Open access • Journal Article • DOI:10.1103/PHYSREVB.79.085119

Explicit tensor network representation for the ground states of string-net models — Source link

Oliver Buerschaper, Miguel Aguado, Guifre Vidal

Institutions: Max Planck Society, University of Queensland

Published on: 23 Feb 2009 - Physical Review B (American Physical Society)

Topics: Tensor contraction and Tensor product of Hilbert spaces

Explicit tensor network representation for the ground states of string-net models

Oliver Buerschaper,[1] Miguel Aguado,[1] and Guifré Vidal[2]
[1]*Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Straße 1, 85748 Garching, Germany*
[2]*School of Physical Sciences, The University of Queensland, Brisbane QLD 4072, Australia*
(Received 30 September 2008; revised manuscript received 27 November 2008; published 23 February 2009)

We provide a simple expression for the ground states of arbitrary string-net models in the form of local tensor networks. These tensor networks encode the data of the fusion category underlying a string-net model and thus represent all doubled topological phases of matter in the infrared limit according to Levin and Wen [Phys. Rev. B **71**, 045110 (2005)]. Furthermore, our construction highlights the importance of the fat lattice equivalence between lattice and continuum descriptions of string-net models.

DOI: 10.1103/PhysRevB.79.085119 PACS number(s): 05.30.−d

I. INTRODUCTION

Topological phases of lattice models have been attracting much interest recently because of their potential use as quantum memories and quantum computers.[1] While it is unclear whether their shortcomings at finite temperature can be satisfactorily resolved,[2–5] these models remain appealing as systems with topological order[6] and as quantum error correcting codes. The first such proposal was Kitaev's introduction of the Abelian toric code[1] as well as a class of non-Abelian generalizations, the quantum double models, with roots in the theory of Hopf algebras. These are examples of gauge models with discrete gauge group.[7,8] More recently, Levin and Wen[9] introduced their string-net models, arguing for string-net condensation as a basic mechanism underlying topological order. Note that, compared to the quantum double models, these string-net models allow for a wider range of topological orders including anyon models with noninteger quantum dimensions.

In order to study these topological phases on the lattice, one naturally seeks an efficient way to describe their quantum states. Accordingly, explicit tensor network constructions (which are an efficient representation) have been presented recently for particular string-net models: in Ref. 10 a *projected entangled-pair state* (PEPS) representation of the toric code and resonating valence bond states is described, whereas a *multiscale entanglement renormalization ansatz* (MERA) representation of all quantum double[11] and all string-net models[12] is also known. Finally, a *double line* tensor network of the toric code and the semion model is discussed in Ref. 13.

In this paper we describe a simple construction expressing the ground state of an arbitrary string-net model as a tensor network built from F symbols. The starting point is the form of the Hamiltonian $H=-\Sigma_i P_i$, where the P_i are commuting projectors, which leads to the realization of the ground level as the +1 eigenspace[14] of the product $\Pi_i P_i$. The tensor network follows in a remarkably straightforward way. Its form is reminiscent of a classical statistical mechanical partition function with local (albeit possibly complex) weights, which is why we call it a Boltzmann weight tensor network (for an early account of the link between partition functions and stabilizer Hamiltonians, the reader is referred to Refs. 15 and 16). The needed ingredients are the data of the underlying fusion category as explained in Ref. 9, i.e., the fusion rules and associated F tensors. The construction is most appropriately understood from the fat lattice perspective.

Our Boltzmann weight tensor network differs from the MERA representation which is also based on F symbols[12] in that it is much simpler than the latter—e.g., it is a two-dimensional network while the MERA spans three dimensions. On the other hand, although conceived independently, for the toric code and semion models our construction coincides with the *double line* tensor network of Ref. 13, where the authors also hint at an unpublished result for generic string-net models.

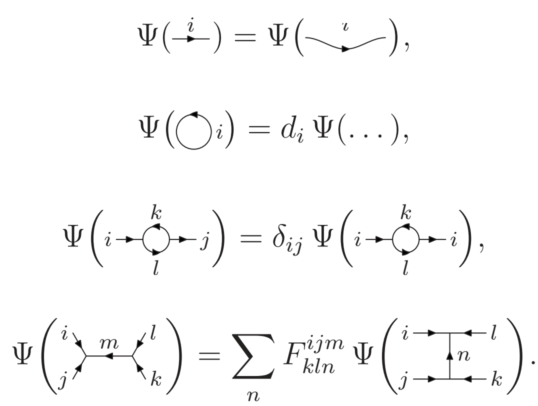
II. STRING-NET MODELS

String-net models were introduced in Ref. 9 in order to encode the universal physical properties of doubled $(2+1)D$ topological phases of matter in quantum lattice models with few-body interactions. The simple structure of their Hamiltonians reflects their conception as infrared fixed points of renormalization-group flows. Explicitly, these Hamiltonians are exactly solvable because they are given by the sum of mutually commuting terms. In the following, we consider those models in Ref. 9 which exhibit a well-defined continuum limit.

These models are defined on a hexagonal lattice Λ. Local degrees of freedom are associated with oriented edges of Λ and elements of the computational basis are labeled by $i \in \{0,1,\ldots,N\}$. These labels may be interpreted as particle species propagating along the edges. For each label i there is a unique label i^* denoting its antiparticle, and reversing the orientation of an edge corresponds to the mapping $i \mapsto i^*$. The label 0 stands for the absence of any particle (vacuum) and one has $0^*=0$. Furthermore, the physical Hilbert space of a string-net model is defined by a set of *fusion rules* δ_{ijk} specifying allowed ($\delta_{ijk}=1$) and forbidden ($\delta_{ijk}=0$) configurations of labels incident to a vertex. Given the set of labels and their fusion rules, one can build a *fusion category* which includes recoupling relations encapsulated in the $6j$ symbol F^{ijm}_{kln}, and an assignment of *quantum dimensions* d_i to the labels. The *total quantum dimension* is given by $\mathcal{D}=(\Sigma_{i=1}^{N} d_i^2)^{1/2}$.

Furthermore there is a natural correspondence between physical configurations on Λ and configurations of string nets in a continuum model defined on the so-called *fat lat-*

tice. The latter is constructed from the physical lattice Λ by puncturing the underlying surface at the center of each plaquette. String nets in this continuum description consist of oriented strings carrying labels in the set $\{0,1,\ldots,N\}$, joined at trivalent branching points in a way that respects the fusion rules δ_{ijk}, and avoiding the punctures. Fixed-point wave functions are constructed using local constraints[9] in order to define equivalence classes of configurations in the fat lattice. These constraints are crafted so as to enforce topological invariance of the wave function (which will eventually be identified with a quantum state in the physical discrete lattice Λ) and are assembled from the objects d_i and F^{ijm}_{kln} introduced above. For a quantum state $|\Psi\rangle = \Sigma_X \Psi(X)|X\rangle$, where $|X\rangle$ is a basis configuration in the discrete lattice Λ and $\Psi(X)$ is its associated amplitude, they explicitly read:



(1)

(2)

(3)

(4)

Intuitively, Eq. (1) ensures invariance under continuous deformations of strings, Eq. (2) describes trading isolated loops for quantum dimensions, Eq. (3) imposes charge conservation, and finally Eq. (4) introduces a recoupling tensor F which generalizes the $6j$ symbols found, e.g., in the theory of angular momentum.

String-net configurations are defined to be equivalent if they can be transformed into each other using the local relations. Equivalence classes are identified with physical configurations. Note that the physical configuration itself can be regarded as a particular string net identical with the physical lattice. We will refer to this particular string net as the canonical representative of the equivalence class, and its uniqueness is ensured by Mac Lane's coherence theorem.[17]

The Hamiltonian on the physical lattice reads

$$H = -\sum_v A_v - \sum_p B_p, \tag{5}$$

where the sums range over the vertices and plaquettes of the lattice. Vertex terms are projectors $A_v = \Sigma_{i,j,k \in v} \delta_{ijk} |ijk\rangle\langle ijk|$ enforcing the fusion rules while plaquette projectors represent the kinetic part of the Hamiltonian and are defined by

$$B_p = \sum_{\alpha_p=1}^{N} \frac{d_{\alpha_p}}{\mathcal{D}^2} B_p^{\alpha_p}, \tag{6}$$

where $B_p^{\alpha_p}$ acts on the plaquette p together with the outer legs of p. Its precise definition is given in Ref. 9, as well as the following simple graphical interpretation on the fat lattice: $B_p^{\alpha_p}$ creates an isolated loop of label α_p around the puncture at plaquette p.

III. TENSOR NETWORKS FOR STRING-NET GROUND STATES

In order to construct an explicit graphical expression for a ground state of a string-net model, we start with the state $|0\ldots0\rangle$ on the physical lattice where all edges carry the vacuum label 0. Note that this can be represented by a completely empty fat lattice. Obviously, this state is an eigenstate of all the A_v operators with eigenvalue +1. Since the Hamiltonian is frustration-free, we end up in the ground level by applying the projection $\Pi_p B_p$. Thus, up to an overall factor, this ground state on the physical lattice is represented by the following string-net state on the fat lattice:

$$|\Psi_0\rangle = \sum_{\alpha_p} \left(\prod_p d_{\alpha_p}\right) |\{\alpha_p\}\rangle, \tag{7}$$

where $|\{\alpha_p\}\rangle$ denotes the string-net configuration shown in Fig. 1(a).

From now on we will use the local relations of the string-net model in order to reduce Eq. (7) to its canonical representative, which can be directly translated into a configuration on the physical lattice. After applying three rounds of recouplings involving F symbols (F moves) to the strings on the fat lattice, one has

$$\begin{aligned}|\Psi_0\rangle = &\sum_{\{\alpha_p\}} \left(\prod_p d_{\alpha_p}\right) \sum_{\{i_p,j_p,k_p\}} \left(\prod_{(p,q)\in E_1} F^{\alpha_p^* \alpha_p 0}_{\alpha_q^* \alpha_q i_p}\right) \\ &\times \left(\prod_{(p,q)\in E_2} F^{\alpha_p^* \alpha_p 0}_{\alpha_q^* \alpha_q j_p}\right)\left(\prod_{(p,q)\in E_3} F^{\alpha_p^* \alpha_p 0}_{\alpha_q^* \alpha_q k_p}\right) \\ &\times |\{\alpha_p, i_p, j_p, k_p\}\rangle,\end{aligned} \tag{8}$$

where $|\{\alpha_p, i_p, j_p, k_p\}\rangle$ denotes the state of the fat lattice as shown in Fig. 1(b). Note that we have decomposed the edge set of the dual lattice Λ^* as $E(\Lambda^*) = \cup_{i=1}^{3} E_i$, where E_1 denotes the set of horizontal edges, E_2 denotes one set of parallel diagonal edges, and E_3 denotes the other one according to Fig. 2. Using the normalization,

$$F^{ii^*0}_{j^*jk} = \sqrt{\frac{d_k}{d_i d_j}}\,\delta_{ijk}, \tag{9}$$

this expression can be simplified in the case of an infinite or periodic lattice to yield:

$$|\Psi_0\rangle = \sum_{\{\alpha_p,i_p,j_p,k_p\}_\star} \left(\prod_p \frac{\sqrt{d_{i_p} d_{j_p} d_{k_p}}}{d^2_{\alpha_p}}\right) |\{\alpha_p, i_p, j_p, k_p\}\rangle. \tag{10}$$

Note that we have omitted the δ symbols and rather restricted the sum to configurations $\{\alpha_p, i_p, j_p, k_p\}_\star$ that respect the branching rules of the particular string-net model.

For a full reduction to the physical lattice, we eventually need to remove the loops at the vertices. This can be done by applying two F moves at each vertex:

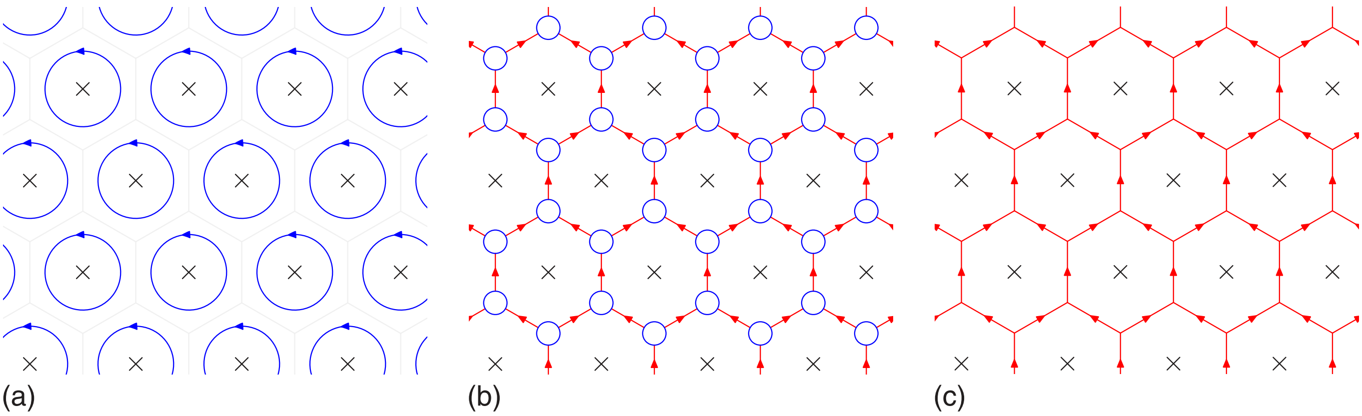


FIG. 1. (Color online) Reducing the fat lattice. In (a) an isolated loop at face p carries a label α_p and all edges $E(\Lambda)$ (gray) of the physical lattice are labeled by the vacuum 1. The overall quantum state is denoted by $|\{\alpha_p\}\rangle$. (b) Three rounds of F moves reduce to the fat lattice configuration which carries both the final physical labels i_p, j_p, and k_p as well as the labels α_p at the vertex loops. This is denoted by $|\{\alpha_p,i_p,j_p,k_p\}\rangle$. In (c) each vertex has been fully reduced to the physical lattice by a sequence of another two F moves each. The corresponding physical quantum state is given by $|\{i_p,j_p,k_p\}\rangle$.

$$|\{\alpha_p,i_p,j_p,k_p\}\rangle = \left(\prod_p \frac{d_{\alpha_p}^2}{\sqrt{d_{i_p}d_{k_p}}}\right)\prod_{v\in\Lambda_1} f(v) \times \prod_{w\in\Lambda_2} g(w)|\{i_p,j_p,k_p\}\rangle. \tag{11}$$

Here Λ_i denote the even and odd sublattices of Λ, respectively, and furthermore one has

$$f(v) = F^{\alpha_p^* j_p \alpha_q}_{k_q \alpha_r i_p}, \tag{12}$$

$$g(w) = F^{\alpha_s^* i_s \alpha_p}_{j_r^* \alpha_r k_s}, \tag{13}$$

where the faces $\{p,q,r\}$ of Λ surround an even vertex v and $\{p,r,s\}$ surround an odd vertex w as indicated in Fig. 2.

At this point the ground state of the string-net model can be written in terms of the physical lattice only:

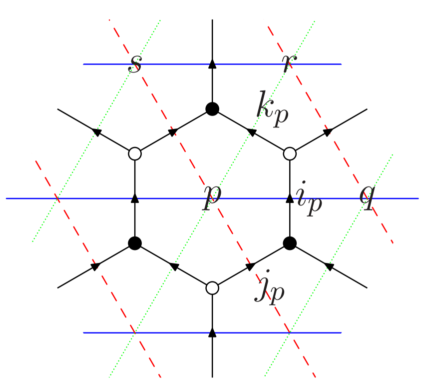


FIG. 2. (Color online) Hexagonal lattice Λ and its dual Λ^*. Horizontal edges (blue, solid line) of Λ^* belong to E_1, diagonal edges (red, dashed line) to E_2, and diagonal edges (green, dotted line) to E_3. The directed edges of Λ are labeled by uniquely associating them to a face. Circled vertices of Λ belong to the even sublattice Λ_1, while filled ones belong to the odd sublattice Λ_2.

$$|\Psi_0\rangle = \sum_{\{\alpha_p,i_p,j_p,k_p\}}\left(\prod_p \sqrt{d_{j_p}}\right)\prod_{v\in\Lambda_1} f(v)\prod_{w\in\Lambda_2} g(w)\times|\{i_p,j_p,k_p\}\rangle. \tag{14}$$

Note that because of the convention for the F symbols in Ref. 9 the branching rules at each vertex are automatically satisfied and we no longer need to restrict the sum. This allows one to isolate the basis coefficients:

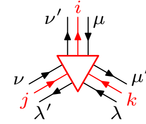
$$\lambda_{\{i_p,j_p,k_p\}} = \left(\prod_p \sqrt{d_{j_p}}\right)\sum_{\{\alpha_p\}}\prod_{v\in\Lambda_1} f(v)\prod_{w\in\Lambda_2} g(w). \tag{15}$$

It is this very expression that we are now going to write in a graphical fashion as a contracted tensor network.

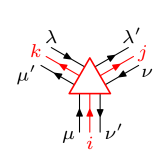
In order to write the coefficients of the string-net model ground state given by Eq. (15) in a graphical fashion, it is instructive to proceed locally. Let us therefore consider an arbitrary face a of Λ together with its next neighbors $b,\ldots,g$. Obviously, the sum over α_a can now be carried out immediately and we obtain the following local expression:

$$\lambda_{\{i_p,j_p,k_p\}} \sim \sqrt{d_{j_a}d_{j_d}}\sum_{\alpha_a} F^{\alpha_a^* j_a \alpha_g}_{k_g \alpha_b i_a} F^{\alpha_d^* j_d \alpha_a}_{k_a \alpha_c i_d} F^{\alpha_e^* j_e \alpha_f}_{k_f \alpha_a i_e} \times F^{\alpha_a^* i_a \alpha_b}_{j_c^* \alpha_c k_a} F^{\alpha_e^* i_e \alpha_a}_{j_d^* \alpha_d k_e} F^{\alpha_f^* i_f \alpha_g}_{j_a^* \alpha_a k_f}. \tag{16}$$

Now define two sets of vertex tensors for the even and odd sublattices of Λ by



$$:= T^{[ijk]}_{\mu\mu'\nu\nu'\lambda\lambda'} := \sqrt{d_j}\, F^{\nu^* j\lambda}_{k\mu i}\,\delta_{\mu\mu'}\delta_{\nu\nu'}\delta_{\lambda\lambda'} \tag{17}$$



$$:= \tilde{T}^{[ijk]}_{\mu\mu'\nu\nu'\lambda\lambda'} := F^{\mu^* i\nu}_{j^*\lambda k}\,\delta_{\mu\mu'}\delta_{\nu\nu'}\delta_{\lambda\lambda'} \tag{18}$$

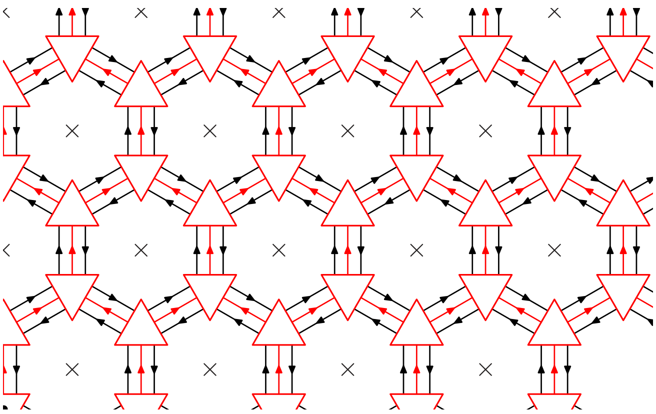
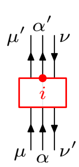


FIG. 3. (Color online) Tensor network describing the ground state of an arbitrary string-net model. Only virtual bonds (black, outer lines) are summed over while physical indices (red, inner lines) are left uncontracted.

and contract them according to the network given in Fig. 3. If we cut out a single face of this network, it can easily be verified that it exactly reproduces the local form of our coefficients as in Eq. (16) up to the factor $\sqrt{d_{j_e}}$ (which can be absorbed, as the summation is extended to the adjacent faces).

Thus we have obtained a simple graphical notation that describes the ground state of an arbitrary string-net model and involves local terms only. In fact, following the arguments of Ref. 9, our graphical calculus encompasses the ground states of all "doubled" topological phases in the infrared limit.

Of course, we may also pull out the indices from the vertex tensors and collect physical indices denoting particle and antiparticle into a single physical index at the edge. This can be done by defining the following tensors:



$$= A^{[i]}_{\alpha\alpha'\mu\mu'\nu\nu'} := \delta_{i\alpha}\delta_{\alpha\alpha'}\delta_{\mu\mu'}\delta_{\nu\nu'} \tag{19}$$

as well as the triangular ones $B_{\alpha\beta\gamma\mu\mu'\nu\nu'\lambda\lambda'} := T^{[\alpha\beta\gamma]}_{\mu\mu'\nu\nu'\lambda\lambda'}$ and $\tilde{B}_{\alpha\beta\gamma\mu\mu'\nu\nu'\lambda\lambda'} := \tilde{T}^{[\alpha\beta\gamma]}_{\mu\mu'\nu\nu'\lambda\lambda'}$, and contracting them according to Fig. 4. Note that the vertex tensors T and B only differ in how their indices are regarded: what used to be a physical index of T has been changed into a virtual one of B.

IV. SUMMARY AND OUTLOOK

In conclusion, in this paper we have derived a remarkably simple tensor network representation for Levin and Wen's string-net ground states. This construction follows directly from the characterization of these states as simultaneous +1 eigenstates of the projectors in the Hamiltonian. It also heavily relies on the notion of the fat lattice. Understanding

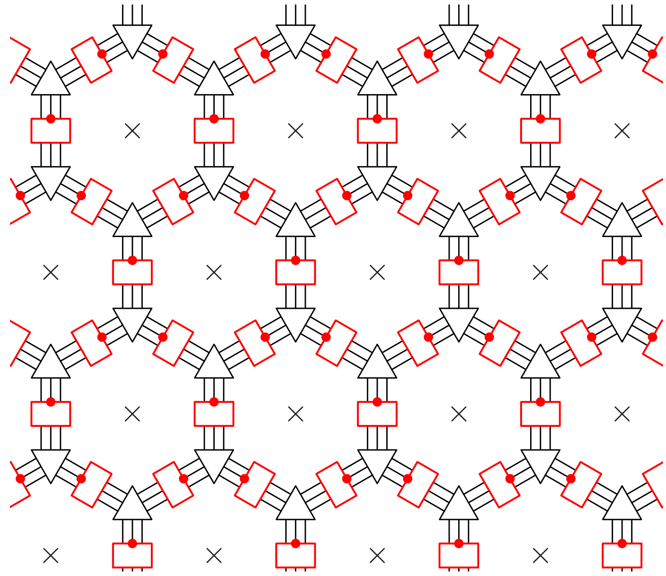


FIG. 4. (Color online) Alternative tensor network for ground states of arbitrary string-net models. Here all bonds are contracted.

string-net models in terms of the mapping from the fat lattice to the physical lattice thus leads to insight and useful results. The tensor network is built from the fusion rules and F tensors of the fusion category underlying the string-net model.

Note that from our Boltzmann weight tensor network, one can trivially build a PEPS representation. In the case of quantum double models, which can be explicitly written as string-net models, dramatic simplifications to this PEPS representation are possible due to their group-theoretical properties. Also, for a general string-net model excited states may possibly be expressed by absorbing their corresponding open string operators into a ground-state tensor network representation.[18]

Due to its simplicity (as compared to the full-fledged specification of all stabilizer operators), this tensor network representation will help study a range of properties of the ground level sector. For example, the topological entanglement entropy[19–21] is one such property hinting at the presence of a new kind of multipartite long-range entanglement that appears to underlie topological order. Along the same lines it may be interesting to establish criteria for a tensor network to represent a ground state with topological order. Of course, for this it would be necessary to extend the present analysis beyond the infrared limit as given by the string-net models.

Recently, it has come to our attention that results similar to ours have been obtained independently by Gu *et al.*[22]

ACKNOWLEDGMENTS

We thank J. I. Cirac and N. Schuch for valuable and inspiring discussions. G. Vidal acknowledges support from the Australian Research Council (Contracts No. FF0668731 and No. DP0878830).

[1] A. Kitaev, Ann. Phys. (N.Y.) **303**, 2 (2003).
[2] C. Castelnovo and C. Chamon, Phys. Rev. B **76**, 184442 (2007).
[3] Z. Nussinov and G. Ortiz, Phys. Rev. B **77**, 064302 (2008).
[4] S. Iblisdir, D. Pérez-García, M. Aguado, and J. K. Pachos, arXiv:0806.1853 (unpublished).
[5] A. Kay, arXiv:0807.0287, Phys. Rev. Lett. (to be published).
[6] X.-G. Wen, Adv. Phys. **44**, 405 (1995).
[7] F. J. Wegner, J. Math. Phys. **12**, 2259 (1971).
[8] M. de Wild Propitius and F. A. Bais, arXiv:hep-th/9511201 (unpublished).
[9] M. A. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005).
[10] F. Verstraete, M. M. Wolf, D. Perez-García, and J. I. Cirac, Phys. Rev. Lett. **96**, 220601 (2006).
[11] M. Aguado and G. Vidal, Phys. Rev. Lett. **100**, 070404 (2008).
[12] R. König, B. W. Reichardt, and G. Vidal, arXiv:0806.4583 (unpublished).
[13] Z.-C. Gu, M. A. Levin, and X.-G. Wen, arXiv:0807.2010 (unpublished).
[14] Contrary to some treatments of stabilizer Hamiltonians, note that in our case the other eigenvalue of each P_i is 0 rather than -1.
[15] M. Van den Nest, W. Dür, and H. J. Briegel, Phys. Rev. Lett. **98**, 117207 (2007).
[16] M. Van den Nest, W. Dür, and H. J. Briegel, Phys. Rev. Lett. **100**, 110501 (2008).
[17] S. Mac Lane, *Categories for the Working Mathematician*, 2nd ed. (Springer, New York, 1998).
[18] O. Buerschaper *et al.*, (unpublished).
[19] A. Hamma, R. Ionicioiu, and P. Zanardi, Phys. Lett. A **337**, 22 (2005).
[20] A. Kitaev and J. Preskill, Phys. Rev. Lett. **96**, 110404 (2006).
[21] M. A. Levin and X.-G. Wen, Phys. Rev. Lett. **96**, 110405 (2006).
[22] Z.-C. Gu, M. Levin, B. Swingle, and X.-G. Wen, preceding paper, Phys. Rev. B **79**, 085118 (2009).