Title
Exploratory Bi-factor Analysis: The Oblique Case

Authors
Robert I. Jennrich
Peter M. Bentler

Publication Date
2011-10-25

Exploratory Bi-factor Analysis: The Oblique Case

Robert I. Jennrich and Peter M. Bentler

University of California at Los Angeles

July 12, 2011

Bi-factor analysis is a form of confirmatory factor analysis originally introduced by Holzinger and Swineford (1937). The bi-factor model has a general factor, a number of group factors, and an explicit bi-factor structure. Jennrich and Bentler (2011) introduced an exploratory form of bi-factor analysis that does not require one to provide an explicit bi-factor structure a priori. They use exploratory factor analysis and a bi-factor rotation criterion designed to produce a rotated loading matrix that has an approximate bi-factor structure. Among other things this can be used as an aid in finding an explicit bi-factor structure for use in a confirmatory bi-factor analysis. They considered only orthogonal rotation. The purpose of this paper is to consider oblique rotation and to compare it to orthogonal rotation. Because there are many more oblique rotations of an initial loading matrix than orthogonal rotations, one expects the oblique results to approximate a bi-factor structure better than orthogonal rotations and this is indeed the case. A surprising result arises when oblique bi-factor rotation methods are applied to ideal data.

Key words: Bi-factor rotation, general factor, group factor, gradient projection algorithms, oblique rotation, orthogonal rotation.

The research of the second author was supported in part by grants 5P01DA01070-36 and 5K05DA00017-33 from the National Institute on Drug Abuse and by grant 5R44CA137841-3 from the National Cancer Institute.

Requests for reprints should be sent to Robert I. Jennrich, 3400 Purdue Ave., Los Angeles, CA, USA. E-mail: rij@stat.ucla.edu

1. Introduction

Bi-factor analysis is confirmatory factor analysis using a factor loading matrix with a structure of the form

$$\Lambda = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \\ * & 0 & * \\ * & 0 & * \\ * & 0 & * \end{pmatrix}$$

More precisely the loadings in the first column are free parameters and after the first column the loading matrix has at most one free parameter in each row. In bi-factor analysis the first factor is called a general factor and the remaining factors are called group factors.

As noted by Jennrich and Bentler (2011) bi-factor analysis is an extensively used form of confirmatory factor analysis. To use bi-factor analysis one must specify a specific bi-factor structure. An appropriate structure, however, may be difficult to find. Model building in bi-factor analysis consists of separating items into groups. This is usually done using prior knowledge from the field under investigation. But this knowledge is not always available or perhaps is not complete. In these cases exploratory methods such as the two-stage Schmid-Leiman (1957) method have been used to aid in defining the required groups.

More recently Jennrich and Bentler (2011) introduced an exploratory form of bi-factor analysis. This does not require the specification of a specific structure, but is designed to give a rotated loading matrix with an approximate bi-factor structure that may be used to suggest a specific structure.

Their approach uses exploratory factor analysis with a rotation criterion that allows arbitrary loadings on the first factor and encourages perfect cluster structure for the loadings on the remaining factors. Their exploratory bi-factor analysis (EBFA) is simply exploratory factor analysis using a bi-factor rotation criterion.

As noted by Jennrich and Bentler (2011) the bi-factor model is more general than the two stage model on which the Schmid-Leiman method is based. Because of this one might expect there are examples where exploratory bi-factor analysis produces a loading matrix that approximates bi-factor structure much better than the approximation produced by the Schmid-Leiman method. Jennrich and Bentler show this is indeed the case.

The Jennrich and Bentler paper considered only orthogonal rotation methods. Here we consider the use of oblique methods. Since the family of oblique rotations of an initial loading matrix is considerably larger than the family of orthogonal rotations, the use of oblique rotations should produce loading matrices that approximate bi-factor structure better than loading matrices produced by orthogonal rotations and in examples this seems to be the case. In the oblique case the group factors are correlated among themselves, but except for very special cases they are uncorrelated with the general factor.

2. Bi-factor rotation

For an introduction to bi-factor rotation see Jennrich and Bentler (2011). Bi-factor rotation uses bi-factor rotation criteria. These are criteria that measure departure from bi-factor structure. More precisely let Λ be an arbitrary $p \times k$ loading matrix. Then B is a bi-factor rotation criterion if $B(\Lambda) \geq 0$ for all Λ and $B(\Lambda) = 0$ whenever Λ has bi-factor structure.

If A is an initial loading matrix and $\hat{\Lambda}$ is a loading matrix that minimizes $B(\Lambda)$ over all rotations Λ of A, then $\hat{\Lambda}$ is called a bi-factor rotation of A.

One can use either orthogonal or oblique rotation. In this paper we will concentrate on the oblique case. In the oblique case the rotation algorithm produces both a loading matrix $\hat{\Lambda}$ and a factor correlation matrix $\hat{\Phi}$.

An analysis using a deviance function to extract an initial loading matrix A from a sample covariance matrix S followed by a bi-factor rotation of A is called an exploratory bi-factor analysis (EBFA) of S. In what follows the deviance function will be the normal deviance function. The gradient projection algorithms of Jennrich (2001, 2002) are used for orthogonal and oblique rotation respectively. To deal with local minima each rotation is the best obtained from 10 random starts, random orthogonal starts in the orthogonal case and random oblique starts in the oblique case.

We will use a specific bi-factor rotation criterion in most of our examples. This criterion is based on the quartimin criterion

$$\mathrm{qmin}(\Lambda) = \frac{1}{4} \sum_i \sum_{r \neq s} \sum \lambda_{ir} \lambda_{is}$$

Let

$$B(\Lambda) = \mathrm{qmin}(\Lambda_2)$$

where Λ_2 is the sub-matrix of Λ containing all but its first column.

To see that this is a bi-factor rotation criterion note that because $\mathrm{qmin}(\Lambda_2) \geq 0$ for all Λ_2, $B(\Lambda) \geq 0$ for all Λ. Moreover if Λ has bi-factor structure Λ_2 has perfect cluster structure. When this is the case $\mathrm{qmin}(\Lambda_2) = 0$ and $B(\Lambda) = 0$. It follows from the definition above that B is a bi-factor rotation criterion.

This criterion is called the bi-quartimin criterion. Note that when the bi-quartimin criterion is used to rotate a initial loading matrix A it is all of A and not just a subset of its columns that are rotated just as would be the case using any other rotation criterion.

3. Real Data Examples

3.1 Bullying data

The National Council for Drug Control at the Ministry of the Interior in Chile provides a public use data set on a variety of variables related to drug use. Among these are 17 variables on bullying and victimization for 1046 school children (CONACE, 2007). [1]

A six factor orthogonal EBFA of thse data gave

$$\hat{\Lambda}_{\text{orth}} = \begin{pmatrix} 0.38 & 0.25 & 0.04 & -0.07 & -0.12 & -0.01 \\ 0.38 & 0.82 & 0.09 & -0.07 & -0.05 & 0.03 \\ 0.57 & 0.29 & -0.06 & -0.20 & -0.27 & -0.17 \\ 0.54 & 0.13 & 0.07 & -0.27 & -0.32 & -0.17 \\ 0.16 & 0.12 & 0.45 & -0.03 & -0.08 & -0.07 \\ 0.25 & 0.16 & 0.63 & -0.07 & -0.01 & -0.03 \\ 0.44 & 0.04 & 0.29 & -0.12 & -0.20 & -0.13 \\ 0.46 & -0.03 & 0.32 & -0.17 & -0.30 & -0.14 \\ 0.60 & -0.11 & -0.08 & 0.61 & 0.04 & 0.03 \\ 0.50 & -0.03 & -0.04 & 0.51 & 0.10 & 0.07 \\ 0.65 & -0.08 & -0.02 & 0.50 & 0.15 & 0.04 \\ 0.70 & -0.07 & -0.09 & 0.17 & 0.49 & 0.10 \\ 0.75 & -0.06 & -0.01 & 0.03 & 0.65 & 0.01 \\ 0.62 & 0.01 & 0.03 & 0.15 & 0.38 & 0.04 \\ 0.57 & -0.03 & -0.05 & 0.07 & -0.01 & 0.68 \\ 0.57 & 0.01 & -0.03 & -0.02 & 0.04 & 0.76 \\ 0.54 & 0.06 & 0.02 & 0.03 & 0.05 & 0.67 \end{pmatrix}$$

To help identify a bi-factor structure corresponding to this all loadings with

[1] We would like to thank Dr. David Huepe for facilitating use of these data.

absolute values less than 0.2 were set to zero. This gives

$$\hat{\Lambda}_{\text{orth r}} = \begin{pmatrix} 0.38 & 0.25 & 0 & 0 & 0 & 0 \\ 0.38 & 0.82 & 0 & 0 & 0 & 0 \\ 0.57 & 0.29 & 0 & 0 & -0.27 & 0 \\ 0.54 & 0 & 0 & -0.27 & -0.32 & 0 \\ 0 & 0 & 0.45 & 0 & 0 & 0 \\ 0.25 & 0 & 0.63 & 0 & 0 & 0 \\ 0.44 & 0 & 0.29 & 0 & 0 & 0 \\ 0.46 & 0 & 0.32 & 0 & -0.30 & 0 \\ 0.60 & 0 & 0 & 0.61 & 0 & 0 \\ 0.50 & 0 & 0 & 0.51 & 0 & 0 \\ 0.65 & 0 & 0 & 0.50 & 0 & 0 \\ 0.70 & 0 & 0 & 0 & 0.49 & 0 \\ 0.75 & 0 & 0 & 0 & 0.65 & 0 \\ 0.62 & 0 & 0 & 0 & 0.38 & 0 \\ 0.57 & 0 & 0 & 0 & 0 & 0.68 \\ 0.57 & 0 & 0 & 0 & 0 & 0.76 \\ 0.54 & 0 & 0 & 0 & 0 & 0.67 \end{pmatrix}$$

An oblique EBFA of the bullying data gave the rounded loading matrix

$$\hat{\Lambda}_{\text{oblq r}} = \begin{pmatrix} 0.41 & 0.22 & 0 & 0 & 0 & 0 \\ 0.43 & 0.80 & 0 & 0 & 0 & 0 \\ 0.65 & 0.24 & 0 & 0 & 0 & 0 \\ 0.64 & 0 & 0 & -0.25 & 0 & 0 \\ 0.21 & 0 & 0.44 & 0 & 0 & 0 \\ 0.30 & 0 & 0.63 & 0 & 0 & 0 \\ 0.50 & 0 & 0.26 & 0 & 0 & 0 \\ 0.55 & 0 & 0.29 & 0 & 0 & 0 \\ 0.54 & 0 & 0 & 0.70 & 0 & 0 \\ 0.43 & 0 & 0 & 0.57 & 0 & 0 \\ 0.57 & 0 & 0 & 0.57 & 0 & 0 \\ 0.54 & 0 & 0 & 0 & 0.60 & 0 \\ 0.57 & 0 & 0 & 0 & 0.85 & 0 \\ 0.51 & 0 & 0 & 0 & 0.47 & 0 \\ 0.48 & 0 & 0 & 0 & 0 & 0.75 \\ 0.46 & 0 & 0 & 0 & 0 & 0.84 \\ 0.45 & 0 & 0 & 0 & 0 & 0.72 \end{pmatrix}$$

In the orthogonal case 3 lines fail to approximate bi-factor structure while in the oblique case all lines approximate bi-factor structure. In the orthogonal case the value of the bi-quartamin criterion is .2837 which is larger than its value .0937 in the oblique case. As expected oblique rotation produces a loading matrix that is closer to bi-factor structure.

The factor correlation matrix in the oblique case was

$$\Phi = \begin{pmatrix} 1.00 & 0.00 & 0.00 & -0.00 & -0.00 & -0.00 \\ 0.00 & 1.00 & 0.26 & -0.19 & -0.05 & 0.06 \\ 0.00 & 0.26 & 1.00 & -0.23 & -0.07 & -0.10 \\ -0.00 & -0.19 & -0.23 & 1.00 & 0.47 & 0.28 \\ -0.00 & -0.05 & -0.07 & 0.47 & 1.00 & 0.32 \\ -0.00 & 0.06 & -0.10 & 0.28 & 0.32 & 1.00 \end{pmatrix}$$

As expected a number of correlations differ significantly from zero. Note that the general factor, however, is uncorrelated with the group factors.

3.2 The OAS Data

The Observer Alexithymia Scale (OAS) is a 33-item 4-point scale (e.g., Haviland, Warren, & Riggs, 2000) recently studied in detail by Reise, Moore and Haviland (2010).[2] A six factor orthogonal EBFA of these data gave the rounded loading matrix.

$$\hat{\Lambda}_{\text{orth r}} = \begin{pmatrix}
0.64 & 0.56 & 0 & 0 & 0 & 0 \\
0.58 & 0 & 0.53 & 0 & 0 & 0 \\
0.42 & 0 & 0 & 0.67 & 0 & 0 \\
0.69 & 0 & 0 & 0 & 0.54 & 0 \\
0.46 & 0 & 0 & 0 & 0 & 0.56 \\
0.54 & 0.68 & 0 & 0 & 0 & 0 \\
0.71 & 0.34 & 0 & 0 & 0 & 0 \\
0.47 & 0 & 0.54 & 0.24 & 0 & 0 \\
0.41 & 0 & 0 & 0.69 & 0 & 0 \\
0.56 & 0 & 0 & 0 & 0.70 & 0 \\
0.29 & 0 & 0 & 0 & 0 & 0.58 \\
0.45 & 0.70 & 0 & 0 & 0 & 0 \\
0.54 & 0 & 0.45 & 0.23 & 0 & 0 \\
0.31 & 0 & 0 & 0.64 & 0 & 0 \\
0.70 & 0 & -0.21 & 0 & 0.41 & 0 \\
0.68 & 0 & 0 & 0 & 0 & 0.45 \\
0.62 & 0.23 & -0.37 & 0 & 0 & 0 \\
0.45 & 0.20 & 0 & -0.27 & 0 & 0 \\
0.47 & 0 & 0.35 & 0 & 0 & 0 \\
0.57 & 0 & 0 & 0 & 0.29 & 0 \\
0.54 & 0 & 0.23 & 0 & 0 & -0.21 \\
0.51 & 0 & 0 & 0.59 & 0 & 0 \\
0.62 & 0 & 0 & 0 & 0.24 & 0.26 \\
0.71 & 0 & 0 & 0 & 0 & 0 \\
0.63 & 0 & -0.34 & 0 & 0 & 0 \\
0.71 & 0 & 0 & 0 & 0 & -0.28 \\
0.44 & -0.20 & 0 & 0.48 & 0 & 0 \\
0.64 & 0 & 0 & 0 & 0 & 0.29 \\
0.54 & 0 & -0.29 & 0 & 0 & 0 \\
0.42 & 0 & 0.25 & 0 & 0 & 0 \\
0.50 & 0 & 0 & 0 & 0 & 0.34 \\
0.67 & 0 & 0 & -0.21 & 0 & -0.28 \\
0.46 & 0 & 0 & 0 & 0 & 0.34
\end{pmatrix}$$

[2]We would like to thank Mark Haviland for making this data available for analysis.

An oblique EBFA of the OAS data gave the rounded loading matrix

$$\hat{\Lambda}_{\text{oblq r}} = \begin{pmatrix}
0.68 & 0.49 & 0 & 0 & 0 & 0 \\
0.58 & 0 & 0.56 & 0 & 0 & 0 \\
0.41 & 0 & 0 & 0.67 & 0 & 0 \\
0.69 & 0 & 0 & 0 & 0.54 & 0 \\
0.42 & 0 & 0 & 0 & 0 & 0.58 \\
0.60 & 0.62 & 0 & 0 & 0 & 0 \\
0.74 & 0.25 & 0 & 0 & 0 & 0 \\
0.47 & 0 & 0.56 & 0 & 0 & 0 \\
0.40 & 0 & 0 & 0.68 & 0 & 0 \\
0.55 & 0 & 0 & 0 & 0.71 & 0 \\
0.24 & 0 & 0 & 0 & 0 & 0.60 \\
0.51 & 0.64 & 0 & 0 & 0 & 0 \\
0.53 & 0 & 0.47 & 0 & 0 & 0 \\
0.30 & 0 & 0 & 0.61 & 0 & 0 \\
0.70 & 0 & 0 & 0 & 0.40 & 0 \\
0.67 & 0 & 0 & 0 & 0 & 0.48 \\
0.63 & 0 & -0.41 & 0 & 0 & 0 \\
0.46 & 0 & -0.22 & -0.20 & 0 & 0 \\
0.46 & 0 & 0.35 & 0 & 0 & 0 \\
0.58 & 0 & 0 & 0 & 0.28 & 0 \\
0.56 & 0 & 0.22 & 0 & 0 & 0 \\
0.50 & 0 & 0 & 0.64 & 0 & 0 \\
0.61 & 0 & 0 & 0 & 0.23 & 0.27 \\
0.72 & 0 & 0 & 0 & 0 & 0 \\
0.64 & 0 & -0.38 & 0 & 0 & 0 \\
0.72 & 0 & 0 & 0 & 0 & -0.24 \\
0.42 & 0 & 0 & 0.42 & 0 & 0 \\
0.64 & 0 & 0 & 0 & 0 & 0.30 \\
0.55 & 0 & -0.32 & 0 & 0 & 0 \\
0.42 & 0 & 0.26 & -0.20 & 0 & 0 \\
0.50 & 0 & 0 & 0 & 0 & 0.37 \\
0.67 & 0 & 0 & -0.22 & 0 & -0.24 \\
0.43 & 0 & 0 & 0 & 0 & 0.36
\end{pmatrix}$$

In the orthogonal case 9 lines fail to approximate bi-factor structure as opposed to 4 lines in the oblique case. In the orthogonal case the value of the bi-quartamin criterion is .5484 which is larger than its value .3010 in the oblique case. As expected oblique rotation produces a loading matrix that is closer to bi-factor structure.

The factor correlation matrix in the oblique case was

$$\Phi = \begin{pmatrix} 1.00 & -0.00 & -0.00 & -0.00 & -0.00 & 0.00 \\ -0.00 & 1.00 & -0.24 & -0.20 & -0.03 & -0.13 \\ -0.00 & -0.24 & 1.00 & 0.38 & -0.18 & -0.03 \\ -0.00 & -0.20 & 0.38 & 1.00 & -0.06 & 0.12 \\ -0.00 & -0.03 & -0.18 & -0.06 & 1.00 & 0.13 \\ 0.00 & -0.13 & -0.03 & 0.12 & 0.13 & 1.00 \end{pmatrix}$$

As expected a number of correlations differ significantly from zero. Note that again the general factor is uncorrelated with group factors.

4. General Factor and Group Factor Correlation

In our examples the general factor is uncorrelated with the group factors. This is often viewed as a desirable property. Why is this the case? To show why this happens in practice consider the following theorem.

Theorem 1: Let Λ be an oblique bi-quartimin rotation of an initial loading matrix A. If $\lambda_{ir} \neq 0$ for some $r > 1$ and the i-th row of Λ_2 has complexity greater than one, then $\phi_{r,1} = 0$.

Proof: Let M_r denote the r-th column of a matrix M and λ_r denote the r-th column of Λ. Let B denote the bi-quartimin criterion. It follows from the stationary condition for oblique rotation (See for example Jennrich, 1973) that

$$\Lambda' \frac{dB}{d\Lambda} = \Delta\Phi \tag{1}$$

for some diagonal matrix Δ. Extracting the $(r, 1)$ element from both sides

$$\lambda_r' \frac{\partial B}{\partial \lambda_1} = \delta_r \phi_{r1}$$

Because B is a bi-factor criterion $\partial B / \partial \lambda_1 = 0$ and hence

$$\delta_r \phi_{r1} = 0 \tag{2}$$

The proof is complete if $\delta_r \neq 0$. Extracting the r-th diagonal element from both sides of (1) gives

$$\lambda_r' \frac{\partial B}{\partial \lambda_r} = \delta_r \tag{3}$$

Let $Q(\Lambda_2)$ be the value of the quartimin criterion at Λ_2. It is shown in Jennrich (2002) that

$$\frac{dQ}{d\Lambda_2} = \Lambda_2 \cdot (\Lambda_2^2 N)$$

where "$\cdot$" denotes an element wise product of two matrices, Λ_2^2 is the element-wise square of Λ_2, and N is a square matrix with zeros on the diagonal and ones elsewhere. Since $r > 1$

$$\frac{\partial B}{\partial \lambda_r} \quad = \quad \frac{\partial Q}{\partial \lambda_r} = (\Lambda_2 \cdot (\Lambda_2^2 N))_{r-1} = \lambda_r \cdot (\Lambda_2^2 N)_{r-1} = \lambda_r \cdot \sum_{\substack{s=2 \\ s \neq r}}^{k} \lambda_s^2$$

Using (3) and the assumption that the i-th row of Λ_2 has complexity greater than one

$$\delta_r = \lambda_r' \frac{\partial B}{\partial \lambda_r} = \sum_{i=1}^{p} \lambda_{ir}^2 \sum_{\substack{s=2 \\ s \neq r}}^{k} \lambda_{is}^2 > 0 \tag{4}$$

It follows from (2) that $\sigma_{r1} = 0$. ■

In our applications no loadings in $\hat{\Lambda}$ were exactly zero. Hence all rows of $\hat{\Lambda}_2$ had complexity greater than one. It follows from Theorem 1 that all correlations between the general and group factors must be zero and they were. We expect this to be the case in general for applied applications.

5. When there is a rotation with bi-factor structure

This section identifies a problem that does not arise in practice. Consider the case when there is a rotation Λ of an initial loading matrix A that has "exact" bi-factor structure. Then there are many rotations of A that have "exact" bi-factor structure. More precisely

Theorem 2: If Λ is an oblique rotation of an initial loading matrix A and Λ has bi-factor structure, then there is a continuum of such rotations.

Proof: Let $\Lambda = A(T')^{-1}$ be an oblique rotation of A that has bi-factor structure. Then $A = \Lambda T'$. Let

$$\tilde{T} = [t_1, \tilde{t}_2, t_3, \cdots, t_k]$$

Where t_r is the r-th column of T and $\tilde{t}_2$ a vector of unit length in the space spanned by t_1 and t_2 chosen so $\tilde{T}$ is nonsingular. In factor terminology this corresponds to rotating the second factor in the space of the first two. There is a continuum of such $\tilde{t}_2$. Let $\tilde{\Lambda} = A(\tilde{T}')^{-1}$ be the oblique rotation of A corresponding to $\tilde{T}$. Then $A = \Lambda\tilde{T}'$. Note that

$$A = \lambda_1 t_1' + \cdots + \lambda_k t_k'$$

and

$$A = \tilde{\lambda}_1 t_1' + \tilde{\lambda}_2 \tilde{t}_2' + \tilde{\lambda}_3 t_3' + \cdots + \tilde{\lambda}_k t_k'$$

Where λ_r and $\tilde{\lambda}_r$ denote the r-th columns of Λ and $\tilde{\Lambda}$ respectively.

Let v be a vector of unit length that is orthogonal to all t_r except t_s for $s > 2$. Note that

$$Av' = \lambda_s a \quad \text{and} \quad Av' = \tilde{\lambda}_s a$$

where $a = t_s' v$. Since T has full column rank $a \neq 0$. Thus $\lambda_s = \tilde{\lambda}_s$ for all $s > 2$.

Let v be a vector of unit length that is orthogonal to all t_r except t_2. Note that

$$Av' = \lambda_2 a \quad \text{and} \quad Av' = \tilde{\lambda}_2 b$$

where $a = t_2' v$ and $b = \tilde{t}_2 v$. Since T and $\tilde{T}$ have full column rank a and b are nonzero. Thus

$$\lambda_2 a = \tilde{\lambda}_2 b$$

It follows that the i-th component of λ_2 is zero if and only if the i-th component of $\tilde{\lambda}_2$ is a zero. This together with the fact that the last $k-2$ columns of Λ and $\tilde{\Lambda}$ are equal implies Λ and $\tilde{\Lambda}$ have the same bi-factor structure. Since there is a continuum of $\tilde{t}_2$ there is a continuum of $\tilde{\Lambda}$ and hence a continuum of oblique rotations of A that have bi-factor structure. ■

The proof of Theorem 2 may not identify all oblique rotations of A that have bi-factor structure. The theorem, however, tells one to expect many of them whenever there is one of them. Since the value of any bi-factor rotation criterion at Λ is zero whenever Λ has bi-factor structure, there is a continuum of rotations of A that minimize the criterion. Thus bi-factor rotation is undefined when A has a rotation that has bi-factor structure.

The following example shows computationally that the behavior predicted by Theorem 2 in fact occurs. Consider

$$A = \begin{pmatrix} 1 & 1 & .25 \\ 2 & 1 & .25 \\ 1 & 1 & .25 \\ 2 & .25 & 1 \\ 1 & .25 & 1 \\ 2 & .25 & 1 \end{pmatrix}$$

An oblique rotation of the last two columns of A produces a matrix with perfect cluster structure and hence there is an oblique rotation of A with bi-factor structure.

Ten oblique bi-quartimin rotations of A using random starts gave ten different rotations Λ of A all with bi-factor structure.

The first random start gave

$$\Lambda = \begin{pmatrix} 1.06 & 1.03 & 0.00 \\ 2.06 & 1.03 & -0.00 \\ 1.06 & 1.03 & 0.00 \\ 1.99 & -0.00 & 1.03 \\ 0.99 & 0.00 & 1.03 \\ 1.99 & -0.00 & 1.03 \end{pmatrix}$$

and the tenth random start gave

$$\Lambda = \begin{pmatrix} 1.38 & 0.59 & 0.00 \\ 2.08 & 0.00 & 0.60 \\ 1.38 & 0.59 & 0.00 \\ 2.11 & -0.59 & 0.00 \\ 1.41 & -0.00 & -0.60 \\ 2.11 & -0.59 & 0.00 \end{pmatrix}$$

The others had structures the same as one of these and hence ten random starts produced ten different rotated loading matrices and identified two different bi-factor structures. The 0.00's in these displays are rounded forms of computed values that are zero to over six decimal places.

The ten random starts gave ten distinct correlation matrices. The correlation matrix produced by the first random start was

$$\Phi = \begin{pmatrix} 1.00 & 0.02 & -0.06 \\ 0.02 & 1.00 & 0.47 \\ -0.06 & 0.47 & 1.00 \end{pmatrix}$$

Note that the general factor is correlated with the group factors. This was true for all ten random starts.

All this is quite different from what was encountered in our real data examples. In these we never encountered an initial loading matrix that could be rotated to bi-factor structure and never encountered a correlation matrix in which the general factor was correlated with a group factor. Moreover, when more than one random start produced the same value of the bi-quartimin criterion they also produced the same loading and correlation matrices. This is what happened in our real data examples and what we expect to see in practice.

6. Another bi-factor rotation criterion

Until now only one specific bi-factor rotation criterion, the bi-quartimin criterion, has been identified. Here we introduce another.

A rotation criterion that can produce better approximate perfect cluster structure then the quartimin criterion is Yates (1987) geomin criterion. One might consider using this to construct a bi-factor rotation criterion. Yates' criterion has the form

$$\text{geomin}(\Lambda) = \sum_{i=1}^{p} \left(\prod_{r=1}^{k} \lambda_{ir}^2 \right)^{1/k}$$

This function is not differentiable when one or more λ_{ir} are zero. To deal with this Browne (2001, eq. 8) has suggested a modified form

$$\text{geomin}(\Lambda) = \sum_{i=1}^{p} \left(\prod_{r=1}^{k} (\lambda_{ir}^2 + \epsilon) \right)^{1/k}$$

where ϵ is a small positive value. We will use Browne's modified form with his suggested $\epsilon = .01$. Let Λ be a $p \times k$ loading matrix and let

$$B(\Lambda) = Q(\Lambda_2)$$

where Λ_2 contains the last $k-1$ columns of Λ and Q is the modified geomin criterion. If ϵ were zero this would be a bi-factor rotation criterion because $Q(\Lambda) \geq 0$ for all Λ and is zero whenever Λ has perfect cluster structure. With $\epsilon = .01$ it is almost a bi-factor rotation criterion which we will assume is good enough. We will call B the bi-geomin criterion.

Applying oblique bi-geomin rotation using the bullying data gave

$$\hat{\Lambda}_r = \begin{pmatrix} 0.43 & 0 & 0 & 0 & 0 & 0 \\ 0.57 & 0.71 & 0 & 0 & 0 & 0 \\ 0.66 & 0 & -0.33 & 0 & 0 & 0 \\ 0.67 & 0 & -0.25 & 0 & 0 & 0 \\ 0.38 & 0 & 0.32 & 0 & 0 & 0 \\ 0.51 & 0 & 0.48 & 0 & 0 & 0 \\ 0.58 & 0 & 0 & 0 & 0 & 0 \\ 0.63 & -0.23 & 0 & 0 & 0 & 0 \\ 0.26 & 0 & 0 & 0.84 & 0 & 0 \\ 0.22 & 0 & 0 & 0.67 & 0 & 0 \\ 0.31 & 0 & 0 & 0.68 & 0 & 0 \\ 0.29 & 0 & 0 & 0 & 0.69 & 0 \\ 0.35 & 0 & 0 & 0 & 0.97 & 0 \\ 0.34 & 0 & 0 & 0 & 0.54 & 0 \\ 0.29 & 0 & 0 & 0 & 0 & 0.82 \\ 0.30 & 0 & 0 & 0 & 0 & 0.91 \\ 0.32 & 0 & 0 & 0 & 0 & 0.78 \end{pmatrix}$$

As before loadings with absolute value less than 0.02 have been set to zero to help identify a bi-factor structure. The last 9 variables define the same groups as in the oblique bi-quartimin case. Those identified by the first 8 variables clearly differ. One wonders if there is something about these variables that may suggest a reason for this.

Different exploratory bi-factor methods may lead to different interpretations of one's data. But this also happens with different exploratory factor analysis methods. Such differences might be resolved by fitting the confirmatory models suggested by the exploratory methods and comparing the fits.

The factor correlation matrix was

$$\Phi = \begin{pmatrix} 1.00 & 0.00 & 0.00 & -0.00 & -0.00 & -0.00 \\ 0.00 & 1.00 & -0.04 & -0.04 & 0.05 & 0.12 \\ 0.00 & -0.04 & 1.00 & -0.09 & -0.02 & -0.08 \\ -0.00 & -0.04 & -0.09 & 1.00 & 0.59 & 0.42 \\ -0.00 & 0.05 & -0.02 & 0.59 & 1.00 & 0.45 \\ -0.00 & 0.12 & -0.08 & 0.42 & 0.45 & 1.00 \end{pmatrix}$$

Note that as in the bi-quartimin case the general factor is uncorrelated with the group factors.

7. Discussion

EBFA is simply exploratory factor analysis using a bi-factor rotation criterion. Since it produces loading matrices that have approximate bi-factor structure an important application is to identify a specific bi-factor structure for use in a confirmatory bi-factor analysis.

It would be reasonable to add bi-factor rotation criteria to libraries of rotation criteria and to add bi-factor rotation as an option in general purpose exploratory factor analysis programs such as those found in SAS, SPSS and STATA. This would make exploratory bi-factor analysis immediately available to data analysts.

Oblique bi-quartimin analysis requires only a small modification of an orthogonal bi-quartimin analysis. Since the bi-quartimin rotation criterion has already been defined in Jennrich and Bentler (2011) the only change required to move from orthogonal to oblique rotation is to use an oblique rotation algorithm rather than an orthogonal rotation algorithm. For the authors this meant changing one line of computer code. The primary reason for this paper is to compare the orthogonal and oblique EBFA both empirically and theoretically.

Empirically we found what we expected. Oblique methods give rotated loading matrices that better approximated bi-factor structure than those using orthogonal methods.

On the theoretical side there were surprises. Oblique bi-factor rotation fails completely in the ideal case. That is when there is an oblique rotation of an initial loading matrix A that has “perfect” bi-factor structure. In this case there is an entire continuum of oblique bi-factor rotations that have “perfect” bi-factor structure. This means oblique bi-factor rotation is not uniquely defined in the ideal case. Our Theorem 2 proves this. As our applications show, however, this is not a problem when oblique bi-quartimin rotation is applied to real data because with the real data there was no oblique rotation of the initial loading matrix A that had "perfect" bi-factor structure.

Another surprise was that the general factor and the group factors are uncorrelated when oblique bi-quartimin rotation was applied to our real data problems. This is a desirable property, but at first there seems no reason to expect this. Our Theorem 1, however, shows this should happen.

Until now the only specific bi-factor rotation criterion identified has been the bi-quartmin criterion. We have introduced another based on the geomin criterion rather than on the quartimin criterion and compared its performance with that of the bi-quartimin criterion.

We have used only maximum likelihood for initial factor loading extraction. One might also consider least squares and generalized least squares.

The Appendix shows how to use computer code found at http://www.stat.ucla.edu/research/gpa to perform oblique and orthogonal bi-factor rotation. This URL contains general purpose orthogonal and oblique rotation algorithms written in Matlab, R, SAS PROC IML, and SPSS matrix. All that these require is a subroutine that defines the rotation criterion and its gradient.

Appendix

We will show how to use the Matlab code in the URL reference above to perform oblique bi-factor rotation of an initial loading matrix A. The procedure is similar when using the R, S, SAS PROC IML, and SPSS matrix code also given in the URL.

The first step is to write a program to compute the value and gradient of the bi-factor criterion desired. For the bi-quartimin criterion this is

```
function [v,G]=vgQ(L)
[p,k]=size(L);
Lt=L(:,[2:k]);
Lt2=Lt.^2;
N=ones(k-1,k-1)-eye(k-1);
v=sum(sum(Lt2.*(Lt2*N)));
Gt=4*Lt.*(Lt2*N);
G=[zeros(p,1) Gt];
```

Because orthogonal and oblique bi-quartimin rotation use the same rotation criterion this code is identical to that given in Jennrich and Bentler (2011).

The next step is to download the Matlab code for the general purpose oblique rotation program GPFoblq.

The final step is to compute an oblique bi-quartimin rotation Λ of A using

```
L=GPFoblq(A,T)
```

where T is a nonsingular matrix with columns of unit length used to start the rotation algorithm. It must have the same number of columns as A. A common choice for T is an identity matrix. An oblique random start can be generated using

```
X=randn(k,k);
d=sum(X.^2);
T=X*diag(1./sqrt(d));
```

where k is the number of columns of A and the function randn(k,k) generates a $k \times k$ matrix of independent standard normal values. The authors have found that it is a good idea to use the best of several random starts particularly for important applications. As noted above they used 10.

Jennrich and Bentler (2011) show how to perform orthogonal bi-factor rotation. The only difference is downloading GPForth rather than GPFoblq, changing the rotation command to

```
L=GPForth(A,T)
```

and using

```
X=randn(k,k);
[T,R]=qr(X)
```

to generate random orthogonal starts.

References

Browne, M.W. (2001). An overview of analytic rotation in exploratory factor analysis. *Multivariate Behavioral Research, 36*, 111-150.

CONACE (2007). Descargar Septimo estudio nacional de drogas en poblacion escolar de Chile 2007. http://www.conace.cl/portal/index.php?option=com_content&view=article&id=280&Itemid=370

Haviland, M. G., Warren, W. L., & Riggs, M. L. (2000). An observer scale to measure alexithymia. *Psychosomatics, 41*, 385-392.

Holzinger, K.J. & Swineford, S. (1937). The Bi-factor method. *Psychometrika, 47*, 41-54.

Jennrich, R.I. (1973). Standard errors for obliquely rotated factor loadings. *Psychometrika*, 38, 593-604.

Jennrich, R.I. (2001). A simple general procedure for orthogonal rotation. *Psychometrika, 66*, 289-306.

Jennrich, R.I. (2002). A simple general method for oblique rotation. *Psychometrika, 67*, 7-20.

Jennrich, R.I. & Bentler, P.M. (2011). Exploratory bi-factor analysis. *Psychometrika, 76*, ?-?.

Reise, S. P., Moore, T. M., & Haviland, M. G. (2010). Bifactor models and rotations: Exploring the extent to which multidimensional data yield univocal scores. *Journal of Personality Assessment, 92*, 544-559.

Schmid, J. & Leiman, J.M. (1957) The development of hierarchical factor solutions. *Psychometrika, 22*, 53-61.

Yates, A. (1987). *Multivariate exploratory data analysis: A perspective on exploratory factor analysis.* Albany : State University of New York Press.