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Exponential Bounds and Absence of Positive Eigenvalues for N-Body Schrödinger Operators

Richard Froese*,** and Ira Herbst*,***

Institut Mittag-Leffler, Auravägen 17, S-182 62 Djursholm, Sweden

Abstract. For a large class of N-body potentials V we prove that if ψ is an eigenfunction of $-\Delta+V$ with eigenvalue E then $\sup\{\alpha^2+E:\alpha\geqq 0, \exp(\alpha|x|)\psi\in L^2\}$ is either a threshold or $+\infty$. Consequences of this result are the absence of positive eigenvalues and "optimal" L^2-exponential lower bounds.

I. Introduction

In this paper we will be concerned with the N-body Schrödinger operator

$$H=H_0+V, \tag{1.1}$$

$$V=\sum_{1\leqq i<j\leqq N} V_{ij}, \tag{1.2}$$

in $L^2(\mathbb{R}^{\nu(N-1)})$. Here H_0 arises from the operator

$$\tilde{H}_0=-\sum_{i=1}^{N}\Delta_i/2m_i \tag{1.3}$$

by removing the center of mass (see [16, 17] and Sect. II for more details). Each V_{ij} is multiplication by a real-valued function $v_{ij}(x_i-x_j)$, where here $x\in\mathbb{R}^{\nu N}$ is written $x=(x_1,\dots,x_N)$. Let h_0 be $-\Delta$ in $L^2(\mathbb{R}^\nu)$. We assume in what follows that each two-body potential v_{ij} satisfies

(a) $$v_{ij}(1+h_0)^{-1} \quad \text{is compact}, \tag{1.4}$$

(b) $$(1+h_0)^{-1}(y\cdot\nabla v_{ij})(1+h_0)^{-1} \quad \text{is compact}. \tag{1.5}$$

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* Permanent address: Department of Mathematics, University of Virginia, Charlottesville, VA 22903, USA

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From (a) it easily follows that v_{ij} is a tempered distribution. What is meant by (b) is that the tempered distribution $w_{ij}(y) = y \cdot \nabla v_{ij}(y)$ has the property that the sesquilinear form

$$Q_{ij}(f,g) = ((1+h_0)^{-1} f, w_{ij}(1+h_0)^{-1} g) = w_{ij}(\overline{(1+h_0)^{-1} f} \cdot (1+h_0)^{-1} g)$$

extends from $\mathscr{S}(\mathbb{R}^\nu) \times \mathscr{S}(\mathbb{R}^\nu)$ to the form of a compact operator on $L^2(\mathbb{R}^\nu)$.

An important set for our purposes is the set of thresholds. To describe this set we need further notation. Given a subset C of $\{1, 2, \ldots, N\}$ with cardinality $|C| > 1$, let

$$H(C) = H_0(C) + V(C),$$

$$V(C) = \sum_{\substack{i \in C, j \in C \\ i<j}} V_{ij}$$

as operators in $L^2(\mathbb{R}^{\nu(|C|-1)})$. The operator $H_0(C)$ arises from the operator

$$\tilde{H}_0(C) = -\sum_{i \in C} \Delta_i / 2m_i$$

by removing the center of mass. If $|C| = 1$ we define $H(C) = 0$.

We define the set of thresholds, $\mathscr{T}(H)$, associated with H as

$$\mathscr{T}(H) = \{E \in \mathbb{R} : \text{There exists a partition } \{C_1, C_2, \ldots, C_k\} \text{ of } \{1, \ldots, N\} \text{ into } k \geqq 2 \text{ disjoint subsets, and for each } j \text{ an eigenvalue } E_j \text{ of } H(C_j), \text{ such that } E = E_1 + \ldots + E_k\}. \tag{1.6}$$

We will also need the distance function

$$|x| = \left(\sum_{i=1}^{N} 2m_i |x_i - R|^2 \right)^{1/2}, \tag{1.7}$$

$$R = \left(\sum_{i=1}^{N} m_i x_i \right) \Big/ \left(\sum_{i=1}^{N} m_i \right), \tag{1.8}$$

where $|x_i - R|$ denotes the Euclidean distance in $\mathbb{R}^\nu$.

Our first main result is as follows:

Theorem 1.1. *Suppose $H = H_0 + V$ with two-body potentials satisfying* (1.4) *and* (1.5) *above. Suppose $H\psi = E\psi$. Then*

$$\sup\{\alpha^2 + E : \alpha \geqq 0, \exp(\alpha |x|)\psi \in L^2(\mathbb{R}^{\nu(N-1)})\}$$

is either a threshold or $+\infty$.

Under conditions (1.4) and (1.5) on the two-body potentials, Perry et al. [15] have shown that $\mathscr{T}(H)$ is a *closed* countable set. We make implicit use of this fact in stating a corollary of Theorem 1.1:

Corollary 1.2. *Suppose H is as in Theorem 1.1 and $H\psi = E\psi$.*

(i) *Suppose $E \notin \mathscr{T}(H)$ and $\mathscr{T}(H) \cap [E, \infty)$ is not empty. Let t_0 be the first threshold above E (more explicitly $t_0 = \inf[\mathscr{T}(H) \cap [E, \infty))$. Then for all $\beta < \sqrt{t_0 - E}$,*

$$\exp(\beta |x|)\psi \in L^2(\mathbb{R}^{\nu(N-1)}). \tag{1.9}$$

(ii) *Suppose that for some* $\alpha \geqq 0$, $\exp(\alpha|x|)\psi \in L^2(\mathbb{R}^{\nu(N-1)})$, *where* α *is such that* $\mathcal{T}(H) \cap [E+\alpha^2, \infty)$ *is empty. Then* (1.9) *holds for all* $\beta > 0$.

At this point we cannot eliminate the possibility that (ii) occurs (for some $\psi \neq 0$) without introducing further hypotheses on the potentials V_{ij}. One such hypothesis which eliminates the possibility of such unusual behavior in this situation was given in [11]. This is the basis for condition (i) of the following theorem. The union of conditions (i) and (ii) below forms a rather wide class of potentials.

Theorem 1.3. *Suppose* $H = H_0 + V$ *with* V *as in* (1.2) *and each* v_{ij} *is* h_0*-bounded with bound zero. Suppose* $(h_0+1)^{-1} y \cdot \nabla v_{ij} (h_0+1)^{-1}$ *is bounded for each* (ij). *Let* $p = \mathrm{Max}(2, \nu - 1)$. *Suppose either that*

(i) *for each* $\varepsilon > 0$ *and* (i,j) *there is a* c_ε *such that*

$$y \cdot \nabla v_{ij} \leqq \varepsilon h_0 + c_\varepsilon ,$$

or

(ii) *for each* (i,j), $v_{ij} \in L^p(\mathbb{R}^\nu) + L^\infty(\mathbb{R}^\nu)$ *and there is a decomposition* $v_{ij} = v_{ij}^{(1)} + v_{ij}^{(2)}$ *such that* $(1+|y|) v_{ij}^{(1)} \in L^p(\mathbb{R}^\nu) + L^\infty(\mathbb{R}^\nu)$ *and for each* $\varepsilon > 0$ *there is a* c_ε *such that* $y \cdot \nabla v_{ij}^{(2)} \leqq \varepsilon h_0 + c_\varepsilon$.

Suppose that $H\psi = E\psi$ *with* $\exp(\alpha|x|)\psi \in L^2(\mathbb{R}^{\nu(N-1)})$ *for all* α. *Then* $\psi = 0$.

Remark. The condition $y \cdot \nabla v_{ij} \leqq \varepsilon h_0 + c_\varepsilon$ is certainly satisfied for all $\varepsilon > 0$ if $(1+|y|) v_{ij} \in L^q(\mathbb{R}^\nu)$, where $q > \nu$ and $q \geqq 2$ $\Big[$this follows from $y \cdot \nabla v_{ij} = \sum_{k=1}^{\nu} (D_k(y_k v_{ij}) - (y_k v_{ij}) D_k - v_{ij})$, where here $D_k = \partial/\partial y_k$ is considered an operator$\Big]$. But it need not be if v_{ij} is as singular as is allowed in (ii).

From Corollary 1.2 and Theorem 1.3 we have

Corollary 1.4. *Suppose* $H = H_0 + V$ *with two-body potentials* v_{ij} *satisfying* (1.4), (1.5), *and either condition* (i) *or* (ii) *of Theorem 1.3. Then* H *has no positive eigenvalues and if* $H\psi = E\psi$ *with* $\psi \neq 0$ *it follows that*

$$\exp(\alpha|x|)\psi \notin L^2(\mathbb{R}^{\nu(N-1)}); \quad \alpha > \sqrt{-E}. \tag{1.10}$$

Proof. Assume inductively that for $1 \leqq |C| \leqq k$ (where $k < N$) that $H(C)$ has no positive eigenvalues. Then it follows that for $|C_0| = k+1$, $H(C_0)$ has no positive thresholds. Hence if $H(C_0)\psi = E\psi$ with $E > 0$ it follows from (ii) of Corollary 1.2 that $\exp(\beta|x|)\psi \in L^2$ for all $\beta > 0$. However from Theorem 1.3 this is impossible unless $\psi = 0$. Since our inductive assumption is clearly true for $|C| = 1$ we learn that $H(C)$ has no positive eigenvalues for any $|C| \leqq N$. Hence H has no positive eigenvalues (and no positive thresholds).

Now (1.10) follows easily from (ii) of Corollary 1.2 and Theorem 1.3. □

We remark that as shown in [11], the "lower bound" (1.10) is close to being optimal at least if one allows $V(x)$ to be slightly more general than an N-body potential [see (2.45)]. For in the latter paper an example is constructed of an N-body-like potential V such that $-\Delta + V$ has an eigenfunction ψ whose decay rate is controlled by a threshold which is arbitrarily close to zero.

In [11], (1.10) is proved in certain important special cases including for Hamiltonians which describe atomic and molecular systems. We believe that for generic V there are no embedded eigenvalues and that the decay rate of eigenfunctions is controlled by the lowest threshold. A proper formulation and proof of such a result would be very interesting.

Theorem 1.3 is a kind of unique continuation theorem. We expect that conditions (i) or (ii) of that theorem are far from optimal in eliminating arbitrarily rapid exponential decay.

The ideas in this paper are directly descended from [11, 12]. The latter paper proves absence of positive eigenvalues for a large class of one-body potentials using a similar method. The proof of Theorem 1.1 relies heavily on ideas from [11, 12] and on the "Mourre estimate" [14] which was proved for N-body systems by Perry et al. [15]. The unique continuation type argument on which Theorem 1.3 is based is also an extension of ideas which appear in [11, 12]. Indeed under condition (i) of that theorem, the result already appears in [11].

There are of course situations aside from those in [11, 12] in which partial results along the lines of Theorem 1.1 and Corollary 1.4 were previously known. For one-body systems of the type considered here the absence of positive eigenvalues was proved by Kato [13], Agmon [2], and Simon [8] (see the book by Eastham and Kalf [9] for further developments). For N-body systems with potentials dilation-analytic in angle $\theta_0 \geqq \pi/2$, absence of positive eigenvalues was known from the work of Balslev [5] and of Simon [19]. Previous to this work Weidmann [21, 22] had used the virial theorem to prove absence of positive eigenvalues for a class of homogeneous potentials. The work of Agmon [1] is also relevant here. The fact that eigenfunctions corresponding to non-threshold eigenvalues decay exponentially (at some rate) was known for dilation-analytic potentials from the work of Combes and Thomas [6], but as far as we are aware the bound involving the first threshold above E is new, except of course when E is below the essential spectrum, in which case more detailed estimates are available [3, 7]. From the work of Agmon [3], it follows that for even more general two-body potentials, eigenfunctions with eigenvalues $E<0$ must decay exponentially in certain cones even if $E \in \sigma_{\mathrm{ess}}(H)$.

Some further discussion and references can be found in the notes section of [17].

The organization of this paper now follows. In Sect. II we prove a result (Theorem 2.1) from which Theorem 1.1 follows given the Mourre estimate (Theorem 2.3) of Perry et al. [15]. In Sect. III we prove a unique continuation type result (Theorem 3.1) from which Theorem 1.3 follows. Theorems 2.1, 2.3, and 3.1 are given for potentials V which are more general than N-body potentials. The results which generalize Theorem 1.1 and Corollary 1.4 are given in Corollaries 2.4 and 3.2, respectively.

II. The Mourre Estimate and Exponential Upper Bounds

In this section we consider operators of the form

$$H = -\Delta + V \tag{2.1}$$

in $L^2(\mathbb{R}^n)$, where V is multiplication by a real-valued function satisfying

(a) V is Δ-bounded with bound less than one, (2.2)

(b) $(-\Delta+1)^{-1}\, x\cdot\nabla V(-\Delta+1)^{-1}$ is bounded. (2.3)

Let D be the operator in $L^2(\mathbb{R}^n)$ defined by

$$(Df)(x)=\nabla f(x),$$

and denote by A the generator of dilations:

$$A=(x\cdot D+D\cdot x)/2. \tag{2.4}$$

We will also need the projection-valued measure $\{E(\Delta):\Delta \text{ a Borel subset of } \mathbb{R}\}$ associated with the self-adjoint operator H.

We say that the "Mourre estimate" is satisfied at a point $\lambda_0\in\mathbb{R}$ if there exists a non-empty open interval Δ containing λ_0, a constant $c_0>0$ and a compact operator K_0 so that

$$E(\Delta)[H,A]E(\Delta)\geqq c_0E(\Delta)+K_0. \tag{2.5}$$

Clearly the set of λ_0 for which the Mourre estimate is satisfied is open. We denote by $\mathscr{E}(H)$ the complement of the latter set. The estimate (2.5) was introduced by Mourre [14] who proved that it was satisfied at non-threshold points for certain 3-body Hamiltonians, and used it to prove $\sigma_{\text{s.c.}}(H)=\emptyset$. Mourre's result was improved and extended to N-body Hamiltonians by Perry et al. [15] (see Theorem 2.3 below).

We use the notation $[H,A]$ for the quantity $-2\Delta-x\cdot\nabla V$ which is a form on $\mathscr{D}(\Delta)\times\mathscr{D}(\Delta)$.

In this section we will prove the following result and then apply it to N-body systems. We use the notation $|x|=\left(\sum_{i=1}^{n}x_i^2\right)^{1/2}$.

Theorem 2.1. *Let $H=-\Delta+V$ in $L^2(\mathbb{R}^n)$, where V is a real-valued function satisfying* (2.2) *and* (2.3). *Suppose $H\psi=E\psi$. Then*

$$\sup\{\alpha^2+E:\alpha>0, \exp(\alpha|x|)\psi\in L^2(\mathbb{R}^n)\}$$

is either $+\infty$ or in $\mathscr{E}(H)$.

The following lemma, which will be crucial in our proof of Theorem 2.1, was used in [12]. We sketch a proof (different from that in [12]) in an appendix.

Lemma 2.2 *Let H be as in Theorem 2.1 and $H\psi=E\psi$. Let $\varrho(x)=(|x|^2+1)^{1/2}$. For $\varepsilon>0$ and $\lambda>0$ let*

$$F(x)=\lambda\ln(\varrho(1+\varepsilon\varrho)^{-1}),$$

$$\nabla F(x)=xg(x).$$

Let $\psi_F=\exp(F)\psi$ and define the operator

$$H(F)=H-(\nabla F)^2+(D\cdot\nabla F+\nabla F\cdot D);\qquad \mathscr{D}(H(F))=\mathscr{D}(\Delta). \tag{2.6}$$

Then $\psi_F \in \mathscr{D}(A)$ *and*

$$H(F)\psi_F = E\psi_F, \tag{2.7}$$

$$(\psi_F, H\psi_F) = (\psi_F, ((\nabla F)^2 + E)\psi_F), \tag{2.8}$$

$$(\psi_F, [H, A]\psi_F) = -4\|g^{1/2}A\psi_F\|^2 + (\psi_F, \{(x\cdot\nabla)^2 g - x\cdot\nabla(\nabla F)^2\}\psi_F). \tag{2.9}$$

If in addition $\varrho^\lambda \exp(\alpha\varrho)\psi \in L^2(\mathbb{R}^n)$ *for all* λ *and some fixed* $\alpha \geqq 0$, *then the above also holds with*

$$F = \alpha\varrho + \lambda\ln(1 + \gamma\varrho\lambda^{-1}),$$

$$\nabla F = xg,$$

for all $\gamma > 0$ *and* $\lambda > 0$.

Remarks. (i) In case $F = \lambda\ln(\varrho(1+\varepsilon\varrho)^{-1})$, an easy calculation gives

$$g = \lambda\varrho^{-2}(1+\varepsilon\varrho)^{-1}. \tag{2.10}$$

Although in this case we do not know that $\psi_F \in \mathscr{D}(A)$, it easily follows that the function $g^{1/2}A\psi_F$ is in $L^2(\mathbb{R}^n)$ so that $\|g^{1/2}A\psi_F\|$ has an obvious meaning.

(ii) Note that $\lim_{\varepsilon\downarrow 0} \lambda\ln(\varrho(1+\varepsilon\varrho)^{-1}) = \lambda\ln\varrho$, $\lim_{\lambda\to\infty} (\alpha\varrho + \lambda\ln(1+\gamma\varrho\lambda^{-1})) = (\alpha+\gamma)\varrho$, and this is the reason for our choice of F. Clearly the lemma is also true for other choices. The crucial fact which makes (2.9) useful is the positivity of $\|g^{1/2}A\psi_F\|^2$. This is a consequence of our choice of radially symmetric, monotone increasing functions F. For the purpose of understanding why (2.9) can be useful, one should think of the second term on the right side of (2.9) as negligible and compare (2.9) with (2.5).

(iii) Formally (2.9) follows from the equation $(\psi, [H, \exp(F)A\exp(F)]\psi) = 0$.

Proof of Theorem 2.1. Before beginning in earnest we illustrate the strategy of the proof. Suppose $H\psi = E\psi$ and that $\sup\{\alpha^2 + E : \alpha \geqq 0,\ \exp(\alpha|x|)\psi \in L^2\} = \alpha_0^2 + E \notin \mathscr{E}(H)$. Suppose for simplicity that $\varrho^\lambda \exp(\alpha_0\varrho)\psi \in L^2$ for all λ. Then if $\gamma > 0$ and $F = \alpha_0\varrho + \lambda\ln(1+\gamma\varrho\lambda^{-1})$ we clearly have $\lim_{\lambda\to\infty} \|\exp(F)\psi\| = \infty$. Thus the vector $\Psi_\lambda = \exp(F)\psi/\|\exp(F)\psi\|$ leaves every compact set as $\lambda \to \infty$. It turns out that $(H - E - (\nabla F)^2)\Psi_\lambda \to 0$ so that for small γ, Ψ_λ has energy concentrated around $E + \alpha_0^2$. If F were actually equal to $(\alpha_0+\gamma)|x|$, then we would have $(x\cdot\nabla)^2 g - (x\cdot\nabla)(\nabla F)^2 = (\alpha_0+\gamma)|x|^{-1}$ which contributes negligibly to (2.9) as $\lambda\to\infty$. This is not far from true. Since the energy of Ψ_λ is concentrated around $E + \alpha_0^2$ and Ψ_λ converges to zero weakly, the negativity of $(\Psi_\lambda, [\mathrm{H}, \mathrm{A}]\Psi_\lambda)$ which follows from (2.9) contradicts its positively guaranteed by the Mourre estimate.

We now proceed to implement these ideas. We first show that if $E \notin \mathscr{E}(H)$ then $\varrho^\lambda\psi \in L^2(\mathbb{R}^n)$ for $\lambda > 0$. Assume the contrary so that for some $\lambda > 0$, $\varrho^\lambda\psi \notin L^2(\mathbb{R}^n)$. Let $F = \lambda\ln(\varrho(1+\varepsilon\varrho)^{-1})$ and

$$\Psi_\varepsilon = \psi_F/\|\psi_F\|.$$

By the monotone convergence theorem, $\|\Psi_F\|^2 = \int(\varrho/(1+\varepsilon\varrho))^{2\lambda}|\psi|^2 d^n x$ converges to $\int \varrho^{2\lambda}|\psi|^2 d^n x = +\infty$ so that for any bounded set B

$$\lim_{\varepsilon\downarrow 0} \int_B |\Psi_\varepsilon|^2 d^n x = 0. \tag{2.11}$$

By explicit calculation we find

$$(\nabla F)^2 = \lambda^2(1-\varrho^{-2})\varrho^{-2}(1+\varepsilon\varrho)^{-2}, \tag{2.12}$$

so that $|\nabla F| \leqq \lambda\varrho^{-1}$. It follows from this, Eq. (2.8), and the fact that V is Δ-bounded with bound less than 1 that $\|\nabla\Psi_\varepsilon\|$ is bounded as $\varepsilon\downarrow 0$. Using this fact, it similarly follows from (2.6) and (2.7) that $\|(-\Delta+1)\Psi_\varepsilon\|$ is bounded as $\varepsilon\downarrow 0$. Hence from (2.11), $(-\Delta+1)\Psi_\varepsilon$ converges weakly to zero as $\varepsilon\downarrow 0$. From the compactness of $\varrho^{-1}D(-\Delta+1)^{-1}$ we have as $\varepsilon\downarrow 0$

$$\begin{aligned}\|\nabla F\cdot D\Psi_\varepsilon\| &\leqq \lambda\|\varrho^{-1}D\Psi_\varepsilon\| \\ &= \lambda\|\varrho^{-1}D(-\Delta+1)^{-1}\,(-\Delta+1)\Psi_\varepsilon\| \\ &\to 0\,.\end{aligned}$$

Similarly $\|(\nabla F)^2\Psi_\varepsilon\|$ and $\|(D\cdot\nabla F)\Psi_\varepsilon\|$ converge to zero so that from (2.7) we have

$$\lim_{\varepsilon\downarrow 0}\|(H-E)\Psi_\varepsilon\| = 0\,. \tag{2.13}$$

By definition of $\mathscr{E}(H)$, (2.5) holds for some Δ containing E, some $c_0>0$ and some compact operator K_0. Without loss of generality we can assume $\Delta=[E-\delta, E+\delta]$ for some $\delta>0$. Since Ψ_ε converges weakly to zero, we thus have

$$\begin{aligned}\liminf_{\varepsilon\downarrow 0}(\Psi_\varepsilon, E(\Delta)[H,A]E(\Delta)\Psi_\varepsilon) &\geqq c_0\liminf_{\varepsilon\downarrow 0}\|E(\Delta)\Psi_\varepsilon\|^2 \\ &= c_0>0\,,\end{aligned} \tag{2.14}$$

where the equality in (2.14) follows from

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\|E(\mathbb{R}\backslash\Delta)\Psi_\varepsilon\| &\leqq \lim_{\varepsilon\downarrow 0}\|(H-E)\delta^{-1}E(\mathbb{R}\backslash\Delta)\Psi_\varepsilon\| \\ &\leqq \delta^{-1}\lim_{\varepsilon\downarrow 0}\|(H-E)\Psi_\varepsilon\| = 0\,.\end{aligned} \tag{2.15}$$

We now use (2.9) to derive a contradiction to (2.14). First by explicit calculation [using (2.10) and (2.12)] we find

$$|(x\cdot\nabla)^2 g - (x\cdot\nabla)(\nabla F)^2| \leqq c_1\varrho^{-2}$$

for some c_1 independent of ε, so that from (2.9)

$$\limsup_{\varepsilon\downarrow 0}(\Psi_\varepsilon, [H,A]\Psi_\varepsilon) \leqq 0\,, \tag{2.16}$$

we now claim that

$$\lim_{\varepsilon\downarrow 0}\|(-\Delta+1)E(\mathbb{R}\backslash\Delta)\Psi_\varepsilon\| = 0\,. \tag{2.17}$$

To see this we use (2.13) and (2.15) to get

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\|(H+i)E(\mathbb{R}\backslash\Delta)\Psi_\varepsilon\| &\leqq \lim_{\varepsilon\downarrow 0}\|E(\mathbb{R}\backslash\Delta)(H-E)\Psi_\varepsilon\| \\ &\quad + \lim_{\varepsilon\downarrow 0}|E+i|\,\|E(\mathbb{R}\backslash\Delta)\Psi_\varepsilon\| = 0\,,\end{aligned}$$

which implies (2.17) because V is Δ-bounded with bound less than one.

We have

$$(\Psi_\varepsilon, [H,A]\Psi_\varepsilon) = (\Psi_\varepsilon, E(\Delta)[H,A]E(\Delta)\Psi_\varepsilon) + f_1(\varepsilon) + f_2(\varepsilon)\,, \tag{2.18}$$

where

$$f_1(\varepsilon)=(\Psi_\varepsilon, E(\mathbb{R}\backslash\Delta)[H,A]\Psi_\varepsilon),$$
$$f_2(\varepsilon)=(\Psi_\varepsilon, E(\Delta)[H,A]E(\mathbb{R}\backslash\Delta)\Psi_\varepsilon).$$

From (2.17) and (2.3) we have

$$\lim_{\varepsilon\downarrow 0}|f_1(\varepsilon)|\leqq\lim_{\varepsilon\downarrow 0}\{\|(-\Delta+1)E(\mathbb{R}\backslash\Delta)\Psi_\varepsilon\|\cdot\|(\Delta+1)^{-1}[H,A](-\Delta+1)^{-1}\|\cdot\|(-\Delta+1)\Psi_\varepsilon\|\}=0.$$

(Here we have used $[H,A]=-2\Delta-x\cdot\nabla V$.) Similarly $\lim_{\varepsilon\downarrow 0}f_2(\varepsilon)=0$, so that from (2.18) and (2.16), we have

$$\limsup_{\varepsilon\downarrow 0}(\Psi_\varepsilon, E(\Delta)[H,A]E(\Delta)\Psi_\varepsilon)\leqq 0. \tag{2.19}$$

This contradicts (2.14) so we have shown that if $E\notin\mathscr{E}(H)$, then $\varrho^\lambda\psi\in L^2(\mathbb{R}^n)$ for all λ.

Suppose now that the theorem is false so that

$$\sup\{\alpha^2+E:\exp(\alpha\varrho)\psi\in L^2(\mathbb{R}^n),\alpha>0\}=\alpha_1^2+E, \tag{2.20}$$

where $\alpha_1\geqq 0$ and $\alpha_1^2+E=\Lambda\notin\mathscr{E}(H)$. Again we know that (2.5) holds with $\Delta=[\Lambda-\delta,\Lambda+\delta]$ for some $\delta>0$, $c_0>0$, and K_0 compact. If $\alpha_1=0$, set $\alpha=\alpha_1=0$. If $\alpha_1>0$, then choose $\alpha\in[0,\alpha_1)$ so that

$$\alpha^2+E\in[\Lambda-\delta/2,\Lambda+\delta/2]. \tag{2.21}$$

In either case we have for all $\lambda>0$

$$\varrho^\lambda\exp(\alpha\varrho)\psi\in L^2(\mathbb{R}^n). \tag{2.22}$$

Suppose $\gamma>0$ is such that $\alpha+\gamma>\alpha_1$. Then by (2.20) we have

$$\|\exp((\alpha+\gamma)\varrho)\psi\|=\infty. \tag{2.23}$$

We will obtain a contradiction for sufficiently small $\gamma>0$. In the following we also assume $\gamma\in(0,1]$. Let $F=\alpha\varrho+\lambda\ln(1+\gamma\varrho\lambda^{-1})$ and $\psi_F=\exp(F)\psi$, $\Psi_\lambda=\psi_F/\|\psi_F\|$. As in the previous argument we conclude from (2.23) that for any bounded set $B\subset\mathbb{R}^n$,

$$\lim_{\lambda\to\infty}\int_B|\Psi_\lambda|^2d^nx=0. \tag{2.24}$$

In the following we denote by b_j, $j=1,2,\ldots$ constants which are independent of α, γ, and λ. By direct computation we have

$$\nabla F=\varrho^{-1}x(\alpha+\gamma(1+\gamma\varrho\lambda^{-1})^{-1}), \tag{2.25}$$

so that

$$|\nabla F|\leqq\alpha+\gamma\leqq b_1,\quad |\Delta F|\leqq b_2. \tag{2.26}$$

It thus follows from (2.8) that $\|\nabla\Psi_\lambda\|\leqq b_3$. Using the latter in conjunction with (2.26) and (2.7) gives

$$\|(-\Delta+1)\Psi_\lambda\|\leqq b_4. \tag{2.27}$$

In particular, (2.27) and (2.24) imply that $(-\Delta+1)\Psi_\lambda$ converges weakly to zero. In addition to (2.24) it follows easily that for any bounded set B

$$\lim_{\lambda\to\infty}\int_B|\nabla\Psi_\lambda|^2d^nx=0. \tag{2.28}$$

We claim that

$$\lim_{\lambda\to\infty}\|(H-E-(\nabla F)^2))\Psi_\lambda\|=0. \tag{2.29}$$

To see (2.29) note first that from (2.7)

$$\limsup_{\lambda\to\infty}\|(H-E-(\nabla F)^2)\Psi_\lambda\|=\limsup_{\lambda\to\infty}\|(D\cdot\nabla F+\nabla F\cdot D)\Psi_\lambda\|. \tag{2.30}$$

Since $\nabla F=xg$, we find

$$D\cdot\nabla F+\nabla F\cdot D=2gA+x\cdot\nabla g, \tag{2.31}$$

and compute from (2.25)

$$g=\varrho^{-1}(\alpha+\gamma(1+\gamma\varrho\lambda^{-1})^{-1})\leqq(\alpha+\gamma)\varrho^{-1},$$
$$|x\cdot\nabla g|\leqq b_5\varrho^{-1}. \tag{2.32}$$

From (2.30), (2.32), and (2.24) we have

$$\limsup_{\lambda\to\infty}\|(H-E-(\nabla F)^2)\Psi_\lambda\|=\limsup_{\lambda\to\infty}2\|gA\Psi_\lambda\|. \tag{2.33}$$

By direct calculation and a simple estimate, we have

$$(x\cdot\nabla)^2g-(x\cdot\nabla)(\nabla F)^2\leqq b_6\varrho^{-1}+\gamma(\alpha+\gamma)/2, \tag{2.34}$$

so that from (2.9) we conclude

$$(\Psi_\lambda,[H,A]\Psi_\lambda)\leqq-4\|g^{1/2}A\Psi_\lambda\|^2+b_6(\Psi_\lambda,\varrho^{-1}\Psi_\lambda)+\gamma(\alpha+\gamma)/2. \tag{2.35}$$

Since $(-\Delta+1)^{-1}[H,A](-\Delta+1)^{-1}$ is bounded, and (2.27) holds, we have

$$\|g^{1/2}A\Psi_\lambda\|^2\leqq b_7. \tag{2.36}$$

From (2.32) we have

$$\|gA\Psi_\lambda\|^2\leqq(\alpha+\gamma)\|\varrho^{-1/2}g^{1/2}A\Psi_\lambda\|^2.$$

If χ_N is the characteristic function of $\{x:\varrho\leqq N\}$, we thus have

$$\|gA\Psi_\lambda\|^2\leqq(\alpha+\gamma)\|\chi_Ng^{1/2}A\Psi_\lambda\|^2+(\alpha+\gamma)N^{-1}b_7,$$

so that from (2.24) and (2.28) we have

$$\limsup_{\lambda\to\infty}\|gA\Psi_\lambda\|^2\leqq(\alpha+\gamma)N^{-1}b_7.$$

Since N is arbitrarily large, $\lim_{\lambda\to\infty}\|gA\Psi_\lambda\|=0$, so that (2.33) implies (2.29). From (2.29) we conclude that

$$\limsup_{\lambda\to\infty}\|(H-E-\alpha^2)\Psi_\lambda\|\leqq 2\gamma\alpha+\gamma^2, \tag{2.37}$$

and thus (from (2.21))

$$\limsup_{\lambda\to\infty} \|E(\mathbb{R}\backslash\Delta)\Psi_\lambda\| \leqq \limsup_{\lambda\to\infty} \|(H-E-\alpha^2)(2/\delta)E(\mathbb{R}\backslash\Delta)\Psi_\lambda\| \leqq b_8\gamma, \tag{2.38}$$

$$\limsup_{\lambda\to\infty} \|(H+i)E(\mathbb{R}\backslash\Delta)\Psi_\lambda\| \leqq |E+\alpha^2+i|b_8\gamma + \limsup_{\lambda\to\infty} \|(H-E-\alpha^2)\Psi_\lambda\| \leqq b_9\gamma. \tag{2.39}$$

From (2.39) it follows that

$$\limsup_{\lambda\to\infty} \|(-\Delta+1)E(\mathbb{R}\backslash\Delta)\Psi_\lambda\| \leqq b_{10}\gamma. \tag{2.40}$$

From (2.5) and the fact that Ψ_λ converges weakly to zero we have

$$\liminf_{\lambda\to\infty}(\Psi_\lambda, E(\Delta)[H,A]E(\Delta)\Psi_\lambda) \geqq c_0 \liminf_{\lambda\to\infty} \|E(\Delta)\Psi_\lambda\|^2 \geqq c_0(1-(b_8\gamma)^2). \tag{2.41}$$

From (2.35) however,

$$\limsup_{\lambda\to\infty}(\Psi_\lambda, [H,A]\Psi_\lambda) \leqq \gamma(\alpha+\gamma)/2 \leqq b_{11}\gamma. \tag{2.42}$$

As in previous argument (2.27), (2.40), and (2.42) imply

$$\limsup_{\lambda\to\infty}(\Psi_\lambda, E(\Delta)[H,A]E(\Delta)\Psi_\lambda) \leqq b_{12}\gamma. \tag{2.43}$$

Since c_0 is a fixed positive number, (2.43) contradicts (2.41) for all small enough $\gamma>0$. Thus the theorem is proved. □

To apply Theorem 2.1 to operators of the form (1.1), let us first introduce some notation [3]. Define the inner product

$$\langle x, y\rangle = \sum_{i=1}^{N} 2m_i x_i \cdot y_i \tag{2.44}$$

on $\mathbb{R}^{\nu N}$. Here $x=(x_1, \ldots, x_N)$, $y=(y_1, \ldots, y_N)$, and $x_i \cdot y_i$ indicates the usual inner product in $\mathbb{R}^\nu$. Given a point $x\in\mathbb{R}^{\nu N}$, let the center of mass of x be given by

$$R(x) = \left(\sum_{i=1}^{N} m_i x_i\right) \Big/ \left(\sum_{i=1}^{N} m_i\right).$$

Define the subspace $X \subset \mathbb{R}^{\nu N}$ by

$$X = \{x\in\mathbb{R}^{\nu N} : R(x)=0\},$$

and the projections $\Pi_{ij}:\mathbb{R}^{\nu N}\to\mathbb{R}^{\nu N}$

$$(\Pi_{ij}x)_k = \begin{cases} x_k-(m_ix_i+m_jx_j)/(m_i+m_j); & k=i \text{ or } j \\ 0; & \text{otherwise}. \end{cases}$$

Note $(\Pi_{ij}x)_i = m_j(x_i-x_j)/(m_i+m_j)$, $(\Pi_{ij}x)_j = m_i(x_j-x_i)/(m_i+m_j)$. It is easy to check that Π_{ij} is an orthogonal projection relative to the inner product $\langle\cdot,\cdot\rangle$ and that

$\Pi_i: X \to X$. The reason that (2.44) is natural is that if

$$\tilde{H}_0 = -\sum_{i=1}^{N} \Delta_{x_i}/2m_i,$$

then in fact $-\tilde{H}_0$ is the Laplace-Beltrami operator for $\mathbb{R}^{\nu N}$ with inner product $\langle \cdot, \cdot \rangle$. In other words if we introduce an orthogonal basis $\{e_1, \ldots, e_{\nu N}\}$ in $\mathbb{R}^{\nu N}$ and define the coordinates $\{x^\alpha : \alpha = 1, \ldots, \nu N\}$ of a point x by writing $x = \sum_{\alpha=1}^{\nu N} x^\alpha e_\alpha$, then

$$\tilde{H}_0 = -\sum_{\alpha=1}^{\nu N} \partial^2/(\partial x^\alpha)^2 \equiv -\Delta.$$

Removal of the center of mass motion in this language can be understood by writing

$$\Delta = \Delta_X + \Delta_{X^\perp},$$

where $X \oplus X^\perp = \mathbb{R}^{\nu N}$ and Δ_X is the Laplace-Beltrami operator for the subspace X with inner product (2.44). The operator H_0 ($\tilde{H}_0$ with center of mass removed) is just $-\Delta_X$ so that (1.1) can be written

$$H = -\Delta_X + V.$$

The potentials $(V_{ij}(x) = v_{ij}(x_i - x_j))$ clearly satisfy $V_{ij}(x) = V_{ij}(\Pi_{ij} x)$ and thus can be considered functions on Range $\Pi_{ij} \subset X$. We are thus led to consider operators of the form (1.1) on $L^2(\mathbb{R}^n)$, where

$$V(x) = \sum_{i=1}^{M} V_i(\Pi_i x). \tag{2.45}$$

Here Π_i is a non-zero orthogonal projection. We assume that each V_i is a real-valued function such that (denoting $\nu_i = \dim$ Range Π_i, Δ_i = Laplacian in $L^2(\mathbb{R}^{\nu_i})$)

(a) $$V_i(-\Delta_i + 1)^{-1} \text{ is compact on } L^2(\mathbb{R}^{\nu_i}). \tag{2.46}$$

(b) $$(-\Delta_i + 1)^{-1} y \cdot \nabla V_i (-\Delta_i + 1)^{-1} \text{ is compact on } L^2(\mathbb{R}^{\nu_i}). \tag{2.47}$$

The statement of the Mourre estimate for these more general operators requires a definition of $\mathcal{T}(H)$. Let $\mathcal{M} = \{1, 2, \ldots, M\}$. For each non-empty $I \subset \mathcal{M}$, let

$$\mathcal{V}_I = \left\{ x \in \mathbb{R}^n : x = \sum_{i \in I} u_i, \text{ with } u_i \in \text{Range } \Pi_i \right\},$$

and let $\mathcal{V}_\emptyset = \{0\}$. Let $\mathcal{F}$ be the family of subspaces of $\mathbb{R}^n$ given by

$$\mathcal{F} = \{\mathcal{V}_I : I \subset \mathcal{M}\}. \tag{2.48}$$

For $\mathcal{V} \in \mathcal{F}$ with $\mathcal{V} \neq \{0\}$, let

$$H_{\mathcal{V}} = -\Delta_{\mathcal{V}} + \sum_{\text{Range } \Pi_i \subset \mathcal{V}} V_i(\Pi_i x), \tag{2.49}$$

where $\Delta_{\mathcal{V}}$ is the Laplace operator for the subspace $\mathcal{V}$ and $H_{\mathcal{V}}$ is an operator in $L^2(\mathbb{R}^k)$ with $k = \dim \mathcal{V}$. If $\mathcal{V}$ is $\{0\}$ we define $H_{\mathcal{V}} = 0$ on $\mathbb{C}$. We can now define

$$\mathcal{T}(H) = \{E : E \text{ is an eigenvalue of } H_{\mathcal{V}} \text{ for some } \mathcal{V} \in \mathcal{F} \text{ with } \mathcal{V} \neq \mathbb{R}^n\}. \tag{2.50}$$

Theorem 1.1 follows from Theorem 2.1 and the following result (Theorem 2.3) of Perry et al. [15]. Actually in the latter paper the Mourre estimate is only an intermediate result. The authors consistently make assumptions stronger than (2.46) and (2.47) which they need in order to prove absence of singular continuous spectrum, although these stronger assumptions are not needed to prove the Mourre estimate. In addition in [15] the Mourre estimate is only proved when V is an N-body potential of the form (1.2) and not in the more general case where V is given by (2.45). However the authors explicitly state that their method works for these more general potentials with a suitable definition of thresholds. In [10] we give an alternative (and we believe, simpler) proof of the following result:

Theorem 2.3 [15]. *Suppose $H=-\Delta+V$ in $L^2(\mathbb{R}^n)$, where V is given by* (2.45) *with V_i real-valued multiplication operators satisfying* (2.46) *and* (2.47). *Define $\mathscr{T}(H)$ by* (2.50). *Then $\mathscr{T}(H)$ is a closed countable set and*

$$\mathscr{E}(H)\subset\mathscr{T}(H).$$

It is not difficult to see that the set $\mathscr{T}(H)$ as defined in (2.50) coincides with the set of thresholds defined in Sect. I if V is an N-body potential of the form (1.2).

Combining Theorems 2.1 and 2.3 we have

Corollary 2.4. *Suppose H is as in Theorem 2.3 and $\mathscr{T}(H)$ is given by* (2.50). *Suppose $H\psi=E\psi$. Then*

$$\sup\{\alpha^2+E:\alpha\geqq 0, \exp(\alpha|x|)\psi\in L^2(\mathbb{R}^n)\}$$

is either $+\infty$ or in $\mathscr{T}(H)$.

We end this section with an example which shows that the Mourre estimate is valid for more general potentials than those satisfying (2.46) and (2.47). Our example may seem impossible at first glance because it involves the von Neumann and Wigner [23] potential which has a positive energy bound state:

Lemma 2.5. *Let $H=-d^2/dx^2+V(x)$, where $V(x)=V_1(x)+\alpha(\sin kx)/x$ and*

(a) *α is real, $k>0$,*

(b) *V_1 is real and $V_1(-d^2/dx^2+1)^{-1}$ is compact,*

(c) *$(-d^2/dx^2+1)^{-1}(x\cdot\nabla V_1)(-d^2/dx^2+1)^{-1}$ is compact.*

Then $\mathscr{E}(H)\subset\{0,k^2/4\}$. In addition if $|\alpha|<k$, then $\mathscr{E}(H)\subset\{0\}$.

Proof. We follow [11, 12] except for one important difference. Let $H_0=-d^2/dx^2$ and suppose f is a real-valued function in $C_0^\infty(\mathbb{R})$. Then it is easily seen that $(H_0+1)(f(H)-f(H_0))$ is compact. Thus

$$\begin{aligned} f(H)[H,A]f(H) &= f(H_0)[H,A]f(H_0)+\text{compact} \\ &= f(H_0)(2H_0-(x\cdot\nabla V_1)+\alpha(\sin kx)/x-k\alpha\cos kx)f(H_0) \\ &= 2H_0(f(H_0))^2-k\alpha f(H_0)(\cos kx)f(H_0)+\text{compact}. \end{aligned} \tag{2.51}$$

If we write $p=-id/dx$ and $\cos kx=(e^{ikx}+e^{-ikx})/2$, we have

$$f(H_0)\cos kx\, f(H_0)=1/2\{f(p^2)e^{ikx}f(p^2)+f(p^2)e^{-ikx}f(p^2)\}.$$

Since $e^{ikx}pe^{-ikx}=p-k$, we have

$$f(p^2)e^{ikx}f(p^2)=f(p^2)f((p-k)^2)e^{ikx}.$$

Suppose $f(p^2)=g(p)$, where g has support in $\{p:|p-p_0|<\varepsilon \text{ or } |p+p_0|<\varepsilon\}$ for some $p_0>0$, and $p_0 \neq k/2$. Then if ε is chosen so that $\varepsilon<k/2$ and $\varepsilon<|p_0-k/2|$, the reader can easily check that $f(p^2)f((p-k)^2)=0$. Thus $f(H_0)\cos kx\, f(H_0)=0$ and

$$f(H)[H,A]f(H)=2H_0(f(H_0))^2+\text{compact}.$$

If ε is small enough $2H_0f(H_0)^2 \geqq c_0 f(H_0)^2$ for some $c_0>0$, so that again using the compactness of $f(H_0)^2-f(H)^2$ the Mourre estimate follows for positive λ_0 ($\lambda_0=p_0^2$). For Δ a compact interval contained in $(-\infty,0)$, $E(\Delta)$ is compact so that for negative λ_0, the Mourre estimate is trivial.

To prove the last statement of the Lemma, according to (2.51) it is sufficient to show that if $|\alpha|<k$,

$$2H_0f(H_0)^2-k\alpha f(H_0)\cos kx\, f(H_0) \geqq c_0 f(H_0)^2 \tag{2.52}$$

for some $c_0>0$ and some f which is 1 in an interval containing $k^2/4$. Suppose $\delta\in(0,1)$. Let χ_+ be the indicator function of

$$[(1-\delta)k/2,(1+\delta)k/2] \quad \text{and} \quad \chi_-(x)=\chi_+(-x).$$

Then with $f(H_0)=\chi_+(p)+\chi_-(p)$, we have

$$2H_0f(H_0)^2 \geqq ((1-\delta)^2k^2/2)f(H_0)^2. \tag{2.53}$$

We will show that

$$\|f(H_0)\cos kx\, f(H_0)\|=1/2, \tag{2.54}$$

so that from (2.53) and the fact that $f(H_0)^2=f(H_0)$

$$2H_0f(H_0)^2-k\alpha f(H_0)\cos kx\, f(H_0) \geqq f(H_0)^2((1-\delta)^2(k^2/2)-|k\alpha|/2). \tag{2.55}$$

If δ is small enough (2.55) implies (2.52), so that it only remains to prove (2.54). Using $\chi_-e^{ikx}\chi_+=\chi_+e^{ikx}\chi_+=\chi_-e^{ikx}\chi_-=0$, it is easy to see that

$$B\equiv f(H_0)\cos kx\, f(H_0)=1/2\{\chi_+e^{ikx}\chi_-+\chi_-e^{ikx}\chi_+\}. \tag{2.56}$$

We have

$$\begin{aligned}\|B\psi\|^2&=(1/4)\{\|\chi_+e^{ikx}\chi_-\psi\|^2+\|\chi_-e^{-ikx}\chi_+\psi\|^2\}\\ &\leqq(1/4)\{\|\chi_-\psi\|^2+\|\chi_+\psi\|^2\}=(1/4)\|\psi\|^2,\end{aligned}$$

so that $\|B\|\leqq 1/2$. Clearly if $\chi_+\psi=\psi$, then $B\psi=1/2e^{-ikx}\psi$ and hence $\|B\|=1/2$. This gives (2.54). □

Corollary 2.6. *Let H be as in Lemma 2.5. Then H has no positive eigenvalues except possibly at $k^2/4$. If $|\alpha|<k$ then H has no positive eigenvalues.*

Proof. According to Theorem 2.1 and Lemma 2.5, if $H\psi=E\psi$ with $E>0$, then if $|\alpha|<k$ or $E\neq k^2/4$ we must have $\exp(\alpha x)\psi\in L^2(\mathbb{R})$ for some $\alpha>0$. This contradicts Theorem II.1 of [11] unless $\psi=0$.

Remark. Results of this type have been proved by O.D.E. techniques [4, 8, 16] in the case where $|V_1(x)\leqq c(1+|x|)^{-1-\varepsilon}$ for some $\varepsilon>0$. In fact in [4, 16] it is shown that if $|\alpha|>k$, a positive eigenvalue can indeed occur. (With a short range potential V_1, the borderline case $|\alpha|=k$ does not produce a positive eigenvalue [4, 16].)

III. Exponential Lower Bounds

In this section we will consider self-adjoint operators of the form

$$H = -\Delta + V \tag{3.1}$$

in $L^2(\mathbb{R}^n)$, where as in the last part of Sect. II, V is a function of the form

$$V(x) = \sum_{i=1}^{M} V_i(\Pi_i x). \tag{3.2}$$

Here $\Pi_i : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal projection (with respect to the usual inner product). We will sometimes abuse notation and consider V_i to be a function on $\mathbb{R}^{\nu_i}$, $\nu_i = \dim(\mathrm{Range}\,\Pi_i) > 0$.

We state the following result whose proof is the subject of this section. (The first part of the theorem is given in [11].)

Theorem 3.1. *Suppose H is of the form* (3.1) *and V is given by* (3.2), *where V_i is a real-valued measurable function. Let $p_i = \mathrm{Max}(2, \nu_i - 1)$. Suppose either that*

(i) *V is Δ-bounded with bound less than one, $(-\Delta+1)^{-1} x \cdot \nabla V (-\Delta+1)^{-1}$ is bounded, and for some b_1 and b_2 with $b_1 < 2$, we have*

$$x \cdot \nabla V \leqq -b_1 \Delta + b_2,$$

or

(ii) *for each i, $V_i \in L^{p_i}(\mathbb{R}^{\nu_i}) + L^\infty(\mathbb{R}^{\nu_i})$ and there is a decomposition $V_i = V_i^{(1)} + V_i^{(2)}$, where $(1+|y|)V_i^{(1)} \in L^{p_i}(\mathbb{R}^{\nu_i}) + L^\infty(\mathbb{R}^{\nu_i})$ and for each $\varepsilon > 0$, $y \cdot \nabla V_i^{(2)} \leqq -\varepsilon \Delta_i + b_\varepsilon$ for some b_ε.*

Suppose that $H\psi = E\psi$ with $\exp(\alpha|x|)\psi \in L^2(\mathbb{R}^n)$ for all α. Then $\psi = 0$.

We refer the reader to the remark made after Theorem 1.3 for a comment about the relationship of conditions (i) and (ii).

Proof. That (i) implies the result follows from [11]. We do not repeat the proof here although the astute reader will be able to reconstruct such a proof from what follows. Thus assume that (ii) holds.

For simplicity we first consider the case $n \geqq 3$ and indicate the necessary modifications for $n \leqq 2$ later. Suppose that $\psi \neq 0$ and let $\psi_\alpha = \exp(\alpha r)\psi$, $r = |x|$, and define $\Psi_\alpha = \psi_\alpha / \|\psi_\alpha\|$. Then as in Lemma 2.2 we find $\Psi_\alpha \in \mathscr{D}(H)$,

$$(-\Delta - \alpha^2 - E + \alpha B)\Psi_\alpha = -V\Psi_\alpha, \tag{3.3}$$

$$B = r^{-1}(x \cdot D) + (D \cdot x) r^{-1}, \tag{3.4}$$

$$(\Psi_\alpha, H\Psi_\alpha) = \alpha^2 + E, \tag{3.5}$$

$$(\Psi_\alpha, [H, A]\Psi_\alpha) = -4\alpha \|A r^{-1/2} \Psi_\alpha\|^2. \tag{3.6}$$

These equations all appear in [11]. They are not difficult to obtain from those in Lemma 2.2. The singularity at $r=0$ is not harmful if $n \geqq 3$. Taking the norm of both sides of (3.3) gives

$$\|(-\Delta - \alpha^2 - E)\Psi_\alpha\|^2 + \alpha^2 \|B\Psi_\alpha\|^2 + \alpha(\Psi_\alpha, [B, \Delta]\Psi_\alpha) = \|V\Psi_\alpha\|^2. \tag{3.7}$$

A computation shows that as a quadratic form on $\mathscr{D}(\Delta)\times\mathscr{D}(\Delta)$

$$[B,\Delta]=-4\sum_{i,j}D_iQ_{ij}D_j+(n-1)(n-3)r^{-3}, \tag{3.8}$$

where

$$Q_{ij}(x)=r^{-1}(\delta_{ij}-x_ix_j/r^2). \tag{3.9}$$

Thus from (3.7), (3.8), and (3.9) it follows that

$$\|(-\Delta-\alpha^2-E)\Psi_\alpha\|\leqq\|V\Psi_\alpha\|. \tag{3.10}$$

Here we have used the fact that the matrix (Q_{ij}) is non-negative. Let

$$K_\alpha=((-\Delta-\alpha^2-E)^2+\alpha^2)^{1/2}.$$

We claim that

$$\lim_{\alpha\to\infty}\|VK_\alpha^{-1}\|=0. \tag{3.11}$$

Given (3.11) it follows from (3.10) that for large enough α

$$\|(-\Delta-\alpha^2-E)\Psi_\alpha\|\leqq 1/2\|K_\alpha\Psi_\alpha\|\leqq 1/2\|(-\Delta-\alpha^2-E)\Psi_\alpha\|+1/2\alpha,$$

so that

$$\|(-\Delta-\alpha^2-E)\Psi_\alpha\|\leqq\alpha,$$

and thus

$$\|K_\alpha\Psi_\alpha\|\leqq 2\alpha. \tag{3.12}$$

From (3.5) it follows that for all large α and some $c_1>0$

$$\|\nabla\Psi_\alpha\|\geqq c_1\alpha. \tag{3.13}$$

Let $W_1=x\cdot\nabla V^{(1)}$, $W_2=x\cdot\nabla V^{(2)}$. By assumption we have $-\Delta-W_2\geqq -b$ for some b. Using (3.6) and (3.13) gives

$$0\geqq(\Psi_\alpha,(-2\Delta-W_2-W_1)\Psi_\alpha)\geqq -b+c_1^2\alpha^2-(\Psi_\alpha,W_1\Psi_\alpha). \tag{3.14}$$

Since from (3.12)

$$(\Psi_\alpha,W_1\Psi_\alpha)\leqq\|K_\alpha^{-1}W_1K_\alpha^{-1}\|\cdot(4\alpha^2), \tag{3.15}$$

if we can show that

$$\lim_{\alpha\to\infty}\|K_\alpha^{-1}W_1K_\alpha^{-1}\|=0, \tag{3.16}$$

(3.14) will provide a contradiction for large α. Thus we must prove (3.11) and (3.16).

To see (3.11) first note that $\|K_\alpha^{-1}\|\to 0$ so that to prove $\|V_i(\Pi_ix)K_\alpha^{-1}\|\to 0$, we can assume that $V_i\in L^{p_i}(\mathbb{R}^{\nu_i})$. Since we can always write $V_i=f_\varepsilon+g_\varepsilon$ with $g_\varepsilon\in L^\infty(\mathbb{R}^{\nu_i})$, $\|f_\varepsilon\|_{p_i}<\varepsilon$ it suffices to show that

$$\limsup_{\alpha\to\infty}\|V_i(\Pi_ix)K_\alpha^{-1}\|\leqq c\|V_i\|_{p_i}. \tag{3.17}$$

To see (3.17) we factor $L^2(\mathbb{R}^n)=L^2(\mathbb{R}^{\nu_i})\otimes L^2(\mathbb{R}^{n-\nu_i})$ and write

$$V_i(\Pi_ix)K_\alpha^{-1}=V_i(y)((-\Delta_y-\Delta_\perp-\alpha^2-E)^2+\alpha^2)^{-1/2}, \tag{3.18}$$

where Δ_y is the v_i dimensional Laplacian in the variable y and $\Delta_\perp$ involves orthogonal coordinates. To prove (3.17) it thus suffices to show that for each $t\in[0,\infty)$

$$\|V_i(y)((-\Delta_y+t-\alpha^2-E)^2+\alpha^2)^{-1/2}\|\leqq c\|V_i\|_{p_i}, \tag{3.19}$$

where c is independent of t and α and the norm on the left in (3.19) is in $L^2(\mathbb{R}^{v_i})$. To prove (3.19) we use the estimate [20]

$$\|V_i(y)((-\Delta_y+t-\alpha^2-E)^2+\alpha^2)^{-1/2}\|\leqq\|V_i\|_{p_i}\|f_{t,\alpha}\|_{p_i}(2\pi)^{-n/p_i}, \tag{3.20}$$

where $f_{t,\alpha}(y)=((|y|^2+t-\alpha^2-E)^2+\alpha^2)^{-1/2}$. We claim that

$$\|f_{t,\alpha}\|_{p_i}\leqq c', \tag{3.21}$$

where c' is independent of t and α for large α. Clearly once (3.21) is proved we will have shown (3.11). To estimate $\|f_{t,\alpha}\|_{p_i}$ we assume that $t-\alpha^2-E=-\beta^2\leqq-1$. Otherwise (3.21) is easy. We have with $f=f_{t,\alpha}$

$$\int|f|^{p_i}dy^{v_i}=\int_{|y|^2<2\beta^2}|f|^{p_i}dy^{v_i}+\int_{|y|^2>2\beta^2}|f|^{p_i}dy^{v_i}. \tag{3.22}$$

The first integral in (3.22) can be estimated by

$$c\beta^{v_i-1}\int_0^\infty((x^2-\beta^2)^2+\alpha^2)^{-p_i/2}dx. \tag{3.23}$$

We use $(x^2-\beta^2)^2\geqq\beta^2(x-\beta)^2$, where $\beta\geqq1$ to show that (3.23) is less than

$$2c\beta^{v_i-1}\int_0^\infty(\beta^2(x-\beta)^2+\alpha^2)^{-p_i/2}dx\leqq4c(\beta^{v_i-2}/\alpha^{p_i-1})\int_0^\infty(s^2+1)^{-p_i/2}ds$$
$$\leqq\text{const}.$$

The second term in (3.22) is easily shown to be bounded and hence (3.21) follows. We must now prove (3.16). Since $x\cdot\nabla V_i(\Pi_i x)=(\Pi_i x)\cdot(\nabla V_i)(\Pi_i x)$, it suffices to show that

$$\lim_{\alpha\to\infty}\|((-\Delta_y+t-\alpha^2-E)^2+\alpha^2)^{-1/2}y\cdot\nabla V^{(1)}(y)((-\Delta_y+t-\alpha^2-E)^2+\alpha^2)^{-1/2}\|=0 \tag{3.24}$$

uniformly in t for $t\geqq0$. Note that $y\cdot\nabla V_i^{(1)}=[y\cdot D,V_i^{(1)}]=D\cdot(yV_i^{(1)})-(yV_i^{(1)})\cdot D-v_iV_i^{(1)}$. By our previous estimate we already know that

$$\lim_{\alpha\to\infty}\|(1+|y|)V_i^{(1)}((-\Delta_y+t-\alpha^2-E)^2+\alpha^2)^{-1/2}\|=0$$

uniformly in t, so we need only show that

$$\sup_{t\geqq0,\alpha\geqq1}\|D_y((-\Delta_y+t-\alpha^2-E)^2+\alpha^2)^{-1/2}\|<\infty. \tag{3.25}$$

Inequality (3.25) follows from the numerical estimate

$$\sup\{|x|((|x|^2+t-\alpha^2-E)^2+\alpha^2)^{-1/2}:x\in\mathbb{R}^{v_i},t\geqq0,\alpha\geqq1\}<\infty,$$

which is easy to prove. Hence the proof of Theorem 3.1 is complete in the case $n\geqq3$.

To handle the case $n \leqq 2$ we introduce a cutoff function $\eta \in C^\infty(\mathbb{R}^n)$ which is zero in a neighborhood of $x=0$ and one in a neighborhood of infinity. Assuming that $\psi \neq 0$ we can choose η so that $\eta\psi \neq 0$. Defining Ψ_α as before, we have

$$(-\Delta - \alpha^2 - E + \alpha B)\eta\Psi_\alpha = -V\eta\Psi_\alpha + g_\alpha,$$

where

$$g_\alpha = -(\Delta\eta)\Psi_\alpha - 2\nabla\eta \cdot \nabla\Psi_\alpha + 2\alpha(x \cdot \nabla\eta)\Psi_\alpha r^{-1}.$$

By choosing the support of $1-\eta$ sufficiently small we can arrange that

$$\|g_\alpha\| \leqq \exp(-\delta\alpha)\|\eta\Psi_\alpha\|$$

for some $\delta > 0$. Since $(n-1)(n-3)(\eta\Psi_\alpha, r^{-3}\eta\Psi_\alpha) \geqq -c\|\eta\Psi_\alpha\|^2$ for some c (depending on η), we easily find the estimate

$$\|K_\alpha \eta\Psi_\alpha\| \leqq c'\alpha\|\eta\Psi_\alpha\| \tag{3.26}$$

in the same way as before. Similarly we can arrange

$$(\eta\Psi_\alpha, H\eta\Psi_\alpha) = (E + \alpha^2 + O(\exp(-\delta\alpha)))\|\eta\Psi_\alpha\|^2, \tag{3.27}$$

and

$$(\eta\Psi_\alpha, [H, A]\eta\Psi_\alpha) \leqq O(\exp(-\delta\alpha))\|\eta\Psi_\alpha\|^2, \tag{3.28}$$

for some $\delta > 0$ by choosing the support of $1-\eta$ sufficiently small.

Proceeding as before using (3.26) through (3.28) yields the result. □

We give the result analogous to Corollary 1.4 for the more general potential of the type given in (3.2) in the following corollary:

Corollary 3.2. *Suppose $H = -\Delta + V$ with V of the form* (3.2). *Suppose each V_i is real and satisfies* (2.46) *and* (2.47). *Let $p_i = \mathrm{Max}(2, \nu_i - 1)$. Suppose in addition that either*

(i) *for each i and $\varepsilon > 0$, $y \cdot \nabla V_i(y) \leqq -\varepsilon\Delta_i + b_\varepsilon$ for some b_ε,*

or

(ii) *for each i, $V_i \in L^{p_i}(\mathbb{R}^{\nu_i}) + L^\infty(\mathbb{R}^{\nu_i})$ and there is a decomposition $V_i = V_i^{(1)} + V_i^{(2)}$, where $(1+|y|)V_i^{(1)} \in L^{p_i}(\mathbb{R}^{\nu_i}) + L^\infty(\mathbb{R}^{\nu_i})$ and for each $\varepsilon > 0$, $y \cdot \nabla V_i^{(2)} \leqq -\varepsilon\Delta_i + b_\varepsilon$ for some b_ε.*

Then H has no positive eigenvalues and if $H\psi = E\psi$ with $\psi \neq 0$, it follows that

$$\exp(\alpha|x|)\psi \notin L^2(\mathbb{R}^n); \quad \alpha > \sqrt{-E}.$$

The proof of this result is very similar to the proof of Corollary 1.4. The induction is now on the family of subspaces $\mathscr{F}$ defined in (2.48). We omit the details.

Appendix: Proof of Lemma 2.2

In this appendix we sketch the proof of Lemma 2.2 using a method which is different from that in [11] or [12].

Let F be either of the two functions given in the lemma, and let $\xi = \exp(F)$. Suppose $\varphi \in C_0^\infty(\mathbb{R}^n)$. Then the following formula is not difficult to derive

$$(\varphi, [\xi A \xi, -\Delta]\varphi) = (\xi\varphi, [A, -\Delta]\xi\varphi) - 4\|g^{1/2}A\xi\varphi\|^2 + (\xi\varphi, G\xi\varphi), \tag{A.1}$$

where

$$G(x)=(x\cdot\nabla)^2 g - x\cdot\nabla(\nabla F)^2\,. \tag{A.2}$$

By definition of the distribution $x\cdot\nabla V$, it is easy to see that

$$\begin{aligned}(-\xi A\xi\varphi, V\varphi)-(V\varphi,\xi A\xi\varphi) &= -(A(\xi\varphi), V(\xi\varphi))-(V\xi\varphi, A(\xi\varphi))\\ &=(\xi\varphi, x\cdot\nabla V\xi\varphi)\,.\end{aligned} \tag{A.3}$$

Here if W is the distribution $x\cdot\nabla V$, that is

$$W(f)=-\int V(x)\left(\sum_{l=1}^{n}\partial_l(x_l f)\right)d^n x\,,$$

then $(\xi\varphi, x\cdot\nabla V\xi\varphi)$ means $W(|\xi\varphi|^2)$. We have assumed that $(-\Delta+1)^{-1}x\cdot\nabla V(-\Delta+1)^{-1}$ extends to a bounded operator, so we will continue to use the notation $(f,(x\cdot\nabla V)f)$ for $f\in\mathscr{D}(\Delta)$.

Hence from (A.1) we have

$$-2\,\mathrm{Re}(\xi A\xi\varphi,(H-E)\varphi)=(\xi\varphi,[A,H]\xi\varphi)-4\|g^{1/2}A\xi\varphi\|^2+(\xi\varphi, G\xi\varphi)\,. \tag{A.4}$$

Suppose $F=\lambda\ln(\varrho(1+\varepsilon\varrho)^{-1})$. Then ξ is a bounded function in $C^\infty(\mathbb{R}^n)$ and we have $g\leqq\text{const}\,\varrho^{-3}$, $|G|\leqq\text{const}$, $|\nabla F|\leqq\text{const}$, $|\Delta F|\leqq\text{const}$.

Let $H(F)$ and ψ be as in the lemma. Then clearly $H(F)$ is a closed operator on $\mathscr{D}(\Delta)$ with $C_0^\infty(\mathbb{R}^n)$ a core. In addition $H(F)\psi_F=E\psi_F$ in the sense of distributions ($\psi_F=\xi\psi$), so that since C_0^∞ is a core for $H(F)$ we must have $\psi_F\in\mathscr{D}(\Delta)$ and $H(F)\psi_F=E\psi_F$ as vectors in $L^2(\mathbb{R}^n)$. Equation (2.8) thus follows by writing

$$\begin{aligned}(\psi_F, H(F)\psi_F) &= E\|\psi_F\|^2\\ &=\mathrm{Re}(\psi_F, H(F)\psi_F)\\ &=(\psi_F(H-(\nabla F)^2)\psi_F)\,.\end{aligned}$$

To prove (2.9) we first note that if χ is in $C_0^\infty(\mathbb{R}^n)$, it is easy to prove (A.4) for $\varphi=\chi\psi$. Let $\chi(x)=\chi_m(x)=\chi_1(x/m)$, where $\chi_1\in C_0^\infty(\mathbb{R}^n)$ is one in a neighborhood of the origin. We have

$$(1+\varrho)(H-E)\chi_m\psi=(1+\varrho)(-(\Delta\chi_m)\psi-2\nabla\chi_m\cdot\nabla\psi)\,. \tag{A.5}$$

Clearly the right side of (A.5) converges pointwise to zero and

$$(1+\varrho)|(\Delta\chi_m)|+(1+\varrho)|\nabla\chi_m|\leqq\text{const}\,,$$

independent of m so by the dominated convergence theorem

$$\lim_{m\to\infty}\|(1+\varrho)(H-E)\chi_m\psi\|=0\,. \tag{A.6}$$

It is easy to see that $\|(1+\varrho)^{-1}\xi A\xi\chi_m\psi\|$ is bounded as $m\to\infty$ so that the left side of (A.4) converges to zero. Similarly the right side converges and we obtain (2.9).

The lemma is even more easily proved with $F=\alpha\varrho+\lambda\ln(1+\gamma\varrho\lambda^{-1})$ because it follows from the assumptions that $\varrho^k\exp(F)\psi=\varrho^k\xi\psi$ is in L^2 for all k. We first rewrite

$$(\xi A\xi\varphi,(H-E)\varphi)=(A\xi\varphi,(H(F)-E)\xi\varphi)\,.$$

It follows as above that $\psi_F \in \mathscr{D}(\Delta)$ and $(H(F)-E)\psi_F=0$. Using the same approximation scheme as above, it follows that $(H(F)-E)\chi_m\psi_F \to 0$ and $A\chi_m\psi_F$ is bounded (the latter because it easily follows that $\varrho^k A\psi_F \in L^2$ for all k). We omit the details of the proof.

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