Exploratory Numerical Study of Discrete Quantum Gravity

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With reliance on Regge calculus and the Regge-Einstein action, models of four-dimensional Euclidean gravity are simulated numerically. The scale $a = l_0$ is set by fixing the expectation value of a length. Monte Carlo calculations support the following results: In the infinite-volume limit, canonical dimensions are realized and a finite action density is obtained.

Classical gravity can be formulated as field theory and in a geometric way. One may, therefore, regard quantum gravity as the problem of formulating a quantum field theory or one may try a geometric approach. The "or" is not exclusive. Most work is done in the first direction, as it allows a reliance on the established calculus of perturbative field theory. At distances of the order of the Planck length, violent fluctuations in geometry are expected and a perturbative quantum field theory sheds little light on the understanding of quantum space and quantum time. The *a priori* starting point would always be a smooth and flat space-time continuum. The present paper attempts an approach from the very opposite limit, namely, a heavily fluctuating space-time.

In quantum gravity one may extend the functional integral of field theory to

$$Z = \int \mathcal{D} \, [\text{space}] \left\{ \int \mathcal{D} \, [\text{fields}] \exp \left[- \int_V d^4x \, \sqrt{g} \, \text{action(space, fields)} \right] \right\}. \tag{1}$$

We weight over some class of (curved) spaces and define on each space a conventional field theory coupling gravity and matter fields. The metric tensor g and the curvature R, for example, would get their values from $\int \mathcal{D}$ [space], whereas fermion and gauge fields, etc., would get their values from $\int \mathcal{D}$ [fields]. Following Hawking,[1] the theory is formulated for the Euclidean gravity.

Using Regge calculus,[2] I shall illustrate these remarks by working out a four-dimensional (4D) example. Let us consider a decomposition of a torus into N_p pentahedra p. (A pentahedron is a four-simplex connecting five sites by ten links.) To each link l a link length x_l is assigned and l is contained in a number of pentahedra. We now reassign all link lengths under the constraint that each pentahedron remains constructable in flat Euclidean space. This defines a *Regge skeleton* space. The interior of each pentahedron is flat. Curvature is concentrated on *triangles* (i.e., two-simplexes) t and involves *deficit angles* α_t (which are obtained by calculation of the parallel transport around each triangle t). The Regge-Einstein action is given by

$$S = S_{RE} = \sum_t \alpha_t A_t, \tag{2a}$$

where A_t is the area of triangle t and

$$\alpha_t = 2\pi - \sum \text{angles}(t).$$

In the continuum limit,[2–4] the Regge-Einstein action is equivalent to the Einstein action

$$S_E = \int d^4x \, \sqrt{g} \, R. \tag{2b}$$

Let us keep the number of pentahedra, N_p, fixed. We

would like to calculate vacuum expectation values with respect to the partition function

$$Z = \int_V \mathcal{D} \, [\text{space}] e^{m_P^2 S_{RE}}. \tag{3}$$

The measure $\mathcal{D}$ [space] on the 4D Regge skeleton will be defined below. The sign of the action is chosen to give negative modes and zero modes for small fluctuations around flat space.[3,5] However, it is well known[1] that the continuum Einstein action is unbounded and a naive simulation with Eq. (3) is expected to give divergent results. An obvious divergence comes from dilatations. Let us denote the link length of link l by x_l and rescale all links: $x_l' = \lambda x_l$. The action becomes $S' = \lambda^2 S$ and we are in trouble because some $S > 0$ exists. Dilatations are avoided by keeping, in addition to N_p, the total volume V fixed. The analog of a lattice spacing is defined by fixing the expectation value of a length $a = l_0$:

$$l_0 = v_0^{1/4}, \tag{4}$$

with $v_0 = \langle v_p \rangle$ and $V = N_p v_0$. Here v_p is the volume of pentahedron p.

My numerical procedure consists of proposing single new links and accepting or rejecting them according to the Metropolis algorithm. Changing a single link length $x_l \rightarrow x_l'$ will in general change the volume $V \rightarrow V'$. By rescaling all links with the factor $\lambda = (V/V')^{1/4}$ the volume is kept constant. The rescaling is consistent with detailed balance[6]

$$\frac{W(x_l \rightarrow x_l')}{W(x_l' \rightarrow x_l)} = \exp\{m_P^2 [S(x_l' \ldots) - S(x_l \ldots)]\}$$

if and only if $\mathscr{D}$ [space] is scale invariant. For simplicity reasons (to allow for an efficient numerical algorithm) I make a product *Ansatz:*

$$\mathscr{D}\,[\text{space}] = \prod_l df(x_l) \prod_p F_\theta(x_{p_1}, \ldots, x_{p_{10}}). \quad (5a)$$

Here $\prod_l$ extends over all links l and $\prod_p$ over all pentahedra p. The function $F_\theta(x_1, \ldots, x_{10})$ takes care of the pentahedra constraints. Scale invariance yields

$$f(x_l) = c\,\ln(x_l). \quad (5b)$$

The numerical simulation is done under the following two aspects: (1) Are canonical dimensions realized? (2) Do we obtain a finite action density ($m_{\mathrm{P}}^2 \neq 0$)?

(1) By Eq. (4) the length scale is introduced in a rather arbitrary way. This is satisfactory if other length scales are equivalent:

$$l_0' = \langle x_l \rangle, \quad l_0'' = \langle A \rangle^{1/2}, \quad \text{etc.} \quad (6)$$

$\langle x_l \rangle$ is the expectation value for the link length and $\langle A \rangle$ the expectation value for the area of the triangles. Equivalent means $l_0' = c' l_0$ and $l_0'' = c'' l_0$ for $N_p \to \infty$. In other words, we have canonical dimensions and no Hausdorff dimensions.

(2) Even after setting the scale by means of Eq. (4) S_{RE} may still be unbounded for $N_p \to \infty$, but a finite action *density* could be possible as a result of the entropy of $\mathscr{D}$ [space]. A numerical study can decide these questions.

For calculating expectation values numerically (in the limit $N_p \to \infty$) one has to define a model. The only feasible way that I can see at present is to define in flat space a Regge skeleton by a prescription of gluing links at sites together. The model is then set up by use of the partition function (3) and integration with the measure (5). The most natural skeleton in flat space is the random lattice of Christ, Friedberg, and Lee.[7] In the present exploratory study, however, I want to keep the computational effort small. For practical computer reasons I insist on handling the local topology of the model by one table (incidence matrix). I therefore use the hypercubic model, previously investigated by Rocek and Williams,[5] and a simplicial variant of it.

To my knowledge the hypercubic model, called *model 1* henceforth, provides the simplest way for defining a 4D Regge skeleton. One partitions a regular hypercubic lattice of $N = N_1 N_2 N_3 N_4$ sites into $(4!)N$ pentahedra by drawing one appropriate diagonal for each square, cube, and hypercube. At each site thirty links meet and we have to store fifteen links per hypercube into the computer memory. For these fifteen links, initial link length x_l and number of pentahedra n_p connected to each link are (1) $x_l = 1$, $n_p = 24$ for links 1–4; (2) $x_l = \sqrt{2}$, $n_p = 12$ for links 5–10; (3) $x_l = \sqrt{3}$, $n_p = 12$ for links 11–14; and (4) $x_l = 2$, $n_p = 24$ for link 15.

Model 2 is a simplicial variant of model 1. I omit the hypercube diagonal and add a new site at the center of each hypercube, which is connected by sixteen new links to the sixteen corners of the hypercube. This partitions each hypercube into 48 pentahedra. Model 2 has two types of sites: sites at which 44 links meet, and new sites, where sixteen links meet. We have to store thirty links per hypercube into the computer memory. The initial configuration is given by (1) $x_l = 1$, $n_p = 36$ for links 1–4; (2) $x_l = \sqrt{2}$, $n_p = 16$ for links 5–10; (3) $x_l = \sqrt{3}$, $n_p = 12$ for links 11–14; (4) $x_l = 1$, $n_p = 24$ for links 15 and 16; (5) $x_l = 1$, $n_p = 12$ for links 17–24; and (6) $x_l = 1$, $n_p = 8$ for links 25–30. Two models are used to allow a check of possibly universal features.

The Monte Carlo (MC) calculation is done by scanning through all links of the lattice and proposing for each link l a new link length $x_l' = e^{-\epsilon} x_l$. Here ϵ is a uniformly distributed random number in the range $-0.4 \leq \epsilon \leq 0.4$. If the new link length is not consistent with the (up to 36) pentahedra constraints, it is rejected. Otherwise it is accepted or rejected according to the Metropolis algorithm.[6] Numbers carrying dimensions are expressed in system units (4). The square of the Planck mass, m_{P}^2, is a constant in the Boltzmann factor $\exp\{m_{\mathrm{P}}^2 [S(x_l', \ldots) - S(x_l, \ldots)]\}$, and $S(x_l', \ldots)$ is of course taken after rescaling with $\lambda = (V'/V)^{1/4}$. We have established a Markov process and asymptotically we will sample configurations according to the partition function (3).

I have carried out MC calculations for $m_{\mathrm{P}}^2 = 0$, ± 0.3. Systems of size $N = 2^4$ and $N = 3^4$ sites are used and measurements are performed after each sweep. (A sweep is defined by application of the upgrading procedure once to each link.) In view of partition function (3) $m_{\mathrm{P}}^2 = 0$ resembles the zero-order strong-coupling limit of lattice gauge theories. On the other hand, a Planck length $l_{\mathrm{P}} = m_{\mathrm{P}}^{-1} = \infty$ is an analog to the spin-wave limit of lattice gauge theories, if we like the Planck length l_{P} to be proportional to a correlation length. Taking a dimensionful coupling seriously makes quantum gravity very different from a lattice gauge theory.

$m_{\mathrm{P}}^2 = 0$ results are entirely due to the entropy of the measure (5). They are collected in Table I. The first 200 sweeps, for reaching equilibrium, are omitted. The approach to equilibrium is (for model 1 and 3^4 sites) depicted in Fig. 1. $\langle x_l^i \rangle$ ($i = 1$–4) are the restrictions of the link-length expectation value $\langle x_l \rangle$ to subclasses of links with different initial length x_l as defined above. After about sixty sweeps equilibrium is reached and the system has completely lost any memory of the original configuration in flat space. Final link-length averages depend slightly on the numbers of pentahedra sharing the link. Particularly

TABLE I. Numerical results with $m_P^2 = 0$. The statistics is given in sweeps. Error bars are calculated with respect to the indicated number of bins. (The numbers in parentheses are statistical errors in the last digits.)

Statistics	Model 1, 2^4 6×2000	Model 1, 3^4 5×400	Model 2, 2^4 8×1000	Model 2, 3^4 10×200
$\langle x_l \rangle$	2.8779(17)	2.8651(16)	2.8605(16)	2.8557(18)
$\langle A \rangle$	3.3814(23)	3.3609(23)	3.3571(08)	3.3475(11)
$\langle \alpha \rangle$	−0.0142(06)	−0.0148(04)	−0.0136(06)	−0.0147(06)
$\langle S \rangle$	−0.283(04)	−0.280(03)	−0.338(08)	−0.326(10)

interesting is the negative average curvature $\langle S \rangle$. Furthermore, the expectation values $\langle x_l \rangle$ and $\langle A \rangle$ are nearly identical in both models. Finite-size effects $(2^4 \rightarrow 3^4)$ are small. This strongly supports canonical dimensions. In other words, setting the scale by means of Eq. (4) or one of Eqs. (6) is identical for $N_p \rightarrow \infty$.

Of major interest is the analysis of $\langle S \rangle$ as a function of m_P^2. Unfortunately $m_P^2 \neq 0$ slows down the computational speed by a factor of order ≈ 40. So far I have only carried out calculations for model 1 and $m_P^2 = \pm 0.3$. The results collected in Table II rely on roughly 200 h of IBM 3081 computer time.

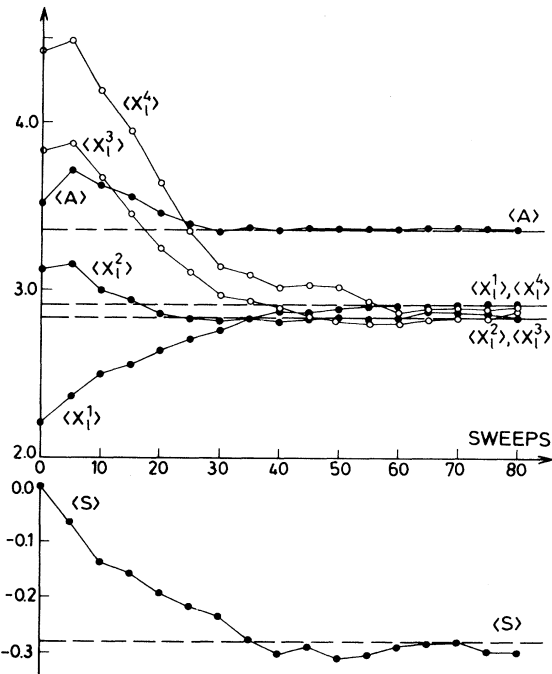Qualitatively, Table II exhibits similar behavior for

FIG. 1. First 80 sweeps, approach to equilibrium for various measured quantities (model 1, 3^4 system). Each dot represents an average over (the previous) five measurements. The solid lines are for guiding the eyes. Asymptotic averages are indicated by dashed, straight lines.

both m_P^2 values. Naively the correct sign is $m_P^2 > 0$, but the connection to Minkowskian gravity is not well understood. Therefore, one should first of all investigate Euclidean gravity in itself and consider the whole range $-\infty < m_P^2 < \infty$. Finite-size effects $(2^4 \rightarrow 3^4)$ are now strong, but clearly indicate finite ratios in the limit $N_p \rightarrow \infty$. I conclude again in favor of canonical dimensions. Most strikingly, the curvature expectation value $\langle S \rangle$ follows this pattern, implying a *finite action density* for $N_p \rightarrow \infty$ or at least a metastable state with finite action density.

A surprising result from Table II is that the deficit-angle expectation value, $\langle \alpha \rangle$, and the action density, $\langle S \rangle$, are different in sign (anticorrelated). This means that the final value of the action density is due to correlations between large triangles and appropriate deficit angles. This fits well with the observation that the total length of links and the total area of triangles are now much larger than in the $m_P^2 = 0$ case. In view of the anticorrelation of $\langle \alpha \rangle$ and $\langle S \rangle$, the action density is conjectured to be finite as a result of the entropy of the measure $\mathscr{D}$ [space].

Guided by the achieved qualitative numerical understanding, I would like to discuss questions concerning the continuum limit and universality. The region of physical interest is

$$l_P << L = V^{1/4}. \qquad (7)$$

Here $l_P = |m_P^{-1}|$ is the Planck length and L the edge length of the finite system.

I will consider two relevant scenarios. The conventional picture is to send $l_0(l_P) \rightarrow 0$ in units (l_P) of the Planck length. This means that $m_P^2 \rightarrow 0$ in system units $(m_0 = l_0^{-1})$ Eq. (4). In physical units (l_P^2) a finite action density can only be obtained if in system units (l_0), $\langle S \rangle \rightarrow 0$ for $m_P^2 \rightarrow 0$. In the present two models this is not the case.

Lee[8] advocates a fundamental length, which may provide a natural cutoff for ultraviolet divergencies. This means that $l_0 \propto l_P$ and is very attractive in view of the canonical dimensions. By fine tuning of m_P^2 one could fix $\langle S \rangle$ to any requested value, for instance $\langle S \rangle = 0$. Carrying out the limit $N_p \rightarrow \infty$ leads to

TABLE II. Model 1 MC results for $m_P^2 = \pm 0.3$. Error bars are calculated as in Table I. They may, however, be unreliable as a result of metastable states and limited statistics. "Equilibrium" gives the number of sweeps omitted for reaching equilibrium.

	$m_P^2 = +0.3$, 2^4 300 5×1200	$m_P^2 = +0.3$, 3^4 700 8×400	$m_P^2 = -0.3$, 2^4 600 11×600	$m_P^2 = -0.3$, 3^4 1100 11×400
Equilibrium Statistics				
$\langle x_l \rangle$	4.33(05)	3.86(02)	4.29(04)	3.99(06)
$\langle A \rangle$	6.43(12)	5.22(04)	6.62(09)	5.74(02)
$\langle \alpha \rangle$	−1.066(05)	−1.015(04)	0.129(03)	0.111(02)
$\langle S \rangle$	5.01(11)	3.42(04)	−5.70(18)	−5.15(03)

$L \propto l_0 (N_p)^{1/4} \to \infty$. The cosmological constant is exactly zero, because $\langle S \rangle$ is by construction volume independent.

The final results have to be universal: They are not allowed to reflect short-distance artifacts of the models used. Nonuniversal features of the models are the prescriptions for gluing links at sites together. On the other hand, the situation is very different from lattice gauge theory, where the O(4) invariance is broken down to the hypercubic group. In lattice gravity the Regge-Einstein action is invariant under general coordinate transformations[3] and a lattice does not really exist: The link lengths are dynamical variables. Further clarification of universality is desirable.

The situation is different for the random lattice, which could be a fundamental concept in itself and no universality with respect to other models would be needed. According to Ref. 8 unitarity of the S matrix can be proven, if the links are appropriately relinked. Unfortunately, this is very difficult to implement in a computer simulation. The present models can, however, be regarded as a first qualitative approximation of the random lattice.

A problem is to understand the possible outcome of flat space. In the present models flat space can only be realized in the case of a fundamental length and by fine tuning of the Planck mass. The random lattice would naturally solve the problem, if it gives $\langle S \rangle = 0$ for $m_P^2 = 0$. This question should be investigated in connection with coupling to matter fields, because empty space does not exist in nature. The simplest realistic case is to consider a SU(2) gauge theory on the Regge skeleton. The formalism has already been worked out in Refs. 7 and 8 and I expect the MC simulation to slow down only by a factor of order 2.

In conclusion, the present investigation is a starting point for numerical work on discrete Euclidean quantum gravity. Presently the main results are canonical dimensions between dimensionful quantities, including the finite, perhaps metastable, action density. An obvious next step is to investigate in more detail the

m_P^2 dependence of $\langle S \rangle$. Future work may concentrate on coupling gravity with an asymptotically free field theory and on using the random lattice. The final aim is to understand Euclidean quantum gravity qualitatively.

Note added.—Further investigations indicate that at least one of the $m_P^2 \neq 0$ results ($m_P^2 = +0.3$) is only metastable. For smaller $|m_P^2|$ values a stable finite action density is obtained. Between large and small $|m_P^2|$ seems to be a phase transition. A detailed publication is in progress.

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