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Explicit Solutions to the Semi-discrete Modified KdV Equation and Motion of Discrete Plane Curves

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Abstract

We construct explicit solutions to continuous motion of discrete plane curves described by a semi-discrete potential modified KdV equation. Explicit formulas in terms the τ function are presented. Bäcklund transformations of the discrete curves are also discussed. We finally consider the continuous limit of discrete motion of discrete plane curves described by the discrete potential modified KdV equation to motion of smooth plane curves characterized by the potential modified KdV equation.

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1 Introduction

As is well known, many integrable partial differential equations (integrable systems) have close relationship to differential geometry. In fact, surfaces of specific curvature property in 3-dimensional space forms have sine-Gordon type equation as the integrability condition of surfaces. More generally, harmonic maps of conformal 2-manifolds into semi-Riemannian symmetric spaces are constructed by solutions to 2-dimensional Toda lattice equation (2DTL).

Transformation of solutions to integrable systems have origins in classical differential geometry. The Bäcklund transformation of the sine-Gordon equation are originally formulated as transformations of pseudo-spherical surfaces in Euclidean 3-space.

On the other hand, substantial progress has been made in the study of discretization of integrable systems preserving "integrable structure". Motivated by extensive study on discrete integrable systems, discretizations of curves and surfaces have been recently studied actively.

This paper concerns with geometry of discrete curves in terms of semi-discrete integrable systems. In [3–5], Doliwa and Santini introduced continuous motion of discrete curves in 3-sphere described by the Ablowitz-Ladik hierarchy [2]. The semi-discrete potential mKdV equation was deduced as the simplest case. Hoffmann and Kutz [13] introduced the notion of discrete curvature for plane discrete curves. Using the discrete curvature, they deduced the semi-discrete mKdV equation from continuous motion of plane discrete curves.

In our previous works [14, 17], we have studied discrete motions of plane discrete curves in purely Euclidean geometric manner. The compatibility condition of a discrete motion is the discrete potential mKdV equation proposed by Hirota [8]. In discrete differential geometric setting, the primal geometric object is the potential function rather than curvature (see [17]). Note that potential function coincides with the turning angle function in smooth curve theory. We have constructed explicit solutions of discrete motions of plane discrete curves in [14].

As a continuation of the previous works, in this paper we study continuous motions of plane discrete curves in terms of potential function. The purpose of the present paper is to construct explicit solutions to continuous motions of plane discrete curves by using the so-called τ function. Moreover we shall give Bäcklund transformations of continuous motions of plane discrete curves. The discrete curvature functions and the semi-discrete mKdV equation discussed in [13] are recovered from our results.

We have been working on three categories of curves motions: (1) continuous motions of plane smooth curves, (2) continuous motions of plane discrete curves and (3) discrete motions of plane discrete curves. In this paper we investigate the relationship of these three motions, and show that these motions are connected by appropriate continuous limiting procedure.

This paper is organized as follows. After recalling the requisite facts on the geometry of plane discrete curves and their continuous motion in Section 2, we prepare a representaion formula for continuous motion of plane discrete curves in terms of τ function. This representation enable us to give explicit parametrization of motions determined by multi-solitons as well as multi-breathers in the next Section 4.

As we have mentioned before, Bäcklund transformation is a fundamental and effective tool for construction of solutions. In Section 5, we extend Bäcklund transformations of plane discrete curves studied in our previous work [14] to those of continuous motions. In particular, we give a new formula for Bäcklund transformations on the semi-discrete potential mKdV equation.

In the final section, we shall discuss continuous limits of motions of plane discrete curves. More precisely, first we shall investigate continuous limits of discrete motions of plane discrete curves to continuous motion of those. Next we study continuous limits of continuous motions of plane discrete curves to continuous motions of plane smooth curves. It should be emphasized that these limiting procedure preserve solutions of equations. More precisely, we shall show that these limiting procedure preserve soliton type solutions. This is confirmed by careful analysis of τ functions. Appendix will be devoted to detailed computations of bilinear equations for our use.

In a separate publication [6], we study discrete hodograph transformations and apply those to obtain discretizations of some integrable systems associated with continuous motions of plane smooth curves.

2 Continuous Motion of Plane Discrete Curves

We start with the following definition.

Definition 2.1 A map $\gamma : \mathbb{Z} \to \mathbb{R}^2;\ l \mapsto \gamma_l$ is said to be a *discrete curve* of constant segment length ϵ if

$$\left|\frac{\gamma_{l+1} - \gamma_l}{\epsilon}\right| = 1. \tag{2.1}$$

We introduce the *angle function* ψ_l of a discrete curve γ by

$$\frac{\gamma_{l+1} - \gamma_l}{\epsilon} = \begin{bmatrix} \cos\psi_l \\ \sin\psi_l \end{bmatrix}. \tag{2.2}$$

A discrete curve γ satisfies

$$\frac{\gamma_{l+1} - \gamma_l}{\epsilon} = R(K_l)\,\frac{\gamma_l - \gamma_{l-1}}{\epsilon}, \tag{2.3}$$

for $K_l = \psi_l - \psi_{l-1}$, where $R(K_l)$ denotes the rotation matrix given by

$$R(K_l) = \begin{bmatrix} \cos K_l & -\sin K_l \\ \sin K_l & \cos K_l \end{bmatrix}. \tag{2.4}$$

We consider the following motion of discrete curves:

$$\frac{d\gamma_l}{ds} = \frac{1}{\cos\frac{K_l}{2}}\, R\left(-\frac{K_l}{2}\right) \frac{\gamma_{l+1} - \gamma_l}{\epsilon}. \tag{2.5}$$

Then from the compatibility condition of (2.3) and (2.5), there exists a potential function θ_l such that

$$\psi_l = \frac{\theta_{l+1} + \theta_l}{2}, \qquad K_l = \frac{\theta_{l+1} - \theta_{l-1}}{2}, \tag{2.6}$$

and it follows that from the isoperimetric condition (2.1) that θ_l satisfies

$$\frac{d\theta_l}{ds} = \frac{2}{\epsilon}\tan\left(\frac{\theta_{l+1} - \theta_{l-1}}{4}\right). \tag{2.7}$$

Equation (2.7) is called the *semi-discrete potential modified KdV (mKdV) equation*.

Hoffmann and Kutz [12, 13] introduced (2.5) as the edge tangential flow of discrete plane curves, which was deduced by discretizing the curvature function of motion of plane smooth curves. We note that Doliwa and Santini formulated in [3–5] the integrable motion of discrete curves in 3-sphere described by the Ablowitz-Ladik hierarchy, where the semi-discrete potential mKdV equation (2.7) arises as the simplest case. Their formulation includes the motion of plane curves as a limiting case.

3 Representation Formula in terms of τ Function

In this section, we present a representation formula for curve motions in terms of τ function. We also give explicit τ functions which correspond to soliton and breather solutions.

Let $\tau_l = \tau_l(s; y)$ be a complex function dependent on the discrete variable l and two continuous variables s and y, satisfying the following system of bilinear equations:

$$D_s\, \tau_l \cdot \tau_l^* = \frac{1}{2\epsilon}\left(\tau_{l-1}^*\tau_{l+1} - \tau_{l+1}^*\tau_{l-1}\right), \tag{3.1}$$

$$\tau_l\tau_l^* = \frac{1}{2}\left(\tau_{l-1}^*\tau_{l+1} + \tau_{l+1}^*\tau_{l-1}\right), \tag{3.2}$$

$$\frac{1}{2}D_sD_y\, \tau_l \cdot \tau_l = -\tau_{l+1}^*\tau_{l-1}^*, \tag{3.3}$$

$$D_y\, \tau_{l+1} \cdot \tau_l = -\epsilon\tau_{l+1}^*\tau_l^*. \tag{3.4}$$

Here, * denotes the complex conjugate, D_s, D_y are the Hirota's *bilinear differential operators* (D-operators) defined by

$$D_s^iD_y^j\, f \cdot g = (\partial_s - \partial_{s'})^i(\partial_y - \partial_{y'})^j\, f(s,y)g(s',y')\Big|_{s=s',y=y'}\ . \tag{3.5}$$

We refer to [9] for calculus of D-operators. The functions satisfying the bilinear equations are called the τ *functions*.

Theorem 3.1 *Let τ_l be a solution to eqs.(3.1)–(3.4). Define a real function $\theta_l(s; y)$ and an $\mathbb{R}^2$-valued function $\gamma_l(s; y)$ by*

$$\theta_l(s; y) := \frac{2}{\sqrt{-1}}\log\frac{\tau_l}{\tau_l^*}, \tag{3.6}$$

$$\gamma_l(s; y) := \begin{bmatrix} -\dfrac{1}{2}\left(\log \tau_l\tau_l^*\right)_y \\ \dfrac{1}{2\sqrt{-1}}\left(\log\dfrac{\tau_l}{\tau_l^*}\right)_y \end{bmatrix}. \tag{3.7}$$

Then for any $s, y \in \mathbb{R}$ and $l \in \mathbb{Z}$, the functions $\theta_l = \theta_l(s; y)$ and $\gamma_l = \gamma_l(s; y)$ satisfy (2.1), (2.3) (2.5) and (2.7).

Proof. Express $\gamma_l = {}^t(X_l, Y_l)$. From (3.4) and its complex conjugate we have

$$\left(\log\frac{\tau_{l+1}}{\tau_l}\right)_y = -\epsilon\,\frac{\tau_{l+1}^*\tau_l^*}{\tau_{l+1}\tau_l}, \qquad \left(\log\frac{\tau_{l+1}^*}{\tau_l^*}\right)_y = -\epsilon\,\frac{\tau_{l+1}\tau_l}{\tau_{l+1}^*\tau_l^*}. \tag{3.8}$$

Adding these two equations we obtain

$$(\log\tau_{l+1}\tau_{l+1}^*)_y - (\log\tau_l\tau_l^*)_y = -\epsilon\left(\frac{\tau_{l+1}^*\tau_l^*}{\tau_{l+1}\tau_l} + \frac{\tau_{l+1}\tau_l}{\tau_{l+1}^*\tau_l^*}\right), \tag{3.9}$$

which yields

$$\frac{X_{l+1} - X_l}{\epsilon} = \cos\psi_l, \quad \psi_l = \frac{1}{\sqrt{-1}}\log\left(\frac{\tau_{l+1}\tau_l}{\tau_{l+1}^*\tau_l^*}\right) = \frac{\theta_{l+1} + \theta_l}{2}. \tag{3.10}$$

Subtracting the second equation from the first equation in eq.(3.8) we have

$$\frac{Y_{l+1}-Y_l}{\epsilon} = \sin\psi_l.$$

Therefore we obtain

$$\frac{\gamma_{l+1}-\gamma_l}{\epsilon} = \begin{bmatrix} \cos\psi_l \\ \sin\psi_l \end{bmatrix}. \tag{3.11}$$

which gives eq.(2.1). Next, from eq.(3.11) we see that

$$\frac{\gamma_{l+1}-\gamma_l}{\epsilon} = R(\psi_l-\psi_{l-1})\,\frac{\gamma_l-\gamma_{l-1}}{\epsilon}, \quad \psi_l-\psi_{l-1} = \frac{\theta_{l+1}-\theta_{l-1}}{2} = K_l, \tag{3.12}$$

which is nothing but eq.(2.3). In order to show (2.5), we identify $\mathbb{R}^2$ as $\mathbb{C}$. Then by using (2.2) and (2.6), we see that (2.5) is rewritten as

$$\cos\frac{K_l}{2}\,\dot{\gamma}_l = e^{-\sqrt{-1}\frac{K_l}{2}}\,\frac{\gamma_{l+1}-\gamma_l}{\epsilon} = e^{\sqrt{-1}\frac{\theta_{l+1}+2\theta_l+\theta_{l-1}}{4}}. \tag{3.13}$$

We have

$$\begin{aligned}\cos\frac{K_l}{2} &= \frac{1}{2}\left[e^{\sqrt{-1}\frac{\theta_{l+1}-\theta_{l-1}}{4}} + e^{-\sqrt{-1}\frac{\theta_{l+1}-\theta_{l-1}}{4}}\right] = \frac{1}{2}\left[\left(\frac{\tau_{l+1}\tau_{l-1}^*}{\tau_{l+1}^*\tau_{l-1}}\right)^{1/2} + \left(\frac{\tau_{l+1}^*\tau_{l-1}}{\tau_{l+1}\tau_{l-1}^*}\right)^{1/2}\right] \\ &= \frac{1}{2}\,\frac{\tau_{l+1}\tau_{l-1}^*+\tau_{l-1}\tau_{l+1}^*}{|\tau_{l+1}\tau_{l-1}|} = \frac{\tau_l\tau_l^*}{|\tau_{l+1}\tau_{l-1}|},\end{aligned} \tag{3.14}$$

where we have used (3.2). Noticing that

$$\gamma_l = X_l + \sqrt{-1}Y_l = -\left(\log\tau_l^*\right)_y, \tag{3.15}$$

the left hand side of (3.13) can be rewritten by using (3.3) as

$$\begin{aligned}\cos\frac{K_l}{2}\,\frac{d\gamma_l}{ds} &= \frac{\tau_l\tau_l^*}{|\tau_{l+1}\tau_{l-1}|}\times(-1)\left(\log\tau_l^*\right)_{ys} = -\frac{\tau_l\tau_l^*}{|\tau_{l+1}\tau_{l-1}|}\,\frac{\frac{1}{2}D_sD_y\,\tau_l^*\cdot\tau_l^*}{\left(\tau_l^*\right)^2} = \frac{\tau_{l+1}\tau_{l-1}}{|\tau_{l+1}\tau_{l-1}|}\,\frac{\tau_l}{\tau_l^*} \\ &= e^{\sqrt{-1}\frac{\theta_{l+1}+2\theta_l+\theta_{l-1}}{4}},\end{aligned}$$

which implies (3.13). Finally, the semi-discrete potential mKdV equation (2.7) can be derived by dividing (3.1) by (3.2). □

4 Explicit Solutions

We now present explicit formulas for the τ function which correspond to multi-soliton and multi-breather solutions to the bilinear equations, respectively.

Theorem 4.1 *For $N \in \mathbb{Z}_{\geq 0}$, consider the τ function*

$$\tau_l(s;y) = \exp\left[-(s+\epsilon l)\,y\right]\ \det\left(f_{j-1}^{(i)}\right)_{i,j=1,\ldots,N}, \tag{4.1}$$

$$f_n^{(i)} = \alpha_i p_i^n (1-\epsilon p_i)^{-l} e^{\frac{p_i}{1-\epsilon^2p_i^2}s+\frac{1}{p_i}y} + \beta_i(-p_i)^n(1+\epsilon p_i)^{-l} e^{-\frac{p_i}{1-\epsilon^2p_i^2}s-\frac{1}{p_i}y}, \tag{4.2}$$

where α_i, β_i and p_i $(i=1,\ldots,N)$ are parameters.

(1) *Choosing the parameters as*

$$p_i,\ \alpha_i \in \mathbb{R}, \quad \beta_i \in \sqrt{-1}\mathbb{R} \quad (i = 1, \ldots, N), \tag{4.3}$$

then τ_l satisfies the bilinear equations (3.1)–(3.4). This gives the N-soliton solution to (2.7).

(2) *Taking $N = 2M$, and choosing the parameters as*

$$\begin{aligned} &p_i,\ \alpha_i,\ \beta_i \in \mathbb{C} \quad (i = 1, \ldots, 2M), \quad p_{2r} = p_{2r-1}^* \quad (r = 1, \ldots, M), \\ &\alpha_{2r} = \alpha_{2r-1}^*, \quad \beta_{2r} = -\beta_{2r-1}^* \quad (r = 1, \ldots, M), \end{aligned} \tag{4.4}$$

then τ_l satisfies the bilinear equations (3.1)–(3.4). This gives the M-breather solution to (2.7).

In order to prove Theorem 4.1, we first consider the following "generic" τ function and system of bilinear equations. Then Theorem 4.1 is derived by applying the reduction.

Proposition 4.2 *Let $\sigma_{l,m}^k = \sigma_{l,m}^k(u, v; y)$ is a function depending on three discrete independent variables $k, l, m \in \mathbb{Z}$ and three continuous independent variables $u, v, y \in \mathbb{R}$ defined by*

$$\sigma_{l,m}^k(u, v; y) = \det\left(f_{k+j-1}^{(i)}(l, m)\right)_{i,j=1,\ldots,N}, \tag{4.5}$$

$$f_k^{(i)}(l, m) = \alpha_i p_i^k (1 - ap_i)^{-l}(1 - bp_i)^{-m} e^{\frac{u}{1-ap_i} + \frac{v}{1-bp_i} + \frac{1}{p_i}y} + \beta_i q_i^k (1 - aq_i)^{-l}(1 - bq_i)^{-m} e^{\frac{u}{1-aq_i} + \frac{v}{1-bq_i} + \frac{1}{q_i}y} \tag{4.6}$$

where a, b, α_i, β_i, p_i and q_i $(i = 1, \ldots, N)$ are parameters. Then $\sigma_{l,m}^k$ satisfies the following bilinear equations:

$$(D_u - 1)\ \sigma_{l,m}^{k-1} \cdot \sigma_{l,m}^k = -\sigma_{l+1,m}^k \sigma_{l-1,m}^{k-1}, \tag{4.7}$$

$$(D_v - 1)\ \sigma_{l,m}^{k-1} \cdot \sigma_{l,m}^k = -\sigma_{l,m+1}^k \sigma_{l,m-1}^{k-1}, \tag{4.8}$$

$$b\sigma_{l,m+1}^{k+1}\sigma_{l+1,m}^k - a\sigma_{l+1,m}^{k+1}\sigma_{l,m+1}^k + (a - b)\sigma_{l+1,m+1}^{k+1}\sigma_{l,m}^k = 0, \tag{4.9}$$

$$\frac{1}{2}D_u D_y\ \sigma_{l,m}^k \cdot \sigma_{l,m}^k = a(\sigma_{l,m}^k)^2 - a\sigma_{l+1,m}^{k+1}\sigma_{l-1,m}^{k-1}, \tag{4.10}$$

$$\frac{1}{2}D_v D_y\ \sigma_{l,m}^k \cdot \sigma_{l,m}^k = b(\sigma_{l,m}^k)^2 - b\sigma_{l,m+1}^{k+1}\sigma_{l,m-1}^{k-1}, \tag{4.11}$$

$$\left(D_y - a\right)\ \sigma_{l+1,m}^k \cdot \sigma_{l,m}^k = -a\sigma_{l+1,m}^{k+1}\sigma_{l,m}^{k-1}. \tag{4.12}$$

Proof of Theorem 4.1 We show that Theorem 4.1 holds from Proposition 4.2. We impose the reduction conditions on $\sigma_{l,m}^k$ as

$$\sigma_{l+1,m+1}^k = B\sigma_{l,m}^k, \tag{4.13}$$

$$\sigma_{l,m}^{k+1} = C\sigma^{*k}_{l,m}, \quad C \in \mathbb{R} \tag{4.14}$$

where B, C are constants. Then putting $b = -a$, the bilinear equations (4.7)–(4.12) are reduced to

$$(D_u - 1)\ \sigma_l^* \cdot \sigma_l = -\sigma_{l+1}\sigma_{l-1}^*, \tag{4.15}$$

$$(D_v - 1)\ \sigma_l^* \cdot \sigma_l = -\sigma_{l-1}\sigma_{l+1}^*, \tag{4.16}$$

$$\sigma_{l-1}^*\sigma_{l+1} + \sigma_{l+1}^*\sigma_{l-1} - 2\sigma_l^*\sigma_l = 0, \tag{4.17}$$

$$\frac{1}{2}D_u D_y\ \sigma_l \cdot \sigma_l = a(\sigma_l)^2 - a\sigma_{l+1}^*\sigma_{l-1}^*, \tag{4.18}$$

$$\frac{1}{2}D_v D_y\ \sigma_l \cdot \sigma_l = -a(\sigma_l)^2 + a\sigma_{l-1}^*\sigma_{l+1}^*, \tag{4.19}$$

$$\left(D_y - a\right)\ \sigma_{l+1} \cdot \sigma_l = -a\sigma_{l+1}^*\sigma_l^*, \tag{4.20}$$

respectively. Here we have used (4.13) and (4.14) to eliminate the m- and k-dependence, respectively, and denoted $\sigma_{l,m}^k = \sigma_l$. We next consider the specialization of continuous independent variables

$$u = cs, \quad v = -cs, \quad c \in \mathbb{R}. \tag{4.21}$$

Then, subtracting (4.16) from (4.15) we have

$$D_s\ \sigma_l^* \cdot \sigma_l = c\left(\sigma_{l-1}\sigma_{l+1}^* - \sigma_{l+1}\sigma_{l-1}^*\right). \tag{4.22}$$

Similarly, we get from (4.18) and (4.19)

$$D_s D_y\ \sigma_l \cdot \sigma_l = 4ac\left\{(\sigma_l)^2 - \sigma_{l-1}^*\sigma_{l+1}^*\right\}. \tag{4.23}$$

Putting

$$a = \epsilon, \quad c = \frac{1}{2\epsilon}, \tag{4.24}$$

and introducing τ_l by

$$\tau_l = e^{-(s+\epsilon l)y}\sigma_l, \tag{4.25}$$

the bilinear equations (4.22), (4.17), (4.23), (4.20) are reduced to (3.1), (3.2), (3.3), (3.4), respectively. Let us next realize the reduction conditions (4.13) and (4.14) by imposing suitable restriction on parameters of solution. We put

$$q_i = -p_i \quad (i = 1, \ldots, N), \quad b = -a. \tag{4.26}$$

Then it is easy to verify that the entries of the determinant satisfy

$$f_k^{(i)}(l+1, m+1) = \frac{1}{1 - a^2 p_i^2}\ f_k^{(i)}(l, m), \tag{4.27}$$

so that the condition (4.13) is realized as

$$\sigma_{l+1,m+1}^k = \prod_{i=1}^{N} \frac{1}{1 - a^2 p_i^2}\ \sigma_{l,m}^k. \tag{4.28}$$

As for the condition (4.14), we have to consider the cases (1) and (2) in Theorem 4.1 separately:
Case (1). We impose the condition (4.3). Then we see that

$$f_{k+1}^{(i)}(l,m) = p_i\, f_k^{(i)*}(l,m), \tag{4.29}$$

and so

$$\sigma_{l,m}^{k+1} = C\,\sigma_{l,m}^{k}, \quad C = \prod_{i=1}^{N} p_i \in \mathbb{R}. \tag{4.30}$$

Case (2). We impose the condition (4.4). Then we see that

$$f_{k+1}^{(2r)}(l,m) = p_{2r-1}^{*} f_k^{(2r-1)*}(l,m), \quad f_{k+1}^{(2r-1)}(l,m) = p_{2r}^{*} f_k^{(2r)*}(l,m), \tag{4.31}$$

and so

$$\sigma_{l,m}^{k+1} = C\,\sigma_{l,m}^{*k}, \quad C = (-1)^M \prod_{r=1}^{M} |p_{2r}|^2 \in \mathbb{R}. \tag{4.32}$$

Finally, putting $m = 0$ without loss of generality and applying the specialization (4.21) and (4.24), (4.6) is rewritten as

$$\begin{aligned} f_k^{(i)}(l,0) &= \alpha_i p_i^k (1-\epsilon p_i)^{-l} e^{\frac{s}{2\epsilon}\left(\frac{1}{1-\epsilon p_i} - \frac{1}{1+\epsilon p_i}\right) + \frac{1}{p_i} y} + \beta_i (-p_i)^k (1+\epsilon p_i)^{-l} e^{\frac{s}{2\epsilon}\left(\frac{1}{1+\epsilon p_i} - \frac{1}{1-\epsilon p_i}\right) - \frac{1}{p_i} y} \\ &= \alpha_i p_i^k (1-\epsilon p_i)^{-l} e^{\frac{p_i}{1-\epsilon^2 p_i^2} s + \frac{1}{p_i} y} + \beta_i (-p_i)^k (1+\epsilon p_i)^{-l} e^{-\frac{p_i}{1-\epsilon^2 p_i^2} s - \frac{1}{p_i} y}, \end{aligned}$$

which is equivalent to (4.2). Therefore we have derived Theorem 4.1 from Proposition 4.2. □

The bilinear equations in Proposition 4.2 are reduced to the quadratic identities of determinants (Plücker relations). In particular, (4.9) and (4.12) have already appeared in [14]. Moreover, by the symmetry between the set of variables (l,u) and (m,v) in $\sigma_{l,m}^{k}$, it suffices to show only (4.7) and (4.10). These bilinear equations will be proved in the Appendix.

Remark 4.3 In the τ function in Theorem 4.1, the parameter dependence of the time evolution in entries of the Casorati determinant have singularities different from 0 and ∞. These types of singularities can be seen in the solutions of equation of principal chiral fields, *i.e.*, harmonic maps of conformal 2-manifolds into compact Lie groups [15, 23, 24] and Maxwell-Bloch equation [16].

Remark 4.4 By introducing $u_l := \frac{\epsilon}{2}\frac{d\theta_l}{ds}$, the semi-discrete potential mKdV equation (2.7) can be transformed to the semi-discrete mKdV equation

$$\frac{du_l}{ds'} = \left(1 + u_l^2\right)(u_{l+1} - u_{l-1}), \tag{4.33}$$

where we put $s = 2\epsilon s'$ for convenience. An auxiliary linear problem for (4.33) is given by [3]

$$\Phi_{l+1} = \frac{1}{\sqrt{1+u_l^2}} \begin{bmatrix} \lambda & \lambda^{-1} u_l \\ -\lambda u_l & \lambda^{-1} \end{bmatrix} \Phi_l, \quad \frac{d}{ds'}\Phi_l = \begin{bmatrix} \frac{\lambda^2 - \lambda^{-2}}{2} & u_l + \lambda^{-2} u_{l-1} \\ -u_l - \lambda^2 u_{l-1} & -\frac{\lambda^2 - \lambda^{-2}}{2} \end{bmatrix} \Phi_l. \tag{4.34}$$

Apparently, the dispersion relation suggested from the linear problem is different from the one in Theorem 4.1. However, putting

$$p_i = \frac{1}{\epsilon}\frac{\lambda_i^2 - 1}{\lambda_i^2 + 1} \tag{4.35}$$

in (4.2), then $f_n^{(i)}$ can be rewritten as

$$f_n^{(i)} \doteq \alpha_i \left(\frac{1}{\epsilon}\frac{\lambda_i^2 - 1}{\lambda_i^2 + 1}\right)^n \lambda_i^l e^{\frac{1}{2}(\lambda_i^2 - \lambda_i^{-2})s' + \frac{\lambda_i^2+1}{\lambda_i^2-1}\epsilon y} + \beta_i \left(-\frac{1}{\epsilon}\frac{\lambda_i^2 - 1}{\lambda_i^2 + 1}\right)^n \lambda_i^{-l} e^{-\frac{1}{2}(\lambda_i^2 - \lambda_i^{-2})s' + \frac{\lambda_i^2+1}{\lambda_i^2-1}\epsilon y},$$

in which the dispersion relation with respect to l and s' is consistent with (4.34). We have chosen the parametrization as in (4.2) so that the continuous limits explained in Section 6 become simpler.

5 Bäcklund Transformations

In this section we discuss the Bäcklund transformation of the continuous motion of plane discrete curves. The Bäcklund transformation of the plane discrete curves has already been formulated in [14]:

Proposition 5.1 *Let γ_l be a discrete curve of segment length ϵ. Let θ_l be the potential function defined by*

$$\frac{\gamma_{l+1} - \gamma_n}{\epsilon} = \begin{bmatrix} \cos\psi_l \\ \sin\psi_l \end{bmatrix}, \quad \psi_l = \frac{\theta_{l+1} + \theta_l}{2}. \tag{5.1}$$

For a nonzero constant λ, take a solution $\widetilde{\theta}_n$ to the following equation

$$\tan\left(\frac{\widetilde{\theta}_{l+1} - \theta_l}{4}\right) = \frac{\frac{1}{\lambda} + \epsilon}{\frac{1}{\lambda} - \epsilon} \tan\left(\frac{\widetilde{\theta}_l - \theta_{l+1}}{4}\right), \tag{5.2}$$

then

$$\widetilde{\gamma}_l = \gamma_l + \frac{1}{\lambda} R\left(\frac{\widetilde{\theta}_l - \theta_{l+1}}{2}\right) \frac{\gamma_{l+1} - \gamma_l}{\epsilon} \tag{5.3}$$

is a discrete curve with the potential function $\widetilde{\theta}_l$.

We next extend the Bäcklund transformation to that of motion of discrete curves. In order to do so, we first present the Bäcklund transformation to the semi-discrete potential mKdV equation:

Lemma 5.2 *Let θ_l be a solution to the semi-discrete potential mKdV equation (2.7). A function $\widetilde{\theta}_l$ satisfying the following system of equations*

$$\left(\frac{1}{\lambda} - \epsilon\right) \tan\frac{\widetilde{\theta}_{l+1} - \theta_l}{4} = \left(\frac{1}{\lambda} + \epsilon\right) \tan\frac{\widetilde{\theta}_l - \theta_{l+1}}{4}, \tag{5.4}$$

$$\left(\frac{1}{\lambda} + \epsilon\right) \frac{\widetilde{\theta}_l'}{4\cos^2\frac{\widetilde{\theta}_l - \theta_{l+1}}{4}} + \left(\frac{1}{\lambda} - \epsilon\right) \frac{\theta_l'}{4\cos^2\frac{\widetilde{\theta}_{l+1} - \theta_l}{4}} = \tan\frac{\widetilde{\theta}_l - \theta_{l+1}}{4} + \tan\frac{\widetilde{\theta}_{l+1} - \theta_l}{4}, \tag{5.5}$$

gives another solution to eq.(2.7). We call $\widetilde{\theta}_l$ a Bäcklund transform of θ_l.

Proof. First compute addition of $(5.5)_{l-1}$ and the derivative of $(5.4)_{l-1}$. Then, by using (5.4), eliminate λ from this equation and (5.5) respectively. Adding those two equations yields

$$\begin{aligned}&\left(\frac{\epsilon}{2}\widetilde{\theta}_l' \cos\frac{\widetilde{\theta}_{l+1}-\widetilde{\theta}_{l-1}}{4} - \sin\frac{\widetilde{\theta}_{l+1}-\widetilde{\theta}_{l-1}}{4}\right)\sin\frac{\widetilde{\theta}_{l+1}+\widetilde{\theta}_{l-1}-2\theta_l}{4}\\ &= \left(\frac{\epsilon}{2}\theta_l' \cos\frac{\theta_{l+1}-\theta_{l-1}}{4} - \sin\frac{\theta_{l+1}-\theta_{l-1}}{4}\right)\sin\frac{\theta_{l+1}+\theta_{l-1}-2\widetilde{\theta}_l}{4},\end{aligned} \tag{5.6}$$

which implies Lemma 5.2. □

Proposition 5.3 *Let γ_l be a motion of discrete curve. Take a Bäcklund transform $\widetilde{\theta}_l$ of θ_l defined in Lemma 5.2. Then*

$$\widetilde{\gamma}_l = \gamma_l + \frac{1}{\lambda} R\left(\frac{\widetilde{\theta}_l - \theta_{l+1}}{2}\right)\frac{\gamma_{l+1}-\gamma_l}{\epsilon} \tag{5.7}$$

is a motion of discrete curve with potential function $\widetilde{\theta}_l$. We call $\widetilde{\gamma}_l$ a Bäcklund transform of γ_l.

Proof. It suffices to show that $\widetilde{\gamma}_l$ satisfies eqs.(2.1), (2.3) and (2.5) with potential function $\widetilde{\theta}_l$, but eqs.(2.1) and (2.3) follow from Proposition 5.1 immediately. Because the system (5.4)–(5.5) yields

$$\left(1-\sqrt{-1}\frac{\epsilon}{2}\widetilde{\theta}_l'\right)e^{\sqrt{-1}\,\frac{\widetilde{\theta}_{l+1}-\theta_l}{2}} = \left(1-\sqrt{-1}\frac{\epsilon}{2}\theta_l'\right)e^{\sqrt{-1}\,\frac{\widetilde{\theta}_l-\theta_{l+1}}{2}} + \frac{\sqrt{-1}}{\lambda}\frac{\widetilde{\theta}_l'+\theta_l'}{2},$$

we identify $\mathbb{R}^2$ with $\mathbb{C}$, so that the motion $\widetilde{\gamma}_l$ satisfies

$$\widetilde{\gamma}_l' = e^{\sqrt{-1}\,\frac{\widetilde{\theta}_{l+1}+\widetilde{\theta}_l}{2}}\left(1-\sqrt{-1}\frac{\epsilon}{2}\widetilde{\theta}_l'\right)\gamma_l = \frac{\widetilde{\gamma}_{l+1}-\widetilde{\gamma}_l}{\epsilon}\left(1-\sqrt{-1}\tan\frac{\widetilde{\theta}_{l+1}-\widetilde{\theta}_{l-1}}{4}\right),$$

which implies (2.5) with $2\widetilde{K}_l = \widetilde{\theta}_{l+1} - \widetilde{\theta}_{l-1}$. □

Remark 5.4 In [12, 13], the Bäcklund transformation of the motions of discrete plane curves described in this paper is characterized by the cross ratio of the four points γ_l, γ_{l+1}, $\widetilde{\gamma}_l$ and $\widetilde{\gamma}_{l+1}$ being constant. In fact, we can verify by direct computation that for the Bäcklund transformation given in Proposition 5.3, the cross ratio of those four points is $-\lambda^2\epsilon^2$.

6 Continuous Limits

In [14], the discrete motion of discrete plane curves and the continuous motion of smooth plane curves have been formulated, together with the Bäcklund transformations and the explicit formulas in terms of the τ functions. They are described by the discrete potential modified KdV equation and the potential modified KdV equation, respectively. In this section, we present the two continuous limits: one from the discrete motion of discrete plane curves to their continuous motion discussed in the preceding sections, another one from the continuous motion of discrete plane curves to the continuous motion of smooth plane curves.

We first summarize the formulations of three kinds of curve motions and explicit solutions. For convenience, we identify Euclidean plane $\mathbb{R}^2$ with complex plane $\mathbb{C}$.

(1) Discrete motion of discrete plane curves.

Motion of curves:

$$\left|\frac{\gamma_{n+1}^m - \gamma_n^m}{a_n}\right| = 1, \tag{6.1}$$

$$\frac{\gamma_{n+1}^m - \gamma_n^m}{a_n} = e^{\sqrt{-1}K_n^m}\ \frac{\gamma_n^m - \gamma_{n-1}^m}{a_{n-1}}, \tag{6.2}$$

$$\frac{\gamma_n^{m+1} - \gamma_n^m}{b_m} = e^{\sqrt{-1}W_n^m}\ \frac{\gamma_{n+1}^m - \gamma_n^m}{a_n}. \tag{6.3}$$

Here, $n, m \in \mathbb{Z}$ denote the discrete independent variables corresponding to space and time, respectively. Moreover, a_n, b_m are real arbitrary functions of the indicated variables, which correspond to the segment length of the curves and time interval, respectively.

Potential function:

$$K_n^m = \frac{\theta_{n+1}^m - \theta_{n-1}^m}{2}, \quad W_n^m = \frac{\theta_n^{m+1} - \theta_{n+1}^m}{2}. \tag{6.4}$$

Compatibility condition:

$$\tan\left(\frac{\theta_{n+1}^{m+1} - \theta_n^m}{4}\right) = \frac{b_m + a_n}{b_m - a_n}\tan\left(\frac{\theta_n^{m+1} - \theta_{n+1}^m}{4}\right). \tag{6.5}$$

Explicit formula in terms of τ function:

$$\theta_n^m = \frac{2}{\sqrt{-1}}\log\frac{\tau_n^m}{\tau^{*n}_m}, \quad \gamma_n^m = \begin{bmatrix} -\frac{1}{2}\left(\log\tau_n^m\tau^{*m}_n\right)_y \\ \frac{1}{2\sqrt{-1}}\left(\log\frac{\tau_n^m}{\tau^{*m}_n}\right)_y \end{bmatrix}. \tag{6.6}$$

Soliton type solutions:

$$\tau_n^m = \exp\left[-\left(\sum_{n'}^{n-1} a_{n'} + \sum_{m'}^{m-1} b_{m'}\right)y\right]\ \det\left(f_{j-1}^{(i)}\right)_{i,j=1,\ldots,N}, \tag{6.7}$$

$$f_k^{(i)} = \alpha_i p_i^k \prod_{n'}^{n-1}(1-a_{n'}p_i)^{-1}\prod_{m'}^{m-1}(1-b_{m'}p_i)^{-1}e^{\frac{1}{p_i}y} + \beta_i(-p_i)^k\prod_{n'}^{n-1}(1+a_{n'}p_i)^{-1}\prod_{m'}^{m-1}(1+b_{m'}p_i)^{-1}e^{-\frac{1}{p_i}y}. \tag{6.8}$$

(2) Continuous motion of discrete plane curves.

Motion of curves:

$$\left|\frac{\gamma_{l+1} - \gamma_l}{\epsilon}\right| = 1, \tag{6.9}$$

$$\frac{\gamma_{l+1} - \gamma_l}{\epsilon} = e^{\sqrt{-1}K_l}\ \frac{\gamma_l - \gamma_{l-1}}{\epsilon}, \tag{6.10}$$

$$\frac{d\gamma_l}{ds} = \frac{e^{-\sqrt{-1}\frac{K_l}{2}}}{\cos\frac{K_l}{2}}\ \frac{\gamma_{l+1} - \gamma_l}{\epsilon}. \tag{6.11}$$

Potential function:

$$K_l = \frac{\theta_{l+1} - \theta_{l-1}}{2}. \tag{6.12}$$

Compatibility condition:

$$\frac{d\theta_l}{ds} = \frac{2}{\epsilon} \tan\left(\frac{\theta_{l+1} - \theta_{l-1}}{4}\right). \tag{6.13}$$

Explicit formula in terms of τ function:

$$\theta_l = \frac{2}{\sqrt{-1}} \log \frac{\tau_l}{\tau_l^*}, \quad \gamma_l = \begin{bmatrix} -\dfrac{1}{2} (\log \tau_l \tau_l^*)_y \\ \dfrac{1}{2\sqrt{-1}} \left(\log \dfrac{\tau_l}{\tau_l^*}\right)_y \end{bmatrix}. \tag{6.14}$$

Soliton type solutions:

$$\tau_l = \exp\left[-(s + \epsilon l)\, y\right] \ \det\left(f_{j-1}^{(i)}\right)_{i,j=1,\dots,N}, \tag{6.15}$$

$$f_k^{(i)} = \alpha_i p_i^k (1 - \epsilon p_i)^{-l} e^{\frac{p_i}{1-\epsilon^2 p_i^2} s + \frac{1}{p_i} y} + \beta_i (-p_i)^k (1 + \epsilon p_i)^{-l} e^{-\frac{p_i}{1-\epsilon^2 p_i^2} s - \frac{1}{p_i} y}. \tag{6.16}$$

(3) Continuous motion of smooth plane curves.

Motion of curves:

$$|\gamma'| = 1, \tag{6.17}$$

$$\frac{\partial}{\partial x}\gamma' = \sqrt{-1}\kappa\, \gamma', \tag{6.18}$$

$$\frac{\partial}{\partial t}\gamma' = -\sqrt{-1}\left(\kappa'' + \frac{\kappa^3}{2}\right) \gamma'. \tag{6.19}$$

Here $\gamma = \gamma(x,t) \in \mathbb{R}^2 \simeq \mathbb{C}$ is arc-length parametrized curve, x and t denote arc-length and time, respectively, and $' = \partial_x$. Moreover, $\kappa = \kappa(x,t)$ is the curvature.

Potential function:

$$\kappa = \theta'. \tag{6.20}$$

Compatibility condition:

$$\theta_t + \frac{1}{2}(\theta_x)^3 + \theta_{xxx} = 0. \tag{6.21}$$

Explicit formula in terms of τ function:

$$\theta = \frac{2}{\sqrt{-1}} \log \frac{\tau}{\tau^*}, \quad \gamma = \begin{bmatrix} -\dfrac{1}{2} (\log \tau \tau^*)_y \\ \dfrac{1}{2\sqrt{-1}} \left(\log \dfrac{\tau}{\tau^*}\right)_y \end{bmatrix}. \tag{6.22}$$

Soliton type solutions:

$$\tau = e^{-xy} \ \det\left(f_{j-1}^{(i)}\right)_{i,j=1,\dots,N}, \tag{6.23}$$

$$f_k^{(i)} = \alpha_i p_i^k e^{p_i x - 4p_i^3 t + \frac{1}{p_i} y} + \beta_i (-p_i)^k e^{-p_i x + 4p_i^3 t - \frac{1}{p_i} y}. \tag{6.24}$$

Theorem 6.1

(1) *Putting*

$$a_n = a\,(const.),\quad b_m = b\,(const.),\quad \delta = \frac{a+b}{2},\quad \epsilon = \frac{a-b}{2},\qquad \frac{s}{\delta} = n+m,\quad l = n-m, \tag{6.25}$$

and taking the limit $\delta \to 0$, the discrete motion of discrete plane curves yields the continuous motion of discrete plane curves.

(2) *Putting*

$$x = \epsilon l + s,\quad t = -\frac{\epsilon^2}{6}s, \tag{6.26}$$

and taking the limit $\epsilon \to 0$, the continuous motion of discrete plane curves yields the continuous motion of smooth plane curves.

Theorem 6.1 can be verified by tedious but straightforward calculations. In fact, the statement (1) can be checked by substituting the parametrization (6.25) into (6.1)–(6.8), expanding in terms of powers of δ and taking the limit $\delta \to 0$. The statement (2) is also checked by a similar manner. We note that the limiting procedures presented in (6.25) and (6.26) have been obtained in [7] and [8] on the level of the equations for θ. Theorem 6.1 claims that the procedure applies to the curve motions and solutions. Also, it should be noted that limiting procedure also applies to the Bäcklund transformations.

In order to demonstrate the calculation, we here discuss the limits of the τ functions corresponding to the soliton type solutions. Substituting (6.25) into (6.8), we have

$$(1-ap_i)^{-n}(1-bp_i)^{-m} = (1-ap_i)^{-\frac{1}{2}\left(\frac{s}{\delta}+l\right)}(1-bp_i)^{-\frac{1}{2}\left(\frac{s}{\delta}-l\right)} = e^{-\frac{s}{2\delta}\log\left[1-2\delta p_i+(\delta^2-\epsilon^2)p_i^2\right]}\left(\frac{1-\epsilon p_i-\delta p_i}{1+\epsilon p_i-\delta p_i}\right)^{-\frac{l}{2}}.$$

Noticing that

$$\log\left[1-2\delta p_i+(\delta^2-\epsilon^2)p_i^2\right] = \log\omega_i - \frac{2p_i}{\omega_i}\delta + O(\delta^2),\quad \omega_i = 1-\epsilon^2p_i^2,$$

we get

$$(1-ap_i)^{-n}(1-bp_i)^{-m} \sim e^{-\frac{\log\omega_i}{2\delta}s}\times e^{\frac{p_i}{\omega_i}s}\left(\frac{1-\epsilon p_i}{1+\epsilon p_i}\right)^{-\frac{l}{2}}.$$

Similarly, we have

$$(1+ap_i)^{-n}(1+bp_i)^{-m} \sim e^{-\frac{\log\omega_i}{2\delta}s}\times e^{-\frac{p_i}{\omega_i}s}\left(\frac{1+\epsilon p_i}{1-\epsilon p_i}\right)^{-\frac{l}{2}}.$$

Therefore $f_k^{(i)}$ yields

$$\begin{aligned} f_k^{(i)} &\sim \alpha_i p_i^k e^{-\frac{\log\omega_i}{2\delta}s}\, e^{\frac{p_i}{\omega_i}s}\left(\frac{1-\epsilon p_i}{1+\epsilon p_i}\right)^{-\frac{l}{2}} e^{\frac{1}{p_i}y} + \beta_i(-p_i)^k e^{-\frac{\log\omega_i}{2\delta}s}\, e^{-\frac{p_i}{\omega_i}s}\left(\frac{1+\epsilon p_i}{1-\epsilon p_i}\right)^{-\frac{l}{2}} e^{-\frac{1}{p_i}y} \\ &= e^{-\frac{\log\omega_i}{2\delta}s}(1-\epsilon^2p_i^2)^{\frac{l}{2}}\left[\alpha_i p_i^k\,(1-\epsilon p_i)^{-l}\,e^{\frac{p_i}{\omega_i}s+\frac{1}{p_i}y} + \beta_i(-p_i)^k\,(1+\epsilon p_i)^{-l}\,e^{-\frac{p_i}{\omega_i}s-\frac{1}{p_i}y}\right], \end{aligned} \tag{6.27}$$

as $\delta \sim 0$. The prefactors of the entries in (6.27) can be factored out of the determinant, and it is easily seen that the overall factor does not affect the solutions, namely, if we remove overall factor from the τ functions, it gives the same θ and γ, as seen from (6.6). This implies that the determinant in (6.7) yields that in (6.15) up to this trivial multiplicative factor. Also, the exponential factor in (6.6) becomes that in (6.14) under the parametrization (6.25). Therefore, we have shown that (6.7) is reduced to (6.15) as $\delta \to 0$.

Similarly, substituting (6.26) into (6.16), we have

$$(1 - \epsilon p_i)^{-l}\, e^{\frac{p_i}{1-\epsilon^2 p_i^2} s} = \exp\left[-\left(\frac{x}{\epsilon} + \frac{6t}{\epsilon^3}\right)\log(1-\epsilon p_i) + \frac{p_i}{1-\epsilon^2 p_i^2} s\right] = \exp\left[\frac{3p_i^2}{\epsilon} t + \left(p_i x - 4p_i^3 t\right) + O(\epsilon)\right],$$

and

$$(1 + \epsilon p_i)^{-l}\, e^{-\frac{p_i}{1-\epsilon^2 p_i^2} s} = \exp\left[\frac{3p_i^2}{\epsilon} t - \left(p_i x - 4p_i^3 t\right) + O(\epsilon)\right],$$

from which we obtain as $\epsilon \sim 0$

$$f_k^{(i)} \sim e^{\frac{3p_i^2}{\epsilon} t} \left[\alpha_i p_i^k e^{p_i x - 4p_i^3 t + \frac{1}{p_i} y} + \beta_i (-p_i)^k e^{-p_i x + 4p_i^3 t - \frac{1}{p_i} y}\right]. \tag{6.28}$$

The prefactor in (6.28) does not affect the solutions. Also, the exponential factor in (6.14) becomes that in (6.22) under the parametrization (6.26). Therefore, we have shown that (6.15) is reduced to (6.23) as $\epsilon \to 0$.

A Derivation of bilinear equations (4.7) and (4.10)

In this appendix we prove Proposition 4.2. As mentioned in Section 3, it suffices to show that the τ function given in (4.5) and (4.6) actually satisfies the bilinear equations (4.7) and (4.10).

A.1 Equation (4.7)

We define the τ function $\sigma_{l,m}^k(u, v; y)$ by

$$\sigma_{l,m}^k(u, v; y) = \det\left(f_{k+j-1}^{(i)}(l, m)\right)_{i,j=1,\ldots,N}, \tag{A.1}$$

where the entries of determinant satisfy the linear relations

$$\frac{f_k^{(i)}(l, m) - f_k^{(i)}(l-1, m)}{a} = f_{k+1}^{(i)}(l, m), \qquad \frac{f_k^{(i)}(l, m) - f_k^{(i)}(l, m-1)}{b} = f_{k+1}^{(i)}(l, m), \tag{A.2}$$

$$\partial_u f_k^{(i)}(l, m) = f_k^{(i)}(l+1, m), \quad \partial_v f_k^{(i)}(l, m) = f_k^{(i)}(l, m+1), \quad \partial_y f_k^{(i)}(l, m) = f_{k-1}^{(i)}(l, m). \tag{A.3}$$

Note that $f_k^{(i)}(l, m)$ given in (4.6) satisfy the above relations. In order to prove (4.7), it is convenient to consider $\rho_{l,m}^k$ defined by

$$\rho_{l,m}^k(u, v; y) = \det\left(f_k^{(i)}(l - j + 1, m)\right)_{i,j=1,\ldots,N}, \tag{A.4}$$

instead of $\sigma^k_{l,m}$. Here, $\sigma^k_{l,m}$ and $\rho^k_{l,m}$ are related as

$$\rho^k_{l,m} = (-a)^{N(N-1)/2}\,\sigma^k_{l,m}, \tag{A.5}$$

which can be easily verified by manipulating the columns of determinant with the first equation in (A.2). We also introduce a notation

$$\rho^k_{l,m} = \left|\mathbf{0}^k_m, \mathbf{1}^k_m, \cdots, \boldsymbol{N-2}^k_m, \boldsymbol{N-1}^k_m\right|, \quad \boldsymbol{j}^k_m = \begin{bmatrix} f^{(1)}_k(l-j,m) \\ f^{(2)}_k(l-j,m) \\ \vdots \\ f^{(N)}_k(l-j,m) \end{bmatrix}. \tag{A.6}$$

It is possible to reduce (4.7) to one of the Plücker relations which are quadratic quadratic identities of determinants whose columns are appropriately shifted. To this end, we construct such formulas that express the determinants in the Plücker relations in terms of derivative or shift of discrete variable of $\rho^k_{l,m}(u,v;y)$ by using the linear relations of the entries. For details of the technique, we refer to [9, 18–21].

Lemma A.1 *The following formulas hold:*

$$\partial_u\,\rho^k_{l,m} = |\,\boldsymbol{-1}, \mathbf{1}, \cdots, \boldsymbol{N-2}, \boldsymbol{N-1}|, \tag{A.7}$$

$$\rho^{k-1}_{l,m} = a^{N-1}\,\left|\mathbf{0}, \mathbf{1}, \cdots, \boldsymbol{N-2}, \boldsymbol{N-1}^{k-1}\right|, \tag{A.8}$$

$$\rho^{k-1}_{l,m} = a^{N-1}\,\left|\mathbf{0}, \mathbf{1}, \cdots, \boldsymbol{N-2}, \boldsymbol{N-2}^{k-1}\right|, \tag{A.9}$$

$$(\partial_u - 1)\,\rho^{k-1}_{l,m} = a^{N-1}\,\left|\boldsymbol{-1}, \mathbf{1}, \cdots, \boldsymbol{N-2}, \boldsymbol{N-1}^{k-1}\right|. \tag{A.10}$$

Note that the superscripts of column vectors are shown only when k is shifted for notational simplicity.

Proof. Equation (A.7) follows from the differential rule of determinants and the fist equation of (A.2). Next, applying the first equation of the difference rule (A.3) to the first column of $\rho^{k-1}_{l,m}$, we have

$$\rho^{k-1}_{l,m} = \left|\mathbf{0}^{k-1}, \mathbf{1}^{k-1}, \cdots, \boldsymbol{N-1}^{k-1}\right| = \left|\mathbf{0}^{k-1} - \mathbf{1}^{k-1}, \mathbf{1}^{k-1}, \cdots, \boldsymbol{N-1}^{k-1}\right| = a\left|\mathbf{0}^{k}, \mathbf{1}^{k-1}, \cdots, \boldsymbol{N-1}^{k-1}\right|.$$

Repeating this procedure for the j-th column ($j = 2, 3, \ldots, N-1$), we get

$$\rho^{k-1}_{l,m} = a^{N-1}\,\left|\mathbf{0}^k, \mathbf{1}^k, \cdots, \boldsymbol{N-2}^k, \boldsymbol{N-1}^{k-1}\right|,$$

which is (A.8). Applying (A.2) to the N-th column of (A.8), we obtain (A.9).

Finally, differentiating (A.9) by u yields

$$\begin{aligned}\partial_u\,\rho^{k-1}_{l,m} &= a^{N-1}\,\left|\boldsymbol{-1}^k, \mathbf{1}^k, \cdots, \boldsymbol{N-2}^k, \boldsymbol{N-1}^{k-1}\right| + a^{N-1}\,\left|\mathbf{0}^k, \mathbf{1}^k, \cdots, \boldsymbol{N-2}^k, \boldsymbol{N-2}^{k-1}\right| \\ &= a^{N-1}\,\left|\boldsymbol{-1}^k, \mathbf{1}^k, \cdots, \boldsymbol{N-2}^k, \boldsymbol{N-1}^{k-1}\right| + \rho^{k-1}_{l,m}\end{aligned}$$

which is equivalent to (A.10). Thus we have proved Lemma A.1. □

Now consider the Plücker relation (see, for example, [21])

$$\begin{aligned}0 =& |-\mathbf{1},\mathbf{0},\mathbf{1},\cdots,\mathbf{N-2}|\times\left|\mathbf{1},\cdots,\mathbf{N-2},\mathbf{N-1},\mathbf{N-1}^{k-1}\right| \\ &+|\mathbf{0},\mathbf{1},\cdots,\mathbf{N-2},\mathbf{N-1}|\times\left|-\mathbf{1},\mathbf{1},\cdots,\mathbf{N-2},\mathbf{N-1}^{k-1}\right| \\ &-\left|\mathbf{0},\mathbf{1},\cdots,\mathbf{N-2},\mathbf{N-1}^{k-1}\right|\times|-\mathbf{1},\mathbf{1},\cdots,\mathbf{N-2},\mathbf{N-1}|.\end{aligned} \tag{A.11}$$

(A.11) is rewritten by using Lemma A.1 as

$$\begin{aligned}0 &= \rho^k_{l+1,m}\times a^{-(N-1)}\rho^{k-1}_{l-1,m}+\rho^k_{l,m}\times a^{-(N-1)}(\partial_u-1)\,\rho^{k-1}_{l,m}-a^{-(N-1)}\rho^{k-1}_{l,m}\times\partial_u\,\rho^k_{l,m} \\ &= a^{-(N-1)}\left[(D_u-1)\ \rho^{k-1}_{l,m}\cdot\rho^k_{l,m}+\rho^k_{l+1,m}\rho^{k-1}_{l-1,m}\right],\end{aligned} \tag{A.12}$$

which implies (4.7).

A.2 Equation (4.10)

We derive (4.10) from (4.7) and (4.12). We first introduce $F^k_{l,m}$ by the subtraction of the right hand side of (4.10) from the left hand side

$$F^k_{l,m} := \frac{1}{2}D_uD_y\ \sigma^k_{l,m}\cdot\sigma^k_{l,m}-a(\sigma^k_{l,m})^2+a\sigma^{k+1}_{l+1,m}\sigma^{k-1}_{l-1,m}, \tag{A.13}$$

and consider

$$\begin{aligned}P &:= F^k_{l,m}\left(\sigma^{k-1}_{l,m}\right)^2-F^{k-1}_{l,m}\left(\sigma^k_{l,m}\right)^2 \\ &= \left[\frac{1}{2}D_uD_y\ \sigma^k_{l,m}\cdot\sigma^k_{l,m}-a(\sigma^k_{l,m})^2+a\sigma^{k+1}_{l+1,m}\sigma^{k-1}_{l-1,m}\right]\ \left(\sigma^{k-1}_{l,m}\right)^2 \\ &-\left(\sigma^k_{l,m}\right)^2\ \left[\frac{1}{2}D_uD_y\ \sigma^{k-1}_{l,m}\cdot\sigma^{k-1}_{l,m}-a(\sigma^{k-1}_{l,m})^2+a\sigma^k_{l+1,m}\sigma^{k-2}_{l-1,m}\right].\end{aligned} \tag{A.14}$$

Equation (A.14) can be rewritten as

$$P = D_y\left(D_x\ \sigma^k_{l,m}\cdot\sigma^{k-1}_{l,m}\right)\cdot\sigma^k_{l,m}\sigma^{k-1}_{l,m}+a\sigma^{k+1}_{l+1,m}\sigma^{k-1}_{l-1,m}\sigma^{k-1}_{l,m}\sigma^{k-1}_{l,m}-a\sigma^k_{l+1,m}\sigma^{k-2}_{l-1,m}\sigma^k_{l,m}\sigma^k_{l,m}, \tag{A.15}$$

where we have used the exchange formula of the D-operator [9]

$$\left(D_uD_y\ f\cdot f\right)g^2-f^2\left(D_uD_y\ g\cdot g\right)=2D_y\ (D_u\ f\cdot g)\cdot fg, \tag{A.16}$$

for arbitrary functions f and g. We manipulate the first term of (A.15) as follows. Using (4.7) and noticing $D_y\ f\cdot f=0$, we have

$$D_y\left(D_x\ \sigma^k_{l,m}\cdot\sigma^{k-1}_{l,m}\right)\cdot\sigma^k_{l,m}\sigma^{k-1}_{l,m} = D_y\left(-\sigma^{k-1}_{l,m}\sigma^k_{l,m}+\sigma^k_{l+1,m}\sigma^{k-1}_{l-1,m}\right)\cdot\sigma^k_{l,m}\sigma^{k-1}_{l,m} = D_y\ \sigma^k_{l+1,m}\sigma^{k-1}_{l-1,m}\cdot\sigma^k_{l,m}\sigma^{k-1}_{l,m}.$$

Then applying another exchange formula

$$D_y\ \alpha\beta\cdot\gamma\delta = \left(D_y\ \alpha\cdot\gamma\right)\beta\delta+\left(D_y\ \beta\cdot\delta\right)\alpha\gamma, \tag{A.17}$$

for arbitrary functions $\alpha, \beta, \gamma, \delta$, we get

$$
\begin{aligned}
&D_y\, \sigma^k_{l+1,m}\sigma^{k-1}_{l-1,m} \cdot \sigma^k_{l,m}\sigma^{k-1}_{l,m} = \left(D_y\, \sigma^k_{l+1,m} \cdot \sigma^k_{l,m}\right)\sigma^{k-1}_{l-1,m}\sigma^{k-1}_{l,m} + \left(D_y\, \sigma^{k-1}_{l-1,m} \cdot \sigma^{k-1}_{l,m}\right)\sigma^k_{l+1,m}\sigma^k_{l,m} \\
&= \left(\sigma^k_{l+1,m}\sigma^k_{l,m} - a\sigma^{k+1}_{l+1,m}\sigma^{k-1}_{l,m}\right)\sigma^{k-1}_{l-1,m}\sigma^{k-1}_{l,m} + \left(-\sigma^{k-1}_{l,m}\sigma^{k-1}_{l-1,m} + a\sigma^k_{l,m}\sigma^{k-2}_{l-1,m}\right)\sigma^k_{l+1,m}\sigma^k_{l,m} \\
&= -a\sigma^{k+1}_{l+1,m}\sigma^{k-1}_{l,m}\sigma^{k-1}_{l-1,m}\sigma^{k-1}_{l,m} + a\sigma^k_{l,m}\sigma^{k-2}_{l-1,m}\sigma^k_{l+1,m}\sigma^k_{l,m}
\end{aligned}
$$

where we have used (4.12) in the second equality. Substituting the above result into (A.15), we see that $P = 0$. Therefore, it follows from (A.14) that

$$
\frac{1}{2}D_uD_y\, \sigma^k_{l,m} \cdot \sigma^k_{l,m} - a(\sigma^k_{l,m})^2 + a\sigma^{k+1}_{l+1,m}\sigma^{k-1}_{l-1,m} = A(u,y,l)(\sigma^k_{l,m})^2, \tag{A.18}
$$

where $A(u,y,l)$ is an arbitrary function. Since $\sigma^k_{l,m} = 1$ (the case of $N = 0$) satisfies (4.7) and (4.12), it should satisfy (A.18) as well. Therefore we see that A must be 0, which implies (4.10).

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References

[1] M. J. Ablowitz, D. J. Kaup and A. C Newell, Coherent pulse propagation, a dispersive, irreversible phenomenon, J. Math. Phys. **15**(1974) 1852–1858.

[2] M. J. Ablowitz and J. F. Ladik, Nonlinear differential-difference equations, J. Math. Phys. **16**(1975) 598–603.

[3] A. Doliwa and P. M. Santini, Integrable dynamics of a discrete curve and the Ablowitz-Ladik hierarchy, J. Math. Phys. **36** (1995)1259–1273

[4] A. Doliwa and P. M. Santini, The integrable dynamic of a discrete curve, Symmetries and Integrability of Difference Equations, D. Levi, L. Vinet and P. Winternitz (eds.), (AMS, Providence 1996) 91–102.

[5] A. Doliwa and P. M. Santini, Geometry of discrete curves and lattices and integrable difference equations, in: *Discrete Integrable Geometry and Physics*, A. Bobenko and R. Seiler (eds.), (Clarendon Press, Oxford 1999) 139–154.

[6] B.-F. Feng, J. Inoguchi, K. Kajiwara, K. Maruno and Y. Ohta, Discrete integrable systems and hodograph transformations arising from motions of discrete plane curves, preprint, 2011 (arXiv:1107.1148).

[7] R. Hirota, Exact N-soliton solution of nonlinear lumped self-dual network equation, J. Phys. Soc. Jpn. **35**(1973) 289–294.

[8] R. Hirota, Discretization of the potential modified KdV equation, J. Phys. Soc. Jpn. **67**(1998) 2234–2236.

[9] R. Hirota, *The direct method in soliton theory*, Cambridge Tracts in Mathematics **155** (Cambridge University Press, 2004)

[10] R. Hirota and J. Satsuma, Nonlinear evolution equations generated from the Bäcklund transformation for the Toda lattice, Progr. Theoret. Phys. **55**(1976) 2037–2038.

[11] R. Hirota and J. Satsuma, A variety of nonlinear network equations generated from the Bäcklund transformation for the Toda lattice, Progr. Theoret. Phys. Suppl. **59**(1976) 64–100

[12] T. Hoffmann, *Discrete differential geometry of curves and surfaces*, COE lecture Notes Vol. 18, Kyushu University (2009).

[13] T. Hoffmann and N. Kutz, Discrete curves in $\mathbb{C}P^1$ and the Toda lattice, Stud. Appl. Math. **113** (2004) 31–55.

[14] J. Inoguchi, K. Kajiwara, N. Matsuura and Y. Ohta, Motion and Bäcklund transformations of plane discrete curves, to appear in Kyushu J. Math. (2011).

[15] M. Jimbo and T. Miwa, Solitons and infinite dimensional Lie algebras, Publ. RIMS. **19**(1983) 943–1001.

[16] S. Kakei and J. Satsuma, Multi-soliton solutions of a coupled system of the nonlinear Schrödinger equation and the Maxwell-Bloch equations, J. Phys. Soc. Jpn. **63**(1994) 885–894.

[17] N. Matsuura, Discrete KdV and discrete modified KdV equations arising from motions of discrete planar curves, to appear in Int. Math. Res. Notices.

[18] K. Maruno, K. Kajiwara and M. Oikawa, Casorati determinant solution for the discrete-time relativistic Toda lattice equation, Phys. Lett. **A241**(1998) 335–343.

[19] K. Maruno and Y. Ohta, Casorati determinant form of dark soliton solutions of the discrete nonlinear Schrödinger equation, J. Phys. Soc. Jpn. **75**(2006) 054002.

[20] Y. Ohta, R. Hirota, S. Tsujimoto and T. Imai, Casorati and discrete Gram type determinant representations of solutions to the discrete KP hierarchy, J. Phys. Soc. Jpn. **62**(1993) 1872–1886.

[21] Y. Ohta, K. Kajiwara, J. Matsukidaira and J. Satsuma, Casorati determinant solution for the relativistic Toda lattice equation, J. Math. Phys. **34**(1993) 5190–5204.

[22] S. Tsujimoto, On a discrete analogue of the two-dimensional Toda lattice hierarchy, Publ. RIMS **38**(2002) 113-133.

[23] K. Uhlenbeck, Harmonic maps into Lie group (classical solutions of the chiral model), J. Differential Geom. **30**(1989) 1—50.

[24] V. E. Zakharov and A. V. Mikhailov, Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method, Sov. Phys. JETP **47** (1978) 1017–1027.

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