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Exploring Trapping Problem on Weighted Heterogeneous Networks Induced by Extended Corona Product

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ABSTRACT Since many dynamical processes can be analyzed in the framework of trapping problem, it is great important to explore trapping problem on diverse complex systems or networks. However, addressing trapping problem on networks generated by graph products is still less touched. In this paper, by taking type of random walks and number of traps into account and compiling four different scenarios, trapping problem is touched more completely on resultant weighted heterogeneous networks of extended corona product and weight reinforcement mechanism. In more detail, we study standard random walk and delayed random walk on the network with a deep trap fixed at one initial node in the first and second scenarios respectively. This two types of random walks are further investigated on the network with three traps placed at initial three nodes in more challenging third and fourth scenarios. In all this four scenarios, the solutions of average trapping time(ATT) are deduced analytically to measure trapping efficiency, which agree well with their corresponding numerical counterparts and show that ATT grows sub-linearly with network size. Besides, expressions of ATT obtained in the second and fourth scenarios indicate that the parameter p governing delayed random walk alters the pre-factor of ATT but has no effect on the leading scaling of ATT. Furthermore, the comparisons between expression of ATT in first and third scenarios, expression of ATT in second and fourth scenarios imply that ATT can be lowered and trapping efficiency can be improved accordingly by introducing more traps. In summary, this work may enrich the clues for understanding trapping issue and modulating trapping process on more general heterogeneous weighted networks.

INDEX TERMS Random walk, average trapping time, heterogeneous weighted network, corona product.

I. INTRODUCTION

In the past few decades, there has been considerable interest in the research for complex networks in many fields [1]–[5]. Triggered by the *BA* scale-free network model [6] and *WS* small-world network model [7], extensive researches show that many different real networks share some striking structural properties [8]–[11]. To this end, considerable deterministic network models that can reproduce common structural properties of real networks had been put forward to serve as test-bed [12]–[15]. Given the observation that massive networks are always composed of small pieces such as communities, modules and motifs [16]–[18], quite a few deterministic networks had been built out of smaller networks by employing graph products such as hierarchical product [19], [20], corona product [21], [22].

In addition, lots of researches devoted themselves to unveil the interplay between structural properties and dynamical processes occurred on networks [4], [23]. As an important dynamical process, random walk has been studied extensively on various deterministic networks such as fractal networks derived from Caley trees [24]–[26], T fractal [27]–[29], Sierpinski gasket [30], [31] and other non-fractal networks [32], [33] since they have potential applications in modeling chemical reactions [34], [35], image segmentation [36], normalized Laplacian spectrum [37]–[39] and so on. Except for standard random walk, the delayed random walk motivated by the observed phenomenon that there is a delay for the new structure to influence dynamical processes taking place on networks has been proposed and explored on some networks recently [40]–[42].

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Since many dynamical processes can be analyzed in the framework of trapping problem [43], trapping problem is a primary issue of random walks. Furthermore, trapping efficiency (transport efficiency) is a preferred readout of trapping problem. Average trapping time, in turn, is a favorite indicator of trapping efficiency [44], [45]. It is noteworthy that in most previous works only one trap is placed at network when studying trapping problem and computing average trapping time [24], [41], while only a few works take multiple traps into account to explore trapping issue [30].

In this paper, we study two types of random walks on a family of weighted heterogeneous networks induced by extended corona product [18], [21], [22] with different numbers of traps. In more detail, we introduce the weighted heterogeneous networks proposed by *Qi* et al in Ref [18] in section 2. We study standard random walk and delayed random walk on the networks with one trap fixed at a initial node and derive the analytical expressions of *ATT* in section 3 and section 4 respectively. Furthermore, this two types of random walks are addressed on the networks with three traps being placed at three initial nodes and expressions of *ATT* are deduced in section 5 and section 6 respectively. All the obtained analytical solutions of *ATT* and the numerical solutions of it completely agree with each other. All these obtained results of *ATT* indicate that it grows sub-linearly with any value of δ controlling networks' iteration. In addition, analytical solutions of *ATT* also imply that the delay parameter steering delayed random walk can only modify the pre-factor of *ATT*, but doesn't impact the leading scaling of *ATT*. Finally, it can be found that introducing more traps could reduce the average trapping time and improve trapping efficiency to some extent.

II. CONSTRUCTION RULE OF WEIGHTED HETEROGENEOUS NETWORK

The family of weighted heterogeneous network employed as test-bed here was proposed by *Qi* et al in ref [18]. It was produced iteratively by integrating extended corona product with weight reinforcement mechanism and parameterized by δ(δ is a positive integer). Let κ_1 be a weighted graph with two vertices connected by one edge with unit weight. Then the heterogeneous weighted networks G_n is constructed as follows. For $n = 0$, G_0 is a triangle in which each pair of nodes is linked by a edge with unit weight. For $n \geq 1$, G_n is obtained from G_{n-1} by performing series of operations depicted in Fig.1: generate a weight network $\kappa_1 \circledcirc G_{n-1}$ by applying the extended corona product of G_{n-1} and κ_1, increase weight of each edge in G_{n-1} by δ times. It's noteworthy that G_n will reduce to Koch network if $\delta = 0$. Since the properties of Koch network have been extensively studied, here we only consider the case of $\delta > 0$. As a result, for node i created at generation n_i, the weight of the edge from node i to node j at generation n is $\omega_{ij}(n) = (1+\delta)^{n-n_i}$ and the strength of node i at generation n is [18]

$$S_i(n) = \sum_j \omega_{ij}(n) = 2(2+\delta)^{n-n_i} \tag{1}$$

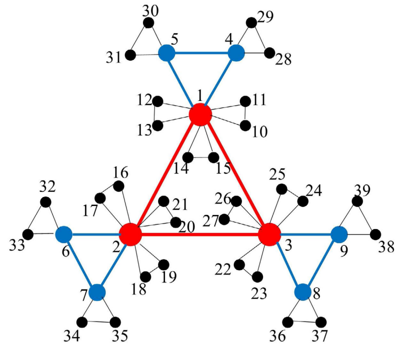

FIGURE 1. The first three generations of weighted heterogeneous networks when $\delta = 1$. Here, nodes and lines created at generations 0, 1 and 2 are marked with red, blue and black respectively. Besides, nodes are counted counterclockwise.

III. ANALYTICAL SOLUTION OF ATT FOR STANDARD RANDOM WALKS WITH A SINGLE TRAP

After introducing the iterative rule of the weighted heterogeneous networks, we will study the standard random walk on this weighted network with a single trap.

A. NUMERICAL FORMULATIONS OF TRAPPING PROBLEMS

At each time step, the walker starting from its current position i except the trap moves to any of its nearest neighbors j with the transition probability $p_{ij} = \frac{\omega_{ij}}{S_i}$, which constitutes entry of the transition matrix $P = S^{-1}W$, where S is the diagonal strength matrix and W is weighted adjacency matrix respectively. In this work, we study the problem with an immobile trap located at node 1(due to the symmetry, the trap can be also located at node 2 or 3). Let T_i^n denotes the trapping time, which is the expected time for a walker from the node i to first reach the trap, we have

$$\begin{aligned} T_i^n &= \sum_{j=1}^{N_n} p_{ij}(n)(T_j^n + 1) \\ &= \sum_{j=1}^{N_n} p_{ij}(n)T_j^n + \sum_{j=1}^{N_n} p_{ij}(n) \\ &= \sum_{j=2}^{N_n} p_{ij}(n)T_j^n + p_{i1}(n)T_1^n + 1 \\ &= \sum_{j=2}^{N_n} p_{ij}(n)T_j^n + p_{i1}(n) \times 0 + 1 \\ &= \sum_{j=2}^{N_n} p_{ij}(n)T_j^n + 1 \end{aligned} \tag{2}$$

where $i \neq 1$, which can be recast in matrix form as:

$$\bar{T} = \bar{T}\bar{P} + \bar{e} \tag{3}$$

where $\bar{P}$ is a sub-matrix of transition matrix with the row and the column corresponding to trap being removed, $\bar{T}$ is

(N_n-1)-dimensional vector denoting the trapping time of all nodes except trap's location and N_n is the total number of nodes in this weighted network. Here $\bar{e}$ is the (N_n-1)-dimensional vector formed by removing the first column of the N_n-dimensional vector e, where $e=(1,1,\ldots,1)$. Solving *Eq.*(3) leads to

$$\bar{T}=(\bar{I}-\bar{P})^{-1}\bar{e} \tag{4}$$

in which $\bar{I}$ is the $(N_n-1)\times(N_n-1)$ identity matrix.

Next to, let $\langle T\rangle_n$ denotes the *ATT*, which is the average of T_i^n over all initial nodes in G_n except the trap. Thus, utilizing *Eq.*(4), we obtain

$$\langle T\rangle_n=\frac{1}{N_n-1}\sum_{i=2}^{N_n}T_i^n=\frac{1}{N_n-1}\sum_{i=2}^{N_n}\sum_{j=2}^{N_n}\tau_{ij} \tag{5}$$

Eqs.(4) and (5) show that the problem of calculating T_i^n and $\langle T\rangle_n$ is reduced to find the sum of elements of matrix $(\bar{I}-\bar{P})^{-1}$. However, for large n, it's difficult to obtain the numerical results of *Eq.*(4) by direct calculating. So, we use it to carry out some simple work. Let $\delta=1$, we can calculate T_i^n for the first several generations using *Eq.*(4), which were summarized in Table 1.

TABLE 1. Trapping time for various n when $\delta=1$. Notice that owing to the obvious symmetry, some nodes have the same trapping time. All the values are calculated straightforwardly from *Eq.*(4).

n\i	(2,3)	(4,5)	(6,7,8,9)	(10,11,12,13,14,15)
0	2			
1	5	2	7	
2	25/2	5	35/2	2
3	125/4	25/2	175/4	5
4	625/8	125/4	875/8	25/2
5	3125/16	625/8	4375/16	125/4

B. EXPLICIT EXPRESSIONS OF AVERAGE TRAPPING TIME

Let M denotes the first passage time (*FPT*) for starting from node i to any of its old neighbors, and let N represents the *FPT* for going from any of the new neighbors of i to one of its old neighbors. Based on the structure of the network, we have the following relations

$$\begin{cases} M &= \dfrac{\delta+1}{\delta+2}\times 1+\dfrac{1}{\delta+2}(1+N) \\ N &= \dfrac{1}{2}(1+M)+\dfrac{1}{2}(1+N) \end{cases} \tag{6}$$

which lead to $M=\frac{4+\delta}{1+\delta}$. Therefore, we have the following rule

$$T_i^{n+1}=\frac{4+\delta}{1+\delta}T_i^n \tag{7}$$

Since the evolutionary rule of *FPT* depicted by *Eq.*(7) is basic quantity for deriving *ATT*, it is relevant to verify it by some ways. It is noteworthy that the numerical results listed in Table 1 obey the equation $T_i^{n+1}=\frac{5}{2}T_i^n$, which coincides with the special case of *Eq.*(7) when $\delta=1$. Therefore, the evolutionary rule of *FPT* depicted by *Eq.*(7) was verified initially.

Let Λ_n denotes the set of nodes belonging to G_n, and $\bar{\Lambda}_n$ be the set of nodes created at generation n. Obviously, $\Lambda_n=\Lambda_{n-1}\bigcup\bar{\Lambda}_n$. In order to evaluate $\langle T\rangle_n$, we define two quantities for $g\le n$,

$$T_{g,tot}^n=\sum_{i\in\Lambda_g}T_i^n,\ \bar{T}_{g,tot}^n=\sum_{i\in\bar{\Lambda}_g}T_i^n \tag{8}$$

Then, $T_{n,tot}^n$ can be computed as

$$T_{n,tot}^n=T_{n-1,tot}^n+\bar{T}_{n,tot}^n=\frac{4+\delta}{1+\delta}T_{n-1,tot}^{n-1}+\bar{T}_{n,tot}^n \tag{9}$$

Next we will explicitly determine the quantity $T_{n,tot}^n$. To this end, we should first determine $\bar{T}_{n,tot}^n$.

Under the condition of $\delta=1$, $T_4^1=\frac{1}{2}(1+T_1^1)+\frac{1}{2}(1+T_5^1)$, $T_5^1=\frac{1}{2}(1+T_1^1)+\frac{1}{2}(1+T_4^1)$, $T_6^1=\frac{1}{2}(1+T_2^1)+\frac{1}{2}(1+T_7^1)$, $T_7^1=\frac{1}{2}(1+T_2^1)+\frac{1}{2}(1+T_6^1)$, $T_8^1=\frac{1}{2}(1+T_3^1)+\frac{1}{2}(1+T_9^1)$, $T_9^1=\frac{1}{2}(1+T_3^1)+\frac{1}{2}(1+T_8^1)$. Thus,

$$\begin{aligned}\bar{T}_{1,tot}^1&=\sum_{i\in\bar{\Lambda}_1}T_i^1\\&=T_4^1+T_5^1+T_6^1+T_7^1+T_8^1+T_9^1\\&=12+2\bar{T}_{0,tot}^1\end{aligned} \tag{10}$$

similarly, for the case of $n=2$, we have

$$\bar{T}_{2,tot}^2=\sum_{i\in\bar{\Lambda}_2}T_i^2=\sum_{i=10}^{39}T_i^2=60+2\bar{T}_{1,tot}^2+6\bar{T}_{0,tot}^2 \tag{11}$$

Finally,

$$\begin{aligned}\bar{T}_{n,tot}^n&=\sum_{i\in\bar{\Lambda}_n}T_i^n=12\times 5^{n-1}+2\bar{T}_{n-1,tot}^n\\&\quad+2\times 3\bar{T}_{n-2,tot}^n+\ldots+2\times 3^{n-1}\bar{T}_{0,tot}^n\end{aligned} \tag{12}$$

In the same way, we can derive formulas for any δ,

$$\begin{aligned}\bar{T}_{n,tot}^n&=12\times(\delta+4)^{n-1}+2\bar{T}_{n-1,tot}^n+2(\delta+2)\bar{T}_{n-2,tot}^n\\&\quad+\ldots+2(\delta+2)^{n-2}\bar{T}_{1,tot}^n+2(\delta+2)^{n-1}\bar{T}_{0,tot}^n\end{aligned} \tag{13}$$

and

$$\begin{aligned}\bar{T}_{n+1,tot}^{n+1}&=12\times(\delta+4)^{n}+2\bar{T}_{n,tot}^{n+1}+2(\delta+2)\bar{T}_{n-1,tot}^{n+1}\\&\quad+\ldots+2(\delta+2)^{n-1}\bar{T}_{1,tot}^{n+1}+2(\delta+2)^{n}\bar{T}_{0,tot}^{n+1}\end{aligned} \tag{14}$$

where $12\times(\delta+4)^{n-1}$ and $12\times(\delta+4)^n$ are the double of numbers of nodes created at generations n and $n+1$ respectively. *Eq.*(14) minus *Eq.*(13) times $\frac{(\delta+4)(\delta+2)}{\delta+1}$ and making use of the relation $T_i^{n+1}=\frac{\delta+4}{\delta+1}T_i^n$, we get

$$\begin{aligned}&\bar{T}_{n+1,tot}^{n+1}-12\times(\delta+4)^n\\&=2\bar{T}_{n,tot}^{n+1}\\&\quad+\frac{(\delta+4)(\delta+2)}{\delta+1}(\bar{T}_{n,tot}^n-12\times(\delta+4)^{n-1})\end{aligned} \tag{15}$$

which may be rewritten as

$$\bar{T}^{n+1}_{n+1,tot} = \frac{(\delta+4)^2}{\delta+1}\bar{T}^n_{n,tot} - \frac{12}{\delta+1}(\delta+4)^n \tag{16}$$

Using $\bar{T}^1_{1,tot} = 12+2(T^1_2+T^1_3)$, $Eq.$(16) is solved inductively as

$$\bar{T}^n_{n,tot} = \frac{16\delta+40}{\delta+1}[\frac{(\delta+4)^2}{\delta+1}]^{n-1} + 4(\delta+4)^{n-1} \tag{17}$$

Substituting $Eq.$(17) for $\bar{T}^n_{n,tot}$ into $Eq.$(9), we have

$$\begin{aligned} T^n_{n,tot} &= \frac{4+\delta}{1+\delta}T^{n-1}_{n-1,tot} \\ &+ \frac{16\delta+40}{\delta+1}[\frac{(\delta+4)^2}{\delta+1}]^{n-1} + 4(\delta+4)^{n-1} \end{aligned} \tag{18}$$

Considering the initial condition $T^0_{0,tot} = 4$, $Eq.$(18) is resolved by induction to yield

$$T^n_{n,tot} = (\frac{A}{B})^n[\frac{(16\delta+40)(A^n-1)}{A(\delta+3)} + \frac{4B(B^n-1)}{A\delta} + 4] \tag{19}$$

where $A = \delta+4$, $B = \delta+1$. From [18], it is known that

$$N_n = \frac{6(\delta+4)^n + 3\delta + 3}{\delta+3} \tag{20}$$

Dividing both sides of $Eq.$(19) by $N_n - 1$, we arrive at the accurate formula for the average trapping time at the trap located at node 1 on the weighted heterogeneous network.

$$\begin{aligned} \langle T\rangle_n &= \frac{1}{N_n-1}T^n_{n,tot} \\ &= \frac{(8\delta+20)(A^n-1)}{A(3A^n+\delta)}(\frac{A}{B})^n \\ &+ \frac{2(A-1)(B^n-1)}{(3A^n+\delta)\delta}(\frac{A}{B})^{n-1} + \frac{2(A-1)}{3A^n+\delta}(\frac{A}{B})^n \end{aligned} \tag{21}$$

To verify analytical expression of ATT provided by $Eq.$(21) comprehensively, it was compared with numerical solutions provided by $Eq.$(5). Both of them agree well with each other, which was summarized in Fig.2.

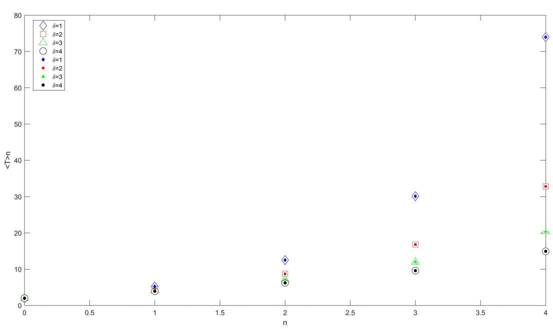

FIGURE 2. **Average trapping time for four different values of δ on standard random walk with one trap. The filled symbols represent numerical results and the empty symbols denote values of analytical expression given by *Eq.*(21).**

From $Eq.$(20), we have

$$\begin{aligned} A^n &= \frac{(\delta+3)N_n - 3\delta - 3}{6} \\ (\frac{A}{B})^n &= [\frac{(\delta+3)N_n - 3\delta - 3}{6}]^{\frac{ln\frac{A}{B}}{lnA}} \end{aligned} \tag{22}$$

For large networks, combining $Eq.$(21) with $Eq.$(22) leads to

$$\langle T\rangle_n \approx (\frac{A}{B})^n \sim N_n^{\frac{ln\frac{A}{B}}{lnA}} \tag{23}$$

$Eq.$(23) shows that the ATT grows as a power-law function of network size with exponent $\theta(\delta) = \frac{ln\frac{A}{B}}{lnA}$. The exponent $\theta(\delta)$is always smaller than 1, thus the average trapping time grows sub-linearly with the growth of the network regardless of the value of weight parameter δ.

IV. ANALYTICAL SOLUTION OF ATT FOR DELAYED RANDOM WALKS WITH A SINGLE TRAP

In the preceding section, we have obtained the explicit expressions of average trapping time in the context of standard random walk, next to, we will study the delayed random walk [40] on this weighted network with a single trap.

A. NUMERICAL FORMULATIONS OF TRAPPING PROBLEMS

Before defining delayed random walk on this weighted network, it is beneficial to introduce some relevant definitions. We have defined Λ_n and $\bar{\Lambda}_n$ in Sec.3. To this end, if a neighbour of node i on this weighted network belongs to Λ_{n-1}, it's called old neighbors of node i, otherwise it is called new neighbors of node i. During the process of delayed random walks in G_n, if the walker currently locates at an old node i, it's allowed to jump to any of node i's old neighbors or any neighbors, with their respective probabilities p and $(1-p)(0 \leq p \leq 1)$. If the walker locates at a new node now, it can move to any neighbors in G_n. Concretely, the transition probability p_{ij} is defined by:

$$p_{ij} = \begin{cases} p \times \frac{\omega_{ij}(n)}{\sum_{k\in\Lambda_{n-1}}\omega_{ik}(n)} + (1-p)\times\frac{\omega_{ij}(n)}{S_i(n)}, i,j\in\Lambda_{n-1} \\ (1-p)\times\frac{\omega_{ij}(n)}{S_i(n)}, i\in\Lambda_{n-1}, j\in\bar{\Lambda}_n \\ \frac{\omega_{ij}(n)}{S_i(n)}, i\in\bar{\Lambda}_n \end{cases} \tag{24}$$

At each time step, the walker starting from its current position i moves to any of its neighboring nodes j with this transition probability p_{ij}. In this section, we study the delayed random walk with a fixed trap located at node 1. Of course, the results will be the same if the trap is placed at node 2 or 3. Let F^n_i denotes the trapping time in G_n, we have

$$\begin{aligned} F^n_i &= \sum_{j=1}^{N_n} p_{ij}(n)(F^n_j + 1) \\ &= \sum_{j=1}^{N_n} p_{ij}(n)F^n_j + \sum_{j=1}^{N_n} p_{ij}(n) \end{aligned}$$

$$= \sum_{j=2}^{N_n} p_{ij}(n)F_j^n + p_{i1}(n)F_1^n + 1$$
$$= \sum_{j=2}^{N_n} p_{ij}(n)F_j^n + p_{i1}(n) \times 0 + 1$$
$$= \sum_{j=2}^{N_n} p_{ij}(n)F_j^n + 1 \tag{25}$$

which can be written in matrix formulation as:

$$\bar{F} = \bar{F}\bar{P} + \bar{e} \tag{26}$$

where $\bar{P}$ represents the transition probability matrix for delayed random walks in G_n. Transforming *Eq.*(26) leads to

$$\bar{F} = (\bar{I} - \bar{P})^{-1}\bar{e} \tag{27}$$

We use $\langle F\rangle_n$ to denote the *ATT*. From *Eq.*(27), we further obtain

$$\langle F\rangle_n = \frac{1}{N_n - 1}\sum_{i=2}^{N_n} F_i^n = \frac{1}{N_n - 1}\sum_{i=2}^{N_n}\sum_{j=2}^{N_n} \tau_{ij} \tag{28}$$

Although computing the inverse of a matrix means heavy computational burden for large networks, we can use *Eq.*(28) to check the analytical results. In subsequent sections, the analytical solution of average trapping time in the context of delayed random walk will be derived and compared with numerical solution provided by *Eq.*(28).

B. EXPLICIT EXPRESSIONS OF AVERAGE TRAPPING TIME

In the following, we will deduce the *ATT* in the context of delayed random walk. In order to evaluate $\langle F\rangle_n$, we define two quantities for $g \leq n$,

$$F_{g,tot}^n = \sum_{i\in\Lambda_g} F_i^n, \bar{F}_{g,tot}^n = \sum_{i\in\bar{\Lambda}_g} F_i^n \tag{29}$$

Then, the *ATT* can be written as follows:

$$\langle F\rangle_n = \frac{1}{N_n - 1}F_{n,tot}^n \tag{30}$$

Standard random walk studied in Sec.3 corresponds to the special case of delayed random walk [42] when $p = 0$. To be more convenience of deduction, we introduce $F_i^n, F_{g,tot}^n, \bar{F}_{g,tot}^n, \langle F\rangle_n$ to denote $T_i^n, T_{g,tot}^n, \bar{T}_{g,tot}^n, \langle T\rangle_n$ respectively.

Before computing $F_{n,tot}^n$, we first determine the evolution rule of F_i^n. Similar to the standard random walks in G_n, the quantities M and N for delayed random walks satisfy the following relations:

$$\begin{cases} M &= p + (1-p)[\frac{\delta+1}{\delta+2} \times 1 + \frac{1}{\delta+2}(1+N)] \\ N &= \frac{1}{2}(1+N) + \frac{1}{2}(1+M) \end{cases} \tag{31}$$

which give $M = \frac{\delta+4-2p}{\delta+1+p}$. Therefore, we have the following rule

$$F_i^{n+1} = \frac{\delta+4-2p}{\delta+1+p}T_i^n \tag{32}$$

which means from iteration $n-1$ to iteration n, the *FPT* for any node i increases by a factor of $\frac{\delta+4-2p}{\delta+1+p}$. For $p = 0$, *Eq.*(32) becomes *Eq.*(7). Using *Eq.*(32), $F_{n,tot}^n$ can be computed as

$$F_{n,tot}^n = F_{n-1,tot}^n + \bar{F}_{n,tot}^n = \frac{\delta+4-2p}{\delta+1+p}T_{n-1,tot}^{n-1} + \bar{F}_{n,tot}^n \tag{33}$$

For new nodes $u, v \in \bar{\Lambda}_n$, both of them are adjacent and have a common neighbor $x \in \Lambda_{n-1}$, so trapping times of these three nodes obey the equations

$$\begin{cases} F_u^n = \frac{1}{2}(1+F_v^n) + \frac{1}{2}(1+F_x^n) \\ F_v^n = \frac{1}{2}(1+F_u^n) + \frac{1}{2}(1+F_x^n) \end{cases} \tag{34}$$

where $F_u^n = F_v^n$. So we obtain

$$F_u^n = F_v^n = 2 + F_x^n \tag{35}$$

Summing up the trapping times of all nodes in $\bar{\Lambda}_n$, we obtain

$$\begin{aligned} \bar{F}_{n,tot}^n &= 2|\bar{\Lambda}_n| + \sum_{i\in\Lambda_{n-1}} [S_i(n-1) \times F_i^n] \\ &= 2|\bar{\Lambda}_n| + \sum_{i\in\Lambda_{n-1}} [S_i(n-1) \times \frac{\delta+4-2p}{\delta+1+p}T_i^{n-1}] \end{aligned} \tag{36}$$

Similarly, $\bar{T}_{n,tot}^n$ could also be determined as follows:

$$\begin{aligned} \bar{T}_{n,tot}^n &= 2|\bar{\Lambda}_n| + \sum_{i\in\Lambda_{n-1}} [S_i(n-1) \times T_i^n] \\ &= 2|\bar{\Lambda}_n| + \sum_{i\in\Lambda_{n-1}} [S_i(n-1) \times \frac{\delta+4}{\delta+1}T_i^{n-1}] \end{aligned} \tag{37}$$

where $|\bar{\Lambda}_n| = 6(\delta+4)^{n-1}$.

Combining *Eqs.*(36) and (37), we obtain

$$\frac{\bar{T}_{n,tot}^n - 2|\bar{\Lambda}_n|}{\frac{\delta+4}{\delta+1}} = \frac{\bar{F}_{n,tot}^n - 2|\bar{\Lambda}_n|}{\frac{\delta+4-2p}{\delta+1+p}} \tag{38}$$

from which we further derive

$$\bar{F}_{n,tot}^n = \frac{\delta+1}{\delta+4} \times \frac{\delta+4-2p}{\delta+1+p}\bar{T}_{n,tot}^n + \frac{36p(\delta+2)(\delta+4)^{n-2}}{\delta+1+p} \tag{39}$$

On the other hand,

$$\bar{T}_{n,tot}^n = T_{n,tot}^n - T_{n-1,tot}^n = T_{n,tot}^n - \frac{\delta+4}{\delta+1}T_{n-1,tot}^{n-1} \tag{40}$$

Substituting *Eqs.*(39) and(40) into *Eq.*(33) leads to

$$F_{n,tot}^n = \frac{\delta+1}{\delta+4} \times \frac{\delta+4-2p}{\delta+1+p}T_{n,tot}^n + \frac{36p(\delta+2)(\delta+4)^{n-2}}{\delta+1+p} \tag{41}$$

Combining *Eq.*(41) with *Eq.*(30), we arrive at the explicit expression for *ATT* is

$$\langle F\rangle_n = \frac{\delta+1}{\delta+4} \times \frac{\delta+4-2p}{\delta+1+p}\langle T\rangle_n + \frac{18p(\delta+2)(\delta+3)(\delta+4)^{n-2}}{(\delta+1+p)[3(4+\delta)^n+\delta]}$$

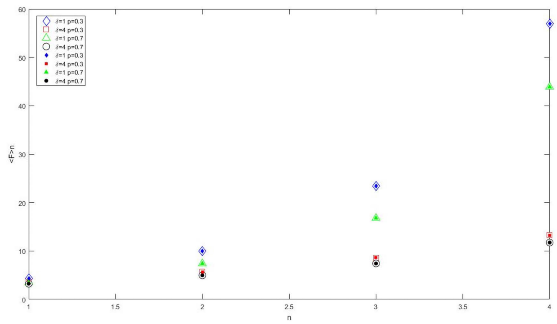

FIGURE 3. **Average trapping time on delayed random walk with one trap. Here, numerical results are denoted by filled symbols while the empty symbols represent the values of *ATT* computed analytically by *Eq.*(42).**

$$
\begin{aligned}
&= \frac{F}{G}[\frac{(8\delta+20)(A^n-1)}{A(3A^n+\delta)}(\frac{A}{B})^{n-1} \\
&\quad +\frac{2(A-1)(B^n-1)}{(3A^n+\delta)\delta}(\frac{A}{B})^{n-2}+\frac{2(A-1)}{3A^n+\delta}(\frac{A}{B})^{n-1}] \\
&\quad +\frac{18p(A-1)(B+1)A^{n-2}}{G(3A^n+\delta)} \qquad (42)
\end{aligned}
$$

where $A = \delta+4, B = \delta+1, F = \delta+4-2p, G = \delta+1+p$.

Eq.(42) shows that when $p = 0$, $\langle F\rangle_n$ reduces to the expression of *ATT* obtained in Sec.3, which initially verified the expression of *ATT* deduced here. We also further comprehensively verified the exact expressions of *ATT* given by *Eq.*(42) by comparing it with numerical solution provided by equation *Eq.*(28). For different combinations of values of δ and p, the analytical and numerical results agree well with each other, which was summarized in Fig.3.

Eq.(42) shows that for trapping process in G_n, the *ATT* of the network is controlled by parameters p and δ. In the following, we will analyze how they impact the leading behavior and pre-factor of *ATT*.

Since the total number of nodes in G_n is *Eq.*(20), we get

$$
\begin{aligned}
A^{n-1} &= \frac{(\delta+3)N_n-3\delta-3}{6(\delta+4)} \\
(\frac{A}{B})^{n-1} &= [\frac{(\delta+3)N_n-3\delta-3}{6(\delta+4)}]^{\frac{ln\frac{A}{B}}{lnA}} \qquad (43)
\end{aligned}
$$

For large networks and $n \to \infty$, plugging *Eq.*(43) into *Eq.*(42) leads to

$$
\langle F\rangle_n \approx \frac{F}{G}(\frac{A}{B})^{n-1} \sim N_n^{\frac{ln\frac{A}{B}}{lnA}} \qquad (44)
$$

We can observe from *Eq.*(44) that the average trapping time grows sub-linearly with the growth of the network regardless of the value of weight parameter. The reason is that the exponent $\frac{ln\frac{A}{B}}{lnA}$ is always smaller than 1 when $\delta > 0$ in the range of $0 \le p \le 1$. Simultaneously, the delayed parameter p keeps the leading scaling unchanged. However, as shown in *Eq.*(44), the parameter p can significantly modify the pre-factor of *ATT*. When p grows from 0 to 1, the pre-factor continuously drops, which implies that the delay parameter p can improve the trapping efficiency to some extent.

V. ANALYTICAL SOLUTION OF ATT FOR STANDARD RANDOM WALKS WITH THREE TRAPS

In this section, we will explore standard random walk on this weighted network with three traps being fixed at node 1,2, 3 and calculate the average trapping time.

A. NUMERICAL FORMULATIONS OF TRAPPING PROBLEMS

Let D_i^n denotes the trapping time, we have

$$
\begin{aligned}
D_i^n &= \sum_{j=1}^{N_n} p_{ij}(n)(D_j^n+1) \\
&= \sum_{j=1}^{N_n} p_{ij}(n)D_j^n + \sum_{j=1}^{N_n} p_{ij}(n) \\
&= \sum_{j=4}^{N_n} p_{ij}(n)D_j^n + \sum_{j=1}^{3} p_{ij}(n)D_j^n + 1 \\
&= \sum_{j=4}^{N_n} p_{ij}(n)D_j^n + 1 \qquad (45)
\end{aligned}
$$

which can be written in matrix form as:

$$
\bar{D} = \bar{D}\bar{P} + \bar{e} \qquad (46)
$$

Solving *Eq.*(46) leads to

$$
\bar{D} = (\bar{I} - \bar{P})^{-1}\bar{e} \qquad (47)
$$

We use $\langle D\rangle_n$ to denote the *ATT*. From *Eq.*(47), we further obtain [30]

$$
\langle D\rangle_n = \frac{1}{N_n-3}\sum_{i=4}^{N_n} D_i^n = \frac{1}{N_n-3}\sum_{i=4}^{N_n}\sum_{j=4}^{N_n}\tau_{ij} \qquad (48)
$$

*Eq*s.(47) and (48) show that the problem of calculating D_i^n and $\langle D\rangle_n$ are reduced to find the sum of elements of matrix $(\bar{I}-\bar{P})^{-1}$.

B. EXPLICIT EXPRESSIONS OF AVERAGE TRAPPING TIME

Since the procedure for deducing *ATT* with three traps being fixed at the network is similar to that in Sec.3, so we simplify the description of deduction process.

In order to evaluate $\langle D\rangle_n$, we define two quantities for $g \le n$,

$$
D_{g,tot}^n = \sum_{i\in\Lambda_g} D_i^n, \bar{D}_{g,tot}^n = \sum_{i\in\bar{\Lambda}_g} D_i^n \qquad (49)
$$

Then, the $\langle D\rangle_n$ can be written as follows:

$$
\langle D\rangle_n = \frac{1}{N_n-3}D_{n,tot}^n \qquad (50)
$$

From *Eq.*(15), we know

$$
\bar{D}_{n+1,tot}^{n+1} = \frac{(\delta+4)^2}{\delta+1}\bar{D}_{n,tot}^n - \frac{12}{\delta+1}(\delta+4)^n \qquad (51)
$$

In the case of three traps, $\bar{D}^1_{1,tot} = 12 + 2(D^1_2 + D^1_3) = 12$, *Eq*.(51) is solved inductively as

$$\bar{D}^n_{n,tot} = 8[\frac{(\delta+4)^2}{\delta+1}]^{n-1} + 4(\delta+4)^{n-1} \tag{52}$$

On the another way,

$$D^n_{n,tot} = D^n_{n-1,tot} + \bar{D}^n_{n,tot} = \frac{\delta+4}{\delta+1} D^{n-1}_{n-1,tot} + \bar{D}^n_{n,tot} \tag{53}$$

Substituting *Eq*.(52) into *Eq*.(53), we have

$$D^n_{n,tot} = \frac{4+\delta}{1+\delta} D^{n-1}_{n-1,tot} + 8[\frac{(\delta+4)^2}{\delta+1}]^{n-1} + 4(\delta+4)^{n-1} \tag{54}$$

Considering the initial condition $T^0_{0,tot} = 0$, *Eq*.(54) is resolved by induction to yield

$$D^n_{n,tot} = (\frac{A}{B})^n [\frac{(8\delta+8)(A^n-1)}{A(\delta+3)} + \frac{4B(B^n-1)}{A\delta}] \tag{55}$$

where $A = \delta+4$, $B = \delta+1$. Plugging *Eq*.(55) into *Eq*.(50), we arrive at the accurate formula for the *ATT* at the traps located at nodes 1,2 and 3 on this weighted heterogeneous network.

$$\langle D \rangle_n = (\frac{A}{B})^{n-1} [\frac{4}{3} + \frac{(A-1)(B^n-1)}{(6A^n-6)\delta}] \tag{56}$$

we also verify the exact expressions given by *Eq*.(56) with numerical ones given by *Eq*.(48), which agree well with each other and summarized in Fig.4. At the same time, we could find $\langle D \rangle_n$ is quantitatively less than the analytical expression of ATT given in *Eq*.(21) for any n, which indicates that introducing more traps on this weighted network in the context of standard random walk may improve trapping efficiency to some extent.

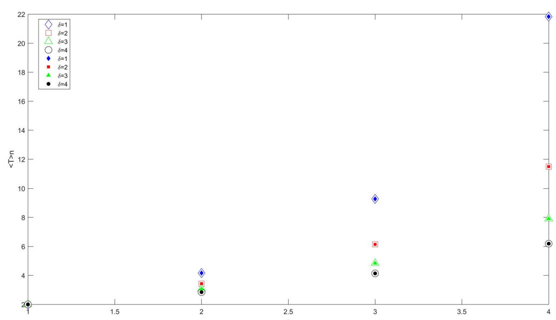

FIGURE 4. Average trapping time on standard random walk with three traps. The filled symbols and the empty symbols represent numerical results and the analytical values of *ATT* respectively.

When $n \to \infty$, plugging *Eq*.(43) into *Eq*.(56) results in

$$\langle D \rangle_n \approx \frac{4}{3} (\frac{A}{B})^{n-1} \sim N_n^{\frac{ln \frac{A}{B}}{lnA}} \tag{57}$$

From this equation we find that the *ATT* grows as a power-law function of network size and exponent is same as that in the case of introducing one trap. Therefore, average trapping time in the case of introducing three traps onto the network in the context of standard random walk also grows sub-linearly with the growth of the network regardless of the value of weight parameter δ.

VI. ANALYTICAL SOLUTION OF ATT FOR DELAYED RANDOM WALKS WITH THREE TRAPS

In this section, we will introduce three traps onto the network in the context of delayed random walk and deduce average trapping time analytically.

A. NUMERICAL FORMULATIONS OF TRAPPING PROBLEMS

As for the transition probability p_{ij}, it fully complies with *Eq*.(24). In addition, the numerical formulations of trapping problems could be in accordance with the Sec.5 except the transition probability.

Let L^n_i denotes the trapping time, $\langle L \rangle_n$ denote the *ATT*, we could obtain from Ref [30] that

$$L^n_i = \sum_{j=4}^{N_n} p_{ij}(n) L^n_j + 1$$

$$\langle L \rangle_n = \frac{1}{N_n - 3} \sum_{i=4}^{N_n} L^n_i = \frac{1}{N_n - 3} \sum_{i=4}^{N_n} \sum_{j=4}^{N_n} \tau_{ij} \tag{58}$$

Although computing the inverse of a matrix means heavy computational burden for large networks, we can use *Eq*.(58) to check the analytical results.

B. EXPLICIT EXPRESSIONS OF AVERAGE TRAPPING TIME

In the following, we will deduce the *ATT* in the context of delayed random walk with three traps. In order to evaluate $\langle L \rangle_n$, we define two quantities for $g \le n$,

$$L^n_{g,tot} = \sum_{i \in \Lambda_g} L^n_i, \bar{L}^n_{g,tot} = \sum_{i \in \bar{\Lambda}_g} L^n_i \tag{59}$$

Similarly in Sec.4, we introduce $L^n_i, L^n_{g,tot}, \bar{L}^n_{g,tot}, \langle L \rangle_n$ to denote the values of $D^n_i, D^n_{g,tot}, \bar{D}^n_{g,tot}, \langle D \rangle_n$ respectively.

Similarly to *Eq*.(32), we obtain

$$L^{n+1}_i = \frac{\delta+4-2p}{\delta+1+p} D^n_i \tag{60}$$

Using *Eq*.(60), $L^n_{n,tot}$ can be computed as

$$L^n_{n,tot} = L^n_{n-1,tot} + \bar{L}^n_{n,tot} = \frac{\delta+4-2p}{\delta+1+p} D^{n-1}_{n-1,tot} + \bar{L}^n_{n,tot} \tag{61}$$

Summing up the trapping times of all nodes in $\bar{\Lambda}_n$, we obtain

$$\begin{aligned} \bar{L}^n_{n,tot} &= 2|\bar{\Lambda}_n| + \sum_{i \in \Lambda_{n-1}} [S_i(n-1) \times L^n_i] \\ &= 2|\bar{\Lambda}_n| + \sum_{i \in \Lambda_{n-1}} [S_i(n-1) \times \frac{\delta+4-2p}{\delta+1+p} D^{n-1}_i] \end{aligned} \tag{62}$$

Similarly, $\bar{D}^n_{n,tot}$ could also be determined as follows:

$$\begin{aligned}\bar{D}^n_{n,tot} &= 2|\bar{\Lambda}_n| + \sum_{i\in\Lambda_{n-1}} [S_i(n-1)\times D^n_i] \\ &= 2|\bar{\Lambda}_n| + \sum_{i\in\Lambda_{n-1}} [S_i(n-1)\times \frac{\delta+4}{\delta+1} D^{n-1}_i] \end{aligned} \quad (63)$$

where $|\bar{\Lambda}_n| = 6(\delta+4)^{n-1}$.

Combining *Eqs*.(62) and (63), we obtain

$$\frac{\bar{D}^n_{n,tot} - 2|\bar{\Lambda}_n|}{\frac{\delta+4}{\delta+1}} = \frac{\bar{L}^n_{n,tot} - 2|\bar{\Lambda}_n|}{\frac{\delta+4-2p}{\delta+1+p}} \quad (64)$$

from which we further derive

$$\bar{L}^n_{n,tot} = \frac{\delta+1}{\delta+4}\times\frac{\delta+4-2p}{\delta+1+p}\bar{D}^n_{n,tot} + \frac{36p(\delta+2)(\delta+4)^{n-2}}{\delta+1+p} \quad (65)$$

On the other hand,

$$\bar{D}^n_{n,tot} = D^n_{n,tot} - D^n_{n-1,tot} = D^{(n)}_{n,tot} - \frac{\delta+4}{\delta+1} D^{n-1}_{n-1,tot} \quad (66)$$

Substituting *Eqs*.(65) and(66) into *Eq*.(61) leads to

$$L^n_{n,tot} = \frac{\delta+1}{\delta+4}\times\frac{\delta+4-2p}{\delta+1+p} D^n_{n,tot} + \frac{36p(\delta+2)(\delta+4)^{n-2}}{\delta+1+p} \quad (67)$$

Dividing both sides of *Eq*.(67) by $N_n - 3$, we arrive at a formula for the ATT, which leads to

$$\begin{aligned}\langle L\rangle_n &= \frac{1}{N_n-3} L^n_{n,tot} \\ &= \frac{\delta+1}{\delta+4}\times\frac{\delta+4-2p}{\delta+1+p}\langle D\rangle_n + \frac{18p(\delta+2)(\delta+3)(\delta+4)^{n-2}}{(\delta+1+p)(3A^n-3)} \\ &= \frac{F}{G}[\frac{4}{3}(\frac{A}{B})^{n-2} + \frac{(A-1)(B^n-1)}{(6A^n-6)\delta}(\frac{A}{B})^{n-2}] \\ &\quad + \frac{6p(A-1)(B+1)A^{n-2}}{G(A^n-1)} \end{aligned} \quad (68)$$

where $A = \delta+4$, $B = \delta+1$, $F = \delta+4-2p$, $G = \delta+1+p$.

We still compared analytical results obtained from *Eq*.(68) with the numerical solution of ATT and summarized the comparison in Fig.5. It can be seen from Fig.5 that both analytical results of ATT and numerical ones still coincide with each other. In addition, by comparing *Eq*.(68) with *Eq*.(42), we also find the value of $\langle L\rangle_n$ decreases significantly for any n, which implies that placing more traps on the network in the context of delayed random walk also can elevate trapping efficiency to some extent.

From *Eq*.(20), it can be deduced that

$$\begin{aligned} A^{n-2} &= \frac{(\delta+3)N_n - 3\delta - 3}{6(\delta+4)^2} \\ (\frac{A}{B})^{n-2} &= [\frac{(\delta+3)N_n - 3\delta - 3}{6(\delta+4)^2}]^{\frac{\ln\frac{A}{B}}{\ln A}} \end{aligned} \quad (69)$$

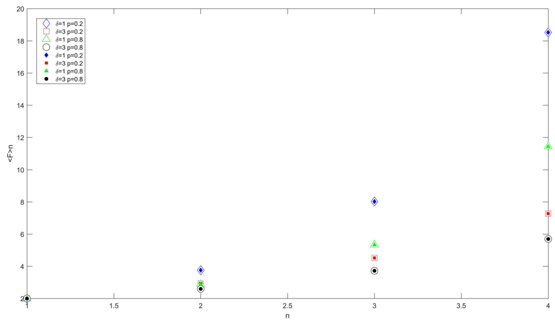

FIGURE 5. Average trapping time with four different combinations of values of p and δ on delayed random walk with three traps. Here, we use the empty symbols to represent the values of ATT computed analytically and the numerical results of ATT are denoted by filled symbols.

Plugging *Eq*.(69) into *Eq*.(68) and taking $n \to \infty$ into account brings about

$$\langle L\rangle_n \approx \frac{F}{G}(\frac{A}{B})^{n-2} \sim N_n^{\frac{\ln\frac{A}{B}}{\ln A}} \quad (70)$$

From *Eq*.(70), we found that the average trapping time grows sub-linearly with the growth of the network regardless of the value of parameter δ. In addition, the delayed parameter p keeps the leading scaling unchanged, but it can significantly modify the pre-factor of ATT.

VII. CONCLUSION

In this paper, we studied both standard random walk and delayed random walk on a family of weighted heterogeneous networks with one or three deep traps and analytically computed the average trapping time(ATT) as indicator of trapping efficiency in four distinct schemes respectively. In all the four cases, the analytical expressions of ATT coincide with corresponding numerical solution of ATT. Furthermore, the analytical solutions of ATT obtained collectively lead to the following conclusions. Firstly, ATT grows sub-linearly with network size no matter what type of random walk employed and how many deep traps fixed on the network. Secondly, parameter p steering delayed random walk only modifies the pre-factor of ATT and leaves the leading scaling of ATT unchanged. Thirdly, introducing more traps can lower ATT and improve trapping efficiency. To some extent, conclusions made here and in previous works such as [30], [40], [41], [46]–[48] may collectively serve as pieces of puzzles for illuminating trapping issue and related dynamical processes on more general networks or systems.

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