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Exploring a Syntactic Notion of Modal Many-Valued Logics*

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Abstract

We propose a general semantic notion of modal many-valued logic. Then, we explore the difficulties to characterize this notion in a syntactic way and analyze the existing literature with respect to this framework.

Keywords: modal fuzzy logic, modal many-valued logic.

1 Introduction

The purpose of this paper is the search for a syntactic notion of *modal many-valued logic* that generalizes the notion of classical (normal) modal logic [8, 3]. In particular, modal fuzzy logics[1] will be inside this class. The addition of modal operators to fuzzy formalisms has been previously considered in the literature for several semantic purposes; for instance, in [34, 12, 26, 30, 23, 13, 21].

This paper is also motivated by semantic issues since we understand modal many-valued logics as logics defined by Kripke frames (possibly with many-valued accessibility relations) where every world follows the rules of a many-valued logic given by a residuated lattice, this many-valued logic being the same for every world. We point out that the semantics of this general framework is doubly many-valued because both the accessibility relation and the non-modal fragment are many-valued. We could have considered the logics introduced either by many-valued accesibility relations over classical logic (for instance [20, 9, 30, 13, 35]) or by classical accessibility relations over many-valued logics (for instance [24, 22, 25]); but we prefer to keep to the general semantics because these other two logics are extensions of the general one; indeed, the general logic is minimal.

Unfortunately, within this general framework we have not been able to find a syntactic characterization of the notion of modal many-valued logic that works in all

*The present paper is a revised and slightly extended version of [5].
[1]In the sense of considering $[0, 1]$ as the set of truth values.

the cases. The difficulties arise from the fact that we do not know a general method to axiomatize the modal many-valued logics given by the class of all Kripke frames (i.e., the minimal ones). It turns out that the only minimal logics axiomatized in the literature, as we will stress later, are the ones where the many-valued counterpart is given by a finite Heyting algebra [17], the standard (infinite) Gödel algebra [7] or a finite Łukasiewicz algebra [6].

The reader will find the details of the considered framework in Section 2. In Section 3 we will show the reader the difficulties of this search and we will try to specify which conditions should satisfy this syntactic notion. After, in Section 4 we will also review the works in the literature that fits inside our framework. The paper ends with some of the main open problems in the field. Throughout the paper we will assume basic knowledge on many-valued (and fuzzy) logics and their algebraic semantics, see [22, 32] for basic references.

2 A semantic approach

In this section we start by giving the definition of the modal many valued logic $Log_{\Box}(\mathbf{A}, \mathsf{F})$ associated with an algebra $\mathbf{A}$ and a class of $\mathbf{A}$-valued Kripke frames F.

The language of this new logic is, by definition, the propositional language generated by a set Var of propositional variables[2] together with the connectives given by the algebraic signature of $\mathbf{A}$ expanded with a new unary connective: the necessity[3] operator $\Box$. The set of formulas of the resulting language will be denoted by $Fm_{\Box}$.

We point out that the intended meaning of the universe A is a set of truth-values. This set could be endowed with a very general algebraic structure in order to develop our framework. However, in order to simplify our discussion we will assume in this paper that $\mathbf{A}$ is a complete residuated lattice (in the sense of a complete FL_{ew}-algebra [32]). Hence, the algebraic language of $\mathbf{A}$ exactly consists of meet $\wedge$, join $\vee$, truth constant 1, truth constant 0, implication $\rightarrow$ and fusion $\odot$.

We stress that these conditions are quite weak and a lot of well-known algebras satisfy them, for instance, complete MTL-algebras [14] and complete BL-algebras [22]. Hence, in particular we can consider that $\mathbf{A}$ is any of the three basic continuous t-norm (standard) algebras: *Lukasiewicz algebra* $[\mathbf{0},\mathbf{1}]_{\mathbf{Ł}}$, *product algebra* $[\mathbf{0},\mathbf{1}]_{\mathbf{\Pi}}$ and *Gödel algebra* $[\mathbf{0},\mathbf{1}]_{\mathbf{G}}$. We also note that due to the fact that the free algebra with countable generators (i.e., the Lindenbaum-Tarski algebra) of any of the previous varieties of algebras is not complete, it is not included in our framework.

An $\mathbf{A}$-*valued Kripke frame* is a pair $\mathfrak{F} = \langle W, R \rangle$ where W is a set (of *worlds*) and R is a binary relation valued in A (i.e., $R : W \times W \longrightarrow A$) called *accessibility relation*. It is said that the Kripke frame is *classical* in case that the range of R is included in $\{0, 1\}$[4]. Whenever $\mathbf{A}$ is fixed, we will denote by Fr and CFr the classes of all $\mathbf{A}$-valued Kripke frames and all $\mathbf{A}$-valued classical Kripke frames. For the

[2] In most cases it is assumed that $Var = \{p_0, p_1, p_2, \ldots\}$.

[3] Later on we will give some ideas about how to develop these ideas with the possibility operator $\Diamond$.

[4] Here 0 means the minimum of $\mathbf{A}$ and 1 its maximum.

rest of the paper we will mainly focus on these two classes since they provide in some sense minimal logics.

Before introducing $Log_\Box(\mathbf{A}, \mathsf{F})$ we need to define what is an **A**-valued Kripke model. An **A**-*valued Kripke model* is a triple $\mathfrak{M} = \langle W, R, e\rangle$ where $\langle W, R\rangle$ is an **A**-valued Kripke frame and e is a map, called *valuation*, assigning to each variable in Var and each world in W an element of A. The map e can be uniquely extended to a map $\bar{e} : Fm_\Box \times W \longrightarrow A$ satisfying that:

- in its first component $\bar{e}$ is an algebraic homomorphism for the connectives in the algebraic signature of **A**, and
- $\bar{e}(\Box\varphi, w) = \bigwedge\{R(w, w') \to \bar{e}(\varphi, w') : w' \in W\}$.

Although the functions e and $\bar{e}$ are different there will be no confusion between them, and so sometimes we will use the same notation e for both.

Following the same definitions than in the Boolean modal case [8, 3] it is clear how to define *validity* of a $Fm_\Box$-formula in an **A**-valued Kripke model and in an **A**-valued Kripke frame.

Now we are ready to introduce the *modal many-valued logic* $Log_\Box(\mathbf{A}, \mathsf{F})$. It is defined as the set of formulas $\varphi \in Fm_\Box$ satisfying that for every **A**-valued Kripke model $\langle W, R, e\rangle$ over a frame $\langle W, R\rangle$ in F and for every world w in W, it holds that $e(\varphi, w) = 1$.

Remark 1. *For the sake of simplicity in this paper we restrict ourselves to adding the necessity operator $\Box$, but analogously we could have considered a possibility operator ruled by the condition*[5]

$$e(\Diamond\varphi, w) = \bigvee\{R(w, w') \odot e(\varphi, w') : w' \in W\}.$$

Remark 2. *We stress that for the case that* **A** *is the Boolean algebra* **2** *of two elements all previous definitions correspond to the standard terminology in the field of modal logic (cf. [8, 3]). As far as the authors know the first one to talk about this way of extending the valuation e into the modal many-valued realm was* M. Fitting *in [16]. Indeed, the same standard translation of classical modal logic into first-order classical logic (see [3])) is also an embedding of the modal many-valued logic based on* **A** *into the first-order many-valued logic based on* **A**.

Remark 3. *A natural generalization of the previous definition is to introduce the definition of $Log_\Box(\mathsf{K}, \mathsf{F})$, where K is a class of algebras, as the intersection of $Log_\Box(\mathbf{A}, \mathsf{F})$ where* **A** *belongs to the class K. We are not going to pursue the study of these logics here but we point out that later we will see that in general the modal logic given by* **A** *does not coincide with the modal logic given by the variety generated by this algebra.*

Up to now we have considered a logic as a set of formulas. Besides this way to consider logics, it is also common to consider them as consequence relations, e.g., [4]. Following this approach next we define two different consequence relations.

[5] The connective $\odot$ is what sometimes is called in the literature fuzzy conjunction, fusion, multiplicative conjunction, etc. (see [1, 22, 32]).

The *modal many-valued local consequence* $\models_{l(\mathbf{A},\mathsf{F})}$ associated with an algebra $\mathbf{A}$ and a class of $\mathbf{A}$-valued Kripke frames F is defined by the following equivalence:

$$\Gamma \models_{l(\mathbf{A},\mathsf{F})} \varphi \quad \text{iff}$$

for every $\mathbf{A}$-valued Kripke model $\langle W, R, e\rangle$ over a frame $\langle W, R\rangle$ in F and for every world w in W, it holds that if $e(\gamma, w) = 1$ for every $\gamma \in \Gamma$, then $e(\varphi, w) = 1$.

The *modal many-valued global consequence* $\models_{g(\mathbf{A},\mathsf{F})}$ associated with an algebra $\mathbf{A}$ and a class of $\mathbf{A}$-valued Kripke frames F is given by the following definition:

$$\Gamma \models_{g(\mathbf{A},\mathsf{F})} \varphi \quad \text{iff}$$

for every $\mathbf{A}$-valued Kripke model $\langle W, R, e\rangle$ over a frame $\langle W, R\rangle$ in F, it holds that if $e(\gamma, w) = 1$ for every $\gamma \in \Gamma$ and every world w in W, then $e(\varphi, w) = 1$ for every world w in W.

We point out that the set of theorems of both consequence relations is precisely the set $Log_\Box(\mathbf{A}, \mathsf{F})$.

3 Differences with the modal Boolean case

General Considerations. Throughout the paper we assume that we have fixed a complete FL_{ew}-algebra $\mathbf{A}$ (i.e., a complete residuated lattice). We remind the reader that a *complete FL_{ew}-algebra* [32] is an algebra $\mathbf{A} = \langle A, \wedge, \vee, 1, 0, \to, \odot\rangle$ of type $(2, 2, 0, 0, 2, 2)$ such that:

- $\langle A, \wedge, \vee, 1, 0\rangle$ is a bounded complete lattice,
- $\langle A, \odot, 1\rangle$ is a commutative monoid,
- for every $a, b, c \in A$, it holds that $b \odot a \leq c$ iff $a \leq b \to c$ (*residuation law*).

We point out that in particular all MTL-algebras [14] and BL-algebras [22] are FL_{ew}-algebras.

In order to find a successful syntactic definition of the notions introduced in the previous section[6] first of all we would need to settle a completeness theorem for the logics introduced in the previous section. In particular, we should know how to axiomatize the *minimal* logic $Log_\Box(\mathbf{A}, \mathsf{Fr})$. What formulas must we add to an axiomatization of the many-valued logic defined by $\mathbf{A}$ in order to obtain a complete axiomatization of $Log_\Box(\mathbf{A}, \mathsf{Fr})$? This paper mainly focus on exploring this minimal logic $Log_\Box(\mathbf{A}, \mathsf{Fr})$. The reason for doing so is that in our opinion any adequate notion of modal many-valued logic (over $\mathbf{A}$) should be introduced as an extension of this minimal logic, and hence the key step in our search of a syntactic notion of modal many-valued logic is precisely this minimal logic. Thus, we will mainly

[6] In the modal Boolean case it is well-known the existence of modal logics that are Kripke frame incomplete. Hence, the searched definition of modal many-valued logic will have to include more logics that the ones introduced in Section 2.

focus on the class of all Kripke frames and not on particular subclases (except for the class of all classical Kripke frames).

The fact that the famous modal axiom (K) (sometimes called *normality* axiom)

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \tag{K}$$

does not in general belong to $Log_\Box(\mathbf{A}, \mathsf{Fr})$ is what makes difficult even to suggest an axiomatization of $Log_\Box(\mathbf{A}, \mathsf{Fr})$. As a simple counterexample we can consider the logic $Log_\Box([\mathbf{0},\mathbf{1}]_{\text{Ł}}, \mathsf{Fr})$ and the Kripke model given by $W = \{a, b\}$, $R(a, a) = 1$, $R(a, b) = 1/2$, $e(p_0, a) = 1$, $e(p_0, b) = 1/2$, $e(p_1, a) = 1$ and $e(p_1, b) = 0$. Then, $e(\Box(p_0 \to p_1) \to (\Box p_0 \to \Box p_1), a) = 1/2$.

It is easy to check that the *necessity rule*, from φ follows $\Box\varphi$, holds for $Log_\Box(\mathbf{A}, \mathsf{Fr})$. Another property that holds for $Log_\Box(\mathbf{A}, \mathsf{Fr})$ is the monotonicity of the necessity operator, i.e., if $\varphi \to \psi$ is in the logic then also $\Box\varphi \to \Box\psi$ is in the logic. Moreover, it is possible to see that

$$(\Box\varphi \wedge \Box\psi) \leftrightarrow \Box(\varphi \wedge \psi) \tag{1}$$

is valid under our semantics. The truth of this last statement is a consequence of the validity of the equation $x \to (y \wedge z) \approx (x \to y) \wedge (x \to z)$ in any residuated lattice. We notice that in the Boolean case the normality axiom (K) follows from (1). However, this does not hold in the many-valued case. Indeed, the formula that results from (1) replacing conjunction with fusion is in general false, that is,

$$(\Box\varphi \odot \Box\psi) \to \Box(\varphi \odot \psi)$$

and

$$(\Box(\varphi \to \psi) \odot \Box\varphi) \to \Box\psi$$

could fail. The reader can notice that the last formula is equivalent to (K) thanks to the residuation law.

Although in general (K) does not belong to $Log_\Box(\mathbf{A}, \mathsf{Fr})$, let us remark two particular cases where (K) holds. For these two cases the problem of finding an axiomatization for the minimal logic is easier, and indeed it has been successfully solved very often as we will see in examples of Section 4. The first one is when the operations $\odot$ and $\wedge$ coincide in the algebra $\mathbf{A}$. The proof of this case is an easy consequence of the validity of (1). A particular application of this first case is that (K) belongs to $Log_\Box([\mathbf{0},\mathbf{1}]_{\mathbf{G}}, \mathsf{Fr})$. And the second case is when F is the class of classical Kripke frames CFr, i.e., for any algebra $\mathbf{A}$ all $\mathbf{A}$-valued classical Kripke frames satisfy the normality condition. In particular this means that (K) belongs to $Log_\Box([\mathbf{0},\mathbf{1}]_{\text{Ł}}, \mathsf{CFr})$ and $Log_\Box([\mathbf{0},\mathbf{1}]_{\mathbf{\Pi}}, \mathsf{CFr})$. The proof of this second case is based on the following lemma.

Lemma 4. *The formula* (K) *is valid in a Kripke frame* $\langle W, R\rangle$ *iff for every* $w, w' \in W$ *it holds that* $R(w, w') = R(w, w') \odot R(w, w')$ *(i.e.,* $R(w, w')$ *is idempotent).*

Proof. Let us start proving the implication to the right. Hence, let us consider a Kripke frame $\langle W, R\rangle$ and let us assume that there is an element $a \in A$ such that $a \odot a < a$ and $a = R(w, w')$ for certain worlds w, w'. We want to see that the formula (K) is not valid in our Kripke frame. We define a valuation V by the conditions: (i) $V(p, w') = a$, and (ii) $V(p, x) = 1$ for every world $x \neq w'$. Then, $e(\Box p, w) = 1$, $e(\Box(p \to p \odot p), w) = 1$ and $e(\Box(p \odot p), w) \leq a \to a \odot a \neq 1$. Therefore, $\langle W, R, V \rangle$ is not a model of (K) because $e(\Box p \to (\Box(p \to p \odot p) \to \Box(p \odot p)), w) \neq 1$.

The other direction is an straightforward consequence of the validity of the quasi-equation

$$x \approx x \odot x \quad \Rightarrow \quad (x \to y) \odot (x \to (y \to z)) \leq x \to z$$

in all residuated lattices. $\square$

$$\varphi \Rightarrow \varphi$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$\varphi_1 \to \varphi_2, \varphi_2 \to \varphi_3 \Rightarrow \varphi_1 \to \varphi_3$$

$$\frac{\Gamma, t_i \to \varphi \Rightarrow \Delta, t_i \to \psi \quad \text{for every } i \in \{1, \ldots, n\}}{\Gamma \Rightarrow \Delta, \varphi \to \psi}$$

$$\frac{\Gamma, \psi \to t_i \Rightarrow \Delta, \varphi \to t_i \quad \text{for every } i \in \{1, \ldots, n\}}{\Gamma \Rightarrow \Delta, \varphi \to \psi}$$

$$\Rightarrow \varphi \wedge \psi \to \varphi$$
$$\Rightarrow \varphi \wedge \psi \to \psi$$
$$\varphi_3 \to \varphi_1, \varphi_3 \to \varphi_2 \Rightarrow \varphi_3 \to (\varphi_1 \wedge \varphi_2)$$

$$\Rightarrow \varphi \to \varphi \vee \psi$$
$$\Rightarrow \psi \to \varphi \vee \psi$$
$$\varphi_1 \to \varphi_3, \varphi_2 \to \varphi_3 \Rightarrow (\varphi_1 \vee \varphi_2) \to \varphi_3$$

$$\Rightarrow t_i \to t_j \quad , \text{if } t_i \leq t_j$$
$$t_i \to t_j \Rightarrow \quad , \text{if } t_i \not\leq t_j$$

$$(\varphi_1 \wedge \varphi_2) \to \varphi_3 \Rightarrow \varphi_1 \to (\varphi_2 \to \varphi_3)$$
$$\varphi_1 \to (\varphi_2 \to \varphi_3) \Rightarrow (\varphi_1 \wedge \varphi_2) \to \varphi_3$$

$$\varphi \Rightarrow (\mathit{true} \to \varphi)$$
$$(\mathit{true} \to \varphi) \Rightarrow \varphi$$

$$\frac{\Rightarrow \varphi}{\Rightarrow \Box \varphi}$$

$$\Rightarrow \Box(\varphi \to \psi) \to (\Box \varphi \to \Box \psi)$$
$$\Rightarrow (t_i \to \Box \varphi) \equiv \Box(t_i \to \varphi), \text{ if } i \in \{1, \ldots, n\}$$

Table 1: Fitting systems ($A = \{t_1, \ldots, t_n\}$)

Our last remark here is about the algebraic semantics for modal fuzzy logics. We remind that it is well-known that all logics are complete with respect to an algebraic semantics (considering an adequate set of designated elements over the algebras). Hence, there is also an algebraic semantics for our modal fuzzy logics. However, we point out that modal fuzzy logics do not enjoy, in general, an algebraic completeness with respect to chains. This is a big difference with the framework of fuzzy logics, and the reader must be aware of it in order to avoid mistakes. Of course it is possible to force chain completeness adding new axioms to our minimal fuzzy modal logics, as is done in [10], but then the new framework is not so general as ours.

Transfer Properties. We are going to show with three counterexamples that in general metalogical properties are lost when we move from the modal Boolean case to the modal many-valued one. This implies that in order to attack future problems for modal many-valued logics we will need to introduce new machinery, what makes this new field a really exciting and appealing one.

First of all we point out that the fact that two algebras $\mathbf{A}$ and $\mathbf{B}$ generate the same variety does not imply that $Log_\Box(\mathbf{A}, \mathsf{CFr}) = Log_\Box(\mathbf{B}, \mathsf{CFr})$. As a counterexample we can consider $\mathbf{A}$ as the standard Gödel algebra $[\mathbf{0}, \mathbf{1}]_\mathbf{G}$ and $\mathbf{B}$ as its subalgebra of universe $\{0\} \cup [1/2, 1]$. It is not hard to see that $\Box\neg\neg p \to \neg\neg\Box p$ belongs to $Log_\Box(\mathbf{B}, \mathsf{CFr})$ while fails to belong to $Log_\Box(\mathbf{A}, \mathsf{CFr})$. Indeed the previous claim is a particular case of the following lemma, which is also related to results in [2, 7].

Lemma 5. *Let* $\mathbf{A}$ *be a Gödel chain. Then, the following statements are equivalent:*

1. $\Box\neg\neg p \to \neg\neg\Box p$ *belongs to* $Log_\Box(\mathbf{A}, \mathsf{CFr})$.
2. *the set* $A \setminus \{0\}$ *has a minimum element.*

Proof. Let us start proving by contradiction the implication $(1 \Rightarrow 2)$. If there is no minimum element of $A \setminus \{0\}$, then there is an infinite decreasing sequence $a_0 > a_1 > \ldots > a_n > a_{n+1} > \ldots > 0$. We consider the classical Kripke model given by $W = \{0\} \cup \{a_n : n \in \omega\}$, $R = \{0\} \times \{a_n : n \in \omega\}$ (i.e., it is classical) and for every $x \in W$, $e(p, x) = x \in A$. Then, $e(\Box p, 0) = 0$, $e(\neg\neg\Box p, 0) = 0$ and $e(\Box\neg\neg p, 0) = 1$. Hence, $\Box\neg\neg p \to \neg\neg\Box p$ is not valid in the classical Kripke frame $\langle W, R, V\rangle$.

Let us prove the other direction. Let us denote by a the minimum of $A \setminus \{0\}$. First of all we note that the formulas $\Box\neg\neg p$ and $\neg\neg\Box p$ can only take classical values (i.e., 0 and 1) due to the behaviour of the Gödel negation and the fact that we are considering classical Kripke frames. So, it is enough to see that $e(\Box\neg\neg p, w) = 1$ and $e(\neg\neg\Box p, w) = 0$ never happens. Let us assume $e(\Box\neg\neg p, w) = 1$ and $e(\neg\neg\Box p, w) = 0$. Then $1 = e(\Box\neg\neg p, w) = \bigwedge\{R(w, w') \to e(\neg\neg p, w') : w' \in W\}$, i.e., for every $w' \in W$, $R(w, w') \leq e(\neg\neg p, w')$. Thus, for every $w' \in W$, if $R(w, w') \neq 0$ then $e(p, w') \neq 0$. Hence, for every $w' \in W$, if $0 \neq R(w, w')$ then $a \leq e(p, w')$. Thus, $a \leq \bigwedge\{R(w, w') \to e(p, w') : w' \in W\} = e(\Box p, w)$. Therefore, $e(\neg\neg\Box p, w) = \neg 0 = 1$, which is a contradiction. $\Box$

Secondly we notice that it can happen that two classes F_1 and F_2 of classical Kripke frames have different modal many-valued logics for an algebra $\mathbf{A}$ while for the case of the Boolean algebra of two elements they share the same logic. Why? It is well-known that the modal logic $S4$ is generated both by the class F_1 of finite quasi-orders (perhaps fails the antisymmetric property) and the class F_2 of infinite partial orders. However, as a consequence of Lemma 5, $\Box\neg\neg p \to \neg\neg\Box p$ belongs to $Log_\Box([\mathbf{0},\mathbf{1}]_\mathbf{G}, \mathsf{F}_1)$ while it does not belong to $Log_\Box([\mathbf{0},\mathbf{1}]_\mathbf{G}, \mathsf{F}_2)$.

Lastly we remark that it is possible to have that $Log_\Box(\mathbf{2}, \mathsf{F})$ enjoys the finite Kripke frame property while $Log_\Box(\mathbf{A}, \mathsf{F})$ does not. A counterexample is given by the standard Gödel algebra $[\mathbf{0},\mathbf{1}]_\mathbf{G}$ and the class F of classical quasi-orders. The failure of the finite Kripke frame property of $Log_\Box([\mathbf{0},\mathbf{1}]_\mathbf{G}, \mathsf{F})$ is again witnessed, for instance, by the formula $\Box\neg\neg p \to \neg\neg\Box p$.

4 Examples in the literature

In the last years there has been a growing number of papers about combining modal and many-valued logics. Some approaches do not fit in our framework, like [10, 31], but others stay as particular cases of our framework. Among the ones that fit in our framework we can cite [16, 17, 24, 18, 19, 25, 7].

Next we will discuss the known axiomatizations in the literature of logics of the form $Log_\Box(\mathbf{A}, \mathsf{F})$ where $\mathbf{A}$ is non Boolean and F is the class of all $\mathbf{A}$-valued Kripke frames or the class of all $\mathbf{A}$-valued classical Kripke frames[7]. Indeed, the only known ones are for logics satisfying axiom (K), i.e., the authors are unaware of any axiomatization for a case where axiom (K) fails[8]. This remains as a challenge.

A is a finite Heyting algebra. This case was considered by M. Fitting in [17, Section6]. The language includes constants t_i for every element of the fixed algebra $\mathbf{A}$ (i.e., for every truth value), what simplifies the proofs and allows to give a unified presentation of the calculus to axiomatize $Log_\Box(\mathbf{A}, \mathsf{Fr})$. The last statement refers to the fact that all these calculi share the same schemes without constants. The calculus is given using sequents and can be found in Table 1. Completeness of this calculus means that $Log_\Box(\mathbf{A}, \mathsf{Fr})$ coincides with the set of formulas $\varphi \in Fm_\Box$ such that the sequent $\Rightarrow \varphi$ is derivable using the calculus in Table 1. We notice that using the constants it is very easy to see that $Log_\Box(\mathbf{A}, \mathsf{Fr}) \neq Log_\Box(\mathbf{A}, \mathsf{CFr})$. Other papers that study these cases are [28, 29, 27]. We notice that for any finite algebra $\mathbf{A}$ (not only Heyting ones) the logic $Log_\Box(\mathbf{A}, \mathsf{CFr})$ has been studied in [15] (indeed their notion of modality covers more cases than ours), but their approach does not seem to be fruitful for the case of $Log_\Box(\mathbf{A}, \mathsf{Fr})$.

[7] Hence, we do not consider cases where the class of frames satisfy some extra conditions, e.g., reflexive and $\odot$-transitive frames. The reason why we do not talk about them is because we want to consider the minimal logics.

[8] Indeed, in [6] it is given an axiomatization for the case $Log_\Box(\mathbf{A}, \mathsf{F})$ where $\mathbf{A}$ is a finite Łukasiewicz chain, but the reason why this works is that because the authors are able to interdefine the modality $\Box$ with other modalities that really satifiy the axiom (K). Hence, essentially the same difficulties remain in the case solved in [6].

$$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$$
$$\varphi \to (\psi \to \varphi)$$
$$(\varphi \wedge \psi) \to (\psi \wedge \varphi)$$
$$(\varphi \wedge (\psi \wedge \chi)) \to ((\psi \wedge \varphi) \wedge \chi)$$
$$(\varphi \to (\psi \to \chi)) \leftrightarrow ((\varphi \wedge \psi) \to \chi)$$
$$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$$
$$0 \to \varphi$$
$$\varphi \to (\varphi \wedge \varphi)$$
$$\neg\varphi \leftrightarrow (\varphi \to 0)$$

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$
$$\neg\neg\Box\varphi \to \Box\neg\neg\varphi$$

From φ and $\varphi \to \psi$, infer ψ
From φ infer $\Box\varphi$

Table 2: Inference Rules of $Log_\Box([\mathbf{0},\mathbf{1}]_\mathbf{G}, \mathsf{Fr})$

$\mathbf{Log}_\Box([\mathbf{0},\mathbf{1}]_\mathbf{G}, \mathsf{Fr})$. This case has been studied by X. Caicedo and R. Rodríguez in [7]. They have proved that this logic is axiomatized by the calculus given in Table 2. The proof is based on the construction of a canonical model[9], which indeed is classical. From here it follows that for every class of Kripke frames F, it holds that $Log_\Box([\mathbf{0},\mathbf{1}]_\mathbf{G}, \mathsf{Fr}) = Log_\Box([\mathbf{0},\mathbf{1}]_\mathbf{G}, \mathsf{CFr})$[10]. Therefore, for the case of $[\mathbf{0},\mathbf{1}]_\mathbf{G}$ we already know how to introduce the notion of *modal many-valued logic*: it is any set of $Fm_\Box$-formulas that contains the formulas in Table 2 and is closed under the rules in Table 2.

$\mathbf{Log}_\Box([\mathbf{0},\mathbf{1}]_\text{Ł}, \mathsf{CFr})$. The recent paper [25] by G. Hansoul and B. Teheux axiomatizes the normal modal logic $Log_\Box([\mathbf{0},\mathbf{1}]_\text{Ł}, \mathsf{CFr})$ with the infinite calculus given in Table 3. The proof is based on the construction of a classical canonical model. Surprisingly this proof does not need the presence in the language of constants for every truth value. The trick to avoid the introduction of constants is based on a result of [33] (see [25, Definition 5.3]).

$\mathbf{Log}_\Box(\text{Ł}_{\mathbf{n-1}}, \mathsf{Fr})$. Here the notation $\text{Ł}_{\mathbf{n-1}}$ (see [11]) refers to the n-valued Łukasiewicz chain algebra. Since a proper handling of the resulting non normal modality $\Box$ is so far missing, the authors of [6] manage to axiomatize the logic $Log_\Box(\text{Ł}_{\mathbf{n-1}}, \mathsf{Fr})$ by adding a truth constant in the language for each element in $\text{Ł}_{\mathbf{n-1}}$. The axiomatization given in [6] is shown in Table 4. Indeed, the presence of constants

[9]This technique also gives strong completeness in the sense that the caculus in Table 2 axiomatizes $\models_{l([\mathbf{0},\mathbf{1}]_\mathbf{G}, \mathsf{Fr})}$.

[10]We stress that this does not contradict facts like that $\Box p \to p$ and $\Box\neg p \to \neg p$ define, over $[\mathbf{0},\mathbf{1}]_\mathbf{G}$, the same class of classical Kripke frames while they define different classes of Kripke frames. The formula $\Box p \to p$, over the standard Gödel algebra, defines the class of (classical) Kripke frames $\langle W, R\rangle$ satisfying that $R(w,w) = 1$ for every $w \in W$. And the formula $\Box\neg p \to \neg p$, over the standard Gödel algebra, defines the class of Kripke frames $\langle W, R\rangle$ satisfying that $R(w,w) \neq 0$ for every $w \in W$.

$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
$\varphi \to (\psi \to \varphi)$
$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$
$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$
$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
$\Box(\varphi \odot \varphi) \leftrightarrow (\Box\varphi \odot \Box\varphi)$
$\Box(\varphi \oplus \varphi) \leftrightarrow (\Box\varphi \oplus \Box\varphi)$
$\Box(\varphi \oplus \varphi^m) \leftrightarrow (\Box\varphi \oplus (\Box\varphi)^m)$ for every $m \geq 0$
From φ and $\varphi \to \psi$, infer ψ
From φ infer $\Box\varphi$

Table 3: Inference Rules of $Log_\Box([\mathbf{0},\mathbf{1}]_{\text{Ł}}, \mathsf{CFr})$

allows for the interdefinability of $\Box$ with the graded modalities $\Box_t$ (where $t \in \mathbf{A}$) corresponding to the cuts of the many-valued accessibility relation, i.e., defining

$$e(\Box_t\varphi, w) = \bigwedge\{e(\varphi, w') : R(w, w') \geq t\}$$

to extend the valuation. Then, it is easy to notice that all modalities $\Box_t$ are normal in contrast with the modality $\Box$. The interdefinability is given by the valid equivalence

$$(\Box\varphi) \leftrightarrow \bigwedge\{t \to \Box_t\varphi : t \in \text{Ł}_{n-1}\}.$$

We notice that in some particular cases, axiomatizations for these graded modalities have been found in the literature (see for instance [13, 35, 6]).

5 Main Open Problems

In opinion of the authors the main open problems in this field are the search of axiomatizations for $Log_\Box([\mathbf{0},\mathbf{1}]_{\text{Ł}}, \mathsf{Fr})$ and $Log_\Box([\mathbf{0},\mathbf{1}]_{\mathbf{\Pi}}, \mathsf{Fr})$ in case they are recursively axiomatizable. The main difficulties here are the lack of normality of these logics.

Once there is an axiomatization for them (if any) it seems easy to find the right definition of modal many-valued logic. And once we know the definition of the class of modal many-valued logics the next step will be their study with all possible techniques: algebras[11], Kripke frames, Kripke models, sequent calculus, etc.

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[11]Remember that extensions of algebraizable logics are algebraizable [4].

Notation

For $t \neq 0$, $\Box_t \varphi$ stands for $\bigwedge \left\{ \left(t \to \neg\Box\neg((\varphi \leftrightarrow t')^{n-1}) \right)^{n-1} \to t' : t' \in Ł_n \right\}$

$$m.\varphi := \varphi \oplus \overset{m}{\ldots} \oplus \varphi$$
$$\varphi^m := \varphi \odot \overset{m}{\ldots} \odot \varphi$$

Axioms

$$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$$
$$\varphi \to (\psi \to \varphi)$$
$$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$$
$$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$$
$$(\varphi \wedge \psi) \leftrightarrow (\varphi \odot (\varphi \to \psi))$$
$$(\varphi \vee \psi) \leftrightarrow \left(((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi) \right)$$
$$(\varphi \odot \psi) \leftrightarrow \neg(\varphi \to \neg\psi)$$
$$n.\varphi \to (n-1).\varphi$$
$$(m.\varphi^{m-1})^n \leftrightarrow (n.\varphi^m), \quad 2 \leq m \leq n-2 \text{ and } m \nmid (n-1)$$
$$(t_i \to t_j) \leftrightarrow t_k, \quad \text{if } t_k = t_i \to t_j$$

$$\Box_t(\varphi \to \psi) \to (\Box_t\varphi \to \Box_t\psi)$$
$$\Box_{t_i}\varphi \to \Box_{t_j}\varphi, \quad \text{if } t_i \leq t_j$$
$$\Box_{t_i}(t_j \to \varphi) \leftrightarrow (t_j \to \Box_{t_i}\varphi)$$
$$(\Box\varphi) \leftrightarrow \left(\bigwedge \left\{ t \to \Box_t\varphi : t \in Ł_n, t \neq 0 \right\}\right)$$

Rules

$$\varphi, \varphi \to \psi \vdash \psi$$
$$\text{If } \emptyset \vdash \varphi \text{ then } \emptyset \vdash \Box_t\varphi$$

Table 4: Inference Rules of $Log_\Box(\mathbf{Ł_{n-1}}, \mathsf{Fr})$

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