Differential and Integral Equations, Volume 4, Number 3, May 1991, pp. 519–525.

EXPONENTIAL ASYMPTOTIC STABILITY FOR SCALAR LINEAR VOLTERRA EQUATIONS

SATORU MURAKAMI
Department of Applied Mathematics, Okayama University of Science
Ridai-cho 1–1, Okayama 700, Japan

(Submitted by: C. Corduneanu)

Abstract. Exponential asymptotic stability of the zero solution of the scalar linear Volterra equation

$$\dot{x}(t) = Ax(t) + \int_0^t B(t-s)x(s)\,ds$$

is studied. Roughly speaking, it is proved that the exponential asymptotic stability can be characterized by a growth condition on $B(t)$. The result answers the problem posed by Corduneanu and Lakshmikantham.

In this article, we shall be concerned with a linear Volterra equation with an integrable kernel B,

$$\dot{x}(t) = Ax(t) + \int_0^t B(t-s)x(s)\,ds \quad t \geq 0, \tag{E}$$

and study the exponential asymptotic stability of the zero solution of (E) in conjunction with the exponential behavior of $|B(t)|$ as $t \to \infty$. The subject deeply relates to the paper [1] due to Corduneanu and Lakshmikantham. In fact, they have posed the following problem [1, pp. 845–848]: If the zero solution of (E) is uniformly asymptotically stable, then is it of exponential type? To analyze the problem, we shall focus our attention to the case where (E) is a scalar equation, and show (Theorem 1) that if B satisfies a growth condition, then the above problem can be solved in the affirmative. Furthermore, under the restriction on B, we shall investigate the converse of our Theorem 1, too. In fact, under the assumption that $B(t)$ does not change sign on $[0, \infty)$, we prove (Theorem 2) that if the zero solution of (E) possesses the stability property of exponential type, then B satisfies a growth condition. Thus (roughly speaking) the stability of exponential type for the zero solution of (E) can be characterized by a growth condition on B (Theorem 3). Hence, the stability of exponential type can never be realized for the equation with a kernel B which does not satisfy a growth condition, and the problem posed above can be solved in the negative.

Received April 11, 1990.
AMS Subject Classifications: 34D20, 45M10.

Now, we shall explain the notations employed throughout this article, and give some definitions. Let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}^+ = [0, \infty)$. For any compact interval $J \subset \mathbb{R}$, we denote by $C(J)$ the space of all continuous real-valued functions on J, and set $|\phi|_J = \sup\{|\phi(s)| : s \in J\}$. Also, we denote by $L^1(\mathbb{R}^+)$ the space of all Lebesgue integrable real-valued functions on $\mathbb{R}^+$. For any function $f : \mathbb{R}^+ \to \mathbb{R}$, the value of the Laplace transform $\hat{f}$ at s, a complex number, is defined as

$$\hat{f}(s) = \int_0^\infty f(t)e^{-st}dt.$$

If $|f(t)|e^{-\alpha t} \in L^1(\mathbb{R}^+)$ for a real constant α, then $\hat{f}(s)$ exists and is continuous in s for $\operatorname{Re} s \geq \alpha$, and moreover it is analytic on the domain $\operatorname{Re} s > \alpha$. Especially, if f is an absolutely continuous function which satisfies $|f(t)| \leq ae^{bt}$ on $\mathbb{R}^+$ for some constants a and b, then inversion formula

$$f(t) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{c-iT}^{c+iT} \hat{f}(s)e^{st}ds := \int_{(c)} \hat{f}(s)e^{st}ds, \quad t > 0,$$

holds for all $c > b$ ([6]).

Consider a scalar linear Volterra equation

$$\dot{x}(t) = Ax(t) + \int_0^t B(t-s)x(s)\,ds \quad t \geq 0, \tag{E}$$

where A is a constant and $B(t)$ is continuous in $t \in \mathbb{R}^+$ with $B \in L^1(\mathbb{R}^+)$. For any $\sigma \in \mathbb{R}^+$ and $\phi \in C([0,\sigma])$, there is one and only one function $x(t)$ which satisfies equation (E) on $[\sigma, \infty)$ and $x(t) \equiv \phi(t)$ on $[0, \sigma]$ (cf. [2]). Such a function $x(t)$ is called a solution of (E) on $[\sigma, \infty)$ through (σ, ϕ), and is denoted by $X(t; \sigma, \phi)$. Clearly, $x(t) \equiv 0$ is a solution of (E), which is called the zero solution of (E).

Definition. The zero solution of (E) is said to be

(i) uniformly stable (US), if for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that $\sigma \in \mathbb{R}^+$ and $\phi \in C([0,\sigma])$ with $|\phi|_{[0,\sigma]} < \delta(\epsilon)$ imply $|x(t;\sigma,\phi)| < \epsilon$ for all $t \geq \sigma$;

(ii) uniformly asymptotically stable (UAS), if it is US and moreover, if there is a $\delta_0 > 0$ with the property that for each $\epsilon > 0$ there exists a $T(\epsilon) > 0$ such that $\sigma \in \mathbb{R}^+$ and $\phi \in C([0,\sigma])$ with $|\phi|_{[0,\sigma]} < \delta_0$ imply $|x(t;\sigma,\phi)| < \epsilon$ for all $t \geq \sigma + T(\epsilon)$;

(iii) exponentially asymptotically stable (Ex AS), if there exist positive constants K and α such that

$$|x(t;\sigma,\phi)| \leq Ke^{-\alpha(t-\sigma)}|\phi|_{0,\sigma]}, \quad t \geq \sigma \geq 0,$$

for any $\phi \in C([0,\sigma])$.

By the definition, it is clear that Ex AS implies UAS. Denote by $X(t)$ the principal solution of (E); that is, $X(t)$ is the solution of (E) with $X(0) = 1$. Then the solution $x(t;\sigma,\phi)$ is expressed by the variation of parameters formula as

$$x(t;\sigma,\phi) = X(t-\sigma)\phi(\sigma) + \int_\sigma^t X(t-u)\left\{\int_0^\sigma B(u-\theta)\phi(\theta)\,d\theta\right\}du. \tag{1}$$

In fact, $X(t)$ is identical with the one called the resolvent for (E) in [3]. Therefore, the zero solution of (E) is UAS if and only if $X(t) \in L^1(\mathbb{R}^+)$ and that one of the above conditions is equivalent to the condition $H(s) := s - A - \hat{B}(s) \neq 0$ for $\operatorname{Re} s \geq 0$ ([3]).

Now, assume that the zero solution of (E) is UAS. Then $\hat{X}(s)$ exists and is continuous in s for $\operatorname{Re} s \geq 0$. Considering the Laplace transform of both sides of (E), and integrating the first term by parts, one obtains $H(s)\hat{X}(s) = 1$ for $\operatorname{Re} s > 0$; and hence,

$$H(s)\hat{X}(s) = 1 \quad \text{for } \operatorname{Re} s \geq 0 \tag{2}$$

by the continuity of $H(s)$ and $\hat{X}(s)$. Thus, we have

$$\hat{X}(s) = H^{-1}(s) \quad \text{for } \operatorname{Re} s \geq 0. \tag{3}$$

Theorem 1. *Suppose that*

$$\int_0^\infty |B(t)|e^{\gamma t}dt < \infty \tag{4}$$

for a positive constant $\gamma > 0$. If the zero solution of (E) *is UAS, then it is Ex AS.*

Proof: To establish the theorem, we shall employ the idea in [4, Theorem 1.5.2]. Since the zero solution of (E) is UAS, $|X(t)|$ is bounded on $\mathbb{R}^+$. Therefore, from (3) and inversion formula it follows that

$$X(t) = \int_{(\alpha)} H^{-1}(s)e^{st}ds, \quad t > 0, \tag{5}$$

for any $\alpha > 0$.

Claim 1. $X(t) = \int_{(-\epsilon)} H^{-1}(s)e^{st}ds$ $(t > 0)$ for some $\epsilon > 0$.

To verify the claim, we first observe by (4) that $\hat{B}(s)$ is defined for $\operatorname{Re} s \geq -\gamma$ and $|\hat{B}(s)| \to 0$ as $|s| \to \infty$ uniformly for $\operatorname{Re} s \geq -\gamma$. Hence, one can choose a constant $T_0 > 0$ so that $H(s) \neq 0$, $-\gamma \leq \operatorname{Re} s \leq 0$, if $|\operatorname{Im} s| \geq T_0$. Let

$$D = \{s : -\gamma/2 \leq \operatorname{Re} s \leq 0, \ \ |\operatorname{Im} s| \leq T_0\}$$

and

$$c_0 = \max\{\operatorname{Re} s : s \in D, \ H(s) = 0\}.$$

Since $H(s)$ is analytic on the domain $\operatorname{Re} s > -\gamma$, it has at most a finite number of zeros in the set D; hence $c_0 < 0$. Take a constant $\epsilon > 0$ so that $\epsilon < -c_0$. We consider the integration of the function $H^{-1}(s)e^{st}$ around the boundary of the box $\{\lambda + i\tau : -\epsilon \leq \lambda \leq \epsilon, \ -T \leq \tau \leq T\}$. Since $H(s)$ has no zeros in this box, it follows that the integral over the boundary is zero; that is,

$$\left(\int_{\epsilon-iT}^{\epsilon+iT} + \int_{\epsilon+iT}^{-\epsilon+iT} + \int_{-\epsilon+iT}^{-\epsilon-iT} + \int_{-\epsilon-iT}^{\epsilon-iT}\right) H^{-1}(s)e^{st}ds = 0.$$

Therefore, the claim will be verified by (5) if we show that

$$\lim_{T\to\infty} \int_{-\epsilon+iT}^{\epsilon+iT} H^{-1}(s)e^{st}ds = 0 \quad \text{and} \quad \lim_{T\to\infty} \int_{-\epsilon-iT}^{\epsilon-iT} H^{-1}(s)e^{st}ds = 0.$$

Let $s = \lambda + iT$, $-\epsilon \le \lambda \le \epsilon$. Since

$$|\hat{B}(s)| \le \int_0^\infty |B(t)|e^{\gamma t}dt := \ell < \infty$$

and

$$H(s) = s\{1 - \frac{A + \hat{B}(s)}{\lambda + iT}\},$$

we have

$$H^{-1}(s) = \frac{1}{s}\sum_{n=0}^{\infty}\{\frac{A + \hat{B}(s)}{\lambda + iT}\}^n$$

for large T. Hence

$$|H^{-1}(s)e^{st}| \le \frac{e^{\epsilon t}}{T}\sum_{n=0}^{\infty}(M/T)^n = e^{\epsilon t}/(T - M),$$

where $M = |A| + \ell$. Thus,

$$\left|\int_{-\epsilon+iT}^{\epsilon+iT} H^{-1}(s)e^{st}ds\right| \le 2\epsilon e^{\epsilon t}/(T - M)$$

for large T, which proves

$$\lim_{T\to\infty}\int_{-\epsilon+iT}^{\epsilon+iT} H^{-1}(s)e^{st}ds = 0.$$

Similarly, we obtain

$$\lim_{T\to\infty}\int_{-\epsilon-iT}^{\epsilon-iT} H^{-1}(s)e^{st}ds = 0.$$

Claim 2. $|X(t)| \le K_1 e^{-\epsilon t}$ $(t > 0)$ for a constant K_1. Indeed, if we set

$$g(s) = H^{-1}(s) - (s - c_0)^{-1} = \big(A + \hat{B}(s) - c_0\big)H^{-1}(s)(s - c_0)^{-1},$$

then $\sup_{\tau\in\mathbb{R}} \tau^2|g(-\epsilon+i\tau)| < \infty$; consequently, $\int_{(-\epsilon)} |g(s)|\, ds := K_2 < \infty$. Therefore, we obtain

$$\begin{aligned}|X(t)| &= \left|\int_{(-\epsilon)} H^{-1}(s)e^{st}ds\right| \le \left|\int_{(-\epsilon)} g(s)e^{st}ds\right| + \left|\int_{(-\epsilon)} \frac{e^{st}}{s - c_0}ds\right| \\ &\le (K_2 + 1)e^{-\epsilon t}\end{aligned}$$

by claim 1, where we used the fact that $\int_{(-\epsilon)} \frac{e^{st}}{s-c_0}ds = e^{c_0 t}$. This completes the proof of the claim with $K_1 = K_2 + 1$.

Finally, we shall show that the zero solution of (E) is Ex AS. To do this, it suffices to certify by Claim 2 and formula (1) that

$$\left|\int_\sigma^{t+\sigma} X(t + \sigma - u)\int_0^\sigma B(u - \theta)\phi(\theta)\, d\theta\, du\right| \le Ke^{-\epsilon t}|\phi|_{[0,\sigma]}$$

for any $\phi \in C[0,\sigma]$, where K is a constant. Since $\int_0^\infty |B(t)|e^{\epsilon t}dt \le \ell$, we obtain

$$\begin{aligned}
&\left|\int_\sigma^{t+\sigma} X(t+\sigma-u)\int_0^\sigma B(u-\theta)\phi(\theta)\,d\theta\,du\right| \\
&\le \int_\sigma^{t+\sigma} K_1^{-\epsilon(t+\sigma-u)}\int_0^\sigma |B(u-\theta)|\,d\theta\,du\cdot|\phi|_{0,\sigma]} \\
&\le K_1|\phi|_{[0,\sigma]}\int_0^\sigma\int_\sigma^{t+\sigma} e^{-\epsilon(t+\sigma-u)}|B(u-\theta)|\,du\,d\theta \\
&\le K_1|\phi|_{[0,\sigma]}\int_0^\sigma\left\{\int_\sigma^{t+\sigma}|B(u-\theta)|e^{\epsilon(u-\theta)}du\right\}e^{\epsilon(\theta-t-\sigma)}d\theta \\
&\le K_1|\phi|_{[0,\sigma]}\int_0^\sigma\left\{\int_\theta^{\infty}|B(u-\theta)|e^{\epsilon(u-\theta)}du\right\}e^{\epsilon(\theta-t-\sigma)}d\theta \\
&\le (K_1\ell/\epsilon)e^{-\epsilon t}|\phi|_{[0,\sigma]}
\end{aligned}$$

by Claim 2. Hence, we may set $K = K_1\ell/\epsilon$ to complete the proof.

Next, we shall study the converse of Theorem 1. Though we do not know whether or not the converse of Theorem 1 holds in general, we can obtain the following result under a restriction on B.

Theorem 2. *Suppose that $B \in L^1(\mathbb{R}^+)$ and that $B(t)$ does not change sign on $\mathbb{R}^+$. If $|X(t)| \le Ke^{-\alpha t}$, $t \in \mathbb{R}^+$, for some positive constants K and α, then there exists a constant $\gamma > 0$ such that condition* (4) *holds.*

Proof: Note that $\hat{X}(s)$ exists and is analytic in s for $\operatorname{Re} s > -\alpha$ and satisfies relation (2). In particular, $\hat{X}(0) \ne 0$. By the continuity of $\hat{X}(s)$ at $s = 0$, we can find a (small) open neighborhood U of 0 such that $\hat{X}(s) \ne 0$ on U. Thus, $\hat{X}^{-1}(s)$ is analytic on U; hence the function $F(s) := s - A - \hat{X}^{-1}(s)$ is analytic on U and satisfies

$$\hat{B}(s) = F(s) \quad \text{for } \operatorname{Re} s \ge 0. \tag{6}$$

Now, we claim that $tB(t) \in L^1(\mathbb{R}^+)$. If this is not the case, then there is a constant $T > 1$ such that

$$\int_0^T |B(t)|(t-1)\,dt > k,$$

where $k = |F'(0)|$. Note that

$$\begin{aligned}
\sup_{0\le t\le T}\left|\frac{1-e^{-ht}}{h} - t\right| &= \sup_{0\le t\le T}\left|\frac{ht^2}{2!} - \frac{h^2t^3}{3!} + \frac{h^3t^4}{4!} - \cdots\right| \\
&\le |h|\left(\frac{T^2}{2!} + \frac{T^3}{3!} + \frac{T^4}{4!} + \cdots\right) \le |h|e^T \le 1
\end{aligned}$$

if $|h| < \min(1, e^{-T}) := \delta_T$. Thus, if $0 < h < \delta_T$, then

$$\frac{1-e^{-ht}}{h} \ge t-1 \quad \text{for } t \in [0,T];$$

and consequently, by (6), we obtain

$$-\frac{F(h)-F(0)}{h} = -\frac{\hat{B}(h)-\hat{B}(0)}{h} = \int_0^\infty B(t)\frac{1-e^{-ht}}{h}dt$$

or

$$\left|\frac{F(h)-F(0)}{h}\right| = \int_0^\infty |B(t)|\frac{1-e^{-ht}}{h}dt \geq \int_0^T |B(t)|(t-1)\,dt$$

for $0 < h < \delta_T$, because $B(t)$ does not change sign on $\mathbb{R}^+$. Letting $h \to 0^+$, we get

$$k = |F'(0)| \geq \int_0^T |B(t)|(t-1)\,dt > k,$$

a contradiction. Hence, we must have that $tB(t) \in L^1(\mathbb{R}^+)$.

Next, we shall prove that

$$F'(s) = -\int_0^\infty te^{-st}B(t)\,dt \quad \text{for } s \geq 0.$$

Indeed, from the inequality $1 - x \leq e^{-x}$ for $x \geq 0$, it follows that

$$\left|\frac{1-e^{-ht}}{h}e^{-st}B(t)\right| \leq t|B(t)| \quad \text{on } \mathbb{R}^+$$

for any $h > 0$ and $s \geq 0$. Hence, Lebesgue's dominated theorem implies that

$$\begin{aligned} F'(s) &= \lim_{h\to 0+} \frac{F(s+h)-F(s)}{h} = \lim_{h\to 0+} \frac{\hat{B}(s+h)-\hat{B}(s)}{h} \\ &= \lim_{h\to 0+} \int_0^\infty \frac{e^{-ht}-1}{h}e^{-st}B(t)\,dt = -\int_0^\infty te^{-st}B(t)\,dt \end{aligned}$$

for $s \geq 0$. Now, applying the argument for $B(t)$ in the foregoing paragraphs to the function $tB(t)$, we obtain that $t^2B(t) \in L^1(\mathbb{R}^+)$ and $F''(s) = \int_0^\infty t^2e^{-st}B(t)\,dt$ for $s \geq 0$. Repeat this procedure to obtain

$$t^nB(t) \in L^1(\mathbb{R}^+) \quad \text{and} \quad F^{(n)}(s) = (-1)^n \int_0^\infty t^ne^{-st}B(t)\,dt, \quad s \geq 0, \tag{7}$$

for $n = 1, 2, \ldots$. Since F is analytic on U, Maclaurin's series $\sum_{n=0}^\infty \frac{F^{(n)}(0)}{n!}s^n$ is absolutely convergent on a closed disk of center zero with a radius $\gamma > 0$. Hence,

$$\sum_{n=0}^\infty \frac{|F^{(n)}(0)|}{n!}\gamma^n < \infty \quad \text{or} \quad \sum_{n=0}^\infty \frac{\gamma^n}{n!}\int_0^\infty t^n|B(t)|\,dt < \infty$$

by (7). Thus,

$$\int_0^\infty e^{\gamma t}|B(t)|\,dt = \int_0^\infty \left(\sum_{n=0}^\infty \frac{(\gamma t)^n}{n!}\right)|B(t)|\,dt = \sum_{n=0}^\infty \frac{\gamma^n}{n!}\int_0^\infty t^n|B(t)|\,dt < \infty,$$

where we used Fatou's theorem. This completes the proof.

Combining Theorem 1 and Theorem 2, we have the following result.

Theorem 3. *Suppose that $B \in L^1(\mathbb{R}^+)$ and that $B(t)$ does not change the sign on $\mathbb{R}^+$. If the zero solution of* (E) *is UAS, then the following three statements are equivalent:*

(i) *The zero solution of* (E) *is Ex AS;*

(ii) $|X(t)| \le Ke^{-\alpha t}$, $t \ge 0$, *for some positive constants K and α;*

(iii) $\int_0^\infty |B(t)|e^{\gamma t}dt < \infty$ *for a constant $\gamma > 0$.*

It may be hoped that Theorem 3 is extended to the non-scalar case. In fact, it is straightforward to see that the proof of Theorem 1 is extendable to the non-scalar case by a slight modification. However, the author has not succeeded in extending Theorem 2 to the non-scalar case.

Corduneanu and Lakshmikantham [1, pp. 845-848] have posed a problem: "Whenever the zero solution of (E) is UAS, does $|X(t)|$ decay exponentially as $t \to \infty$?" The following example answers the problem in the negative (also, see [5], where a counterexample is given for the equation with purely discrete delays).

Example. Consider a scalar equation

$$\dot{x}(t) = -x(t) + \int_0^t B(t-s)x(s)\,ds, \tag{8}$$

where B is a nonnegative continuous function with $\int_0^\infty B(t)\,dt < 1$. Since $|s+1| \ge 1$ and $|\hat{B}(s)| \le \int_0^\infty B(t)\,dt < 1$ for $\operatorname{Re} s \ge 0$, we have $H(s) \ne 0$ for $\operatorname{Re} s \ge 0$; hence, the zero solution of (8) is UAS by [3, Theorem 3.5]. Therefore, from Theorem 3 it follows that the zero solution of (8) is Ex AS when $B(t) \equiv e^{-bt}$ $(b > 1)$, while the zero solution of (8) is not Ex AS when $B(t) = k/(t+1)^b$ $(0 < k < b-1)$.

Acknowledgement. The author expresses his hearty gratitude to Professor Constantin Corduneanu for his useful comments and advice on the manuscript.

REFERENCES

[1] C. Corduneanu and V. Lakshmikantham, *Equations with unbounded delay: a survey,* Nonlinear Anal., TMA, 4 (1980), 831–877.

[2] R.D. Driver, *Existence and stability of solutions of a delay differential system,* Arch. Rational Mech. Anal., 10 (1962), 401–426.

[3] S.I. Grossman and R.K. Miller, *Nonlinear Volterra integrodifferential equations,* J. Diff. Equations, 13 (1973), 551–566.

[4] J.K. Hale, "Theory of Functional Differential Equations," Applied Math. Sci., 3, Springer, New York, 1977.

[5] F.L. Huang, *Exponential stability for autonomous linear functional differential equations with infinite retardations,* Science in China, 32 (1989), 550–563.

[6] D.V. Widder, "The Laplace Transform," Princeton Univ. Press, Princeton, N.J., 1946.