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Explosive transitions to synchronization in networks of phase oscillators

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The emergence of dynamical abrupt transitions in the macroscopic state of a system is currently a subject of the utmost interest. The occurrence of a first-order phase transition to synchronization of an ensemble of networked phase oscillators was reported, so far, for very particular network architectures. Here, we show how a sharp, discontinuous transition can occur, instead, as a generic feature of networks of phase oscillators. Precisely, we set conditions for the transition from unsynchronized to synchronized states to be first-order, and demonstrate how these conditions can be attained in a very wide spectrum of situations. We then show how the occurrence of such transitions is always accompanied by the spontaneous setting of frequency-degree correlation features. Third, we show that the conditions for abrupt transitions can be even softened in several cases. Finally, we discuss, as a possible application, the use of this phenomenon to express magnetic-like states of synchronization.

M any complex systems operate transitions between different regimes or phases under the action of a control parameter. These transitions can be monitored using a global order parameter, a physical quantity (e.g. scalar, vector, …) accounting for the symmetry of the phases. Phase transitions can be of first or second order according to whether the order parameter varies continuously or discontinuously at a critical value of the control parameter. In complex networks theory[1], phase transitions have been observed in the way the graph collectively organizes its architecture (e.g. percolation[2,3]) and dynamical state (e.g. synchronization[4–6]).

Recently, Achlioptas *et al.*[7] proposed a slight modification of the classical percolation models, leading to a first-order type transition, named "explosive percolation". In the model, the authors consider a *preferential* growth process for a network with fixed number of vertices. Edges are added one by one, by picking each time two randomly selected possible edges, and deciding to keep the one having a lower value of the product of the size of the two components that edge is joining, this way delaying as much as possible the formation of a giant connected component. At the percolation threshold, a macroscopic part of the system (a giant cluster of size O(N)) becomes connected. While the transition is found to be abrupt in Erdös-Renyi (ER) networks, heterogeneous structures as scale-free (SF) networks not always display explosive percolation, which crucially depends, instead, on the degree distribution[8,9]. The continuous or discontinuous nature of explosive percolation is still a matter of debate[10].

The existence of abrupt transitions has also been investigated for the synchronization of networked phase oscillators[5,6]. In this context, a first-order transition has initially been described in the Kuramoto model[11] with a particular realization of a uniform frequency distribution (evenly spaced frequencies) and an all-to-all network topology[12]. However, lately it has been shown that a positive correlation between the node degree and the corresponding oscillator's natural frequency may lead to a first-order transition also for SF networks[13]. These latter results have been further generalized to networked chaotic oscillators, and experimentally proved[14].

In this report, we show that a sharp, discontinuous phase transition is not restricted to the above rather limited and apparently opposite cases, but it constitutes, instead, a *generic feature* of the synchronization of networked phase oscillators. Precisely, we initially give a condition for the transition from unsynchronized to synchronized states to be first-order, and demonstrate how such a condition is easy to attain in many circumstances, and for a wide class of oscillators' initial frequency distributions. We then show how such transitions are always accompanied by the spontaneous emergence of frequency-degree correlation features, and discuss how the considered condition can even be softened in several cases. Finally, we illustrate, as a possible application, the option of expressing *magnetic-like* states of synchronization with the use of such transitions.

Results

Our results are obtained with a network of N Kuramoto[11] oscillators, which is described by:

$$\frac{d\phi_i}{dt} = \omega_i + d \sum_{i=1}^{N} a_{ij} \sin\left(\phi_j - \phi_i\right), \quad (1)$$

where ϕ_i is the phase of the ith oscillator ($i = 1, \dots, N$), ω_i is its associated natural frequency [drawn from a generic frequency distribution $p(\omega)$], d is the coupling strength, and $\{a_{ij}\}$ are the elements of the adjacency matrix that uniquely defines the nodes' interactions.

The classical order parameter for the system given by equation (1) is $r(t) = \frac{1}{N}\left|\sum_{l=1}^{N} e^{i\phi_l(t)}\right|$, and the level of phase synchronization can be monitored by looking at the value of $S = \langle r(t)\rangle_T$, where $\langle\dots\rangle_T$ denotes a time average with $T \gg 1$. Furthermore, for each oscillator i, we denote by $\mathcal{N}(i)$ the set of oscillators linked to it.

As the coupling strength d increases, system (1) undergoes a phase transition from the unsynchronized $(S \sim 1/\sqrt{N})$ to a synchronous $(S \sim 1)$ state, where all oscillators ultimately acquire the same frequency. For this phase transition to display a first-order feature, we have to avoid that any oscillator behave as the core of a clustering process, where its neighbors begin to aggregate to the synchronous state smoothly and progressively, as in the classical routes described in Ref.15.

Along this paper, we realize such a condition by explicitly imposing certain constraints in the frequency differences between each node i and the whole set $\mathcal{N}(i)$ of oscillators belonging to its neighborhood. A first general constraint can be set as follows:

(C1) for each oscillator i, all nodes j belonging to $\mathcal{N}(i)$ satisfy $|\omega_i - \omega_j| > \gamma_c$.

Notice that condition (C1) is tantamount to imposing a minimal value for the frequency difference between linked nodes of the network.

We now fix $N = 500$, and illustrate the synchronization route for several frequency distributions $p(\omega)$, when condition (C1) is met in a frequency gap-conditioned (FGC) random network (see Methods section for the details on the network construction). In particular, we use the adjacency matrix obtained after the FGC network is constructed to simulate system (1), and monitor the state of the network by gradually increasing the value of d in steps $\delta d = 10^{-4}$, from $d = 0$. At each step, a long transient is discarded before the data are acquired for further processing. Moreover, insofar as we are looking for a first-order phase transition (and thus for an expected associated hysteresis), simulations are also performed in the reverse way, i.e. starting from a given d_{max} (where the network is phase synchronized), and gradually decreasing the coupling by δd at each step. In what follows, the two sets of numerical trials are termed *forward* and *backward*, respectively.

Figure 1 reports the results obtained by setting $p(\omega)$ as a uniform frequency distribution in the interval $[0, 1]$. Panels a and b of Fig. 1 show S as a function of d. In particular, panel a (b) illustrates the case of a fixed mean degree $\langle k\rangle = 40$ (of a fixed frequency gap $\gamma = 0.4$), and reports the results of the forward and backward simulations at different values of γ ($\langle k\rangle$). A first important result is the evident first-order character acquired, in all cases, by the transition for sufficiently high values of γ.

A second relevant result is the spontaneous emergence of degree-frequency correlation features associated to the passage from a second- to a first-order phase transition. While such a correlation was imposed *ad hoc* in Refs.13,14, here condition (C1) creates for each oscillator i a frequency barrier around ω_i, where links are forbidden.
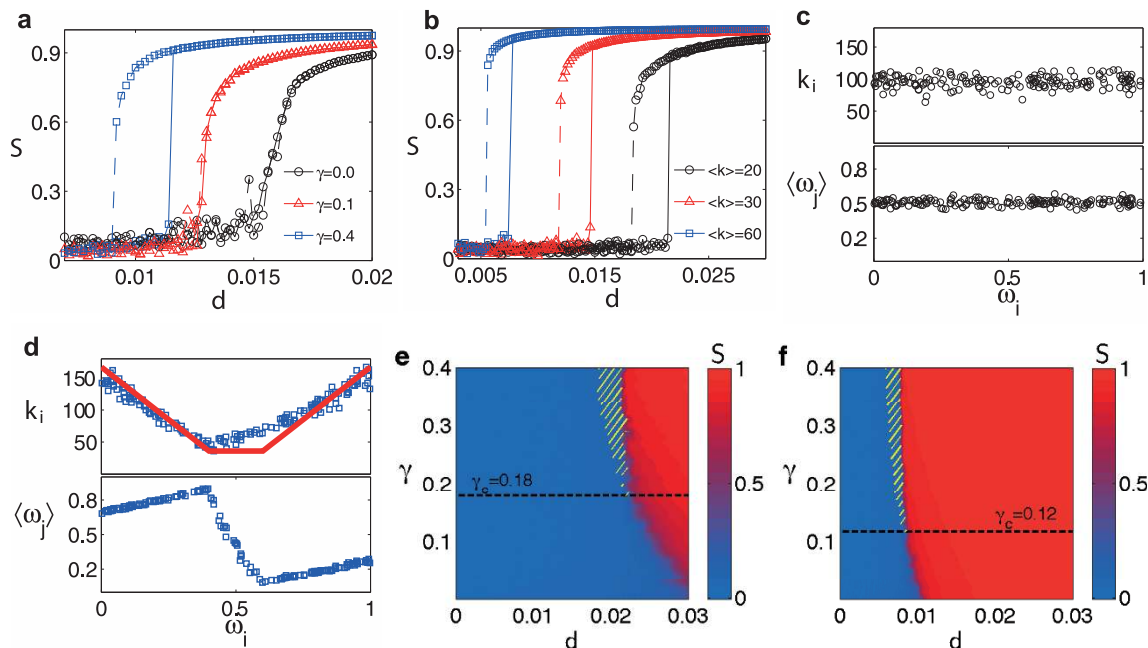
Figure 1 | Explosive transition to synchronization. Phase synchronization level S (see text for definition) *vs.* the coupling strength d, for different values of the gap γ at $\langle k\rangle = 40$ (panel **a**), and for different values of the average degree $\langle k\rangle$ at $\gamma = 0.4$ (panel **b**). In both panels, the continuous (dashed) lines refer to the forward (backward) simulations. See the Methods section for the construction procedure of the networks. In panels **c** and **d**, the degree k_i that each node achieves after the network construction is completed (upper plots) and the average of the natural frequencies $\langle\omega_j\rangle$ for $j \in \mathcal{N}(i)$ (bottom plots) are reported *vs.* the node's natural frequency ω_i, for $\langle k\rangle = 100$ and frequency gaps $\gamma = 0.0$ (panel **c**) and $\gamma = 0.4$ (panel **d**). The red solid line in panel **d** is a sketch of the theoretical prediction $f(\omega)$ (see text). Panels **e** and **f** show S (color coded according to the color bar) in the parameter space (d, γ) for (**e**) $\langle k\rangle = 20$ and (**f**) $\langle k\rangle = 60$. The horizontal dashed lines mark the separation between the region of the parameter space where a second-order transition occurs (below the line) and that in which the transition is instead of the first order type (above the line). The yellow striped area delimits the hysteresis region.

The final degree k_i is proportional to the total probability for that oscillator to receive connections from other oscillators in the network, and therefore to $1 - \int_{\omega_i - \gamma}^{\omega_i + \gamma} p(\omega')d\omega'$. This is shown in panels c and d of Fig. 1, where the degree k_i that each node achieves after completing the FGC network construction is reported as a function of its natural frequency ω_i, for $\langle k \rangle = 100$. Precisely, the upper plot of panel c refers to the case $\gamma = 0$ in which no degree-frequency correlation is present. In the upper plot of panel d, instead, we report the case $\gamma = 0.4$ (a value for which a first-order phase transition occurs) and the (conveniently normalized) function $f(\omega) = 1 - \int_{\omega - \gamma}^{\omega + \gamma} p(\omega')d\omega'$, with $p(\omega) = 1$ for $\omega \in [0, 1]$, and $p(\omega) = 0$ elsewhere, which gives evidence of the emergence of a very pronounced V-shape relationship between the frequency and the degree of the network's nodes. By further inspecting the average frequency of each oscillator's neighborhood, comparison between the lower plots of panels c and d manifests that condition (C1) leads to the emergence of a bipartite-like network where low frequency oscillators are mainly coupled to high frequency oscillators.

Finally, panels e and f of Fig. 1 report S in the $d - \gamma$ space, for $\langle k \rangle = 20$ and $\langle k \rangle = 60$, and show that the rise of a first order phase transition is, indeed, a generic feature in the parameter space. The horizontal dashed lines in panels e and f mark the values of γ_c, separating the two regions where a second-order transition (below the line) and a first-order transition (above the line) occurs. Remarkably, while increasing $\langle k \rangle$ of the network facilitates the occurrence of the explosive transition, as both the values of γ and d for which the first-order phase transition takes place decrease, it also shrinks the width of the hysteresis (the yellow striped area in panels e and f). This is consistent with the results of Ref.12 for an all-to-all connected case, where in the limit $N \to \infty$ a first-order phase transition has been predicted in the absence of hysteresis. It is worth noticing that similar scenarios of synchronization were observed in the past for chains of periodic[16] and chaotic phase coherent[17] oscillators. In this latter cases, synchronization was shown to appear or vanish in two ways: a soft transition (without cluster formation) for chains with very small frequency mismatches between the elements, and a hard transition for rather long chains with relatively large frequency mismatches.

We verified that fulfillment of condition (C1) leads to a first-order transition for a very wide class of distributions of the oscillators' natural frequencies. The panels a and b of Fig. 2 show, for instance, the case of the asymmetrical Rayleigh distribution, conveniently rescaled to the interval [0, 1], given by $p(\omega) = \frac{\omega}{\sigma^2} e^{-\frac{\omega^2}{2\sigma^2}}$ (see the inset in panel a of Fig. 2), with $\sigma = 0.25$. The results highlight the presence of an abrupt transition (Fig. 2, panel a), together with the emergence of a clear frequency-degree correlation, well described, again, by the function $f(\omega)$ (Fig. 2, panel b).

We now move to discussing several ways for even softening condition (C1), thus making it even more generally applicable, while still keeping the abrupt character of the transition. For the first extension, we consider again the case of a uniform frequency distribution in the interval [0, 1]. In this case, condition (C1) can be relaxed as follows:

(C2) for each oscillator i, all nodes j belonging to $\mathcal{N}(i)$ satisfy $|\omega_i - \langle \omega_j \rangle| > \gamma_c$, where $\langle \dots \rangle$ indicates the average value over the ensemble $\mathcal{N}(i)$.

The new condition is tantamount to softening condition (C1) to the *local mean field* of frequency differences in the neighborhood of each network node. The panels c and d of Fig. 2 report the results of networks obtained with a modified construction procedure, in which pairs of randomly selected nodes are now linked if the value of $|\omega_i - \langle \omega_j \rangle|$ (averaged over the set of nodes j already linked to node i, and the one candidate to be further linked), exceeds a gap γ. Again, an explosive transition occurs (Fig. 2, panel c) in correspondence with the emergence of frequency-degree correlations (Fig. 2, panel d).

Furthermore, it is worth noticing that a strict application of condition (C1) for non uniform frequency distributions implies that oscillators at different frequencies would in general have a different number of available neighbors in the network. For this reason, a natural extension of condition (C1) is to consider a frequency-dependent gap $\gamma(\omega)$ defined by $\int_{\omega - \gamma}^{\omega + \gamma} p(\omega')d\omega' = Z$. Panels e and f of Fig. 2 report the case of a Gaussian distribution limited to the frequency range [0, 1], centered at $\omega = 0.5$, and given by $p(\omega) = \frac{1}{\sigma\sqrt{2N}} e^{-\frac{(\omega - 0.5)^2}{2\sigma^2}}$, with $\sigma = 0.13$. The gap condition for the construction of the network is now to fix the value of Z, and accept the pairing of nodes when $|\omega_i - \omega_j| > \frac{1}{2}[\gamma(\omega_i) + \gamma(\omega_j)]$. The generic case is again that an explosive transition is obtained, with pro-
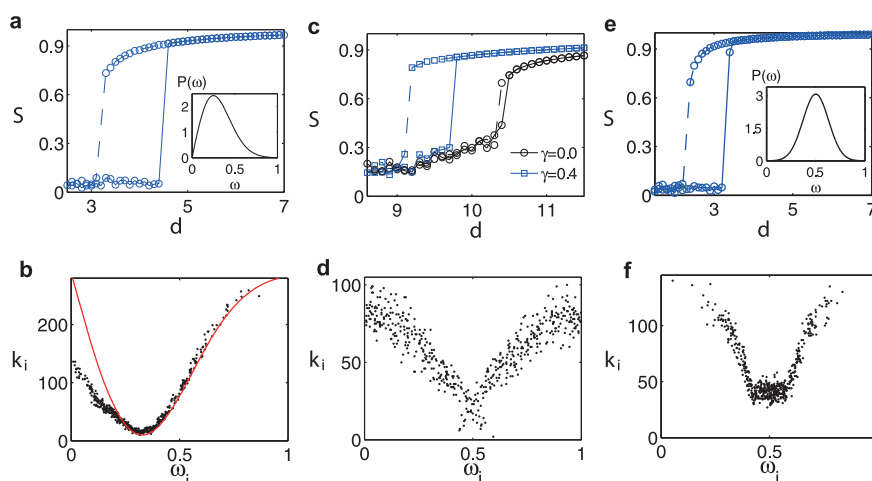
Figure 2 | Extension to different frequency distributions and network construction rules. (Top row) *S vs. d* resulting from the forward (continuous lines) and backward (dashed lines) simulations of system (1) for different frequency distributions or network construction rules, and (bottom row) the corresponding distribution of the final node degree k_i *vs.* the corresponding oscillator's natural frequency. (**a**)-(**b**) Rayleigh distribution for $\gamma = 0.3$. In panel **b**, the red solid line depicts the theoretical prediction $f(\omega)$; (**c**)-(**d**) uniform frequency distribution, but network constructed accordingly to the local mean field condition (see text) for $\gamma = 0$, and $\gamma = 0.4$. In panel **d** $\gamma = 0.4$; (**e**)-(**f**) Gaussian distribution with $Z = 0.7$. The insets in panels **a** and **e** report the corresponding distribution $p(\omega)$. See the text and the Methods section for the details on the specific construction procedure used in each case. In all cases, $\langle k \rangle = 60$.
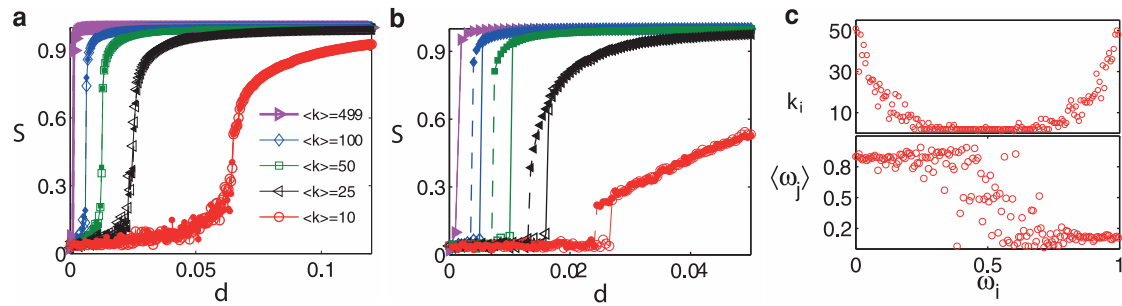
Figure 3 | Explosive synchronization in networks with evenly spaced natural frequencies. S vs. d resulting from the forward (continuous lines) and backward (dashed lines) simulations of system (1), for different values of the average degree $\langle k \rangle$ (see legend in panel **a**) after performing a random pruning (**a**) and a preferential pruning (**b**) of links in an all-to-all connected network (see the text and the Methods section for the networks construction). Natural frequencies are evenly spaced in the interval $[0, 1]$. In panel **c**, the degree k_i that each node achieves after the network pruning is completed (upper plots) and the average of the local natural frequency $\langle \omega_j \rangle$ for $j \in \mathcal{N}(i)$ (bottom plots) are reported vs. the node's natural frequency ω_i for the preferential pruning process with $\langle k \rangle = 10$.

nounced frequency-degree correlation features, as long as $p(\omega)$ is symmetrical.

An alternative way to explore the frontier between first- and second-order transitions is to start from the case of Ref.12 (an all to all network configuration with evenly spaced frequencies of the oscillators), and to study the robustness of explosiveness under removal of links by means of two pruning processes: a random pruning, where links are randomly chosen and removed, and a preferential pruning that removes links with a probability proportional to the inverse of the frequency difference of the connected nodes, so that close frequency oscillators are more probable to be disconnected (see the Methods section for details).

The results are shown in Fig. 3. For the random pruning process (panel a of Fig. 3), one can easily see that, as the connectivity of the network becomes low enough (i.e. as more and more links are pruned in a way that $\langle k \rangle$ progressively decreases) the transition gradually passes from first- to second-order. On the contrary, for the preferential pruning process (panel b of Fig. 3), the system maintains the explosive behavior due to the higher probability to uphold connections associated to large frequency mismatches, although the size of the discontinuity decreases. Once again, inspection of the nodes' degree and of the mean frequencies of the nodes' neighborhoods (panel c of Fig. 3) reveals the spontaneous emergence of correlation between topology and dynamics for the preferential pruning process.

Discussion

In conclusion, we demonstrated the validity and generality of several conditions for the occurrence of abrupt phase transitions in networks of phase oscillators. Our study generalizes all previous results, and extends the possibility of encountering first-order phase transitions to a large variety of network topologies, as well as to a large variety of frequency distributions of the oscillators. This suggests practical methods for engineering networks able to display critical phenomena, and the emergence of dynamical abrupt transitions in their macroscopic states. Furthermore, the evidence for the emergence of frequency-degree correlations in connection with these abrupt transition, may shed light on the mechanisms underlying the relationship between topology and dynamics in many real-world systems.

One interesting application is expressing *magnetic-like* states of synchronization in such an ensemble of networked oscillators, provided that the coupling strength is set inside the hysteresis region of the first-order phase transition. For this purpose, one can again consider the case of an initial uniform distribution of the oscillators' frequencies, and modify system (1) as follows: $\frac{d\phi_i}{dt} = \omega_i + D_p \sin\left(\phi_p - \phi_i\right) + d \sum_{i=1}^{N} a_{ij} \sin\left(\phi_j - \phi_i\right)$, where D_p is the strength of a unidirectional connection to an external pacemaker (equal for all oscillators in the network), and ϕ_p is the phase of the pacemaker
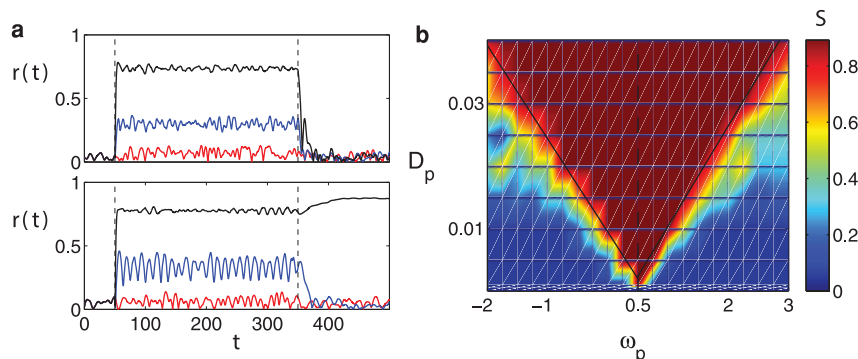
Figure 4 | Magnetic-like states of synchronization. (**a**) Time evolution of the parameter $r(t)$ under a pacemaker forcing for a value $d = 0.004$ outside the hysteresis region (upper plot), and a value $d = 0.009$ within the hysteresis region (bottom plot). $\omega_p = 1.0$ and $D_p = 0.0005$ (bottom red line), $D_p = 0.005$ (middle blue line), and $D_p = 0.02$ (top black line). The pacemaker is active from $t = 50$ to $t = 350$, as marked by the vertical dashed lines. (**b**) Colormap of $S = \langle r(t) \rangle_{t>350}$ (coded as indicated in the color bar), showing the region of the parameter space D_p-ω_p where the *magnetic-like* state of synchronization is maintained after removal of the pacemaker. The initial frequencies of the oscillators are taken from a uniform distribution in the interval $[0, 1]$. $\gamma = 0.49$ and $\langle k \rangle = 40$.

obeying $\frac{d\phi_p}{dt} = \omega_p$. Initially, the system is let to freely evolve into the unsynchronized regime, with $D_p = 0$. The pacemaker is then switched on, and D_p is selected so that all oscillators are entrained to the pacemaker phase. At a subsequent time, the pacemaker is switched off again, and the state of the system is monitored. The results are illustrated in Fig. 4. Precisely, panel a shows that setting d outside (inside) the hysteresis region produces a final state that relaxes to the original unsynchronized behavior (that stays permanently in a synchronized configuration for sufficiently large D_p values). Panel b depicts the regions of the parameter space $D_p - \omega_p$ for which these *magnetic-like* states can ultimately be produced.

Methods

Algorithms for the networks' construction. *Frequency Gap-conditioned (FGC) random networks.* The considered networks result from the following procedure: *i)* we assign natural frequencies $\{\omega_i\}$, drawn from a distribution $p(\omega)$, to the N oscillators; *ii)* we randomly pick a pair (i, j) of oscillators, and form a link between them only if the value of $|\omega_i - \omega_j|$ exceeds a given gap γ; *iii)* we repeat point *ii)* until the desired number of links L in the graph is formed. After a final check on the connectedness of the resulting network, the procedure yields Erdös-Renyi-like topologies with an average degree $\langle k \rangle \equiv \frac{2L}{N}$.

Random and preferential pruning. The considered networks result from the following procedure: *i)* we start from an all-to-all network configuration, and assign to the N oscillators natural frequencies ω_i evenly spaced spanning the interval $[0, 1]$, i.e. $\omega_i = \frac{1}{N-1}(i-1)$; *ii)* from the total number of links $\frac{N}{2}(N-1)$, $\frac{N}{2}(N-1-\langle k \rangle)$ links are pruned to produce a network with desired mean degree $\langle k \rangle$. The links removal can either be performed randomly, or selectively by assigning to each link a pruning probability $p_{ij} = \frac{1}{|\omega_i - \omega_j|}$.

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Author Contributions

IL, AN, ISN and SB devised the model and designed the study. IL and ISN carried out the numerical simulations. IL, ISN, JAA, JMB, DP, and MZ analyzed the data and prepared the figures. AN, MZ, DP, and SB wrote the main text of the manuscript.

Additional information

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