence of reference-price effects $Q_{\text{no-ref}}$ is linear in $p$. It can be shown that the essential properties of the optimal strategy (an introductory stage followed by a steady-state stage, $(p-r)$ does not change its sign in the symmetric case, validity of the two-stage method in the loss-aversive case, etc.) remain valid when the demand rate is monotonically decreasing in $p$ and the profit rate is concave in $p$, i.e.,

$$\frac{d}{dp}Q_{\text{no-ref}}(p) < 0, \qquad \frac{d^2}{dp^2}[(p-c)Q_{\text{no-ref}}(p)] < 0.$$

We now consider the role of nonsmoothness in the model. The nonsmoothness of $\gamma$ as a function of $(p-r)$ in (6) is not based on some solid data, but rather on the methodology of the empirical studies which measured $\gamma$ as a function of the sign of $(p-r)$ rather than as a function of $(p-r)$. Therefore, a reasonable assumption would be to remove the nonsmoothness by replacing $\gamma$ with a smooth function, which we denote by $\gamma_h$, that preserves the properties of (6), namely, $\gamma_h$ is monotonic in $(p-r)$, $\gamma_h \approx \gamma_{\text{gain}}$ when $p \ll r$, and $\gamma_h \approx \gamma_{\text{loss}}$ when $p \gg r$. For example, the functions

$$\gamma_h(p-r) = \frac{\gamma_{\text{gain}}+\gamma_{\text{loss}}}{2} + \frac{\gamma_{\text{loss}}-\gamma_{\text{gain}}}{\pi}\arctan\left(\frac{p-r}{h}\right), \qquad h > 0, \tag{45}$$

satisfy all the above requirements. Note that $\gamma_h$ approaches (6) as $h \to 0$.

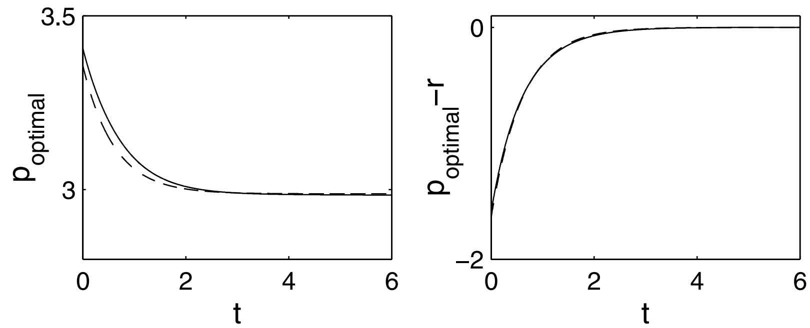
When we replace $\gamma$, as given in (6), with $\gamma_h$, the asymmetric optimization problem becomes smooth and can thus be solved using standard optimization methods. In that case, however, the Euler-Lagrange equation is no longer linear. As a result, it cannot be solved explicitly but only numerically. A typical comparison of simulation results of the smooth asymmetric model with the corresponding explicit solution of the nonsmooth asymmetric model for a monopolistic firm is given in Figure 3. It can be seen that the change of the asymmetry model has a relatively minor effect on the optimal price. In particular, the property that $(p-r)$ does not change its sign seems to be insensitive to to the exact details of the asymmetry modeling.

## 8. EMPIRICAL ILLUSTRATION

In this section, we show how our theoretical results can be applied to a real-life situation, using empirical data for the peanut butter category from Greenleaf (1995). In that study, the demand function in the presence of reference-price effects was estimated as

$$Q_{\text{no-ref}} = 308.3 - 1{,}878.9p, \tag{46}$$

**Figure 3.** (a) Optimal pricing strategy with nonsmooth (Equation (6), dashed line) and with smooth (Equation (45) with $h = 0.5$, solid line) asymmetric reference-price effects. (b) The difference $(p - r)$ in the nonsmooth and smooth asymmetric models.




*Note.* Here, $\gamma_{\text{gain}} = 1$, $\gamma_{\text{loss}} = 1.5$, $r_0 = 5$, $\beta = 2$, $\alpha = 0.05$, $c = 1$, $a = 10$, and $\delta = 2$.

where $p$ is measured in dollars/ounce and $Q$ is measured in ounces. For simplicity, we consider the symmetric case with $\gamma = (\gamma_{\text{gain}} + \gamma_{\text{loss}})/2 = 6{,}709.2$. We use the average price $p_{av} = \$2.57/(28\text{-ounce jar})$ as the initial reference-price $r_0$. A reasonable value for the discounting rate for commercial firms that reflects the excess risk involved in operating in a specific industry (as opposed to the risk-free interest rate), is $\alpha = 0.1/\text{year}$.

Because peanut butter is a mature, well-established product, of most importance are the steady-state price levels. Our explicit solutions allow us to calculate these values directly, using Equations (13), (37), (39), and (42). In Table 1 we present the steady-state price levels under the different pricing strategies for cost levels of \$1.60–\$2.40/(28-ounce jar). It can be seen that the steady-state price of the optimal pricing strategy ($p^{ss}_{\text{optimal}}$) is just below the optimal price in the absence of reference-price effects ($p^{\text{optimal}}_{\text{no-ref}}$), in agreement with the discussion in §3.3, and that the optimal EDLP price $p^{\text{optimal}}_{\text{EDLP}}$ is quite close to these two, in agreement with Equation (44). The steady-state myopic price $p^{ss}_{\text{myopic}}$, in contrast, is much lower than these three prices, in agreement with the discussion in §6.4.

Using data from Briesch et al. (1997), we estimate that $\beta(\text{peanut-butter}) \approx 4.5/\text{year}$. Using expressions (17), (37), (39), and (42), we can write explicitly the different dynamic pricing strategies for peanut butter at any given time during the planning horizon. For example, when $c = \$2$ and the planning horizon is infinite, then $p_{\text{optimal}}(t) = 3.25 - 0.36e^{-0.041t}$, $p^{\text{optimal}}_{\text{no-ref}}(t) = 3.30$, $p_{\text{myopic}}(t) = 2.47 + 0.04e^{-0.61t}$, and $p^{\text{optimal}}_{\text{EDLP}}(t) = 3.22$, where $t$ is measured in weeks.

The explicit expressions we obtain allow us to compare the profits under the different strategies. In Figure 4 we plot the profits under the four pricing strategies for peanut butter as a function of planning horizon $T$. Because maximal profits are attained with $p_{\text{optimal}}$, we present the profits of the three suboptimal strategies relative to the one obtained with $p_{\text{optimal}}$. The most striking result in Figure 4 is that profits under the optimal EDLP strategy are only slightly below those of the optimal pricing strategy. In addition, when $T \gg 1/\beta$, the penalty for ignoring reference price altogether, i.e., $p \equiv p^{\text{optimal}}_{\text{no-ref}}$, is small. Finally, we note that the penalty for adopting the myopic strategy is quite substantial. These results, therefore, confirm the estimates for the profits under the different strategies of §6.4.

## 9. FINAL REMARKS

Asymmetry in marketing models leads to nonsmooth optimization problems. As a result, it was widely believed that it was not possible to obtain explicit solutions in these cases, and that it was necessary to resort to numerical simulations. In this study we have shown that it is possible to obtain explicit expressions in asymmetric models using a two-stage approach, where the solution of the symmetric problem is used to construct a solution in the asymmetric case. Because asymmetry is an important feature of marketing models, our two-stage method is relevant to a variety of other marketing models.

The explicit expression for the optimal pricing strategy in the presence of asymmetric reference price-effects adds to the existing literature on reference price theory as follows:

**Figure 4.** Profits under the three suboptimal pricing strategies: $p(t) \equiv p^{\text{optimal}}_{\text{EDLP}}$ (solid), $p(t) \equiv p^{\text{optimal}}_{\text{no-ref}}$ (dashed) and $p_{\text{myopic}}(t)$ (dash-dot), relative to the profit under the optimal strategy $p_{\text{optimal}}$, as a function of planning horizon $T$.

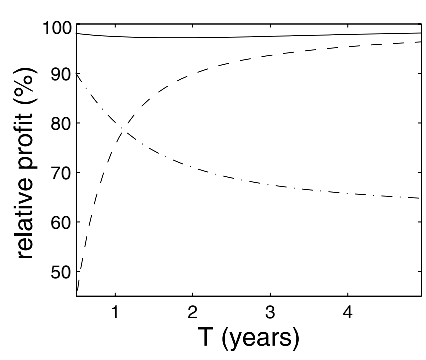


**Table 1.** Steady-state price levels at different cost levels.

| Pricing Strategy | Steady-State | Cost = \$1.6 | Cost = \$1.8 | Cost = \$2 | Cost = \$2.2 | Cost = \$2.4 |
|---|---|---|---|---|---|---|
| No-RP effects | $p_{\text{no-ref}}^{\text{optimal}}$ | \$3.10 | \$3.20 | \$3.30 | \$3.40 | \$3.50 |
| Optimal | $p_{\text{optimal}}^{ss}$ | \$3.04 | \$3.15 | \$3.25 | \$3.35 | \$3.46 |
| EDLP | $p_{\text{EDLP}}^{\text{optimal}}$ | \$3.02 | \$3.12 | \$3.22 | \$3.32 | \$3.42 |
| Myopic | $p_{\text{myopic}}^{ss}$ | \$2.14 | \$2.30 | \$2.47 | \$2.63 | \$2.79 |

1. The two-stage calculation of the optimal pricing strategy is simple and amounts to substitution of parameters in a formula. As a result, unlike calculations using dynamic programming, it is not limited by the number of price changes nor by the number of firms.

2. Analysis of the qualitative features of the model is much simpler because one can directly analyze the solution rather than the equation. Thus, analytical results (e.g., convergence to a steady state) that require considerable analytical effort under previous formulations (Kopalle et al. 1996) become almost trivial once the explicit form of the solution has been obtained.

3. As we obtain an explicit expression for the optimal steady-state price $p_{\text{optimal}}^{ss}$, we are able to fully characterize it, showing, for example, that it is slightly below the optimal price in the absence of reference-price effects.

4. Unlike previous studies that focused on the steady-state stage, we are able to fully characterize the introductory and final stages, both in terms of duration and in terms of penetration or skimming.

5. We show that both open-loop and feedback competition do not add new qualitative features to the model. In particular, they do not lead to high-low pricing strategies.

6. Our two-stage method leads to the surprising result that loss aversion has no effect on the optimal pricing strategy, except for the value of the loss parameter $\gamma$.

7. Our analysis shows that retailers can obtain near-optimal profits with the EDLP strategy. Because this strategy has many other advantages that were not taken into account in our model (simplicity in implementation and analysis, elimination of costs associated with price changes, etc.), our study provides strong support for this approach.

## APPENDIX A: PROOF OF PROPOSITION 1

We verify that $r_{\text{optimal}}(t)$, given by (11), satisfies the following three conditions, which are sufficient for it to be the global maximizer of the variational problem (7)–(8), i.e.,

$$\int_0^T F[t, r_{\text{optimal}}(t), r'_{\text{optimal}}(t)]\, dt \geqslant \int_0^T F[t, r(t), r'(t)]\, dt$$

for all functions $r(t)$ such that $r(0) = r_0$ (see, e.g., Brechtken 1991):

Condition 1. $r_{\text{optimal}}(t)$ is regular, namely

$$F^0_{r'r'} = F_{r'r'}(t, r_{\text{optimal}}(t), r'_{\text{optimal}}(t)) \neq 0, \quad 0 \leqslant t \leqslant T.$$

Proof. From (9) we have that $F^0_{r'r'} = -2(\gamma+\delta)\beta^{-2}\cdot e^{-\alpha t} \neq 0$. □

Condition 2. There is no conjugate point to $t = 0$ for $r_{\text{optimal}}(t)$ in $(0, T]$, i.e., there is no point $t_c \in (0, T]$, such that $R(t_c) = 0$, where $R(t)$ is any nontrivial solution of the Jacobi equation

$$[F^0_{r'r'}R']' - \left[F^0_{rr} - \frac{d}{dt}F^0_{rr'}\right]R = 0, \qquad R(0) = 0. \tag{47}$$

Here $F^0_{rr} = F_{rr}(t, r_{\text{optimal}}(t), r'_{\text{optimal}}(t))$ and $F^0_{rr'} = F_{rr'}(t, r_{\text{optimal}}(t), r'_{\text{optimal}}(t))$.

Proof. The Jacobi equation (47) is simply

$$R'' - \alpha R' - \beta\frac{2\delta(\alpha+\beta)+\alpha\gamma}{2(\gamma+\delta)}R = 0.$$

The solutions of Equation (47) are given by $R(t) = k_1(e^{m_2 t} - e^{-m_1 t})$, where $m_1$ and $m_2$ are defined in (12) and $k_1$ is an arbitrary constant. Therefore, because $m_2 > -m_1$, there is no $t_c \in (0, T]$ for which $R(t_c) = 0$. □

Condition 3. The Weierstrass $\epsilon$-function

$$\epsilon(t, r, r', q) = F(t, r, q) - F(t, r, r') - F_{r'}(t, r, r')(q - r')$$

is negative for every $(t, r, r', q)$ such that $t \in [0, T]$, $q \in \mathcal{R}$ and $q \neq r'$.

Proof. Some technical calculations yield $\epsilon(t, r, r', q) = -(\gamma+\delta)\beta^{-2}e^{-\alpha t}(r' - q)^2 < 0$. □

## APPENDIX B: MYOPIC STRATEGY

We differentiate $\pi$, given by (38), with respect to $p$ and equate it to zero to get

$$p_{\text{myopic}}(t) = \frac{a + (\gamma+\delta)c + \gamma r_{\text{myopic}}(t)}{2(\gamma+\delta)}. \tag{48}$$

Substituting $p_{\text{myopic}}(t)$ from (48) into the relation (2) yields the equation for $r_{\text{myopic}}(t)$:

$$r'_{\text{myopic}} + \beta r_{\text{myopic}}\left(1 - \frac{\gamma}{2(\gamma+\delta)}\right) = \frac{a + (\gamma+\delta)c}{2(\gamma+\delta)}\beta,$$

$$r(0) = r_0.$$

Solving this equation and substituting $r_{\text{myopic}}(t)$ into (48) gives (39).

## APPENDIX C: PROOF OF PROPOSITION 2

Let us denote the profits under a price strategy $p(t)$ by

$$g(\gamma_{\text{gain}}, \gamma_{\text{loss}};\; p(t)) = \int_0^\infty e^{-\alpha t}(p(t)-c)Q(t)\,dt,$$

where $Q$ is given by (5), $r$ is given by (3), and $\gamma$ is given by (6). Then $g$ increases monotonically in $\gamma_{\text{gain}}$ and decreases monotonically in $\gamma_{\text{loss}}$. In particular,

$$g(\gamma_{\text{gain}}, \gamma_{\text{loss}};\; p(t)) \leqslant g(\tilde{\gamma}, \tilde{\gamma};\; p(t)) \quad \text{for all } \tilde{\gamma} \text{ such that } \gamma_{\text{gain}} \leqslant \tilde{\gamma} \leqslant \gamma_{\text{loss}}. \tag{49}$$

Let us also denote the maximal profit in the asymmetric case by

$$G(\gamma_{\text{gain}}, \gamma_{\text{loss}}) = \sup_{p(t)} g(\gamma_{\text{gain}}, \gamma_{\text{loss}};\; p(t)). \tag{50}$$

Then from (49) and Proposition 1, we have that

$$G(\gamma_{\text{gain}}, \gamma_{\text{loss}}) \leqslant G(\tilde{\gamma}, \tilde{\gamma}) < \infty \quad \text{for all } \tilde{\gamma} \text{ such that } \gamma_{\text{gain}} \leqslant \tilde{\gamma} \leqslant \gamma_{\text{loss}}. \tag{51}$$

1. When $p^{ss}_{\text{optimal}}(\gamma_{\text{gain}}) < r_0$, from (16)–(17) with $\gamma = \gamma_{\text{gain}}$ we have that $p_{\text{optimal}}(t) \leqslant r_{\text{optimal}}(t)$ for all $t \geqslant 0$. Therefore, $g(\gamma_{\text{gain}}, \gamma_{\text{loss}};\; p_{\text{optimal}}(t)) = g(\gamma_{\text{gain}}, \gamma_{\text{gain}};\; p_{\text{optimal}}(t)) = G(\gamma_{\text{gain}}, \gamma_{\text{gain}})$. On the other hand, from (50)–(51) we have that

$$g(\gamma_{\text{gain}}, \gamma_{\text{loss}};\; p_{\text{optimal}}(t)) \leqslant G(\gamma_{\text{gain}}, \gamma_{\text{loss}}) \leqslant G(\gamma_{\text{gain}}, \gamma_{\text{gain}}).$$

Combining the last two relations gives that $g(\gamma_{\text{gain}}, \gamma_{\text{loss}};\; p_{\text{optimal}}(t)) = G(\gamma_{\text{gain}}, \gamma_{\text{loss}})$, which is what we wanted to prove.

2. When $p^{ss}_{\text{optimal}}(\gamma_{\text{loss}}) \leqslant r_0 \leqslant p^{ss}_{\text{optimal}}(\gamma_{\text{gain}})$, then $p_{\text{optimal}}(t) \equiv r_{\text{optimal}}(t) \equiv r_0$ for all $t \geqslant 0$. Let $\tilde{\gamma}$ be such that $p^{ss}_{\text{optimal}}(\tilde{\gamma}) = r_0$. From Lemma 3.1 we have that $\gamma_{\text{gain}} \leqslant \tilde{\gamma} \leqslant \gamma_{\text{loss}}$. Therefore, $g(\gamma_{\text{gain}}, \gamma_{\text{loss}}; p_{\text{optimal}}(t)) = g(\tilde{\gamma}, \tilde{\gamma};\; p_{\text{optimal}}(t)) = G(\tilde{\gamma}, \tilde{\gamma})$. On the other hand, $g(\gamma_{\text{gain}}, \gamma_{\text{loss}};\; p_{\text{optimal}}(t)) \leqslant G(\gamma_{\text{gain}}, \gamma_{\text{loss}}) \leqslant G(\tilde{\gamma}, \tilde{\gamma})$. Combining the last two relations gives $g(\gamma_{\text{gain}}, \gamma_{\text{loss}}; p_{\text{optimal}}(t)) = G(\gamma_{\text{gain}}, \gamma_{\text{loss}})$.

3. The proof is the same as in part (1).

## APPENDIX D: PROOF OF PROPOSITION 3

We give here only the sketch of the proof, as it is similar to the one in the monopolistic case. Using (5), we can write the optimization problem (7) as

$$\max_{q_n(t)} \int_0^\infty e^{-\alpha t} \frac{1}{\delta+\gamma}(a - Q + \gamma r - (\delta+\gamma)c_n)q_n\,dt,$$

subject to $\quad r' = \dfrac{\beta}{\delta+\gamma}(a - Q - \delta r).$

The current value Hamiltonian for the $n$th firm is given by

$$H_n = \frac{1}{\delta+\gamma}(a - Q + \gamma r - (\delta+\gamma)c_n)q_n + \lambda_n \frac{\beta}{\delta+\gamma}(a - Q - \delta r).$$

Therefore, the first-order conditions for equilibrium are

$$\frac{\partial H_n}{\partial q_n} = 0, \qquad \lambda_n' = \alpha\lambda_n - \frac{\partial H_n}{\partial r}, \qquad n = 1, \ldots, N,$$

subject to the free-boundary conditions $\lim_{t\to\infty} e^{-\alpha t}\lambda_n(t) = 0$ for $n = 1, \ldots, N$, and $r(0) = r_0$. Solution of these equations yields the desired results.

## APPENDIX E: PROOF OF PROPOSITION 5

From (5) we have that

$$p = \frac{a - Q + \gamma r}{\delta+\gamma}. \tag{52}$$

In light of (52), the right-hand side of (28) is quadratic and concave in $q_n$. Hence, its *global* maximum with respect to $q_n$ is attained at

$$\hat{q}_n = a + \gamma r - \widehat{Q} - (\delta+\gamma)c - \beta\frac{dV_n(r)}{dr}, \tag{53}$$

where $\widehat{Q} = \sum_{n=1}^N \hat{q}_n$. Summing this relation over all $n$ yields

$$\widehat{Q} = \frac{N}{N+1}\left(a + \gamma r - (\delta+\gamma)c - \frac{\beta}{N}V_r\right), \tag{54}$$

where

$$V(r) = \sum_{n=1}^N V_n(r), \qquad V_r(r) = \sum_{n=1}^N \frac{dV_n(r)}{dr}.$$

Thus, we have by (52) and (54) that

$$\hat{p} = \frac{a + \gamma r + (\delta+\gamma)Nc + \beta V_r(r)}{(\delta+\gamma)(N+1)}. \tag{55}$$

Summing (28) over all $n$ gives $\alpha V = (\hat{p} - c)\widehat{Q} + \beta V_r \cdot (\hat{p} - r)$. Substituting $\widehat{Q}$ and $\hat{p}$ from (54)–(55) in the last equation yields

$$\alpha V = \frac{N}{(\delta+\gamma)(N+1)^2}[a + \gamma r - (\delta+\gamma)c + \beta V_r] \cdot \left[a + \gamma r - (\delta+\gamma)c - \frac{\beta}{N}V_r\right] + \frac{\beta V_r}{(\delta+\gamma)(N+1)} \cdot [a + (\delta+\gamma)Nc + \beta V_r - (\delta+\gamma)(N+1)r + \gamma r]. \tag{56}$$

The solution of (56) is given by

$$V(r) = Ar^2 + Br + C, \tag{57}$$

where

$$A = (\alpha+2\beta)\frac{(\delta+\gamma)(N+1)^2}{8\beta^2 N} - \frac{\gamma}{2\beta} \pm \sqrt{\left((\alpha+2\beta)\frac{(\delta+\gamma)(N+1)^2}{8\beta^2 N} - \frac{\gamma}{2\beta}\right)^2 - \frac{\gamma^2}{4\beta^2}}. \tag{58}$$

$B$ is given by (31) and $C$ is a constant whose value does not affect the equilibrium strategies. The following lemma rules out the larger value of $A$ in (58).

LEMMA E.1. *The equilibrium strategies are stable if and only if $A$ is given by* (30).

PROOF. From (55) and (57) we have that

$$\hat{p} = \frac{a + \gamma r + (\delta+\gamma)Nc + \beta(2Ar + B)}{(\delta+\gamma)(N+1)}. \qquad (59)$$

Substituting $\hat{p}$ into (2) yields

$$r' = \beta \frac{a + (\delta+\gamma)Nc + \beta B + r(\gamma + 2A\beta - (\delta+\gamma)(N+1))}{(\delta+\gamma)(N+1)}.$$

The solution of this equation, subject to $r(0) = r_0$, is given by (34). Because $r(t)$ should remain bounded as $t \to \infty$, the stability condition is $m_{\text{feedback}} > 0$, or $\gamma + 2A\beta < (\delta+\gamma)(N+1)$. Using (58), the stability condition can be rewritten as

$$(\alpha+2\beta)\frac{(\delta+\gamma)(N+1)^2}{4\beta N} \pm \sqrt{\left((\alpha+2\beta)\frac{(\delta+\gamma)(N+1)^2}{4\beta N} - \gamma\right)^2 - \gamma^2} < (\delta+\gamma)(N+1).$$

The lemma is thus proved if we can show that

$$\sqrt{\left((\alpha+2\beta)\frac{(\delta+\gamma)(N+1)^2}{4\beta N} - \gamma\right)^2 - \gamma^2} > (\delta+\gamma)(N+1)\left|1 - (\alpha+2\beta)\frac{(N+1)}{4\beta N}\right|.$$

Taking the square of both sides yields, after some manipulations,

$$\frac{\delta(\alpha+2\beta)}{2\beta N} > -\alpha(\delta+\gamma),$$

which is always true because all parameters are positive. □

Substitution of $V(r)$ from (57) in (54) gives the equilibrium value of $Q$:

$$\widehat{Q} = \frac{N}{N+1}\left(a + \gamma r - (\delta+\gamma)c - \frac{\beta}{N}(2Ar + B)\right). \qquad (60)$$

We now prove that the equilibrium strategies are symmetric. Therefore, dividing (60) by $N$ yields the equilibrium strategies (29).

LEMMA E.2.

$$V_n(r) = \frac{1}{N}V(r) \quad n = 1, \ldots, N.$$

PROOF. Substitution of (53) in (28) yields a set of $N$ differential equations

$$\alpha V_n = (\hat{p} - c)\hat{q}_n + \frac{dV_n}{dr}\beta(\hat{p} - r) \quad n = 1, \ldots, N. \qquad (61)$$

Eliminating $\hat{p}$ and $\hat{q}_n$ from (61) and using (53), (59)–(60) yields the linear differential equations

$$\beta(c - r)\frac{dV_n}{dr} - \alpha V_n = H_1 r^2 + H_2 r + H_3, \qquad (62)$$

where $\{H_i\}_{i=1}^3$ are constants that are independent of $n$. The solutions of (62) are of the form

$$V_n(r) = A_n r^2 + B_n r + C_n. \qquad (63)$$

Let us assume that there exist $k \neq j$ such that $V_k \neq V_j$. From (62) it follows that

$$\alpha(V_j - V_k) = \beta(c - r)\frac{d}{dr}(V_j - V_k).$$

Therefore, $V_j - V_k = E(c - r)^{\alpha/\beta}$, where $E \neq 0$ is a constant. However, by (63), we have that $V_j - V_k$ is of the form $V_j - V_k = (A_j - A_k)r^2 + (B_j - B_k)r + (C_j - C_k)$, which is a contradiction. □

## APPENDIX F: PROOF OF PROPOSITION 7

To calculate the feedback strategies in the asymmetric (nonsmooth) case, it is useful to work with the integral relation (Definition 5.2), rather than with the differential one (Hamilton-Jacobi-Bellman equations). Let us denote the profits of the $n$th player under a strategy $q_n$, given $Q_{-n} = \sum_{k \neq n} q^k_{\text{sub-game}}$, by

$$g_n(\gamma_{\text{gain}}, \gamma_{\text{loss}};\ q_n, Q_{-n}) = \int_{t_0}^{\infty} e^{-\alpha t}(p(t) - c)q_n(t)\,dt.$$

Then $g_n$ is monotonically increasing in $\gamma_{\text{gain}}$ and monotonically decreasing in $\gamma_{\text{loss}}$. The rest of the proof is exactly the same as in Appendix C.

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