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Larry A. Shepp
University of Pennsylvania

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EXPLICIT SOLUTIONS TO SOME PROBLEMS OF OPTIMAL STOPPING

By L. A. Shepp

Bell Telephone Laboratories, Inc., Murray Hill

Suppose we are allowed to view successively as many terms as we please of a sequence $X_1, X_2, \cdots$ of independent random variables with common distribution F. We can decide to stop viewing at any time and if we decide to stop at time n, we receive the payoff $(X_1 + \cdots + X_n)/n$. How should we choose a stopping rule in order to maximize the expected payoff? This problem was introduced [5] in the context of inducing an illusory bias by selectively stopping an ESP experiment.

Based on their general theory of optimal stopping rules, Y. S. Chow and Herbert Robbins [9] succeeded in proving that an optimal rule exists when F is a two point distribution. They also proved the intuitively obvious but nontrivial fact that the unique minimal optimal rule is to stop at the first n at which $X_1 + \cdots + X_n \geqq \beta_n$, where $\beta_1, \beta_2, \cdots$ is a sequence of numbers, and gave a way to calculate β_n in principle. Aryeh Dvoretzky [14], and also H. Teicher and J. Wolfowitz [25] then proved that the same results hold for any F with finite second moment (the β's depend on F, of course). Dvoretzky also showed that if F has zero mean and unit variance then $0.32 < \beta_n/n^{\frac{1}{2}} < 4.06$ for n sufficiently large, and conjectured that $\lim \beta_n/n^{\frac{1}{2}}$ exists.

We prove the conjecture and find the value of the limit (which is independent of F as long as F has zero mean and unit variance) as the root $\alpha = 0.83992 \cdots$ of (1.3). The method is to use as an approximation the analogous continuous time problem, for which we can obtain the explicit optimal rule.

In the continuous time problem, also considered by Dvoretzky, the Wiener process $W(t), t \geqq 0$ is sampled continuously and stopping at time t gets the payoff $W(t)/(a + t)$. Dvoretzky pointed out that if $a > 0$ there exists an optimal stopping time. We show that there is a unique optimal stopping time and we find it explicitly: it is the first time τ that $W(\tau) = \alpha(a + \tau)^{\frac{1}{2}}$ (the same α as above). The expected payoff under the optimal rule is also given explicitly (Theorem 1). Except for the constant α, the parabolic form of the boundary determining τ is easily guessed by using the invariance of W under a change of scale. α can then be determined by using the "principle of smooth fit," due to Herman Chernoff and others for various special problems and treated carefully and in some generality by B. I. Grigelionis and A. N. Shiryaev [18]. However, to prove the optimality of τ rigorously we use a different approach, based on the fundamental Wald identity and on the work of Chow, Robbins, and Dvoretzky.

The continuous time problem discussed above is basically similar to the familiar Wald sequential probability ratio problem where, again on heuristic considerations of homogeneity, the optimal stopping rule is given in terms of a pair of

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straightline stopping boundaries. The close analogy to Wald's problem is discussed in some detail in Section 6. Other applications of the homogeneity principle in stopping rule problems can be found in [2], [7], [13], and [24].

Our methods also apply and give explicit solutions to a number of related problems (Section 6) including the problem of maximizing $EW(t)$ over stopping times $t \leqq 1$, where $W(1)$ is assumed known in advance. However, these problems are rather special, and slight changes in formulation make them very difficult. On the other hand, typical stopping time problems are not effectively treated by any of the available methods and it is hoped that the elementary cases considered here will be useful as illuminating guides to more complex problems. As an example of this we refer to the work [2] of John Bather and Herman Chernoff, who used the elementary solution of a very similar problem to obtain tight bounds on an optimal stopping boundary.

We give the more difficult parts of proofs in Section 7 and Section 8, an extension of the Wald identity in Section 9, and some speculations on the $(X_1 + \cdots + X_n)/n$ problem when the X's have infinite variance in Section 10.

1. Continuous time; a precise formulation of the results. Let $W(t) = W(t, \omega)$, $0 \leqq t < \infty$, $\omega \varepsilon \Omega$, be a standard Wiener process, continuous in t for fixed ω and with $EW(t) = 0$, $EW^2(t) = t$. A nonnegative random variable $T(\omega) \leqq +\infty$ is called a stopping time if it does not anticipate the future in the sense that $\{T > t\} \varepsilon \mathfrak{F}_t = \mathfrak{S}[W(s):s < t]$, $0 \leqq t < \infty$. For given u, $-\infty < u < \infty$ and $b > 0$ the expected payoff under T is defined as

$$V(u, b, T) = E(u + W(T(\omega), \omega))/(b + T(\omega)) \tag{1.1}$$

where the integrand (or expectand) in (1.1) is taken to be zero at points ω where $T(\omega) = \infty$. We wish to find $V(u, b) = \sup V(u, b, T)$, the sup being taken over all stopping times T for which the expectation in (1.1) is defined. Our original problem was the special case $u = 0$.

For $c > 0$, let $\tau_c = \tau_c(u, b) = \min [t:u + W(t) \geqq c(b + t)^{\frac{1}{2}}]$, noting that $\tau_c = 0$ if $u \geqq cb^{\frac{1}{2}}$. That τ_c is almost surely defined (finite) will be seen later; this is also a consequence of known facts about W. We will also see later that as a function of c, $V(u, b, \tau_c)$ takes on its maximum value at a unique value of c. Denote this value of c by α.

THEOREM 1. *For $b > 0$, $\tau_\alpha(u, b)$ is an optimal stopping time in the sense that*

$$V(u, b) = V(u, b, \tau_\alpha). \tag{1.2}$$

Moreover, τ_α is the unique *stopping time (up to changes on a null set) for which (1.2) holds. The number α is independent of both u and b and is the unique real root of*

$$\alpha = (1 - \alpha^2) \int_0^\infty e^{\lambda\alpha - \lambda^2/2} \, d\lambda. \tag{1.3}$$

Further, for $b \geqq 0$, $-\infty < u < \infty$,

$$V(u, b) = (1 - \alpha^2) \int_0^\infty e^{\lambda u - \lambda^2 b/2} \, d\lambda, \qquad u \leqq \alpha b^{\frac{1}{2}}, \tag{1.4}$$

$$V(u, b) = u/b, \qquad u > \alpha b^{\frac{1}{2}}.$$

We remark that (1.4) shows that $V(u, 0) = \infty$ for $u \geqq 0$, agreeing with a remark of Dvoretzky ([14], Remark 8). Note that if follows from (1.4) that for $b > 0$, $V(u, b) > u/b$ if and only if $u < b^{\frac{1}{2}}$.

The proof of Theorem 1 is based on the scale-change invariance property of W. Letting $W^*(t) = b^{-\frac{1}{2}}W(bt)$ and $\tau^* = \tau/b$ we note that τ^* is a stopping time for W^* and so,

$$(1.5)\quad E(u + W(\tau))(b + \tau)^{-1} = b^{-\frac{1}{2}}E(u/b^{\frac{1}{2}} + W^*(\tau^*))(1 + \tau^*)^{-1}.$$

Since W^* is also a Wiener process, taking the supremum over τ in (1.5) gives immediately

$$(1.6)\qquad V(u, b) = b^{-\frac{1}{2}}V(u/b^{\frac{1}{2}}, 1).$$

$V(u, b) \geqq u/b$ since $\tau \equiv 0$ is a stopping time. Equality $V(u, b) = u/b$ holds if and only if $\tau = 0$ is optimal for (u, b). It is intuitive after some thought that the set of x for which $V(x, 1) = x$ holds is a half-line $\{x \geqq c\}$ for some c. It follows from (1.6) that we should stop at $\tau = 0$ if and only if $u \geqq cb^{\frac{1}{2}}$. Because W begins anew at each time it seems clear that we should stop the first time τ that $u + W(\tau) \geqq c(b + \tau)^{\frac{1}{2}}$. But this τ is $\tau_c(u, b)$ by definition. This heuristic argument which gives the insight into (1.2), will be made precise in Section 7 and the actual value of c will be found in Section 3. A similar scale-change or dimensional analysis argument appears in [2] and [7].

The first step in a rigorous proof of the theorem consists in showing (Section 7, Lemma 2) that $V(x, 1) = x$ for x large enough. Paradoxically, for any x, *with probability one* there will be a t for which the payoff $(x + W(t))/(1 + t) > x$. However, *within the class of allowable stopping rules*, there is no way to stop and (almost) always get a payoff larger than x. (This paradox is no doubt well known in a slightly simpler setting: even though max $[W(t):0 \leqq t \leqq 1]$ is almost surely strictly positive, for any stopping time $\tau \leqq 1$, we have [22]

$$(1.7)\qquad EW(\tau) = 0.)$$

On the other hand, if we drop the restriction that τ must be a stopping time and allow the observer to know the future he of course can receive EZ where

$$(1.8)\qquad Z(\omega) = sup_{0 \leqq t < \infty}\, (u + W(t, \omega))(b + t)^{-1}.$$

Since [12], $P\{W(t) \geqq at + b \text{ for some } t \geqq 0\} = \exp\,(-2ab)$, $a \geqq 0$, $b \geqq 0$ we have

$$EZ = \int_0^\infty P\{Z \geqq a\}\, da = \int_0^\infty \exp\,(-2a(ab - u))\, da, \qquad u \leqq 0, b > 0.$$

Since $V(u, b) \leqq EZ$, this gives an upper bound for $V(u, b)$.

2. The fundamental Wald identity in continuous time. We will calculate $V(u, b, \tau_c)$, the payoff under τ_c, by using the Wald identity. The identity can be stated [15], [22] for more general stopping times, but we need it in a special form for first passage times. Let $f(t)$, $t \geqq 0$, be a continuous function and let $T(\omega)$ be the first time t that $W(t, \omega) = f(t)$ (set $T(\omega) = \infty$ if there is no such t). Let $F(t) = P\{T \leqq t\}$.

THEOREM 2. *If* (i) $\lambda > 0$, (ii) $f(0) > 0$, (iii) $f(t)/t \to 0$ *as* $t \to \infty$, *then*

$$\int_0^\infty e^{\lambda f(t)-\lambda^2 t/2}\, dF(t) = 1. \tag{2.1}$$

The following proof is based on the proof [1], p. 17, of the so-called Wald fundamental identity of sequential analysis.

The process $Y(t) = \exp(\lambda W(t) - \lambda^2 t/2)$, $t \geqq 0$, satisfies for each fixed t,

$$EY(t) = 1 \tag{2.2}$$

Let χ_A denote the indicator of the set A and $E_A g = E\chi_A g$. For *any* stopping time T, the properties of the conditional expectation give for fixed $t > 0$

$$\begin{aligned} E_{\{T<t\}} Y(t) &= EE[\chi_{\{T<t\}} Y(t) \mid T, Y(T)] \\ &= E\chi_{\{T<t\}} E[Y(t) \mid T, Y(T)]. \end{aligned} \tag{2.3}$$

The strong Markov property [20] gives $E[Y(t) \mid T, Y(T)] = Y(T)$ and so

$$E_{\{T<t\}} Y(t) = E_{\{T<t\}} Y(T). \tag{2.4}$$

For our T, $T(\omega) \geqq t$ implies $W(t, \omega) \leqq f(t)$ by continuity of f, and since $\lambda > 0$ we have

$$\begin{aligned} E_{\{T\geqq t\}} Y(t) &\leqq E_{\{T \geqq t\}} e^{\lambda f(t)-\lambda^2 t/2} \\ &\leqq e^{\lambda f(t)-\lambda^2 t/2}. \end{aligned} \tag{2.5}$$

As $t \to \infty$, the exponent of the last term in (2.5) tends to $-\infty$ by (iii) and from (2.2) and (2.4) we get

$$E_{\{T<\infty\}} e^{\lambda W(T)-\lambda^2 T/2} = \lim_{t\to\infty} E_{\{T<t\}} Y(T) = 1. \tag{2.6}$$

Since $W(T) = f(T)$ the left sides of (2.6) and (2.1) coincide and so (2.6) is just the assertion of the theorem. In Section 9 we give an extended form of the identity, finding the right-hand side of (2.1) for general f and λ.

We remark that if T is any bounded stopping time (not necessarily a first passage time) and $T \leqq M$ with probability one, then choosing $t > M$ it follows immediately from (2.2) and (2.4) that for any λ, $-\infty < \lambda < \infty$,

$$Ee^{\lambda W(T)-\lambda^2 T/2} = 1, \tag{2.7}$$

which proves equation (1) of [22], stated there without proof.

3. The payoff under τ_c. We now apply (2.1) to find $V(u, b, \tau_c)$. Let $f(t) = c(b + t)^{\frac{1}{2}} - u$ and suppose $u < cb^{\frac{1}{2}}$. Setting $F_c(t) = P\{\tau_c \leqq t\}$, we get from (2.1) for $\lambda > 0$,

$$\int_0^\infty e^{\lambda(c(b+t)^{\frac{1}{2}}-u)-\lambda^2 t/2}\, dF_c(t) = 1. \tag{3.1}$$

As $\lambda \to 0$ the integrand goes to one dominatedly. Thus $P_{\{\tau_c<\infty\}} = \int_0^\infty dF_c(t) = 1$ and τ_c is almost surely defined, as was claimed. Now multiply in (3.1) by c

exp $(\lambda u - \lambda^2 b/2)$ and integrate over λ from 0 to ∞ to obtain

$$c\int_0^\infty d\lambda \int_0^\infty e^{\lambda c(b+t)^{\frac{1}{2}}-\lambda^2(b+t)/2}\, dF_c(t) = c\int_0^\infty e^{\lambda u-\lambda^2 b/2}\, d\lambda. \tag{3.2}$$

Interchanging integrals and replacing λ by $y/(b+t)^{\frac{1}{2}}$, the integrals separate and we get an expression for

$$\int_0^\infty c(b+t)^{-\frac{1}{2}}\, dF_c(t) = Ec(b+\tau_c)^{-\frac{1}{2}} = E(u+W(\tau_c))(b+\tau_c)^{-1}.$$

The latter is just $V(u, b, \tau_c)$ and so we obtain for $u < cb^{\frac{1}{2}}$ that

$$V(u, b, \tau_c) = c\int_0^\infty e^{\lambda u-\lambda^2 b/2}\, d\lambda \Big/ \int_0^\infty e^{yc-y^2/2}\, dy. \tag{3.3}$$

The maximum over c of the right side of (3.3) occurs at $c = \alpha$ given by (1.3), and for $c = \alpha$, (3.3) agrees with (1.4).

The evaluation of V follows the technique used by us in [22]. Alternatively, it could have been obtained by the Laplace transform method of D. A. Darling and A. J. F. Siegert [10], but only with much more calculation.

4. The original problem. Let $S_n = X_1 + \cdots + X_n$, $n = 1, 2, \cdots$, where $X_1, X_2, \cdots$ are independent and have common distribution F. A random variable N, whose values are positive integers or $+\infty$ is called a stopping time if for each $n \geqq 0$, $\{N > n\} \in \mathfrak{S}\{X_1, \cdots, X_n\}$. The theory would be simpler and more complete if we included zero as a stopping time; we exclude it here only for ease of reference to [8] and [14]. There is an obviously equivalent constructive definition of a stopping time N. To each N corresponds a sequence $D_1, D_2, \cdots$ of sets, $D_n \subset E_n$, Euclidean n-space; N takes the value n if and only if $(X_1, \cdots, X_n) \in D_n$ for the first time at n; $N = \infty$ if there is no such n.

Given u and n, $-\infty < u < \infty$, $n \geqq 0$, the payoff under N is defined as

$$v(u, n, N) = E(u + S_N)(n + N)^{-1}. \tag{4.1}$$

Let $v(u, n) = \sup v(u, n, N)$ where the sup is taken over all stopping times N. Assuming that F has a second moment, Dvoretzky proved that $v < \infty$ and that there is a unique minimal optimal stopping time ν defined as follows: The equation

$$\beta_n/n = v(\beta_n, n) \tag{4.2}$$

defines the number β_n uniquely for each $n \geqq 0$ and

$$\nu = \nu(u, n) = \min\,[k : u + S_k \geqq \beta_{n+k}].$$

ν is optimal in the sense that $v(u, n, \nu') \leqq v(u, n, \nu)$ for all stopping times ν'; ν is minimal in the sense that equality holds only if $\nu' \geqq \nu$ almost surely.

THEOREM 3. *If F has mean μ and finite variance σ^2 and if β_n is defined by* (4.2) *then with α given by* (1.3)

$$\lim_{n\to\infty} (\beta_n - \mu n)\sigma^{-1} n^{-\frac{1}{2}} = \alpha. \tag{4.3}$$

It is enough to prove the theorem when $\mu = 0$ and $\sigma = 1$ since the general case reduces to this by a change of scale. We first prove that $\liminf \beta_n/n^{\frac{1}{2}} \geqq \alpha$. If this is false then for a sequence of values of n, $\beta_n/n^{\frac{1}{2}} \to \gamma < \alpha$. By (4.2) we have for any stopping time N

$$(4.4) \quad \beta_n/n^{\frac{1}{2}} \geqq v(\beta_n, n, N)n^{\frac{1}{2}} = E(\beta_n/n^{\frac{1}{2}} + S_N/n^{\frac{1}{2}})(1 + N/n)^{-1}.$$

In particular putting $N = \min [k : n^{\frac{1}{2}}\gamma + S_k \geqq \alpha(n + k)^{\frac{1}{2}}]$ and setting $\tau = \tau(n) = N/n$ we have from (4.4) that

$$(4.5) \qquad \beta_n/n^{\frac{1}{2}} \geqq E(\beta_n/n^{\frac{1}{2}} + S_{n\tau}/n^{\frac{1}{2}})(1 + \tau)^{-1}.$$

As $n \to \infty$, $S_{nt}/n^{\frac{1}{2}}$ converges in distribution to $W(t)$ and $\tau(n)$ converges formally to $\tau_\alpha = \tau_\alpha(\gamma, 1)$ (defined in Section 1). Along the sequence of n for which $\beta_n/n^{\frac{1}{2}} \to \gamma$ we have

$$(4.6) \quad E(\beta_n/n^{\frac{1}{2}} + S_{n\tau}/n^{\frac{1}{2}})(1 + \tau)^{-1} \to E(\gamma + W(\tau_\alpha))(1 + \tau_\alpha)^{-1}.$$

The rigorous proof of (4.6) is based on the invariance principle and is deferred to Section 8. Assuming (4.6) and passing to the limit in (4.5) we get

$$(4.7) \qquad \gamma \geqq E(\gamma + W(\tau_\alpha))(1 + \tau_\alpha)^{-1}.$$

The right side is $V(\gamma, 1)$ by Theorem 1. By the remark after Theorem 1, $V(\gamma, 1) > \gamma$ since $\gamma < \alpha$. This contradicts (4.7) and so $\liminf \beta_n/n^{\frac{1}{2}} \geqq \alpha$.

To prove the inequality $\limsup \beta_n/n^{\frac{1}{2}} \leqq \alpha$, suppose instead that for a sequence Q of values of n, $\beta_n/n^{\frac{1}{2}} \to \gamma > \alpha$. Dvoretzky showed that for $u < \beta_n$, $u/n < v(u, n)$ which can be written

$$(4.8) \qquad \eta < E(\eta + S_\nu/n^{\frac{1}{2}})(1 + \nu/n)^{-1}$$

where $\eta = u/n^{\frac{1}{2}}$ and $\nu = \nu(u, n)$ is defined as above. Fix η, $\alpha < \eta < \gamma$. In Section 8 we will show that (i) along a subsequence of Q, $\nu(\eta n^{\frac{1}{2}}, n)/n$ converges in distribution to a stopping time (call it ξ) for W, (ii) another application of the invariance principle gives

$$(4.9) \qquad E(\eta + S_\nu/n^{\frac{1}{2}})(1 + \nu/n)^{-1} \to E(\eta + W(\xi))(1 + \xi)^{-1}.$$

From (4.8) and (4.9) we have $\eta \leqq V(\eta, 1, \xi)$. Dvoretzky showed that $\beta_n \leqq \beta_{n+1}$ and on the basis of this we will prove in Section 8 that (iii) $\xi \geqq t(\gamma) = \min [t : \eta + W(t) \geqq \gamma]$; in particular $\xi \neq 0$. Since $\eta > \alpha$, Theorem 1 implies that $\tau_\alpha(\eta, 1) = 0$ and so $\xi \neq \tau_\alpha(\eta, 1)$. But by the uniqueness assertion of Theorem 1 we get the strict inequality $\eta = V(\eta, 1, \tau_\alpha) > V(\eta, 1, \xi)$, Since we already showed that $\eta \leqq V(\eta, 1, \xi)$ we have a contradiction. Thus $\limsup \beta_n/n^{\frac{1}{2}} \leqq \alpha$ and so $\lim \beta_n/n^{\frac{1}{2}} = \alpha$.

5. The principle of smooth fit at the optimal boundary. One expects [7], p. 82, the solution of an optimal stopping problem for a Markov process to be given in terms of a so-called continuation set C. Namely, the optimal rule is to continue observing the process while it is in C and to stop and take the payoff (call it g) at the time $\tau(C)$ of first exit from C. One technique [7], credited in [18] to V. S. Mikhalevich, for finding C consists in observing that the payoff V under $\tau(C)$

for the optimal C satisfies a free boundary problem [18]: (1) $\mathcal{A}V = 0$ in C where $\mathcal{A}$ is the generator of the process; (2) V agrees with g in the complement of C; (3) *V fits smoothly along the boundary of C.*

In our problem, we are viewing $(W(t), t)$ and if $W(u) = b$ we can stop and take the payoff $g(u, b) = u/b$. Ignoring the necessary differentiability of V and C, conditions (1)–(3) above become:

$$(\partial V/\partial b) + \tfrac{1}{2}(\partial^2 V/\partial u^2) = 0, \qquad (u, b) \in C, \tag{5.1}$$

$$V = g, \qquad (u, b) \notin C, \tag{5.2}$$

$$\partial V/\partial u = \partial g/\partial u, \quad \partial V/\partial b = \partial g/\partial b, \qquad (u, b) \in \partial C, \tag{5.3}$$

where $V = V(u, b)$, $-\infty < u < \infty$, $b \geqq 0$.

Equations (5.1)–(5.3), which are similar to the Stefan problem in partial differential equations, hopefully would determine V and C uniquely. It is easy to check that (1.4) provides one solution to (5.1)—(5.3) and it is of the form $V(u, b) = h(u/b^{\frac{1}{2}})/b^{\frac{1}{2}}$. That the solution should be of this form is intuitive probabilistically as we have seen in (1.6). Substituting $h(u/b^{\frac{1}{2}})/b^{\frac{1}{2}}$ for V in (5.1) gives a second order equation for h which of course has two linearly independent solutions. One solution gives (1.4), and the other gives

$$\begin{aligned} V^*(u, b) &= (eb)^{-\frac{1}{2}} e^{u^2/2b}, & u &< b^{\frac{1}{2}}, \\ V^*(u, b) &= u/b, & u &\geqq b^{\frac{1}{2}}, \end{aligned} \tag{5.4}$$

which satisfies (5.1)–(5.3) with $C^* = \{u < b^{\frac{1}{2}}\}$. However, $V^*(u, b)$ may be rejected because it does not tend to zero as $u \to -\infty$, and perhaps under this additional assumption (5.1)–(5.3) have the unique solution (1.4). It would be of interest to find enough conditions on V and C for (5.1)–(5.3) to have a unique solution and to derive Theorem 1 by this method.

6. Other sequential problems with simple solutions. a. Suppose we have an urn with m minus ones and p plus ones. We draw at random *without replacement* until we want to stop. We know the values of m and p and are also allowed not to draw at all. Which urns are favorable ? That is to say, for which m and p can we make the expected return positive?

Letting C denote the set of (m, p) urns for which the expected return V_{mp} under optimal stopping is positive, it is clear that we should stop as soon as the depleted urn is not in C. It is intuitive but not trivial to prove that $V_{mp+1} \geqq V_{mp} \geqq V_{m+1p}$. To prove that $V_{mp+1} \geqq V_{mp}$ we may argue as follows: Any strategy for the (m, p) game can be used for the $(m, p + 1)$ game provided we are allowed to single out one of the plus ones and distinguish it. The distinguished plus one is ignored in applying the (m, p) strategy but we get paid for it, thereby getting a bigger payoff if the distinguished one comes up in the course of play and getting the same payoff as in the (m, p) game if the distinguished one does not appear. To get around the objection that singling out one of the plus ones is not legitimate we can proceed as follows: Each time we draw a plus we decide *probabilistically* whether this one was the distinguished one by performing an additional random experiment on the side. It is clear that we can design the random experiment so that the payoff has the same distribution as if we actually had previously distinguished one of the pluses. For those who object to introducing additional randomness we add that it is easy to see that allowing additional randomness does not increase V_{mp+1} and so there is a "proper strategy" which does as well or better than the above randomized strategy. This completes the proof that $V_{mp} \leqq V_{mp+1}$. The proof that $V_{m+1p} \leqq V_{mp}$ is similar and is omitted. W. M. Boyce (to whom we are grateful for obtaining enlightening numerical calculations on a number of stopping rule problems) has proven further that $V_{mp} \leqq V_{m+1,p+1}$ for all m and p, which he had conjectured on the basis of numerical evidence.

The urn problem is computationally simpler than the S_n/n problem (where techniques of backward induction and limits must be applied [21], [24]) because V_{mp} satisfies an easily derived forward recursion formula. However, for large m and p roundoff error again introduces difficulties. Using similar techniques to the S_n/n problem, we can again find the asymptotic boundary of C. Since $V_{m+1,p} \leqq V_{m,p}$, there is a sequence $\beta(1), \beta(2), \cdots$ of integers for which $C = \{(m, p): m \leqq \beta(p)\}$. Of course $\beta(p) \geqq p$ and for large p we have

$$\lim_{p\to\infty} (\beta(p) - p)(2p)^{-\frac{1}{2}} = \alpha, \tag{6.1}$$

where α is again given by (1.3).

b. The method of proof of (6.1) again uses a continuous time approximation with exact solution, as follows: Given u and b, $-\infty < u < \infty$ and $b > 0$ consider the process W^* which is the Wiener process W conditioned (pinned) to pass through $-u$ at $t = b$. Find

$$G(u, b) = \sup EW^*(\tau)$$

where the sup is taken over all stopping times $\tau \leqq b$. This problem can be solved via scale change invariance and the Wald identity, but it is simpler to reduce it directly to the problem of Section 1 by applying the following representation of W^* due to J. L. Doob [12],

$$W^*(s) = -us/b + (1 - s/b)W(s/(1 - s/b)), \qquad 0 \leqq s \leqq b. \tag{6.2}$$

Setting $s/(1 - s/b) = t$ for any stopping time s, we have

$$EW^*(s) = -u + bE(u + W(t))(b + t)^{-1}. \tag{6.3}$$

Since the transformation between s and t is one-one we get immediately from Theorem 1 and (6.3) that

$$G(u, b) = -u + bV(u, b) \tag{6.4}$$

where V is given by (1.4). The optimal stopping time τ corresponds to τ_α for W and from (6.2) comes out to be the first time τ that

$$u + W^*(\tau) \geqq \alpha(b - \tau)^{\frac{1}{2}}. \tag{6.5}$$

It is easy to check that $\tau < b$ almost surely.

To connect up with the urn problem, let $\epsilon_1, \cdots, \epsilon_{m+p}$ be the random sequence of ± 1's obtained by drawing until the end. Fix m and p, and for $0 \leqq k < m + p$ and $k < (m + p)t \leqq k + 1$ define an approximate pinned Wiener process

$$W_{mp}^*(t) = (\epsilon_1 + \cdots + \epsilon_k)(m + p)^{-\frac{1}{2}}, \qquad 0 \leqq t \leqq 1.$$

Let u be defined by

$$-u = (-m + p)(m + p)^{-\frac{1}{2}}. \tag{6.6}$$

Then $W_{mp}^*(1) = -u$ and if u is fixed, $W_{mp}^*(t)$ converges in distribution as $p \to \infty$ to $W^*(t)$ the Wiener process pinned to $-u$ at $t = 1$. Setting $b = 1$ in (6.5), a formal passage to the limit shows that if m and p satisfy (6.6) for u fixed and p large enough then $(m, p) \varepsilon C$ if and only if $u < \alpha$. Since $m + p \sim 2p$ from (6.6), this is equivalent to (6.1).

More generally, (6.4) and the above formal reasoning indicate that for $-\infty < u < \infty$ fixed we have

$$\lim_{p\to\infty} (2p)^{-\frac{1}{2}} V_{[p+(2p)^{\frac{1}{2}}u, p]} = -u + V(u, 1) \tag{6.7}$$

where $V(u, 1)$ is given by (1.4). For $u = 0$ this gives $V_{pp} \sim (1 - \alpha^2)(\pi p)^{\frac{1}{2}}$ as $p \to \infty$. Since $V(u, 1) = u$ for $u \leqq \alpha$, (6.7) is consistent with (6.1). A rigorous proof of (6.7) has not actually been carried out by us but it seems that one similar to the proof of Theorem 3 could be constructed.

c.[1] We next find for $-\infty < u < \infty, 0 < b, 0 < \gamma < 2\delta$,

$$V_{\gamma,\delta}(u, b) = \sup_{\tau\geqq 0} E[(u + W(\tau))^+]^\gamma (b + \tau)^{-\delta} \tag{6.8}$$

where $x^+ = (x + |x|)/2$. The heuristics of Section 1 indicate that the optimal rule is again a τ_c (see Theorem 1) for some $c = c(\gamma, \delta)$. The method of Section 3, slightly modified (in (3.1) multiply also by $(c\lambda)^\beta$, $\beta = 2\delta - \gamma - 1$ and proceed as before), shows that $c = c(\gamma, \delta)$ is the *unique* maximum of $c^\gamma / \int_0^\infty \lambda^\beta \cdot \exp(\lambda c - \lambda^2/2)\, d\lambda$ where $\beta = 2\delta - \gamma - 1$ and indicates that

$$V_{\gamma,\delta}(u, b) = c^\gamma \int_0^\infty \lambda^\beta \exp(\lambda u - \lambda^2 b/2)\, d\lambda / \int_0^\infty \lambda^\beta \exp(\lambda c - \lambda^2/2)\, d\lambda. \tag{6.9}$$

For $\gamma = \delta = 1$, (6.8) and (6.9) reduce to Theorem 1.

Dvoretzky [14], Remark 1, points out that if $0 < \gamma < 2\delta$ and if F has a moment of order max $(2, \gamma)$ then an optimal stopping time N for maximizing $E(S_N^+)^\gamma / N^\delta$ exists and is of the form $N = \min[k: S_k \geqq \beta_k]$ as before. It can then be shown [30] that $\beta_n / n^{\frac{1}{2}} \to c(\gamma, \delta)$.

d. We next find for $-\infty < u < \infty$ and $b > 0$

$$V'(u, b) = \sup_{\tau\geqq 0} E|(u + W(\tau))(b + \tau)^{-1}|. \tag{6.10}$$

The heuristic method of Section 1 now indicates that the optimal rule is of the form $\tau_c' = \min[t: |u + W(t)| \geqq c(b + t)^{\frac{1}{2}}]$. The Wald identity argument of Section 3 again can be used to determine c, but the calculations must be modified as follows. Set $T = \tau_c'$ and note that on $\{T \geqq t\}$ we must have $|u + W(t)| \leqq$

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[1] The remainder of this section has been modified after seeing the thesis of L. H. Walker [30] (see Acknowledgment at end of paper). We originally had treated only the special case $\gamma = \delta$.

$c(b+t)^{\frac{1}{2}}$. Thus in the notation of Section 2 for any λ, $-\infty < \lambda < \infty$,

$$E_{\{T \geqq t\}} Y(t) \leqq E_{\{T \geqq t\}} \exp [|\lambda|(|u| + c(b+t)^{\frac{1}{2}}) - \lambda^2 t/2] \tag{6.11}$$
$$\leqq \exp [|\lambda|(|u| + c(b+t)^{\frac{1}{2}}) - \lambda^2 t/2].$$

Since (2.2) and (2.4) continue to hold we find as $t \to \infty$ for any λ

$$E e^{\lambda W(T) - \lambda^2 T/2} = 1. \tag{6.12}$$

Multiplying in (6.12) by $(\frac{1}{2}) \exp \lambda u$, writing the result also for $-\lambda$, and adding we get

$$\int_0^\infty \cosh \lambda c (b+t)^{\frac{1}{2}} e^{-\lambda^2 t/2} \, dF_c'(t) = \cosh \lambda u \tag{6.13}$$

where $F_c'(t) = P\{\tau_c' \leqq t\}$. Now multiply in (6.13) by $\exp(-\lambda^2 b/2)$ and integrate on λ from 0 to ∞ to obtain an expression for the payoff under τ_c'. It is easy to see that the maximal payoff occurs for $c = 1$ and we obtain

$$V'(u, b) = (be)^{-\frac{1}{2}} e^{u^2/2b}, \qquad |u| < b, \tag{6.14}$$
$$V'(u, b) = |u|/b, \qquad |u| \geqq b.$$

It is easily checked that V' and $C' = \{(u, b) : u < b^{\frac{1}{2}}\}$ satisfy the appropriate free boundary problem.

We remark that it is possible to find explicitly for given $b \geqq 0$ and $-\infty < u < \infty$: (i) for given $0 < \gamma < 2\delta$,

$$V'_{\gamma,\delta}(u, b) = \sup_{\tau \geqq 0} E(|u + W(\tau)|^\gamma (b+\tau)^{-2\delta}). \tag{6.15}$$

(ii) for given $\epsilon > 0$, the maximum in (6.8) (or (6.15)) over $\tau \geqq \epsilon$. (i) is a straightforward extension of (6.10) and (ii) is obtained by observing that the optimal stopping time $\tau \geqq \epsilon$ is simply $\epsilon + \tau_c$ (or $\epsilon + \tau_c'$). The explicit expressions for the maxima are easily obtained. Moments of the stopping times τ_c' were studied in [22], and further information on their distribution is obtainable from [6]. In particular for $c = 1$, which is a borderline case for the first moment, $E\tau_c' = \infty$.

e. Consider next the problem of maximizing $E \int_0^\tau W(u)\,du$ over stopping times $\tau \leqq 1$. H. S. Witsenhausen observed that the principle of scale change invariance indicates that there is a number $c < 0$ for which optimal stopping takes place at the first $t < 1$ at which $W(t) = c(1-t)^{\frac{1}{2}}$ or at $t = 1$ if there is no such $t < 1$. The determination of c seems to be difficult. Using integration by parts and the identity obtained from the Wald identity by expanding in powers of λ and setting the coefficient of λ^3 to zero, it can be shown that for any stopping time $\tau \leqq 1$,

$$E \int_0^\tau W(u)\,du = E\tau W(\tau) = \tfrac{1}{3} E W^3(\tau). \tag{6.16}$$

It follows that the same stopping time solves the problem of maximizing each of the terms in (6.16).

f. The Wald problem of sequential hypothesis testing is similar in many ways to the problem of Section 1. We state the Wald problem in continuous time for purposes of comparison. A parameter μ with a given prior distribution assumes only two values. We observe $W(t) + \mu t$ for as long as we please, finally stopping at τ and deciding on the value of μ. Given $\epsilon > 0$, it is desired to minimize $E\tau$ over all stopping times τ for which the probability of error (using the best decision rule at time τ) is $\leqq \epsilon$. As is well known [3], p. 246, it is easy to see heuristically that the optimal stopping boundary is a pair of straight lines, because *if we have not stopped at any given time we are still in the initial situation except for new (posterior) values of the prior probabilities.* The heights of the two lines can be determined either from Wald's identity [1] or from the principle of smooth fit [26]. The prior probabilities make the problem appear more difficult, but actually ([23]) only amount to a change in the generator of the process.

Both problems are made much more difficult if slight changes are made in their formulations; for example, if (i) the loss is τ^a, $a \neq 1$, in Wald's problem, or

(ii) τ is required to be less than a fixed time in either problem, then the properties of invariance or homogeneity are lost. On the other hand, crude examples indicate that the solutions of these and other optimal stopping problems are stable in the sense that nearly optimal payoffs can be obtained with almost any stopping boundary, reasonably close to the optimal boundary.

7. Proof of Theorem 1. We will be using many of the results of [8] and [14] in the course of the proofs, so that it will be helpful to have these references at hand. In continuous time the simple inductive definition (Section 4) of a stopping time breaks down. We depend instead on Lemma 1 which allows us to approximate any stopping time for W by a stopping time which is (i) discrete-valued and (ii) based on discrete time observations of W.

For $h > 0$, let $\mathfrak{D}(h)$ denote the class of stopping times τ for which (i) $\tau = kh$, $k = 0, 1, 2, \cdots$, and (ii) for all k

$$\{\tau > kh\} \in \mathfrak{F}(h, 2h, \cdots kh) \tag{7.1}$$

where $\mathfrak{F}(t_1, \cdots, t_n)$ is the σ-field generated by $W(t_1), \cdots, W(t_n)$.

LEMMA 1. *Given any stopping time τ, there is a sequence τ_n with $\tau_n \in \mathfrak{D}(2^{-n})$ for which $\tau = \lim \tau_n$ almost surely.*

PROOF. For any $h > 0$ let $\tau(h) = kh$ if k is the first integer for which

$$P\{\tau > kh \mid \mathfrak{F}(h, \cdots, kh)\} \leqq \tfrac{1}{2}. \tag{7.2}$$

It is clear that $\tau(h) \in \mathfrak{D}(h)$. To show that $\tau_n = \tau(2^{-n}) \to \tau$ almost surely, fix $a > 0$. Since $\mathfrak{F}(1 \cdot 2^{-n}, 2 \cdot 2^{-n}, \cdots, [a2^n]2^{-n}) \uparrow \mathfrak{F}_a$ as $n \uparrow \infty$ the martingale theorem gives that almost surely

$$P\{\tau > a \mid \mathfrak{F}(1 \cdot 2^{-n}, 2 \cdot 2^{-n}, \cdots, [a2^n]2^{-n})\} \to P\{\tau > a \mid \mathfrak{F}_a\}. \tag{7.3}$$

But τ is a stopping time and so $\{\tau > a\} \in \mathfrak{F}_a$. On $\{\tau < a\}$ the left side of (7.3) is tending to 0 in n and is therefore eventually less than $\frac{1}{2}$. (We could as well have replaced $\frac{1}{2}$ in (7.2) by any number in (0, 1).) By definition $\tau(2^{-n}) \leqq a$ for n sufficiently large and it follows that $\limsup \tau(2^{-n}) \leqq \tau$. To prove that $\liminf \tau(2^{-n}) \geqq \tau$, suppose instead that for some number a and for a sequence of

n, $\lim \tau(2^{-n}) \leqq a < \tau$. Then (7.2) must hold where $h = 2^{-n}$ and $k = k_n \leqq 2^n a$. Passing to a subsequence for which $k_n 2^{-n} \to b \leqq a$ we see that for $\epsilon > 0$ and n sufficiently large (in the subsequence) that $b + \epsilon > k_n 2^{-n}$ and so

$$(7.4)\quad P\{\tau > b + \epsilon \mid \mathfrak{F}(1\cdot 2^{-n}, \cdots, k_n 2^{-n})\} \leqq P\{\tau > k_n 2^{-n} \mid \mathfrak{F}(1\cdot 2^{-n}, \cdots, k_n 2^{-n})\}.$$

Since the right side of (7.4) is $\leqq \frac{1}{2}$ by (7.2), applying the martingale theorem to the left side gives

$$(7.5)\qquad P\{\tau > b + \epsilon \mid \mathfrak{F}_b\} \leqq \tfrac{1}{2}.$$

Now letting $\epsilon \to 0$ gives $P\{\tau \geqq b \mid \mathfrak{F}_b\} \leqq \frac{1}{2}$. But $P\{\tau \geqq b \mid \mathfrak{F}_b\} = 1$ since we are on the set where $\tau > a \geqq b$. The contradiction shows that $\liminf \tau(2^{-n}) \geqq \tau$ and the lemma is proved.

We can now prove the following analogue of [14], Lemma 8.

LEMMA 2. *If* $u \geqq 5b^{\frac{1}{2}} > 0$ *then* $V(u, b, \tau) \leqq u/b$ *for every stopping time* τ.

PROOF. If not, let τ be a stopping time for which $V(u, b, \tau) > u/b$. Let τ_n be the stopping times of Lemma 1 for τ and define

$$(7.6)\qquad Y_n(\omega) = (u + W(\tau_n(\omega), \omega))(b + \tau_n(\omega))^{-1},$$
$$Y(\omega) = (u + W(\tau(\omega), \omega))(b + \tau(\omega))^{-1}.$$

We have $V(u, b, \tau_n) \to V(u, b, \tau)$ as $n \to \infty$ because (i) $Y_n \to y$ almost surely and (ii) the sequence Y_n is dominated by the integrable Z on (1.8). Thus $V(u, b, \tau_n) > u/b$ for some n. But this is impossible since $\tau_n \varepsilon \mathfrak{D}(2^{-n})$ is an inductively generated stopping time and [14], Lemma 8, is directly applicable to show $V(u, b, \tau_n) \leqq u/b$. The contradiction proves Lemma 2.

Using the continuous time analogue of [14], Lemma 10, which is based on the notion [8] (see also [19]) of a regular stopping time it now follows that there exists a stopping time τ with

$$(7.7)\qquad V(u, b, \tau) = V(u, b).$$

In order to find the form of such a τ, fix b and let S be the set of c for which $c = u/b^{\frac{1}{2}}$ for u satisfying $V(u, b) > u/b$. (1.6) shows that S does not depend on b, and the proof of [14], Lemma 5, carries over easily to show that S is a convex set (a semi-infinite interval). Let γ denote the lub of S (incidentally, Lemma 2 shows that $\gamma < 5$). Let τ_γ be defined as in Section 1. For any stopping time τ, taking conditional expectations show that $V(u, b, \tau) \leqq V(u, b, \tau \wedge \tau_\gamma)$, and it follows that there is at least one optimal stopping time $\leqq \tau_\gamma$. More explicitly, the proof of [8], Corollary 1, is adaptable to show that there is a stopping time $\tau^* \leqq \tau_\gamma$ with

$$(7.8)\qquad V(u, b, \tau^*) = \sup_{\tau \leqq \tau_\gamma} V(u, b, \tau).$$

To prove that $\tau^* = \tau_\gamma$ for any τ^* satisfying (7.8) suppose instead that

$P(\tau^* < \tau_\gamma) > 0$. If at the time τ^*, $u + W(\tau^*) < \gamma(b + \tau^*)^{\frac{1}{2}}$ then

$$\beta = (u + W(\tau^*))/(b + \tau^*)^{\frac{1}{2}} \varepsilon S,$$

so there is a stopping time τ for which

$$(7.9)\quad E[u + W(\tau^*) + (W(\tau^* + \tau) - W(\tau^*))](b + \tau^* + (\tau^* + \tau - \tau^*))^{-1} > (u + W(\tau^*))(b + \tau^*)^{-1}.$$

Then $V(u, b, \tau^* + \tau) > V(u, b)$ and so τ^* was not optimal. It is of course necessary to show that τ can be defined in such a way that $\tau^* + \tau$ is measurable, but this is not difficult.

We have seen that there is a stopping time τ satisfying (7.7) and that τ is of the form τ_c, $c = \gamma$. Since only τ_α achieves the maximum payoff among the τ_c we must have $c = \alpha = \gamma$. The last paragraph also shows that τ_α is the *minimal* optimal stopping time.

It is not yet proved that τ_α is the unique optimal stopping time, since there may be some $\tau > \tau_\gamma$ with $V(u, b, \tau) = V(u, b)$. To prove this fact, which will be needed in Section 8, we note that (by taking conditional expectations) it suffices to prove that if τ satisfies

$$(7.10)\qquad V(u, b, \tau) = u/b$$

for some u and b, $u > \alpha b^{\frac{1}{2}} > 0$ then $P\{t = 0\} = 1$. We shall not need the fact [20], p. 88, that $P\{t = 0\}$ is either 0 or 1. If $P\{\tau = 0\} < 1$ then we have

$$(7.11)\qquad E(b + \tau)^{-1} < b^{-1}$$

and so the derivative with respect to u of the left side of (7.10) is less than that of the right side. Hence there exists u', $u > u' > \alpha b^{\frac{1}{2}}$ for which

$$(7.12)\qquad V(u', b, \tau) > u'/b$$

which contradicts that $\alpha = \gamma$, already proved. Thus τ_α is unique and the proof of Theorem 1 is complete.

8. Completion of the proof of Theorem 3. To prove (4.6), we observe that it is sufficient to show that as $n \to \infty$

$$(8.1)\qquad E(\gamma + S_{n\tau}/n^{\frac{1}{2}})(1 + \tau)^{-1} \to E(\gamma + W(\tau_\alpha))(1 + \tau_\alpha)^{-1}$$

because the left sides of (8.1) and (4.6) differ by at most $|\gamma - \beta_n/n^{\frac{1}{2}}|$. Define $S_0 = 0, S_t = S_{[t]}$, $t > 0$ and for fixed T consider the functional F defined on piecewise continuous functions X on $[0, T]$ by $F[X] = (\gamma + X(\theta))/(1 + \theta)$ where $\theta = T \wedge \inf [t: \gamma + X(t) \geqq \alpha(1 + t)^{\frac{1}{2}}]$. We note that F is continuous in the uniform topology at any $X = X(t)$ having no points of tangency to the curve $\alpha(1 + t)^{\frac{1}{2}} - \gamma$. The set of such X has probability one (under W). The invariance principle [11] therefore applies and we obtain that as $n \to \infty$ for each $T > 0$ and each y, $-\infty < y < \infty$,

$$(8.2)\quad P\{(\gamma + S_{n(\tau\wedge T)}/n^{\frac{1}{2}})(1 + (\tau \wedge T))^{-1} \geqq y\} \rightarrow P\{(\gamma + W(\tau_\alpha \wedge T))(1 + (\tau_\alpha \wedge T))^{-1} \geqq y\}.$$

Standard arguments allow us to integrate on y, pass to the limit $T \rightarrow \infty$, and obtain (8.1).

The proof of (4.9) is more delicate and we give it in more detail. First we note that in the definition of the numbers β_n in (4.2) we can let n be any nonnegative number, not necessarily an integer. Define for $t \geqq 0$,

$$(8.3)\qquad f_n(t) = \beta(n(1 + t))n^{-\frac{1}{2}}$$

where we have written $\beta(n) = \beta_n$. Since $\beta(n)$ increases [14], p. 448, $f_n(t)$ increases in t. For fixed t and n sufficiently large ([14], (49) and (50)) show that $f_n(t)$ is bounded

$$(8.4)\qquad .3(1 + t)^{\frac{1}{2}} \leqq f_n(t) \leqq 5(1 + t)^{\frac{1}{2}}.$$

By choosing a countable, dense set of points t and applying the usual Helly-Bray diagonal argument, it is seen that there is a subsequence Q' of Q (the sequence along which $\beta_n/n^{\frac{1}{2}} \rightarrow \gamma > \alpha$) along which $\lim f_n(t)$ exists for each $t \geqq 0$. Thus there is a left-continuous monotonically increasing function $f(t)$ with $.3 \leqq f(t)/(1 + t)^{\frac{1}{2}} \leqq 5$ for which $f_n(t) \rightarrow f(t)$ as $n \varepsilon Q'$ tends to infinity for each t at which f is continuous. We note that we may take

$$f(0) = \lim f_n(0) = \lim \beta_n/n^{\frac{1}{2}} = \gamma.$$

As in Section 4 choose η, $\alpha < \eta < \gamma$, and let $n \varepsilon Q'$ satisfy $\eta n^{\frac{1}{2}} < \beta_n$. In our new notation, the definition (Section 4) of $\nu_n = \nu(\eta n^{\frac{1}{2}}, n)$ becomes

$$(8.5)\qquad \nu_n/n = \min\,[t{:}\,\eta + S_{nt}/n^{\frac{1}{2}} \geqq f_n(t)].$$

Let ξ be the stopping variable for W defined by

$$(8.6)\qquad \xi = \min\,[t{:}\,\eta + W(t) \geqq f(t)].$$

To prove that $\nu_n/n \rightarrow \xi$ in distribution; we see from (8.5) that for fixed $t > 0$,

$$(8.7)\qquad P\{\nu_n/n \leqq t\} = P\{\eta + S_{n\tau}/n^{\frac{1}{2}} \geqq f_n(\tau), \quad \text{some} \quad \tau \leqq t\}.$$

For any $\epsilon > 0$, if n is large enough we must have[2] $f(\tau) - \epsilon \leqq f_n(\tau) \leqq f(\tau+) + \epsilon$ for all τ, $0 \leqq \tau \leqq t$ and so for large values of n,

$$(8.8)\qquad \begin{aligned} &P\{\eta + S_{n\tau}/n^{\frac{1}{2}} \geqq f(\tau+) + \epsilon, \quad \text{some} \quad \tau \leqq t\} \\ &\leqq P\{\eta + S_{n\tau}/n^{\frac{1}{2}} \geqq f_n(\tau), \quad \text{some} \quad \tau \leqq t\} \\ &\leqq P\{\eta + S_{n\tau}/n^{\frac{1}{2}} > f(\tau) - \epsilon, \quad \text{some} \quad \tau \leqq t\}. \end{aligned}$$

Letting $n \rightarrow \infty$, we may apply the invariance principle to the extreme terms in

[2] NOTE ADDED IN PROOF. We are grateful to Mrs. M. E. Thompson for kindly pointing out that continuity of f is tacitly assumed in (8.8). She elegantly fills this gap by showing directly that f_n are equicontinuous.

(8.8). The first term in (8.8) tends to $P\{\eta + W(\tau) \geqq f(\tau+) + \epsilon$, some $\tau \leqq t\}$ and the last term tends to $P\{\eta + W(\tau) \geqq f(\tau) - \epsilon$, some $\tau \leqq t\}$. As $\epsilon \to 0$, the difference between the latter probabilities is seen to be small and hence the middle term in (8.8) tends to a limit. Since the middle term is simply $P\{\nu_n/n \leqq t\}$ we obtain that

$$(8.9) \qquad P\{\nu_n/n \leqq t\} \to P\{\xi \leqq t\},$$

which proves assertion (i) of Section 4.

To prove (4.9) it is necessary to show that the overshoot of $\eta + S_\nu/n^{\frac{1}{2}}$ over $f_n(\nu/n)$ is small, where we have written ν for ν_n. For $\epsilon > 0$, let $A_n(\epsilon)$ denote the event that

$$(8.10) \qquad |\eta + S_{\nu n}/n^{\frac{1}{2}} - f(\nu_n/n)| < \epsilon$$

and let $B_n(\epsilon)$ denote the complementary event. Set

$$(8.11) \qquad Y_n = (\eta + S_{\nu n}/n^{\frac{1}{2}})(1 + \nu_n/n)^{-1}, \qquad Y = (\eta + W(\xi))(1 + \xi)^{-1}.$$

The difference between the left and right sides of (4.9) is less than

$$(8.12) \qquad |EY_n - EY| \leqq |E(Y_n - f(\nu_n/n)(1 + (\nu_n/n))^{-1})| + |E(f(\nu_n/n))(1 + (\nu_n/n))^{-1} - EY|.$$

It is clear that the second term on the right of (8.12) tends to zero since $\nu_n/n \to \xi$ in distribution and $Y = f(\xi)/(1 + \xi)$. The first term is decomposed into an expectation over $A_n(\epsilon)$ and over $B_n(\epsilon)$. In the integral over $A_n(\epsilon)$, the integrand is everywhere $<\epsilon$ by (8.10). Another application of the invariance principle shows that $P(B_n(\epsilon))$ is small for large n and small ϵ, and it follows that the integral over $B_n(\epsilon)$ goes to zero. This proves (ii) of Section 4. (iii) follows immediately from (8.6) since $\eta < \gamma = f(0) \leqq f(t)$ as was already proved. The proof of Theorem 3 is now complete.

9. The general form of the Wald identity in the Gaussian case. For any f and λ, the proof of Theorem 2 shows that the left side of (2.1) is between zero and one. This suggests that it can be written as a probability.

THEOREM 4. *If $f(t)$, $t \geqq 0$, is continuous and $-\infty < \lambda < \infty$ then*

$$(9.1) \qquad \int_0^\infty e^{\lambda f(t) - \lambda^2 t/2}\, dF(t) = P\{W(t) = f(t) - \lambda t, \text{ for some } t < \infty\}$$

where $F(t) = P\{T \leqq t\}$, and T is defined as in Section 2.

As before, the left side of (9.1) is $E_{\{T<\infty\}} \exp(\lambda W(T) - \lambda^2 T/2)$. It is said that the Wald identity holds for a particular f and λ provided that (9.1) is true with the right side replaced by unity. For given f and λ, Theorem 4 reduces the problem of deciding when the Wald identity holds to a familiar problem, where Kolmogorov's test [20] gives the answer in most cases. To prove the theorem, we have in the notation of Section 2, by continuity of f and W and monotone convergence

$$(9.2) \qquad E_{\{T>t\}} Y(t) = \lim_{n\to\infty} E_{\{W(t_i)<f(t_i),\, i=1,\cdots,n\}} Y(t)$$

where $0 < t_1 < \cdots < t_n < t$ is a sequence of refining partitions which become dense in $[0, t]$. A short calculation shows that the nth term on the right side of (9.2) can be written simply as $P\{W(t_i) + \lambda t_i \leqq f(t_i), i = 1, \cdots, n\}$. Again by monotone convergence, we get

$$(9.3) \qquad E_{\{T>t\}}Y(t) = P\{W(s) + \lambda s \leqq f(s), 0 \leqq s \leqq t\}.$$

Using (2.2) and (2.4) we get from (9.3)

$$(9.4) \qquad 1 = E_{\{T \leqq t\}} e^{\lambda W(T) - \lambda^2 T/2} + P\{W(s) + \lambda s \leqq f(s), 0 \leqq s \leqq t\}.$$

It is known that the last term in (9.4) does not change if the first inequality in the braces is made strict. Letting $T \to \infty$ and noting that $W(T) = f(T)$ we obtain (9.1).

In the special case when the basic distribution is Gaussian[3] we can sharpen the usual Wald identity [1], p. 17, in a similar way. Let $\eta_1, \eta_2, \cdots$ be standard Gaussian and independent and consider the general stopping time $N(\eta_1, \eta_2, \cdots) = \min[n: (\eta_1, \eta_2, \cdots) \varepsilon D_n]$ where D_n is a given set in Euclidean space, $n = 1, 2, \cdots$; set $N(\eta_1, \eta_2, \cdots) = \infty$ if there is no n for which

$$(\eta_1, \eta_2, \cdots, \eta_n) \varepsilon D_n .$$

Suppose $-\infty < \lambda < \infty$, and denote $\phi(\lambda) = \exp(\lambda^2/2)$, $S_n = \eta_1 + \eta_2 + \cdots + \eta_n$, $n = 1, 2, \cdots$. Then

$$(9.5) \quad E_{\{T<\infty\}} e^{\lambda S_N} \phi(\lambda)^{-N} = P\{(\eta_1 + \lambda, \cdots, \eta_n + \lambda) \varepsilon D_n \text{ for some } n\}.$$

This is proved in the same way as Theorem 4 and yields the Wald identity whenever the right side of (9.5) is unity.

Although we have made no use of the results, we have included this section for its own interest. Theorem 4 is closely related to the interesting work of I. V. Girsanov [17] which we intend to discuss elsewhere.

10. A conjecture for the case of infinite second moment. As pointed out by Dvoretzky, the original S_n/n problem makes sense even if the second moment of F is infinite (so long as the first moment exists) and it seems likely that an optimal stopping rule exists [14], Remark 6. It seems reasonable by analogy with Theorem 3 that the following should be true.

If $S_n/n^{1/\alpha}$ converges in distribution to G, a strict sense [16] stable law with exponent α (not to be confused with α of (1.3)) then if $1 < \alpha < 2$: (i) there is a unique minimal optimal stopping time N for $E(u + S_N)/(n + N)$ for each $n \geqq 0$ and $-\infty < u < \infty$, (ii) $N = \min[k: S_k \geqq \beta_k]$ for a sequence β_k, (iii) the limit

$$(10.1) \qquad \lim_{n\to\infty} \beta_n n^{-(1/\alpha)} = c$$

exists, (iv) c satisfies $\beta(b) = cb^{1/\alpha}$, $b > 0$, where $\beta(b)$ is the least u for which

$$(10.2) \qquad E(u + X(\tau))(b + \tau)^{-1} \leqq ub^{-1}$$

[3] It was pointed out to the author by Gus Haggstrom that more general forms of (9.5) have been obtained by R. R. Bahadur [27] and H. D. Millar [28].

for all stopping times $\tau \geqq 0$. In (iv), $X = X(t)$, $t \geqq 0$, is the stable process determined by G, that is X has stationary, independent increments, $X(0) = 0$ and $X(1)$ has distribution G.

Acknowledgment. In a recent paper [29], which came to our attention after this paper was submitted, Howard M. Taylor has explicitly solved several stopping rule problems in continuous time, including the problem of our Theorem 1, by using very elegant potential theoretic methods. Also Le Roy H. Walker, using methods somewhat similar to ours, has independently solved the S_n/n problem and obtained our Theorem 3 among other results. He also extends the results to the case S_n^a/n^b were $0 < a < 2b$

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