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EXPLICIT UPPER BOUNDS FOR THE REMAINDER TERM IN THE DIVISOR PROBLEM

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To the memory of John Selfridge

Abstract. We first report on computations made using the GP/PARI package that show that the error term $\Delta(x)$ in the divisor problem is $= \mathscr{M}(x,4) + O^*(0.35\,x^{1/4}\log x)$ when x ranges $[1\,081\,080, 10^{10}]$, where $\mathscr{M}(x,4)$ is a smooth approximation. The remaining part (and in fact most) of the paper is devoted to showing that $|\Delta(x)| \le 0.397\,x^{1/2}$ when $x \ge 5\,560$ and that $|\Delta(x)| \le 0.764\,x^{1/3}\log x$ when $x \ge 9\,995$. Several other bounds are also proposed. We use this results to get an improved upper bound for the class number of a quadractic imaginary field and to get a better remainder term for averages of multiplicative functions that are close to the divisor function. We finally formulate a positivity conjecture concerning $\Delta(x)$.

1. Introduction

The object of this paper is to study for an explicit viewpoint the remainder term of the summatory function of the τ-function, where $\tau(n)$ denotes the number of (positive) divisors of n, i.e., to study

$$\Delta(x) = \sum_{n\le x} \tau(n) - x(\log x + 2\gamma - 1). \tag{1.1}$$

This function has been extensively studied, and the reader will find a good survey in [7]. It is known in particular that

$$\Delta(x) \ll_\varepsilon x^{131/416+\varepsilon}$$

for any $\varepsilon > 0$. We want to get fully explicit bounds of this shape here, and the best exponent we reach is $1/3$ (see Theorem 1.2 below). Note that $131/416 = 0.314\cdots$ is not so much smaller than $1/3 = 0.333\cdots$. Note further that Theorem 1.1 below gives an upper bound with a worse exponent, but which is better on a large range. The divisor function has been studied from this viewpoint in several papers, and we quote here [12], [19], [4] and [15].

Here are our main results:

Theorem 1.1. *When $x \ge 1$, we have $|\Delta(x)| \le 0.961\,x^{1/2}$.*
When $x \ge 1\,981$, we have $|\Delta(x)| \le 0.482\,x^{1/2}$.
When $x \ge 5\,560$, we have $|\Delta(x)| \le 0.397\,x^{1/2}$.

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These bounds are sharp, since $|\Delta(x)| > 0.5\,x^{1/2}$ when $x = 1\,980$ while $|\Delta(x)| > 0.4\,x^{1/2}$ when $x = 5\,559$.

Theorem 1.2. *When $x \geq 9\,995$, we have $|\Delta(x)| \leq 0.764\,x^{1/3}\log x$.*

This bound is also sharp since $|\Delta(x)| > 0.80\,x^{1/3}\log x$ when $x = 9\,994$. This bound is of course asymptotically better than the one given by Theorem 1.1, but this latter one still prevails when $x \leq 59\,576\,122\,384$.

There are two usual paths to study $\Delta(x)$ that can be broadly described by either using a Voronoï-like formula as in [13], or using the fractional part-function, expanding it in a Fourier series and using exponential sums, and using, for instance, [1, Lemma 8.4] (see also [2] for similar material, as well as [6, section 8]). We use the first technique, but rely on an earlier paper of Voronoï where a very explicit result is proved.

We rely also on some rather extensive computations detailed in section 6 made with the help of the PARI/GP program (see [20]) and its auxiliary GP2C. One of the main problems with such extensive computations is always how to store them, since tables are difficult to use. We again use the Voronoï formula to get such a model and prove the following.

Theorem 1.3. *For all $x \in [3, 10^{10}]$ we have*

$$\Delta(x) = \mathscr{M}(x,4) + O^*\big(0.9\,x^{1/4}\log x\big)$$

and for all $x \in [1\,081\,080, 10^{10}]$, we have

$$\Delta(x) = \mathscr{M}(x,4) + O^*\big(0.35\,x^{1/4}\log x\big).$$

Here

$$\mathscr{M}(x,4) = \frac{x^{1/4}}{\pi\sqrt{2}}\Big(\cos\big(4\pi\sqrt{x} - \tfrac{\pi}{4}\big) + 2^{1/4}\cos\big(4\pi\sqrt{2x} - \tfrac{\pi}{4}\big) + \frac{2}{3^{3/4}}\cos\big(4\pi\sqrt{3x} - \tfrac{\pi}{4}\big) + \frac{3}{4^{3/4}}\cos\big(4\pi\sqrt{4x} - \tfrac{\pi}{4}\big)\Big).$$

Section 6 contains more bounds of this shape. Note that the constant 0.35 is very good and fairly stable, since, for instance,

$$\big|\Delta(x_0) - \mathscr{M}(x_0,4)\big| \geq 0.289\,x_0^{1/4}\log x_0 \quad \text{when } x_0 = 9\,137\,256\,975.$$

A constant of 0.30 would require us to start at least at $2.7\cdot 10^9$, which renders the preliminary computations difficult. It would be valuable to extend Theorem 1.3 to a larger range.

We end this introduction by mentioning a curious conjecture upon which we stumbled:

Conjecture 1.4. For all $T \geq 1$, we have

$$\int_T^\infty \frac{\Delta(u)du}{u^{7/4}} \geq 0.$$

See section 8 for more background on this conjecture.

Notation. We use the Landau-like notation $f = O^*(g)$ to say that $|f| \leq g$. We use $\psi(x) = x - [x] - 1/2$, where $[x]$ is the integer part of x. We shall also need the multiplicative function

$$\tilde{\tau}(n, D) = \sum_{\substack{uv=n,\\ (u,v,D)=1}} 1 \tag{1.2}$$

for some parameter D, where (u, v, D) denotes the gcd of u, v and D.

2. Two applications

An application to number fields. Let $K/\mathbb{Q}$ be a number field of degree n, class number h_K, signature (r_1, r_2) and let d_K be the absolute value of its discriminant. We set b_K to be a real number such that each ideal class contains a nonzero ideal A satisfying $\mathcal{N}(A) \leqslant b_K\sqrt{d_K}$, where $\mathcal{N}$ denotes the ideal-norm operator in K. It is well known that one can take for b_K the Minkoswki bound $(4/\pi)^{r_2} n! n^{-n}$. If K is an imaginary quadratic field, then the better bound $b_K = 3^{-1/2}$, due to Gauss, can be used instead of the Minkowski constant.

It has been shown by the second author of [1] that the inequality

$$h_K \leqslant 2^{2-n} b_K d_K^{1/2} \left(\log\left(b_K^2 d_K\right)\right)^{n-1} \tag{2.1}$$

holds for all number fields K subject to the condition $d_K \geqslant 36 b_K^{-2}$. In the case of real quadratic fields, using Dirichlet's analytic class number formula and precise estimates for $L(1, \chi)$ (where χ is the primitive real Dirichlet character attached to K) and the fundamental unit of K, Maohua Le [11] proved that

$$h_K \leqslant \sqrt{d_K}/2.$$

A simpler proof of this bound has been provided by the third author in [16]. Using Theorem 1.3 we deduce the following slight improvement of (2.1) in the case of imaginary quadratic fields.

Corollary 2.1. *Let $K = \mathbb{Q}\left(\sqrt{-d}\right)$ be an imaginary quadratic field with $d > 0$ squarefree and d_K is the absolute value of its discriminant. If $d_K \geqslant 108$, then we have*

$$h_K \leqslant \sqrt{\frac{d_K}{12}} \log d_K.$$

Examples. In what follows, we set $\mathcal{B}_K = \left[\sqrt{d_K/12}\log d_K\right]$, where K is an imaginary quadratic subfield of the cyclotomic field $\mathbb{Q}(\zeta_d)$ where ζ_d is a primitive d-th root of unity. The computations have been made using PARI system.

d	h_K	$\mathcal{B}_K$
311	19	29
1559	51	83
149159	597	1328
300119	781	1994

An application to averages of multiplicative functions. [15, Lemma 3.2] proposes an automatic way of deriving an explicit bound for averages of multiplicative non-negative functions that are close enough to a given model. The two models proposed are the constant function 1 and the divisor function. In this latter case, using this lemma requires an explicit bound for $\sum_{n\le t}\tau(n)/n$ and the above paper relies on [19, Lemma 1] (this is also the second part of [15, Lemma 3.3]). We improve this lemma to the following.

Corollary 2.2. *We have, for all $t > 0$,*

$$\sum_{n\le t}\frac{\tau(n)}{n} = \tfrac{1}{2}\log^2 t + 2\gamma\log t + \gamma^2 - \gamma_1 + O^*(1.16/t^{1/3})$$

where γ_1 is the second Laurent-Stieljes constant, for instance, [10] *and* [3]. *In particular, we have*

$$\gamma_1 = -0.0728158454836767248605863758749013191377 + O^*(10^{-40}). \tag{2.2}$$

3. Borrowing from Dirichlet

Let us first recall a result of Dirichlet.

Lemma 3.1 (Dirichlet)**.** *When $x \ge 1$ is a real number, we have*

$$\Bigl|\Delta(x) + 2\sum_{n\le\sqrt{x}}\psi(x/n)\Bigr| \le \tfrac{1}{2}.$$

The proof we present is somewhat more complete than that of [1, Lemma 8.1], since we express $\mathcal{R}(x)$ below fully in terms of ψ_2.

Proof. Set $\psi_2(t) = \frac{1}{2}\psi(t)^2$. We first notice that the function $x \mapsto \frac{1}{8} + \int_1^x \psi(t)dt$ is periodic of period 1, and that, when $0 \le y < 1$,

$$\int_1^{1+y}\psi(t)dt + \tfrac{1}{8} = \int_0^y (t - \tfrac{1}{2})dt + \tfrac{1}{8} = \psi_2(y) = \psi_2(1+y)$$

and thus ψ_2 is an antiderivative of ψ. By Dirichlet's Hyperbola Principle and Euler-MacLaurin's Summation Formula we get

$$\begin{aligned}\sum_{n\leqslant x}\tau(n) &= 2\sum_{n\leqslant x^{1/2}}[x/n]-\left[\sqrt{x}\right]^2\\ &= 2x\sum_{n\leqslant x^{1/2}}\frac{1}{n}-2\sum_{n\leqslant x^{1/2}}\psi\left(\frac{x}{n}\right)+\frac{1}{2}-x+2\sqrt{x}\psi\left(\sqrt{x}\right)-\psi\left(\sqrt{x}\right)^2-\frac{1}{4}\\ &= 2x\left(\frac{\log x}{2}+\gamma-\frac{\psi\left(\sqrt{x}\right)}{\sqrt{x}}-\frac{\psi_2\left(\sqrt{x}\right)}{x}+2\int_{\sqrt{x}}^{\infty}\frac{\psi_2(t)}{t^3}\mathrm{d}t\right)\\ &\quad -2\sum_{n\leqslant x^{1/2}}\psi\left(\frac{x}{n}\right)+\frac{1}{4}-x+2\sqrt{x}\psi\left(\sqrt{x}\right)-\psi\left(\sqrt{x}\right)^2\\ &= x\left(\log x+2\gamma-1\right)-2\sum_{n\leqslant x^{1/2}}\psi\left(\frac{x}{n}\right)+\mathcal{R}(x),\end{aligned}$$

where

$$\mathcal{R}(x)=\frac{1}{4}-3\psi_2\left(\sqrt{x}\right)+4x\int_{\sqrt{x}}^{\infty}\frac{\psi_2(t)}{t^3}\mathrm{d}t.$$

The inequality $0\leqslant\psi_2(t)\leqslant 1/8$ implies that

$$\left|\tfrac{1}{4}-3\psi_2\left(\sqrt{x}\right)\right|\leqslant\tfrac{1}{4}$$

and

$$4x\left|\int_{\sqrt{x}}^{\infty}\frac{\psi_2(t)}{t^3}\mathrm{d}t\right|\leqslant\frac{x}{2}\int_{\sqrt{x}}^{\infty}\frac{\mathrm{d}t}{t^3}=\frac{1}{4},$$

which concludes the proof. □

Corollary 3.2. *When $x\geq 1$ is a real number, we have $|\Delta(x)|\leq\sqrt{x}+\frac{1}{2}$.*

4. Auxiliary results

Let us start with a generic formula, valid for any sequence (φ_n). We define an abstract remainder term by

$$\Delta_\varphi(t)=\sum_{n\leq t}\varphi_n-(at\log t+bt)$$

for some real numbers a and b. The following formula holds for any complex number $s\neq 1$:

$$\sum_{n\leq T}\frac{\varphi_n}{n^s}=\frac{aT^{1-s}\log T}{1-s}+\frac{b(1-s)-as}{(1-s)^2}T^{1-s}+\frac{s(a-b(1-s))}{(1-s)^2}+T^{-s}\Delta_\varphi(T)+s\int_1^T\Delta_\varphi(u)du/u^{s+1}. \quad (4.1)$$

This is most readily obtained by summation by parts.

From $\tilde{\tau}(\cdot, D)$ to $\tau(\cdot)$. The gcd condition in $\tilde{\tau}(\cdot, D)$ is easily handled by using the Möbius function. Indeed, on using the following easily proved formula

$$\mathbb{1}_{(u,v,D)=1} = \sum_{\substack{\delta|u,\delta|v,\\ \delta|D}} \mu(\delta), \tag{4.2}$$

we readily get, for $T > 0$,

$$\sum_{n\le T} \frac{\tilde{\tau}(n,D)}{n^s} = \sum_{\delta|D} \mu(\delta) \sum_{\substack{\delta|u,\\ \delta|v,\\ uv\le T}} \frac{1}{(uv)^s} = \sum_{\delta|D} \frac{\mu(\delta)}{\delta^{2s}} \sum_{n\le T/\delta^2} \frac{\tau(n)}{n^s}.$$

On selecting $s = 0$, this leads to the asymptotic formula

$$\sum_{n\le T} \tilde{\tau}(n,D) = A(D)T\log T + B(D)T + \Delta(T,D) \tag{4.3}$$

where $A(D)$ and $B(D)$ are defined by

$$A(D) = \sum_{\delta|D} \frac{\mu(\delta)}{\delta^2}, \quad B(D) = \sum_{\delta|D} \frac{\mu(\delta)}{\delta^2}(2\gamma - 1 - 2\log\delta), \tag{4.4}$$

while $\Delta(\cdot, D)$ is expressed in terms of $\Delta(\cdot)$ by

$$\Delta(T,D) = \sum_{\delta|D} \mu(\delta)\Delta(T/\delta^2). \tag{4.5}$$

Some formulae with $\tilde{\tau}(n,D)$. We select $a = A(D)$, $b = B(D)$, $s = 1/2$ and $s = 3/4$ in formula (4.1) and quote explicitly:

$$\begin{aligned}\sum_{n\le T} \frac{\tilde{\tau}(n,D)}{n^{1/2}} &= 2A(D)T^{1/2}\log T + 2(B(D) - A(D))T^{1/2} \\ &\quad + 2A(D) - B(D) + \frac{\Delta(T,D)}{T^{1/2}} + \tfrac{1}{2}\int_1^T \frac{\Delta(u,D)du}{u^{3/2}},\end{aligned} \tag{4.6}$$

which is the case $s = 1/2$ from above. The case $s = 3/4$ reads

$$\begin{aligned}\sum_{n\le T} \frac{\tilde{\tau}(n,D)}{n^{3/4}} &= 4A(D)T^{1/4}\log T + 4(B(D) - 3A(D))T^{1/4} \\ &\quad + 12A(D) - 3B(D) + \frac{\Delta(T,D)}{T^{3/4}} + \tfrac{3}{4}\int_1^T \frac{\Delta(u,D)du}{u^{7/4}}.\end{aligned} \tag{4.7}$$

A generic integral. We note that, when $s \neq 1, 2$,

$$\int \frac{t(\log t + c) + d}{t^s} dt = \frac{\log t + (s-2)^{-1} + c}{(2-s)t^{s-2}} + \frac{d}{(1-s)t^{s-1}}. \tag{4.8}$$

Proof. Take the derivative of the right-hand side and check that it is the integrand. □

5. Borrowing from Voronoï

The purely elementary method of Voronoï, which improves on the Dirichlet hyperbola formula by using triangles instead of rectangles beneath the hyperbola $mn = x$, yields the following result [21, pages 280, 281].

Lemma 5.1. *When $x \geq 1$, $T \geq 1$ and $D \geq 1$ are real numbers, we have*

$$|\Delta(x)| \leq \frac{19}{12}\sum_{n\leq T}\tilde{\tau}(n,D) + \Big(\frac{\sqrt{x}}{4T}+\frac{\sqrt{T}}{6}\Big)\sum_{n\leq T}\frac{\tilde{\tau}(n,D)}{\sqrt{n}} + \frac{3x^{1/4}}{4}\sum_{n\leq T}\frac{\tilde{\tau}(n,D)}{n^{3/4}} + \frac{T}{6} + \sqrt{\frac{x}{T}} + \frac{7}{4}$$

where $\tilde{\tau}$ is defined in (1.2).

Comparing with [22, page 209, Théorème] and [23, page 429, paragraph 49, théorème I], or with [8] or [13], we see that, in case $D = 1$, one can asymptotically dispense with the first two sums at the cost of a $O_\varepsilon(x^\varepsilon)$ for any $\varepsilon > 0$, and that the constant $3/4$ in front of the third sum can be reduced to $1/(\pi\sqrt{2})$. The advantage of the above lemma relies on its range of validity. The parameter D (or the fact that we can replace the τ-function by the number of coprime divisors) is a distinct feature of the above bound. We shall select $D = 6$, reducing the total bound by a factor about $(1-\frac{1}{4})(1-\frac{1}{9}) = 2/3$.

Proof. The paper [21] contains the required estimates, but the following notes may be helpful to the reader: equation (17) on page 280 contains the function F which is generally defined in equation (1) at the very beginning of the paper; it is also given at the beginning of section 26, page 275. To read equation (17) the reader will need equation (10), page 279, which contains the definition of R. This definition comes in fact from (18), page 271. □

Voronoï continues by bounding $\tilde{\tau}$ by τ (see equation (19) and (20) of [21, pages 280, 281]). On using (4.6) and (4.7) and shortening $A(D)$ and $B(D)$ to A and B, respectively, we reach

$$\begin{aligned}(5.1)\quad |\Delta(x)| \leq{}& \frac{T}{12}(23A\log T + 23B - 19A + 2) + 3(xT)^{1/4}(A\log T + B - 3A)\\ &+ \sqrt{\frac{x}{T}}\Big(A\frac{\log T}{2} + \frac{B-A}{2} + 1\Big) + \frac{36A-9B}{4}x^{1/4}\\ &+ \frac{2A-B}{4}\frac{\sqrt{x}}{T} + \frac{2A-B}{6}\sqrt{T} + G(D,x,T),\end{aligned}$$

with

$$\begin{aligned}(5.2)\quad G(D,x,T) ={}& \tfrac{7}{4} + \big(7 + (xT^{-3})^{1/4} + (xT^{-3})^{1/2}\big)\frac{\Delta(D,T)}{4}\\ &+ \Big(\frac{\sqrt{x}}{8T}+\frac{\sqrt{T}}{12}\Big)\int_1^T\frac{\Delta(D,u)du}{u^{3/2}} + \frac{9x^{1/4}}{16}\int_1^T\frac{\Delta(D,u)du}{u^{7/4}}.\end{aligned}$$

The introduction of the parameter D in Lemma 5.1 will be numerically interesting. We will use only small D's, such as 1, 2 or 6.

6. Numerically comparing Δ with a model

We need to compute values of $\Delta(x)$ for fairly large x. The first idea is to compute it directly, take its absolute value, divide it by $\sqrt{x}$ and look for the point when it is less than a given bound, say 0.5. The drawback of this method is that one would have to redo all the computations with the bound 0.3. To avoid that, one can store the value on short enough ranges, say every $5\cdot 10^7$, but we would have to store these tables and they would be very bulky to use in computations. Musing on this idea, we readily discover that a better idea would be to compare $\Delta(x)$ with a model and bound the resulting error term. This is a very general idea, and one that we have already used in [17, Theorem 2]; the difficulty is always to guess a proper model. However, this issue is easily solved here, since a model is provided to us by the Voronoï formula. We define

$$\mathscr{M}(x,M) = \frac{x^{1/4}}{\pi\sqrt{2}}\sum_{m\le M}\frac{\tau(m)}{m^{3/4}}\cos\Big(4\pi\sqrt{mx}-\tfrac{\pi}{4}\Big). \tag{6.1}$$

We look for numerical bounds for $|\Delta(x)-\mathscr{M}(x,M)|/[x^{1/4}\log x]$ for some small M. Note that $\mathscr{M}(x,M)$ is thus of size $x^{1/4}$ in that case. We have found that, when $M=1$ or $M=4$, the function $x^{1/4}$ is too small to evaluate $|\Delta(x)-\mathscr{M}(x,M)|$ while $x^{1/4}\log x$ seems just too large. The bounds obtained are, however, better when subtracting $\mathscr{M}(x,1)$, and even better when subtracting $\mathscr{M}(x,4)$.

The computations necessitate some care. For $x\in[N,N+1)$, we consider the function

$$f(x) = \Big[\sum_{n\le N}\tau(n) - \big(x\log x+(2\gamma-1)x\big) - \mathscr{M}(x,M)\Big]/[x^{1/4}\log x]. \tag{6.2}$$

We find that, with $S=\sum_{n\le N}\tau(n)$,

$$\begin{aligned} x^{1/4}\log x\, f'(x) = &-\frac{S}{4x}-\frac{S}{x\log x}-\frac{3\log x-6\gamma+3}{4}+\frac{2\gamma-1}{\log x}\\ &+\sum_{m\le M}\frac{\sqrt{2}\tau(m)}{m^{1/4}x^{1/4}}\sin\Big(4\pi\sqrt{mx}-\tfrac{\pi}{4}\Big)\\ &-\frac{1}{\sqrt{2}\pi\log x}\sum_{m\le M}\frac{\tau(m)}{m^{3/4}x^{3/4}}\cos\Big(4\pi\sqrt{mx}-\tfrac{\pi}{4}\Big).\end{aligned}$$

Since $S\ge 2x-1$, we are sure this derivative is nonpositive when

$$\begin{aligned} 3\log x-6\gamma+5 \ge \frac{1}{x}+\frac{4}{x\log x}+\frac{8\gamma+4}{\log x}+\sum_{m\le M}\frac{4\sqrt{2}\tau(m)}{m^{1/4}x^{1/4}}\\ +\frac{2\sqrt{2}}{\pi\log x}\sum_{m\le M}\frac{\tau(m)}{m^{3/4}x^{3/4}}.\end{aligned}$$

The difference between the left-hand side and the right-hand side is an increasing function, from which it follows immediately that there exists an integer $N_0(M)$ such that, when $N\ge N_0(M)$, the function $x\mapsto(\Delta(x)-\mathscr{M}(x,M))/[x^{1/4}\log x]$ is nonincreasing in each interval $[N,N+1)$. The parameter M being fixed, $N_0(M)$ is a fixed (and small) value, and, for instance, $N_0(1)=2$ and $N_0(4)=5$ (we find that, in the case $M=4$, $f'(x)<0$ when $x\ge 11.062$, and is not an integer). Finding the maximum of $|\Delta(x) - \mathscr{M}(x, M)|/[x^{1/4}\log x]$ below this value can be automated, but it is more expedient, as well as less error-prone, to simply plot the function in each of the remaining unit intervals.

Numerical experiments show that $\mathscr{M}(x, 1)$ is already a good model! For small values, we find that

Using the model $\mathscr{M}(x,1)$				
Beginning	End	Max $\leq$	Where	Sum there
9	10 001	0.689848	12	35
10 001	20 001	0.442832	15120-	147800
20 001	30 001	0.440962	25200	259338
30 001	40 001	0.405939	30240-	316597
40 001	50 001	0.400379	49140	538485
50 001	60 001	0.406026	50400	553570
60 001	70 001	0.379055	60480-	675163
70 001	80 001	0.379005	75600-	860836
80 001	90 001	0.382929	83160-	954846
90 001	100 001	0.410340	97020	1129117

When the maximum have been attained at the end of the interval $[N, N+1)$, the program has attached a minus sign at the back of the data "Where". We have used the function `MajoreDelta` between 1 and 10^{10} below.

```
{MajoreDelta(beg, end, bigM = 1, NOM = 2,
             OnFile = 0, verbose = 1, TexFormat = 0, whentotell = 5*10^7) =
   local(maximum = 0, maxloc = 0, ou = beg, ouloc = beg, begloc, endloc,
         startat = 1, sommeou, sommeouloc, side, sideloc, somme = 0,
         aux, coef =  1/Pi/sqrt(2), previouscostimescoef, previousmt);

   whentotell = max(beg + whentotell, NOM) - beg;

   for(n = startat, max(beg, NOM)-1, somme += numdiv(n));

   if(NOM > end,
      print("Range is too low, NO( ", bigM,") being ", NOM);
      return(),);

   for(k = 0, ceil((end-beg)/whentotell-1),
      begloc = max(beg + k*whentotell, NOM);
      endloc = min(beg + (k+1)*whentotell, end)-1;
      maxloc = 0;
      previouscostimescoef = cos(Pi*(4*sqrt(begloc)-0.25))*coef;
      previousmt = begloc*(log(begloc)+(2*Euler-1));

      for(n = begloc, endloc,
         somme += numdiv(n);
         aux =  abs((somme-previousmt)/n^(1/4)
                   - previouscostimescoef)/log(n);
         if(aux > maxloc, maxloc = aux; ouloc = n;
            sommeouloc = somme; sideloc = 1,);

         previousmt = (n+1)*(log(n+1)+(2*Euler-1));
         previouscostimescoef = cos(Pi*(4*sqrt(n+1)-0.25))*coef;
         aux = abs((somme-previousmt)/(n+1)^(1/4)
                   -previouscostimescoef)/log(n+1);
```

```
      if(aux > maxloc, maxloc = aux; ouloc = n+1;
         sommeouloc = somme; sideloc = -1,));

   if(verbose, Output(1, begloc, endloc, maxloc, ouloc,
                      sommeouloc, sideloc, OnFile, TexFormat),);
   if(maxloc > maximum,
      maximum = maxloc; ou = ouloc;
      sommeou = sommeouloc; side = sideloc,);
);
if(verbose, Output(1, max(beg, NOM), end, maximum, ou, sommeou,
                   side, OnFile, TexFormat),);
return([somme, maximum]);}
```

The code for the function `Output` will be easily guessed by the reader. It can also be obtained by sending an e-mail request to the third named author of this paper. We have converted this function into a C-program and have compiled it with GP2C via the command

```
gp2c -g ModeleDelta-special.gp > MajoreDelta-special.gp.c
```

This step speeds the computations by a large factor (about 10). We then started GP with the option `-p 10000000000` and *installed* the compiled functions as described in the GP2C manual.

Below is the table obtained, each entry requiring at the beginning nearly 40 minutes (on a desktop computer).

Using the model $\mathscr{M}(x,1)$				
Beginning	End	Max $\leq$	Where	Sum there
9	50 000 001	0.689848	12	35
50 000 001	100 000 001	0.362373	82882820	1523997698
100 000 001	150 000 001	0.335167	134603040	2540265823
150 000 001	200 000 001	0.340907	165765640	3162894841
200 000 001	250 000 001	0.302913	203898905	3932714293
250 000 001	300 000 001	0.305402	274266920	5371256127
300 000 001	350 000 001	0.324542	302325156	5950196787
350 000 001	400 000 001	0.285504	365148280	7255586684
400 000 001	450 000 001	0.326125	441535536	8857292252
450 000 001	500 000 001	0.311085	479524060	9658927478
500 000 001	550 000 001	0.298151	543810960	11022257029
550 000 001	600 000 001	0.314576	591645600	12041674931
600 000 001	650 000 001	0.301294	639685376	13069360680
650 000 001	700 000 001	0.315219	660261970	13510663499
700 000 001	750 000 001	0.276965	728973036	14988837355
750 000 001	800 000 001	0.272097	772166412	15921409781
800 000 001	850 000 001	0.316275	838474560	17357704112
850 000 001	900 000 001	0.299946	855884040	17735695879
900 000 001	950 000 001	0.294188	921729600	19168468472
950 000 001	1 000 000 001	0.321118	959528080	19993096164

When modeling the error term by $x^{1/4}$, the local maxima happened to be slowly increasing, which is why we multiplied by an additional $\log x$ obtaining these slowly decreasing local maxima.

Increasing M yields better results, though the improvement is slow to become noticeable.

Using the model $\mathscr{M}(x, 4)$				
Beginning	End	Max ≤	Where	Sum there
74	10 001	0.520207	120	602
10 001	20 001	0.436010	15120-	147800
20 001	30 001	0.403803	25200	259338
30 001	40 001	0.377591	30240-	316597
40 001	50 001	0.399680	49140	538485
50 001	60 001	0.392255	50400	553570
60 001	70 001	0.367556	65520	736809
70 001	80 001	0.359261	75240-	856382
80 001	90 001	0.353541	83160-	954846
90 001	100 000	0.397458	98280	1145047

Using the model $\mathscr{M}(x, 4)$				
Beginning	End	Max ≤	Where	Sum there
74	50 000 001	0.520207	120	602
50 000 001	100 000 001	0.332461	82882820	1523997698
100 000 001	150 000 001	0.320852	134603040	2540265823
150 000 001	200 000 001	0.317678	165765640	3162894841
200 000 001	250 000 001	0.289804	232589280	4516702124
250 000 001	300 000 001	0.301569	274266920	5371256127
300 000 001	350 000 001	0.319558	319842688	6312982612
350 000 001	400 000 001	0.271346	365148280	7255586684
400 000 001	450 000 001	0.303091	419237280	8388259211
450 000 001	500 000 001	0.289065	465178560	9355841003
500 000 001	550 000 001	0.288701	522937800	10578721101
550 000 001	600 000 001	0.289808	583222500	11861877982
600 000 001	650 000 001	0.296236	639685376	13069360680
650 000 001	700 000 001	0.292158	678391200	13900010069
700 000 001	750 000 001	0.267957	730296576	15017376156
750 000 001	800 000 001	0.263906	772166412	15921409781
800 000 001	850 000 001	0.306857	838474560	17357704112
850 000 001	900 000 001	0.283255	868746501	18015191334
900 000 001	950 000 001	0.267106	913641302	18992209828
950 000 001	1 000 000 000	0.300615	959528080	19993096164

See section 14 for a detailed output. Here are the main corollaries, beside Theorem 1.3, that arise from these computations:

Corollary 6.1. *For each $x \in [1\,440, 10^{10}]$, we have*

$$\Delta(x) = \mathscr{M}(x,1) + O^*\bigl(0.45x^{1/4}\log x\bigr)$$

and we can replace $\mathscr{M}(x,1)$ by $\mathscr{M}(x,4)$ in this equality. Moreover, for $x \in [2\,017, 10^{10}]$,

$$\Delta(x) = \mathscr{M}(x,4) + O^*\bigl(0.44x^{1/4}\log x\bigr).$$

Here is the counterpart of Theorem 1.3, when using $\mathscr{M}(x,1)$ as a model.

Corollary 6.2. *For each $x \in [4\,221\,010, 10^{10}]$, we have*

$$\Delta(x) = \mathscr{M}(x,1) + O^*\bigl(0.35x^{1/4}\log x\bigr).$$

Corollary 6.3. *For each $x \in [3, 10^{10}]$, we have*

$$\Delta(x) = \mathscr{M}(x,1) + O^*\bigl(x^{1/4}\log x\bigr).$$

Going below $x = 3$ does not make much sense: if we extend the range to cover $[2,3]$, the constant 0.9 when $M = 4$ becomes 1.7, but we cannot reach $x = 1$, because our upper bound vanishes (since $\log 1 = 0$), but not the difference. A similar remark applies to the case $M = 1$.

7. Numerically comparing $\Delta(x)$ to $\sqrt{x}$

It is easy to use the bounds of the previous section to compare $\Delta(x)$ with $\sqrt{x}$ when x is somewhat large. The results are then most easily extended to smaller values of x by short computations. We have used the function `MajoreDelta` with `beg = 1`, and `D = 1` of the following routine:

```
{MajoreDelta(beg, end, OnFile = 0, verbose = 1,
            TexFormat = 0, whentotell = 5*10^7) =
   local(maximum = 0, maxloc = 0, ou = beg, ouloc = beg, aux,
         startat = 1, sommeou, sommeouloc, side, sideloc, somme = 0,
         ad = 1, bd = 2*Euler-1, begloc, endloc);

   for(n = startat, beg-1, somme += numdiv(n));
   for(k = 0, ceil((end-beg)/whentotell-1),
      begloc = beg + k*whentotell;
      endloc = min(begloc + whentotell, end)-1;
      maxloc = 0;
      for(n = begloc, endloc,
         somme += numdiv(n);
         /* The function with 'somme' fixed is decreasing */
         aux =  abs(somme-n*(ad*log(n)+bd))/sqrt(n);
         if(aux > maxloc, maxloc = aux; ouloc = n;
            sommeouloc = somme; sideloc = 1,);
         aux = abs(somme-(n+1)*(ad*log(n+1)+bd))/sqrt(n+1);
         if(aux > maxloc, maxloc = aux; ouloc = n+1;
            sommeouloc = somme; sideloc = -1,));

      if(verbose, Output(begloc, endloc, maxloc, ouloc,
                         sommeouloc, sideloc, OnFile, TexFormat),);
      if(maxloc > maximum,
         maximum = maxloc; ou = ouloc;
         sommeou = sommeouloc; side = sideloc,));
   if(verbose, Output(beg, end, maximum, ou,
                      sommeou, side, OnFile, TexFormat),);
   return([somme, maximum]);}
```

Here is the table obtained, each entry requiring at the beginning about ten minutes and about twenty-five at the end (on a desktop computer).

Beginning	End	Max ≤	Where	Sum there
1	50 000 001	0.960695	12	35
50 000 001	100 000 001	0.070919	82882820	1523997698
100 000 001	150 000 001	0.058336	135408288	2556270358
150 000 001	200 000 001	0.058275	165765640	3162894841
200 000 001	250 000 001	0.048470	219367470	4247106335
250 000 001	300 000 001	0.047795	253159920	4937622542
300 000 001	350 000 001	0.049268	302325156	5950196787
350 000 001	400 000 001	0.041915	353687040	7016569614
400 000 001	450 000 001	0.044068	403507656	8058104197
450 000 001	500 000 001	0.043468	479524060	9658927478
500 000 001	550 000 001	0.039691	529621200	10720648283
550 000 001	600 000 001	0.040632	562282656	11415433396
600 000 001	650 000 001	0.039443	639685376	13069360680
650 000 001	700 000 001	0.041340	660261970	13510663499
700 000 001	750 000 001	0.035375	728973036	14988837355
750 000 001	800 000 001	0.033995	768928275	15851410875
800 000 001	850 000 001	0.037986	838474560	17357704112
850 000 001	900 000 001	0.036950	855884040	17735695879
900 000 001	950 000 001	0.035765	921729600	19168468472
950 000 001	1 000 000 001	0.036828	959528080	19993096164

Here are some more corollaries:

$$\begin{cases} \max_{59\,200<x\le 10^{10}} |\Delta(x)|/\sqrt{x} & \le & 0.175, \\ \max_{7\,880\,000<x\le 10^{10}} |\Delta(x)|/\sqrt{x} & \le & 0.101, \\ \max_{1.8\cdot 10^{7}<x\le 10^{10}} |\Delta(x)|/\sqrt{x} & \le & 0.05. \end{cases} \tag{7.1}$$

Looking for the bound 0.5, we find that

Lemma 7.1. *When* $1981 \le x \le 10^{10}$, *we have* $|\Delta(x)| \le 0.482\, x^{1/2}$.

8. Bounding two integrals with Δ

We consider here, for $\sigma > 1$, the integral

$$I(D,T,\sigma) = \int_1^T \frac{\Delta(D,u)du}{u^\sigma} \tag{8.1}$$

with the aim of bounding $I(D,T,3/2)$ and $I(D,T,7/4)$ explicitly. We abbreviate $I(1,T,\sigma)$ by $I(T,\sigma)$. We define, for $\sigma > 1$,

$$\kappa(D,\sigma) = \sum_{\delta|D} \frac{\mu(\delta)}{\delta^{2(\sigma-1)}} \frac{\zeta(\sigma-1)^2}{\sigma-1} + \sum_{\delta|D} \frac{\mu(\delta)}{\delta^2} \frac{-2\log\delta - \frac{1}{2-\sigma} + 2\gamma - 1}{2-\sigma} \tag{8.2}$$

and

$$I_\sharp(D,T,\sigma) = \frac{1}{2i\pi} \int_{c'-i\infty}^{c'+i\infty} \sum_{\delta|D} \frac{\mu(\delta)}{\delta^{2(s+1-\sigma)}} \frac{\zeta^2(s)T^s ds}{s(s-\sigma+1)} \tag{8.3}$$

for $0 < c' < \sigma - 1$.

Lemma 8.1. *We have, when* $\sigma \in]1,2[$,

$$I(D,T,\sigma) = \kappa(D,\sigma) + \frac{I_\sharp(D,T,\sigma)}{T^{\sigma-1}}.$$

This shows that $I(D,T,\sigma)$ tends to a limit when T goes to infinity (on selecting for instance $c' = (\sigma-1)/2$). Note that $\kappa(1,3/2) = 0.57413324\cdots$, which numerically fits, and that $\kappa(1,7/4) = 0.40765213\cdots$.

Proof. We start with the case $D = 1$. We define

$$I_0(T,\sigma) = \int_1^T \frac{\sum_{n\le u}\tau(n)du}{u^\sigma}. \tag{8.4}$$

We rewrite this function as follows:

$$I_0(T,\sigma) = \sum_{n\le T}\tau(n)\int_n^T \frac{du}{u^\sigma} = \sum_{n\ge 1}\frac{\tau(n)}{n^{\sigma-1}}f_\sigma(n/T),$$

where

$$f_\sigma(v) = \begin{cases} \displaystyle\int_v^1 \frac{dw}{w^{2-\sigma}} = \frac{1-v^{\sigma-1}}{\sigma-1} & \text{when } v\le 1;\\ 0 & \text{when } v \ge 1.\end{cases} \tag{8.5}$$

We consider the Mellin transform of f_σ,

$$\check{f}_\sigma(s) = \int_0^\infty f_\sigma(v)v^{s-1}dv = \frac{1}{s(s+\sigma-1)}, \tag{8.6}$$

which is readily computed so that

$$f_\sigma(v) = \frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty}\check{f}_\sigma(s)v^{-s}ds. \tag{8.7}$$

This gives us

$$\begin{aligned} I_0(T,\sigma) &= \frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty}\sum_{n\ge 1}\frac{\tau(n)}{n^{s+\sigma-1}}\check{f}_\sigma(s)T^s ds \\ &= \frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty}\zeta^2(s+\sigma-1)\frac{T^s ds}{s(s+\sigma-1)}. \end{aligned}$$

The poles of the integrand are in $2-\sigma$ (a double pole), in 0 (a simple pole) and in $1-\sigma$ (a simple pole). Note that, in the vicinity of $s = 2-\sigma$, we have

$$\zeta^2(s+\sigma-1) = \frac{1}{(s+\sigma-2)^2} + \frac{2\gamma}{s+\sigma-2} + O(1)$$

and that

$$\frac{T^s}{s(s+\sigma-1)} = \frac{T^{2-\sigma}}{2-\sigma}\biggl(1 + (s+\sigma-2)\Bigl(\log T - \frac{1}{2-\sigma} - 1\Bigr)\biggr) + O((s+\sigma-2)^2)$$

so that

$$\frac{\zeta^2(s+\sigma-1)T^s}{s(s+\sigma-1)} = \frac{T^{2-\sigma}}{2-\sigma}\biggl(\frac{1}{(s+\sigma-2)^2} + \frac{1}{s+\sigma-2}\Bigl(\log T - \frac{1}{2-\sigma} - 1 + 2\gamma\Bigr)\biggr) + O(1).$$

The Cauchy Residue Theorem yields:

$$I_0(T,\sigma) = \frac{T^{2-\sigma}}{2-\sigma}\Big(\log T - \frac{1}{2-\sigma} - 1 + 2\gamma\Big) + \frac{\zeta(\sigma-1)^2}{\sigma-1} + \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} \zeta^2(s+\sigma-1)\frac{T^s ds}{s(s+\sigma-1)}$$

for any $1-\sigma < c < 0$. We need the condition $c > 1-\sigma$ to ensure the convergence of the integral. Indeed, we know that

$$|\zeta^2(a+ib)| \ll (|b|+2)^{-(1-a)}\log^2(|b|+2)$$

when $0 \le a \le 1$. Better bounds are known, but the size of $|\zeta^2(a+ib)|$ can indeed be as large as $|b|$, and this implies that we can ensure the convergence of the integral only when $c > 1-\sigma$.

Let us remark here that

$$\int_1^T \frac{u(\log u + 2\gamma - 1)}{u^\sigma}du = T^{2-\sigma}\frac{\log T + (\sigma-2)^{-1} + 2\gamma - 1}{2-\sigma} - \frac{(\sigma-2)^{-1} + 2\gamma - 1}{2-\sigma}.$$

The lemma follows readily when $D = 1$. For a general D, we appeal to (4.5), and deduce that

$$I(D,T,\sigma) = \sum_{\delta|D}\frac{\mu(\delta)}{\delta^{2(\sigma-1)}}\Big(I(T/\delta^2,\sigma) - \int_{1/\delta^2}^1 \frac{\log u + 2\gamma - 1}{u^{\sigma-1}}du\Big). \tag{8.8}$$

We notice that

$$\int_{1/\delta^2}^1 \frac{\log u + 2\gamma - 1}{u^{\sigma-1}}du = \frac{(\sigma-2)^{-1} + 2\gamma - 1}{2-\sigma} - \frac{(\sigma-2)^{-1} + 2\gamma - 1 - 2\log\delta}{(2-\sigma)\delta^{2(2-\sigma)}}. \tag{8.9}$$

□

We need to bound $I_\sharp(T,1/2)$ and $I_\sharp(T,3/4)$ explicitly.

Lemma 8.2. *We have*

$$I_\sharp(T,3/2) = I_\sharp(T,7/4) + \tfrac{1}{2} + O^*(\tfrac{9}{2}/T^{0.22}).$$

Proof. Let us first compute the derivative of $I_\sharp(T,\sigma)$ with respect to σ. We readily find that

$$I'_\sharp(T,\sigma) = \frac{-1}{2i\pi}\int_{c'-i\infty}^{c'+i\infty}\frac{\zeta^2(s)T^s ds}{s(s-\sigma+1)^2} = \frac{-1}{4(\sigma-1)^2} - \frac{1}{2i\pi}\int_{-\frac14-i\infty}^{-\frac14+i\infty}\frac{\zeta^2(s)T^s ds}{s(s-\sigma+1)^2}.$$

At this level, we employ the functional equation of the Riemann zeta function in the form

$$\zeta(s) = 2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)\zeta(1-s) \tag{8.10}$$

to get, when $\sigma \in [3/2, 7/4]$, and with $c' = -\delta > -1/4$,

$$\left|\frac{1}{2i\pi}\int_{-\delta-i\infty}^{-\delta+i\infty}\frac{\zeta^2(s)T^s ds}{s(s-\sigma+1)^2}.\right| \le \frac{\zeta(1+\delta)^2}{T^\delta\pi^{3+2\delta)}2^{2\delta}}\int_0^\infty \frac{|\sin(\pi(-\delta+iy)/2)\Gamma(1-\delta+iy)|^2 dy}{|\delta+iy||(\delta-\frac14)+iy|^2}.$$

On selecting $\delta = 0.22$, we compute that

$$I'_\sharp(T,\sigma) = \frac{-1}{4(\sigma-1)^2} + O^*(18/T^{0.22}). \qquad \square$$

First, we use GP to produce the following bounds.

Lemma 8.3. *We have*

$$\max_{1\le T\le 100\,000} T^{1/4}|I(T,7/4)-\kappa(7/4)| \le 0.302$$

and also

$$\max_{1\,260\le T\le 10\,000\,000} T^{1/4}|I(T,7/4)-\kappa(7/4)| \le 0.00979$$

Proof. This is obtained by using the function `MajoreResteJ`. □

Let us now evaluate $I(T,7/4)$ by using Lemma 3.1.

Lemma 8.4. *We have* $\max_{T\ge 1} T^{1/4}|\kappa(7/4)-I(T,7/4)| \le 4.000\,001$.

Proof. We find that, on using (3.1) and noticing that $I(\infty,7/4)=\kappa(7/4)$,

$$|\kappa(7/4)-I(T,7/4)| \le \int_T^\infty \frac{du}{u^{5/4}} + \tfrac{1}{2}\int_T^\infty \frac{du}{u^{7/4}} \le \frac{4+\frac{2}{3}T^{-1/2}}{T^{1/4}}.$$

Lemma 8.3 takes care of the small values of T. □

Once Lemma 9.3 has been established, we will have access to the following improvement:

Lemma 8.5. *We have* $\max_{T\ge 1} T^{1/4}|\kappa(7/4)-I(T,7/4)| \le 1.90$.

See Lemma 11.1 for a further improvement.

Lemma 8.6. *We have* $\max_{T\ge 1}|I(T,7/4)| \le 0.479$.

The computations we ran lead us to think that $I(T,3/4)\le\kappa(3/4)$ is plausible. We formulate the following general question:

Question 8.7. Is it true that, for $\sigma\in[3/2,7/4]$, we have

$$\forall T\ge 1,\quad I(T,\sigma)\le\kappa(\sigma)?$$

This question is surprising as some positivity mechanism seems hidden. A proof (or disproof) assuming GRH would also be welcome. The range $[3/2,7/4]$ may be extended, but $\sigma=2$ seems to have a special status. The reader will understand the conjecture stated in the introduction by noticing that $I(\infty,\sigma)=\kappa(\sigma)$. We mention here the papers [18, (2.2)], [9] and [5] where the Dirichlet series $\int_1^\infty \Delta(u)du/u^s$ is studied.

Proof. A numerical computation using the GP calculator and the function `MajoreJ` below shows that

$$\max_{1\le T\le 10^7}|I(T,7/4)| \le 0.4077$$

and, on using Lemma 8.4, the lemma follows readily. □

Lemma 8.8. *We have* $\max_{T\ge 1}|I(T,3/2)| \le 4.71$.

This bound is fairly poor since we believe that $|I(T,3/2)| \le \kappa(3/2) = 0.574\cdots$. Once Lemma 9.3 will be established, we will have access to the following improvement:

Lemma 8.9. *We have* $\max_{T\ge 1}|I(T,3/2)| \le 2.61$.

Proof. We have, by Lemma 8.1 and 8.2:

$$\begin{aligned} I(T,3/2) &= \kappa(3/2) + \frac{I_\sharp(T,3/2)}{T^{1/2}} \\ &= \kappa(3/2) + \frac{1}{2T^{1/2}} + \frac{I_\sharp(T,7/4)}{T^{1/2}} + O^*(\tfrac{9}{2}/T^{0.22}) \\ &= \kappa(3/2) + \frac{1}{2T^{1/2}} + T^{1/4}\big(I(T,7/4)-\kappa(7/4)\big) + O^*(\tfrac{9}{2}/T^{0.22}). \end{aligned}$$

We appeal to Lemma 8.4 or to Lemma 8.5 to bound the third summand. A numerical computation using the GP calculator shows that

$$\max_{1\le T\le 10^7}|I(T,3/2)| \le \kappa(3/2). \qquad \square$$

9. A first bound

We use Corollary 3.2 with $D=1$ to get

$$\begin{aligned} |G(x,T)| \le \tfrac{7}{4} &+ \big(7+(xT^{-3})^{1/4}+(xT^{-3})^{1/2}\big)\frac{\sqrt{T}+\frac12}{4} \\ &+ \Big(\frac{\sqrt{x}}{8T}+\frac{\sqrt{T}}{12}\Big)\int_1^T\frac{\Delta(u)du}{u^{3/2}} + \frac{9x^{1/4}}{16}\int_1^T\frac{\Delta(u)du}{u^{7/4}}. \end{aligned}$$

We appeal to Lemma 8.6 and 8.8 to get

$$\begin{aligned} |G(x,T)| \le \tfrac{7}{4} &+ \big(7+(xT^{-3})^{1/4}+(xT^{-3})^{1/2}\big)\frac{\sqrt{T}+\frac12}{4} \\ &+ 4.71\Big(\frac{\sqrt{x}}{8T}+\frac{\sqrt{T}}{12}\Big) + 0.479\frac{9x^{1/4}}{16}. \end{aligned}$$

We select

$$T = \Big(\frac{\sqrt{357}}{6}+\frac{3}{2}\Big)^{-4/3}x^{1/3} = cx^{1/3} \tag{9.1}$$

and get

$$\begin{aligned} |G(x,T)| \le \tfrac{7}{4} &+ \big(7+c^{-3/4}+c^{-3/2}\big)\frac{\sqrt{c}x^{1/6}+\frac12}{4} \\ &+ 4.71\Big(\frac{x^{1/6}}{8c}+\frac{\sqrt{c}x^{1/6}}{12}\Big) + 0.479\frac{9x^{1/4}}{16}, \end{aligned}$$

i.e.,

$$|G(x,T)| \le 0.27x^{1/4} + 7.7x^{1/6} + 6.0.$$

The global bound we obtain is

$$|\Delta(x)| \le 1.146x^{1/3}\log x - 10.5x^{1/3} + 8.93x^{1/4} + 11.4x^{1/6} + 5.91.$$

When we divide by $x^{1/3}\log x$, the function first decreases and then increases up to 1.146. It is ≤ 1.146 when $x \ge 379$.

Lemma 9.1. *We have*

$$\max_{14\le x\le 10^6} \frac{|\Delta(x)|}{x^{1/3}\log x} = \frac{|\Delta(36)|}{36^{1/3}\log 36} = \frac{140}{36^{1/3}\log 36} \le 0.4593.$$

Lemma 9.2. *When $x \ge 3$, we have $|\Delta(x)| \le 1.146 x^{1/3}\log x$.*

See also Lemma 10.7. As a consequence we get:

Lemma 9.3. *When $x \ge 121$, we have $|\Delta(x)| \le 0.76 x^{1/2}$.*

See also Lemma 10.6. As a further consequence we get:

Lemma 9.4. *When $x \ge 4\,033$, we have $|\Delta(x)| \le 0.475 x^{1/2}$.*

We have $|\Delta(x)| > 0.48 x^{1/2}$ when $x = 4032$.

10. Taking advantage of D

We can now use Lemma 8.9 and also use the parameter D. A direct computation gives us the following bounds.

Lemma 10.1. *We have $\max_{1\le x\le 10^7} |\Delta(2,x)| \le 0.883\, x^{1/2}$.*
We have $\max_{1\le x\le 10^7} |\Delta(6,x)| \le 0.927\, x^{1/2}$.

Lemma 10.2. *We have $\max_{x\ge 1} |\Delta(2,x)| \le 0.883\, x^{1/2}$.*
We have $\max_{x\ge 1} |\Delta(6,x)| \le 0.950\, x^{1/2}$.

Proof. We use (4.5) together with Lemma 9.4 when available, as well as Lemma 10.1 for the smaller values. □

Lemma 10.3. *We have*

$$\max_{1\le T\le 10^7} |I(2,T,7/4)| \le 0.902,$$
$$\max_{1\le T\le 10^7} |I(6,T,7/4)| \le 0.0945,$$
$$\max_{1\le T\le 10^7} |I(6,T,3/2)| \le 0.131.$$

We also have $I(6,10^7,3/2) = -0.056667 + O^*(10^{-6})$.

Proof. We use the PARI/GP package. □

Lemma 10.4. *We have, for all $T \ge 1$, $|I(2,T,7/4)| \le 0.953$.*
We have, for all $T \ge 1$, $|I(6,T,7/4)| \le 0.163$.

Proof. We use, when $T \ge T_0 = 10^7$ and, on using Lemma 9.4,

$$|I(D,T,7/4)| \le |I(D,T_0,7/4)| + \int_{T_0}^{T} \sum_{\delta|D} 0.475 \frac{du}{\delta u^{5/4}}$$

$$\le |I(D,T_0,7/4)| + 1.90 \frac{\sigma(D)}{D T_0^{1/4}}. \tag{10.1}$$

A numerical application using Lemma 10.3 concludes the proof. □

Lemma 10.5. *We have, for all $T \ge 1$, $|I(2,T,3/2)| \le 3.91$.*
We have, for all $T \ge 1$, $|I(6,T,3/2)| \le 5.98$.

Proof. We reuse (8.8), together with (8.9), to write

$$I(D,T,\sigma)=\sum_{\delta|D}\frac{\mu(\delta)I(T/\delta^2,\sigma)}{\delta^{2(\sigma-1)}}-\sum_{\delta|D}\frac{\mu(\delta)}{\delta^{2(\sigma-1)}}\frac{(\sigma-2)^{-1}+2\gamma-1}{2-\sigma}$$
$$+\sum_{\delta|D}\frac{\mu(\delta)}{\delta^2}\frac{(\sigma-2)^{-1}+2\gamma-1-2\log\delta}{2-\sigma}.$$

On using (4.4), we get

$$I(D,T,\sigma)=\sum_{\delta|D}\frac{\mu(\delta)I(T/\delta^2,\sigma)}{\delta^{2(\sigma-1)}}-\sum_{\delta|D}\frac{\mu(\delta)}{\delta^{2(\sigma-1)}}\frac{(\sigma-2)^{-1}+2\gamma-1}{2-\sigma}$$
$$+B(D)-\frac{A(D)}{(2-\sigma)^2}.$$

This leads to

$$I(D,T,3/2)=\sum_{\delta|D}\frac{\mu(\delta)I(T/\delta^2,3/2)}{\delta}-2(2\gamma-3)\frac{\phi(D)}{D}+B(D)-4A(D).$$

By appealing to Lemma 8.2, we get:

$$\begin{aligned}I(D,T,3/2)&=\kappa(D,3/2)\frac{\phi(D)}{D}+\sum_{\delta|D}\frac{\mu(\delta)I_\#(T/\delta^2,3/2)}{\sqrt{T}\delta}\\&\quad-2(2\gamma-3)\frac{\phi(D)}{D}+B(D)-4A(D)\\&=\frac{\phi(D)}{2D\sqrt{T}}+O^*\Big(\sum_{\delta|D}\frac{1}{\delta^{0.56}T^{0.72}}\Big)+\sum_{\delta|D}\frac{\mu(\delta)I_\#(T/\delta^2,7/4)}{\sqrt{T}\delta}\\&\quad+\kappa(D,3/2)\frac{\phi(D)}{D}-2(2\gamma-3)\frac{\phi(D)}{D}+B(D)-4A(D)\\&=O^*\Big(\sum_{\delta|D}\frac{1}{\delta^{0.56}T^{0.72}}\Big)+\sum_{\delta|D}\frac{\mu(\delta)T^{1/4}(I(T/\delta^2,7/4)-\kappa(7/4))}{\delta}\\&\quad+\kappa(D,3/2)\frac{\phi(D)}{D}+\frac{\phi(D)}{2D\sqrt{T}}-2(2\gamma-3)\frac{\phi(D)}{D}+B(D)-4A(D).\end{aligned}$$

Lemma 8.5 applies. □

Next, we use a direct computation with T and c from (9.1) and get, with $D=6$:

(10.2) $$|\Delta(x)|\le 0.764x^{1/3}\log x-4.505x^{1/3}+4.755x^{1/4}+10.30x^{1/6}+7/4.$$

As a consequence we get:

Lemma 10.6. *When $x\ge 421$, we have $|\Delta(x)|\le 0.688\,x^{1/2}$.*

Proof. Use the above inequality (10.2) when $x\ge 10^9$, Lemma 7.1 when $x\ge 1981$ and `MajoreDelta` otherwise. □

Lemma 10.7. *When $x\ge 9\,995$, we have $|\Delta(x)|\le 0.764\,x^{1/3}\log x$.*

Proof. The right-hand side of inequality (10.2) divided by $x^{1/3}\log x$ is decreasing and then increasing. □

The third bound of Theorem 1.1 is a further consequence of this bound.

11. Second round

We can try to use our better estimates to improve on the final result. The next lemma indeed improves on Lemma 8.5, but the global improvement is of no consequence.

Lemma 11.1. *We have* $\max_{T\geq 1} T^{1/4}|\kappa(7/4) - I(T, 7/4)| \leq 1.83$.

Proof. For $T \leq 10^7 = T_0$, this follows from Lemma 8.3. For larger T's, we use (10.2) to show that $|\kappa(D, 7/4) - I(D, T, 7/4)|$ is not more than

$$\int_T^\infty \Big(0.764u^{1/3}\log u - 4.505u^{1/3} + 4.755u^{1/4} + 10.30u^{1/6} + 7/4\Big)\frac{du}{u^{7/4}}$$

i.e., $T^{1/4}|\kappa(7/4) - I(T, 7/4)|$ is not more than

$$0.764\frac{\frac{12}{5}\log T + (\frac{12}{5})^2}{T^{1/6}} - 4.505\frac{\frac{12}{5}}{T^{1/6}} + 4.755\frac{2}{T^{1/4}} + 10.30\frac{\frac{12}{7}}{T^{1/3}} + \frac{7}{4}\frac{\frac{3}{4}}{T^{1/2}}.$$

This function is decreasing, and takes a value ≤ 1.83 at $T = 10^7$. □

We thus get $\max_{T\geq 1}|I(2, T, 3/2)| \leq 3.79$ and $\max_{T\geq 1}|I(2, T, 3/2)| \leq 5.79$.
We use `MajDelta` with T and c from (9.1) and get, with $D = 6$:

$$|\Delta(x)| \leq 0.764x^{1/3}\log x - 4.505x^{1/3} + 4.755x^{1/4} + 10.11x^{1/6} + 7/4, \tag{11.1}$$

which is a very modest improvement.

12. Proof of Corollary 2.1

Since $\zeta_K(s) \leq \zeta(s)^n$ for every $s > 1$ and every number field of degree n (see [14, Chapter 7, Corollary 3]), we find that (since $n = 2$ here)

$$h_K \leq \sum_{m\leqslant b_K\sqrt{d_K}} \tau(m).$$

On invoking Theorem 1.1 we get

$$h_K \leqslant \sqrt{\frac{d_K}{12}}\log d_K + \sqrt{\frac{d_K}{3}}\left(2\gamma - 1 - \log\sqrt{3} + 0.961\left(\frac{d_K}{3}\right)^{-1/4}\right),$$

and it is easily seen that

$$2\gamma - 1 - \log\sqrt{3} + 0.961\,(d_K/3)^{-1/4} < 0$$

as soon as $d_K \geqslant 108$.

13. Proof of Corollary 2.2

An integration by parts yields

$$\begin{aligned}\sum_{n\leq t}\frac{\tau(n)}{n} &= \sum_{n\leq t}\tau(n)\Big(\int_n^t\frac{dt}{t} + \frac{1}{t}\Big)\\ &= \int_1^t (u\log u + (2\gamma-1)u + \Delta(u))\frac{du}{u^2} + \log t + 2\gamma - 1 + \frac{\Delta(t)}{t}\\ &= \tfrac{1}{2}\log^2 t + A\log t + B + \frac{\Delta(t)}{t} - \int_t^\infty\frac{\Delta(u)du}{u^2}\end{aligned}$$

for constants $A = 2\gamma$ and $B = \gamma^2 - \gamma_1$. By Theorem 1.1, we find that

$$R(t) = t^{1/3}\left|\frac{\Delta(t)}{t} - \int_t^\infty \frac{\Delta(u)du}{u^2}\right| \le 3 \cdot 0.961/t^{1/6},$$

which is not more than 1.16 provided t is larger than 236. We readily write a routine to complete the proof. Below are some partial results.

Interval	$R(t) \le$
[0,1]	1.16
[1,2]	0.60
[2,3]	0.57
[3,4]	0.72
[4,5]	0.48

Interval	$R(t) \le$
[5,6]	0.48
[6,7]	0.74
[7,8]	0.43
[8,9]	0.61
[9,10]	0.52

14. Tables

We give the values obtained at some points, so that future authors can check their and our results. We can also start computations anew from one of these points. These computations have taken about ten days on a decent computer.

Using the model $\mathscr{M}(x, 4)$				
Beginning	End	Max $\le$	Where	Sum there
1 000 000 000	1 050 000 000	0.274960	1033783300	21617363398
1 050 000 000	1 100 000 000	0.300485	1061260200	22219769642
1 100 000 000	1 150 000 000	0.289880	1124565312	23610355396
1 150 000 000	1 200 000 000	0.309673	1183291200	24903544168
1 200 000 000	1 250 000 000	0.281165	1209300625	25477231529
1 250 000 000	1 300 000 000	0.259583	1286477760	27182768219
1 300 000 000	1 350 000 000	0.278165	1349790904	28585396325
1 350 000 000	1 400 000 000	0.287948	1357738256	28761673191
1 400 000 000	1 450 000 000	0.271429	1449339220	30796727408
1 450 000 000	1 500 000 000	0.260179	1493821875	31787089049
1 500 000 000	1 550 000 000	0.283459	1536464160	32737721129
1 550 000 000	1 600 000 000	0.270070	1591890300	33975109938
1 600 000 000	1 650 000 000	0.285854	1619982000	34602998536
1 650 000 000	1 700 000 000	0.292418	1678295250	35907926633
1 700 000 000	1 750 000 000	0.281376	1732250520	37117138632
1 750 000 000	1 800 000 000	0.288213	1774936800	38074990519
1 800 000 000	1 850 000 000	0.269459	1814760150	38969526424
1 850 000 000	1 900 000 000	0.259731	1853948320	39850647721
1 900 000 000	1 950 000 000	0.277342	1919056152	41316379639
1 950 000 000	2 000 000 000	0.243022	1980250000	42696013532

Using the model $\mathscr{M}(x, 4)$				
Beginning	End	Max $\leq$	Where	Sum there
2 000 000 000	2 050 000 000	0.293896	2035173616	43935895580
2 050 000 000	2 100 000 000	0.276613	2067566622	44667854438
2 100 000 000	2 150 000 000	0.251389	2122520400	45910757214
2 150 000 000	2 200 000 000	0.252292	2190178000	47442935997
2 200 000 000	2 250 000 000	0.280737	2242590948	48631324066
2 250 000 000	2 300 000 000	0.248571	2272574080	49311700641
2 300 000 000	2 350 000 000	0.268572	2325892808	50522582467
2 350 000 000	2 400 000 000	0.279156	2366582400	51447477213
2 400 000 000	2 450 000 000	0.256179	2401245000	52235927480
2 450 000 000	2 500 000 000	0.270924	2458573065	53541031206
2 500 000 000	2 550 000 000	0.264865	2545875360	55531071836
2 550 000 000	2 600 000 000	0.269957	2559702020	55846525595
2 600 000 000	2 650 000 000	0.249882	2618708448	57193584643
2 650 000 000	2 700 000 000	0.270260	2670564018	58378495847
2 700 000 000	2 750 000 000	0.300742	2731307040	59767766081
2 750 000 000	2 800 000 000	0.275779	2750075328	60197295267
2 800 000 000	2 850 000 000	0.246828	2814240537	61666736191
2 850 000 000	2 900 000 000	0.263185	2851560000	62522060994
2 900 000 000	2 950 000 000	0.261988	2934660966	64428396764
2 950 000 000	3 000 000 000	0.283013	2987643784	65645054999

Using the model $\mathscr{M}(x, 4)$				
Beginning	End	Max $\leq$	Where	Sum there
3 000 000 000	3 050 000 000	0.273352	3023790600	66475644081
3 050 000 000	3 100 000 000	0.296701	3072928352	67605433767
3 100 000 000	3 150 000 000	0.244745	3130246086	68924291232
3 150 000 000	3 200 000 000	0.279620	3183780600	70157047562
3 200 000 000	3 250 000 000	0.261684	3239964000	71451767900
3 250 000 000	3 300 000 000	0.259188	3277140048	72309009478
3 300 000 000	3 350 000 000	0.246630	3339610560	73750462665
3 350 000 000	3 400 000 000	0.265684	3367538928	74395265220
3 400 000 000	3 450 000 000	0.260423	3413610945	75459468392
3 450 000 000	3 500 000 000	0.263876	3480115590	76996733092
3 500 000 000	3 550 000 000	0.279600	3549873600	78610565016
3 550 000 000	3 600 000 000	0.285749	3576846340	79234940728
3 600 000 000	3 650 000 000	0.263195	3622600800	80294547476
3 650 000 000	3 700 000 000	0.260356	3650296881	80936229741
3 700 000 000	3 750 000 000	0.282580	3726736650	82708325799
3 750 000 000	3 800 000 000	0.247660	3786588436	84096959797
3 800 000 000	3 850 000 000	0.260401	3839553025	85326592376
3 850 000 000	3 900 000 000	0.236244	3883096910	86338060202
3 900 000 000	3 950 000 000	0.250083	3904000500	86823797338
3 950 000 000	4 000 000 000	0.246883	3987985851	88776488468

It is not apparent here, but the maxima have all been attained at the beginning of the intervals $[N, N+1)$, for the program would otherwise have attached a minus sign at the back of the data "Where".

Using the model $\mathscr{M}(x,4)$				
Beginning	End	Max ≤	Where	Sum there
4 000 000 000	4 050 000 000	0.245688	4025648718	89652741254
4 050 000 000	4 100 000 000	0.290455	4096960560	91312823163
4 100 000 000	4 150 000 000	0.257367	4116441888	91766549369
4 150 000 000	4 200 000 000	0.248970	4176455300	93164858229
4 200 000 000	4 250 000 000	0.251872	4214402192	94049464392
4 250 000 000	4 300 000 000	0.248332	4289204400	95794228028
4 300 000 000	4 350 000 000	0.252001	4334643000	96854722988
4 350 000 000	4 400 000 000	0.242011	4372030080	97727660193
4 400 000 000	4 450 000 000	0.263651	4434229920	99180648960
4 450 000 000	4 500 000 000	0.258938	4485181896	100371538069
4 500 000 000	4 550 000 000	0.249314	4500699138	100734334151
4 550 000 000	4 600 000 000	0.257573	4599891522	103054728842
4 600 000 000	4 650 000 000	0.268573	4635160200	103880282398
4 650 000 000	4 700 000 000	0.280269	4651785616	104269535886
4 700 000 000	4 750 000 000	0.271884	4747743000	106517355799
4 750 000 000	4 800 000 000	0.253353	4797640320	107686979253
4 800 000 000	4 850 000 000	0.272364	4843238478	108756281301
4 850 000 000	4 900 000 000	0.265096	4864923000	109264946520
4 900 000 000	4 950 000 000	0.238144	4917146130	110490367074
4 950 000 000	5 000 000 000	0.248641	4973705100	111818154382

Using the model $\mathscr{M}(x,4)$				
Beginning	End	Max ≤	Where	Sum there
5 000 000 000	5 050 000 000	0.286181	5027022945	113070441623
5 050 000 000	5 100 000 000	0.244395	5091750720	114591475202
5 100 000 000	5 150 000 000	0.258298	5119404040	115241550386
5 150 000 000	5 200 000 000	0.261626	5176785636	116590954111
5 200 000 000	5 250 000 000	0.251771	5240781400	118096648865
5 250 000 000	5 300 000 000	0.247590	5262850320	118616068971
5 300 000 000	5 350 000 000	0.253317	5308652478	119694375880
5 350 000 000	5 400 000 000	0.277900	5379593492	121365298487
5 400 000 000	5 450 000 000	0.245229	5449523400	123013321106
5 450 000 000	5 500 000 000	0.264486	5462614192	123321929177
5 500 000 000	5 550 000 000	0.281960	5500150656	124207003555
5 550 000 000	5 600 000 000	0.241436	5560748820	125636390340
5 600 000 000	5 650 000 000	0.253000	5615407644	126926247584
5 650 000 000	5 700 000 000	0.246390	5668548484	128180792385
5 700 000 000	5 750 000 000	0.261162	5746455792	130020921855
5 750 000 000	5 800 000 000	0.261145	5779524000	130802295723
5 800 000 000	5 850 000 000	0.235602	5849192160	132449113120
5 850 000 000	5 900 000 000	0.258207	5869321932	132925096267
5 900 000 000	5 950 000 000	0.243802	5929741468	134354172920
5 950 000 000	6 000 000 000	0.247427	5975287568	135431862589

Using the model $\mathscr{M}(x,4)$				
Beginning	End	Max $\leq$	Where	Sum there
6 000 000 000	6 050 000 000	0.257078	6047581276	137143152604
6 050 000 000	6 100 000 000	0.264247	6076125240	137819065525
6 100 000 000	6 150 000 000	0.237060	6145856660	139470848065
6 150 000 000	6 200 000 000	0.254862	6183777600	140369443674
6 200 000 000	6 250 000 000	0.250717	6240605010	141716492287
6 250 000 000	6 300 000 000	0.258793	6269789344	142408485236
6 300 000 000	6 350 000 000	0.236728	6337831710	144022371582
6 350 000 000	6 400 000 000	0.257214	6367561200	144727750832
6 400 000 000	6 450 000 000	0.234042	6430236132	146215266132
6 450 000 000	6 500 000 000	0.243262	6456122508	146829827370
6 500 000 000	6 550 000 000	0.229275	6529368096	148569291100
6 550 000 000	6 600 000 000	0.262629	6588000720	149962313570
6 600 000 000	6 650 000 000	0.256246	6627458574	150900066265
6 650 000 000	6 700 000 000	0.247609	6686825190	152311411807
6 700 000 000	6 750 000 000	0.276045	6727772700	153285180309
6 750 000 000	6 800 000 000	0.236609	6760131840	154054885524
6 800 000 000	6 850 000 000	0.256445	6822102224	155529366196
6 850 000 000	6 900 000 000	0.251456	6862382760	156488075487
6 900 000 000	6 950 000 000	0.245043	6914587680	157730946631
6 950 000 000	7 000 000 000	0.267887	6960231180	158817929605

Using the model $\mathscr{M}(x,4)$				
Beginning	End	Max $\leq$	Where	Sum there
7 000 000 000	7 050 000 000	0.238635	7045913952	160859238973
7 050 000 000	7 100 000 000	0.261163	7095895040	162050472517
7 100 000 000	7 150 000 000	0.260647	7123107862	162699202748
7 150 000 000	7 200 000 000	0.245588	7153692680	163428442105
7 200 000 000	7 250 000 000	0.272719	7245201600	165611085040
7 250 000 000	7 300 000 000	0.253628	7289919000	166678092011
7 300 000 000	7 350 000 000	0.261667	7329609000	167625369333
7 350 000 000	7 400 000 000	0.246222	7351690752	168152485810
7 400 000 000	7 450 000 000	0.235580	7436388960	170174940560
7 450 000 000	7 500 000 000	0.258627	7453473300	170583004109
7 500 000 000	7 550 000 000	0.236566	7549916010	172887291892
7 550 000 000	7 600 000 000	0.236878	7559867700	173125136377
7 600 000 000	7 650 000 000	0.235268	7611602866	174361811910
7 650 000 000	7 700 000 000	0.267827	7679106060	175975934558
7 700 000 000	7 750 000 000	0.250518	7742196000	177485064361
7 750 000 000	7 800 000 000	0.237627	7794947646	178747294449
7 800 000 000	7 850 000 000	0.240256	7808001006	179059687402
7 850 000 000	7 900 000 000	0.241885	7870262400	180550027446
7 900 000 000	7 950 000 000	0.240581	7905966138	181404884029
7 950 000 000	8 000 000 000	0.276060	7961011704	182723158806

Using the model $\mathscr{M}(x,4)$				
Beginning	End	Max ≤	Where	Sum there
8 000 000 000	8 050 000 000	0.250616	8003807296	183748324280
8 050 000 000	8 100 000 000	0.237212	8055421920	184985053658
8 100 000 000	8 150 000 000	0.224514	8126722674	186694022338
8 150 000 000	8 200 000 000	0.267527	8193921120	188305238294
8 200 000 000	8 250 000 000	0.249990	8222771718	188997157791
8 250 000 000	8 300 000 000	0.239580	8264446302	189996811511
8 300 000 000	8 350 000 000	0.236183	8308550250	191054967561
8 350 000 000	8 400 000 000	0.254539	8375178258	192653972096
8 400 000 000	8 450 000 000	0.250009	8403113964	193324558500
8 450 000 000	8 500 000 000	0.253240	8458325316	194650159683
8 500 000 000	8 550 000 000	0.236732	8547846636	196800293764
8 550 000 000	8 600 000 000	0.231778	8586658080	197732763494
8 600 000 000	8 650 000 000	0.247982	8613789264	198384712006
8 650 000 000	8 700 000 000	0.251100	8669286000	199718536031
8 700 000 000	8 750 000 000	0.244156	8747676300	201603194893
8 750 000 000	8 800 000 000	0.247581	8766483264	202055456509
8 800 000 000	8 850 000 000	0.248287	8825690880	203479517971
8 850 000 000	8 900 000 000	0.234675	8882685504	204850727934
8 900 000 000	8 950 000 000	0.248082	8944167540	206330308775
8 950 000 000	9 000 000 000	0.243354	8951421360	206504901825

Using the model $\mathscr{M}(x,4)$				
Beginning	End	Max ≤	Where	Sum there
9 000 000 000	9 050 000 000	0.275712	9001398276	207707961207
9 050 000 000	9 100 000 000	0.238592	9072415200	209417978179
9 100 000 000	9 150 000 000	0.289532	9137256975	210979790121
9 150 000 000	9 200 000 000	0.242404	9169786080	211763475672
9 200 000 000	9 250 000 000	0.235055	9229445316	213201075792
9 250 000 000	9 300 000 000	0.225647	9269774283	214173095441
9 300 000 000	9 350 000 000	0.246018	9303571200	214987813705
9 350 000 000	9 400 000 000	0.235136	9385928200	216973647295
9 400 000 000	9 450 000 000	0.239579	9432100650	218087297681
9 450 000 000	9 500 000 000	0.265125	9495486000	219616479788
9 500 000 000	9 550 000 000	0.229020	9532008024	220497772162
9 550 000 000	9 600 000 000	0.246084	9562200508	221226435660
9 600 000 000	9 650 000 000	0.241961	9614588560	222490991420
9 650 000 000	9 700 000 000	0.272435	9686476956	224226715659
9 700 000 000	9 750 000 000	0.240663	9712890915	224864607349
9 750 000 000	9 800 000 000	0.240285	9789225600	226708477160
9 800 000 000	9 850 000 000	0.232302	9834292260	227797345066
9 850 000 000	9 900 000 000	0.222418	9880665810	228918004951
9 900 000 000	9 950 000 000	0.223151	9924314400	229973012094
9 950 000 000	10 000 000 000	0.259598	9976913352	231244609722

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