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Exponential Decay for the Semilinear Cauchy-Ventcel Problem with Localized Damping

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ABSTRACT: Le but de ce travail est d' étudier la décroissance exponentielle de l'énergie des solutions losque le temps tend vers l'infini du problème aux limites de Cauchy-Ventcel semi-linéaire dissipatif dans un domaine borné. On donne des conditions suffisantes sur les non linéarités de f et g pour avoir la décroissance exponentielle de l'énergie. Ce problème décrit les vibrations d'un corps élastique avec un raidisseur mince sur le bord. La méthode de démonstration est basée sur les techniques de multiplicateurs et un principe de continuation unique qui permettent d'estimer l'énergie totale des solutions.

Key words: Cauchy-Ventcel problem, Exponential decay.

Contents

1. Introduction

Let Ω be a bounded, open, connected set in $\mathbb{R}^n\,(n \geq 2)$ having a boundary $\Gamma = \partial\Omega$ of class C^2. We denote by ∇_T the tangential-gradient on Γ, by Δ_T the tangential Laplacien on Γ and by ∂_ν the normal derivative towards the exterior of Γ. This paper is devoted to the study of the exponential decay of solutions of the following semilinear damped Cauchy-Ventcel problem:

$$\begin{cases} u_1'' - \Delta u_1 + f(u_1) + a(x)u_1' = 0 & \text{in} \quad \Omega \times]0,\infty[\\ u_2'' + \partial_\nu u_1 - \Delta_T u_2 + g(u_2) + b(x)u_2' = 0 & \text{on} \quad \Gamma \times]0,\infty[\\ u_{1|_\Sigma} = u_2, & \\ u(0) = (u_1(0), u_2(0)) = u'(0) = (u_1'(0), u_2'(0)) = 0 \ \text{ in } \Omega \times \Gamma. & \end{cases} \tag{1}$$

This equation modelise the asymptotic vibrations of an elastic body with a thin of high rigidity of its boundary.

We assume

$$\begin{aligned} &a \in L^\infty(\Omega), \ \ a(x) \geq a_0 > 0, \quad \text{a.e. in} \ \ \omega, \\ &b \in L^\infty(\Gamma), \ \ b(x) \geq b_0 > 0, \quad \text{a.e. in} \ \ \Gamma \end{aligned}$$

where $\omega \subset \Omega$ is an open, non-empty subset of Ω and a_0 and b_0 are constants,we assume also the non-negativity of f and g:

$$\begin{aligned} &f(s)s \geq 0, \\ &g(s)s \geq 0, \quad \forall s \in \mathbb{R}. \end{aligned} \tag{3}$$

We shall distinguish two particular cases where condition (3) is satisfied.

In the first case , we will assume f and g globally Lipschitz i.e.

$$\begin{cases} f, g \in C^1(\mathbb{R}) \text{ and there exist some constants } c_1, c_2 > 0,\ p > 1,\ (n-2)p \leq n \\ \text{such that} \\ |f(s_1) - f(s_2)| \leq c_1(1 + |s_1|^{p-1} + |s_2|^{p-1})\,|s_1 - s_2|\,, \\ |g(s_1) - g(s_2)| \leq c_2(1 + |s_1|^{p-1} + |s_2|^{p-1})\,|s_1 - s_2|\,, \text{ for every } s_1, s_2 \in \mathbb{R}. \end{cases} \tag{4}$$

In the second case, we will consider the case where f and g are super linear, i.e.

$$\begin{cases} \exists\, \delta_1 > 0 : \ f(s)s \geq (2+\delta_1)F(s), \\ \exists\, \delta_2 > 0 : g(s)s \geq (2+\delta_2)G(s), \end{cases} \quad \forall\, s \in \mathbb{R}, \tag{5}$$

with

$$F(z) = \int_0^z f(s)ds, \quad G(z) = \int_0^z g(s)ds,\ \forall z \in \mathbb{R}. \tag{6}$$

This situation can not be treated as a perturbation of the linear case. We shall therefore restrict our attention to the particular case where ω is a neighborhood of the boundary Γ. We shall adapt the multiplier technique developed in J. L. Lions [7] in order to obtain suitable energy estimates. In this case conditions (5) will be sufficient to establish the uniform exponential decay.

This paper is organized as follows. In Section 2 we shall state and prove the main results in the case where f and g are globally Lipschitz. The case where f and g are superlinear will be treated in Section 3.

We set

$$\begin{aligned} &\mathbf{V} = \left\{u = (u_1, u_2) \in H^1(\Omega) \times H^1(\Gamma) \ / \ u_1 \,|_\Gamma = u_2\right\} \\ &\mathbf{H} = L^2(\Omega) \times L^2(\Gamma). \end{aligned}$$

Equipped with the canonical norms

$$\begin{aligned} &|u|^2_{\mathbf{H}} = |u_1|^2_{L^2(\Omega)} + |u_2|^2_{L^2(\Gamma)}\,, \\ &\|u\|^2_{\mathbf{V}} = \|u_1\|^2_{H^1(\Omega)} + \|u_2\|^2_{H^1(\Gamma)}\,, \end{aligned}$$

$\mathbf{V}$ and $\mathbf{H}$ are two Hilbert spaces and $\mathbf{V}$ is dense in $\mathbf{H}$ with continuous injection . Under the conditions above, problem (1) is well posed in the space $\mathbf{V} \times \mathbf{H}$, i.e. for any initial data $\left\{u^0, u^1\right\} \in \mathbf{V} \times \mathbf{H}$, there exists a unique weak solution of (1) which belongs to the space

$$u \in C([0, \infty) ; \mathbf{V}) \cap C^1([0, \infty) ; \mathbf{H}). \tag{7}$$

Let us consider the energy

$$E(t) = \frac{1}{2}\left\{\int_\Omega \left[|u_1'|^2 + |\nabla u_1|^2\right] dx + \int_\Gamma \left[|u_2'|^2 + |\nabla_T u_2|^2\right] d\Gamma\right\} \\ + \int_\Omega F(u_1) dx + \int_\Gamma G(u_2) d\Gamma. \tag{8}$$

For every solution of (1)-(7) the following identity holds

$$E(t_2) - E(t_1) = \\ -\left[\int_{t_1}^{t_2}\int_\Omega a(x) \mid u_1'(x,t) \mid^2 dxdt + \int_{t_1}^{t_2}\int_\Gamma b(x) \mid u_2'(x,t) \mid^2 d\Sigma\right], \\ \forall t_2 > t_1 \geq 0 \tag{9}$$

and therefore the energy is a non increasing function of the variable t . The aim of this paper is to give sufficient conditions on the nonlinearities of f and g and the open subset ω (where the damping term is effective) ensuring the exponential decay of the energy, i.e. the existence of some constants $C > 1$ and $\gamma > 0$ such that

$$E(t) \leq Ce^{-\gamma t}E(0), \ \forall t \geq 0, \tag{10}$$

for every solution of (1)-(7).

2. The case of a globally Lipschitz nonlinearity

In the sequel we set $Q = \Omega \times]0,T[$, $\Sigma = \Gamma \times]0,T[$, and we define for arbitrary x^0 in $\mathbb{R}^n$ arbitrary we define

$$\begin{aligned} m(x) &= x - x^0, \\ \mathbb{R}\left(x^0\right) &= \max_{x\in\overline{\Omega}} |m(x)|, \\ \Gamma\left(x^0\right) &= \{x \in \Gamma \,/\, m(x).\nu(x) > 0\}, \\ \Gamma_*\left(x^0\right) &= \Gamma \backslash \Gamma\left(x^0\right), \\ \Sigma\left(x^0\right) &= \Gamma\left(x^0\right) \times]0,T[, \\ \Sigma_*\left(x^0\right) &= \Gamma_*\left(x^0\right) \times]0,T[. \end{aligned}$$

We define also the operator A by

$$A = \begin{pmatrix} \Delta & 0 \\ -\partial_\nu & \Delta_T \end{pmatrix},$$

and the domain of A by

$$D_A = \{u \in \mathbf{V}, \ \ Au \in \mathbf{H}\}.$$

Let us consider the following linear Cauchy-Ventcel problem

$$\begin{cases} \varphi_{tt} - A\varphi = 0, \\ \varphi(0) = \varphi^0, \, \varphi_t(0) = \varphi^1, \end{cases} \tag{11}$$

then we have the result

Lemma 2.1. *Let Ω be a bounded, open, connected set in $\mathbb{R}^n\,(n \geq 2)$ having a boundary $\Gamma = \partial\Omega$ of class C^2. Let $x^0 \in \mathbb{R}^n$* be arbitrary and $T > T\left(x^0\right) = 2R(x^0)$. Let O be a neighborhood of $\overline{\Gamma\left(x^0\right)}$ and $\omega = O \cap \Omega$. *Then, for every solution* φ of (11), *there exist some constant $C > 0$ such that*

$$\begin{aligned} E_0 \leq C \Big\{ & \textstyle\int_0^T \int_\omega |\varphi_1'(x,t)|^2\, dxdt + \int_\Sigma |\varphi_2'(x,t)|^2\, d\Sigma + \\ & + \textstyle\int_0^T |\nabla\varphi_1|^2_{L^2(\Omega)}\, dt + \int_0^T |\nabla_T\varphi_2|^2_{L^2(\Gamma)}\, dt \Big\},\ \forall(\varphi^0,\varphi^1) \in \mathbf{V}\times\mathbf{H}. \end{aligned} \tag{12}$$

Proof (cf. Appendix).
From A. Ruiz [10] , we have also the following result:
Lemma 2.2. *If $b \in L^\infty(Q)$, $\theta \in H^1(Q)$, then we have*

$$\left.\begin{array}{lll} \theta'' - \Delta\theta + b\theta \ = 0 & \text{in} & Q, \\ \theta = 0 & \text{on} & \omega\times\Sigma, \\ \partial_\nu\theta = 0 & \text{on} & \Sigma_0 \end{array}\right\} \Rightarrow \theta \equiv 0. \tag{13}$$

In the sequel we assume the existence of the following limits

$$\begin{array}{ll} (i) & \underset{s\to+\infty}{Lim} f'(s) = f'(+\infty)\,;\ \ \underset{s\to-\infty}{Lim} f'(s) = f'(-\infty)\,, \\ & \underset{s\to+\infty}{Lim} g'(s) = g'(+\infty)\,;\ \ \underset{s\to-\infty}{Lim} g'(s) = g'(-\infty)\,. \\ (ii) & \underset{s\to+\infty}{Lim} g'(s) = g'(+\infty)\,;\ \ \underset{s\to-\infty}{Lim} g'(s) = g'(-\infty)\,. \end{array}$$

The main result of this section is the following.
Theorem 2.1: *Let f and $g \in C^1(\mathbb{R})$ be such that (3) is satisfied, $f' \in L^\infty(\mathbb{R})$, $g' \in L^\infty(\mathbb{R})$ and the limits (i)-(ii) exist. Assume that*

$$\begin{array}{ll} a \in L^\infty(\Omega),\ \ a(x) \geq a_0 > 0, & \text{a.e. in}\ \ \omega, \\ b \in L^\infty(\Gamma),\ \ b(x) \geq b_0 > 0, & \text{a.e. in}\ \ \Gamma \end{array} \tag{14}$$

for some $a_0 > 0$, $b_0 > 0$ and some open subset $\omega \subset \Omega$ such that (12) and (13) hold.

Then, there exist some constants $C > 1$ and $\gamma > 0$ such that the estimate (10) holds for every solution $u = u(x,t)$ of (1) with initial data $(u^0, u^1) \in \mathbf{V}\times\mathbf{H}$.
Proof: We note that it is sufficient to prove the following estimate

$$E(T) \leq C_0 \left\{ \int_Q a(x) \mid u_1'(x,t) \mid^2 dxdt + \int_\Sigma b(x) \mid u_2'(x,t) \mid^2 d\Sigma \right\}. \tag{15}$$

From (9) and (15), we easily deduce

$$E(T) \leq \frac{C_0}{1+C_0} E(0). \tag{16}$$

This estimate, combined with the semigroup property (cf. J. Rauch et M. Taylor [8]) implies (10) with

$$C = \frac{1}{1+\frac{1}{C_0}}, \quad \gamma = \frac{1}{T}\log\left(\frac{1}{1+\frac{1}{C_0}}\right). \tag{17}$$

In order to prove (15) we write the solution $u = u(x,t)$ of (1) as

$$u = \Phi + \Psi$$

where $\Phi = \Phi(x,t)$ solves (11) with initial data

$$\Phi(0) = u^0,\ \Phi_t(0) = u^1$$

and $\Psi = \Psi(x,t)$ satisfies

$$\begin{cases} \Psi_1'' - \Delta\Psi_1 & = -f(u_1) - a(x)u_1' \quad \text{in} \quad \Omega \times]0,\infty[\,, \\ \Psi_2'' + \partial_\nu\Psi_1 - \Delta_T\Psi_2 & = -g(u_2) - b(x)u_2' \quad \text{on} \quad \Gamma \times]0,\infty[\,, \\ \Psi_{1_\Sigma} = \Psi_{2,} & \\ \Psi(0) = (\Psi_1(0),\Psi_2(0)) = \Psi'(0) = (\Psi_1'(0),\Psi_2'(0)) = 0 \ \text{ on } \Omega\times\Gamma. & \end{cases} \tag{18}$$

Lemma 2.3: *Let Ω be a bounded, open, connected set in $\mathbb{R}^n$ $(n \geq 2)$ having a boundary $\Gamma = \partial\Omega$ of classC^2. Let $x^0 \in \mathbb{R}^n$ be arbitrary, ω a neighbourhood of $\overline{\Gamma(x^0)}$ in Ω and $T > T(x^0)$.*

Then, *for every solution φ* of (11), *there exists some constant $C > 0$ such that*

$$\begin{aligned} E(T) &\leq E(0) \\ &\leq C\left\{\|u^0\|_{\mathbf{V}}^2 + |u^1|_{\mathbf{H}}^2\right\} \\ &\leq C\left\{\int_0^T\int_\omega |\Phi_1'(x,t)|^2\,dxdt + \int_\Sigma |\Phi_2'(x,t)|^2\,d\Sigma + \int_0^T |\Phi|_{\mathbf{H}}^2\,dt\right\}, \end{aligned} \tag{19}$$

$\forall (u^0,u^1) \in \mathbf{V}\times\mathbf{H}$.
Proof: (cf. Appendix).
By using the embedding of $\mathbf{V}$ in $\mathbf{H}$, we obtain

$$\begin{aligned} \int_0^T |\Phi|_{\mathbf{H}}^2\,dt &\leq C\left\{\int_0^T |u|_{\mathbf{H}}^2\,dt + \int_0^T |\Psi|_{\mathbf{H}}^2\,dt\right\} \\ &\leq C_1\left\{\int_0^T |u|_{\mathbf{H}}^2\,dt + \int_0^T \|\Psi\|_{\mathbf{V}}^2\,dt\right\}, \end{aligned}$$

and the standard energy estimates for (18) yield

$$\begin{aligned} &\int_0^T \left[\|\Psi\|_{\mathbf{V}}^2 + |\Psi'|_{\mathbf{H}}^2\right] dt \\ &\leq C\left\{\|f(u_1) + a(x)u_1'\|_{L^1(0,T;L^2(\Omega))}^2 + \|g(u_2) + b(x)u_2'\|_{L^1(0,T;L^2(\Gamma))}^2\right\}. \end{aligned} \tag{20}$$

Combining (19) and (20), we obtain finally

$$E(T) \leq C\left\{\int_Q a(x)\,|u_1'|^2\,dxdt + \int_\Sigma b(x)\,|u_2'(x,t)|^2\,d\Sigma + \int_0^T |u|_{\mathbf{H}}^2\,dt\right\} \tag{21}$$

(we have implicitely used the fact that

$$\begin{aligned} |f(s)| &\leq C_1 |s|, \\ |g(s)| &\leq C_2 |s|, \\ |F(s)| &\leq C |s|^2, \\ |G(s)| &\leq C |s|^2, \forall s \in \mathbb{R} \end{aligned}$$

Remark 3.1: We note that the constant $C > 0$ in (21) depends on $\|f'\|_{L^\infty(\mathbb{R})}$ and $\|g'\|_{L^\infty(\mathbb{R})}$ in a bounded manner .

It remains to prove the estimate

$$\int_0^T |u|_{\mathbf{H}}^2 \, dt \leq C \left\{ \int_Q a(x) |u_1'|^2 \, dxdt + \int_\Sigma b(x) |u_2'(x,t)|^2 \, d\Sigma \right\}. \tag{22}$$

We argue by contradiction. If (22) is not satisfied for some $C > 0$, there exists a sequence of solutions $\{u_n\}$ of (1)-(7) verifying

$$\lim_{n\to\infty} \frac{\int_0^T |u_n|_{\mathbf{H}}^2 \, dt}{\int_Q a(x) |u_{1n}'|^2 \, dxdt + \int_\Sigma b(x) |u_{2n}'(x,t)|^2 \, d\Sigma} = +\infty. \tag{23}$$

We set

$$\lambda_n = \left\{ \int_0^T |u_n|_{\mathbf{H}}^2 \, dt \right\}^{\frac{1}{2}}, \tag{24}$$

and

$$v_n(x,t) = (v_{1n(x,t)}, v_{2n(x,t)}) = (\frac{u_{1n(x,t)}}{\lambda_n}, \frac{u_{2n(x,t)}}{\lambda_n}). \tag{25}$$

The function $v_n(x,t)$ satisfies

$$\left\{ \begin{array}{lll} (v_{1n})_{tt} - \Delta v_{1n} + f_n(v_{1n}) + a(x)(v_{1n})_t & = 0 & \text{in } \Omega \times]0, \infty[, \\ (v_{2n})_{tt} + \partial_\nu v_{1n} - \Delta_T v_{2n} + g_n(v_{2n}) + b(x)(v_{2n})_t & = 0 & \text{on } \Gamma \times]0, \infty[, \\ v_{1n_\Gamma} = v_{2n}, \ \forall t > 0, & & \end{array} \right. \tag{26}$$

where

$$f_n(s) = \frac{1}{\lambda_n} f(\lambda_n s), \quad g_n(s) = \frac{1}{\lambda_n} g(\lambda_n s), \ \forall s \in \mathbb{R}, \ \forall n \in N. \tag{27}$$

On the other hand

$$\int_0^T |v_n|_{\mathbf{H}}^2 \, dt = 1, \tag{28}$$

and

$$\int_Q a(x) |v_{1n}'|^2 \, dxdt + \int_\Sigma b(x) |v_{2n}'(x,t)|^2 \, d\Sigma \to 0. \tag{29}$$

Taking into account the fact that

$$\begin{aligned} \|f_n'\|_{L^\infty(\mathbb{R})} &= \|f'\|_{L^\infty(\mathbb{R})}, \\ \|g_n'\|_{L^\infty(\mathbb{R})} &= \|g'\|_{L^\infty(\mathbb{R})} \end{aligned}$$

and the fact that the estimate (21) is therefore uniform on n, we deduce from (28)-(29) that

$$\{v_n\} \text{ is bounded in } L^\infty(0,\infty;\mathbf{V}) \cap W^{1,\infty}(0,\infty;\mathbf{H}). \tag{30}$$

We extract a subsequence (still denoted by $\{v_n\}$) such that

$$v_n \to v \text{ weakly in } H^1(Q) \times H^1(\Sigma), \tag{31}$$

$$\begin{cases} v_n \to v & \text{strongly in } L^2(Q) \times L^2(\Sigma), \\ v_n \to v & \text{a.e. in } (Q) \times (\Sigma). \end{cases} \tag{32}$$

From (28) and (32), we deduce that

$$\int_0^T |v|^2_{\mathbf{H}}\, dt = 1, \tag{33}$$

and (29) implies

$$(v_1', v_2') = (0,0) \text{ a.e. in } \{\omega \times]0,T[\} \times \Sigma. \tag{34}$$

On the other hand we note that

$$\begin{aligned} f_n(s) &= h_n(s)s = h(\lambda_n s)s, \\ g_n(s) &= I_n(s)s = I(\lambda_n s)s, \quad \forall\, s \in \mathbb{R},\ \forall n \in N, \end{aligned}$$

where

$$h(z) = \frac{f(z)}{z}, \quad I(z) = \frac{g(z)}{z}.$$

Remark that h and $I \in L^\infty(\mathbb{R}), (h$ (resp. I) is bounded by the Lipschitz constant of f' (resp. of g')), $h \geq 0$ and $I \geq 0$. Thus

$$\begin{aligned} f_n(v_{1n}) &= h_n(v_{1n})v_{1n}, \\ g_n(v_{2n}) &= I_n(v_{2n})v_{2n}, \end{aligned}$$

with $\{h_n(v_{1n})\}$ and $\{I_n(v_{2n})\}$ uniformly bounded in $L^\infty(Q)$ and $L^\infty(\Sigma)$. Therefore we may extract some subsequences (still denoted by $\{h_n(v_{1n})\}$and $\{I_n(v_{1n})\}$) such that

$$\begin{aligned} h_n(v_{1n}) &\to p(x,t) \quad \text{in } L^\infty(Q), \quad \text{weak star}, \\ I_n(v_{2n}) &\to q(x,t) \quad \text{in } L^\infty(\Sigma), \quad \text{weak star}, \end{aligned}$$

for some $(p,q) \in L^\infty_+(Q) \times L^\infty_+(\Sigma)$. This allows us to extend to the limit in (26) obtaining (from 34)

$$\begin{cases} (v_1)_{tt} - \Delta v_1 + p(x,t)v_1 = 0 & \text{in } Q, \\ (v_2)_{tt} + \partial_\nu v_1 - \Delta_T v_2 + q(x,t)v_2 = 0 & \text{on } \Sigma, \\ v_{1_\Sigma} = v_2. \end{cases} \tag{35}$$

However, the fact that, in principle the potentials $p(x,t)$ and $q(x,t)$ might depend on t does not allow us to prove directly (from (34) and (35)) that $v \equiv 0$ in order to contradict (33).

In order to solve this difficulty we distinguish the following three situations:

a) there exist a subsequence of $\{\lambda_n\}$ (still denoted $\{\lambda_n\}$) such that

$$\lambda_n \to \lambda \in]0, \infty[$$

.

In this case we easily see that

$$\begin{array}{l} p(x,t)v_1 = \frac{1}{\lambda} f(\lambda v_1), \\ q(x,t)v_2 = \frac{1}{\lambda} g(\lambda v_2). \end{array} \tag{36}$$

Then, $w = v_t$ satisfies:

$$\begin{cases} (w_1)_{tt} - \Delta w_{1n} + f'(\lambda v_1) w_1 & = 0 \quad \text{in } Q, \\ (w_2)_{tt} + \partial_\nu w_1 - \Delta_T w_2 + g'(\lambda v_2) w_2 = 0 & \text{on } \Sigma, \\ w_{1_\Sigma} = w_2, \end{cases} \tag{37}$$

with

$$(w_1, w_2) \equiv (0,0) \qquad \text{a.e. on } \omega \times]0,T[\times \Sigma. \tag{38}$$

b) We are not in situation **(a)** and there exists a subsequence $\{\lambda_n\}$ such that

$$\lambda_n \to 0.$$

In this case

$$\begin{array}{ll} p(x,t) = f'(0) & \text{a.e. in } Q, \\ q(x,t) = g'(0) & \text{a.e. on } \Sigma, \end{array}$$

and $w = v_t$ satisfies, in addition to (38),

$$\begin{cases} (w_1)_{tt} - \Delta w_1 + f'(0) w_1 & = 0 \quad \text{in } Q, \\ (w_2)_{tt} + \partial_\nu w_1 - \Delta_T w_2 + g'(0) w_2 = 0 & \text{on } \Sigma, \\ w_{1_\Sigma} = w_2, \end{cases} \tag{40}$$

c) The sequence $\{\lambda_n\}$ goes to infinity.

In this case we take the derivative of (26) with respect to t and deduce that $w_n = (v_n)_t$ satisfies

$$\begin{cases} (w_{1n})_{tt} - \Delta w_{1n} + f'(\lambda_n v_{1n}) w_{1n} + a(x)(w_{1n})_t & = 0 \quad \text{in } Q, \\ (w_{2n})_{tt} + \partial_\nu w_{1n} - \Delta_T w_{2n} + g'(\lambda_n v_{2n}) w_{2n} + b(x)(w_{2n})_t & = 0 \quad \text{on } \Sigma, \\ w_{1_\Sigma} = w_2. \end{cases} \tag{41}$$

From (31) we know that

$$w_n \to w = v_t \qquad \text{weakly in } L^2(Q) \times L^2(\Sigma). \tag{42}$$

On the other hand $\{f'(\lambda_n v_{1n})\}$(resp. $\{g'(\lambda_n v_{2n})\}$) is uniformly bounded in $L^\infty(Q)$ (resp.in $L^\infty(\Sigma)$) but this does not suffice to extend to the limit in (41). However,

$$\begin{cases} f'(\lambda_n v_{1n})w_{1n} \to z_1(x,t) & \text{weakly in } L^2(Q), \\ g'(\lambda_n v_{2n})w_{2n} \to z_2(x,t) & \text{weakly in } L^2(\Sigma). \end{cases}$$

for some subsequences.

Therefore,

$$\begin{cases} (w_1)_{tt} - \Delta w_1 + z_1(x,t) & = 0 \text{ in } Q, \\ (w_2)_{tt} + \partial_\nu w_1 - \Delta_T w_2 + z_2(x,t) & = 0 \text{ on } \Sigma, \\ w_{1_\Sigma} = w_2 \end{cases} \tag{43}$$

In order to identify the limit (z_1, z_2) we divide the sets $Q = \Omega \times]0,T[$ and $\Sigma = \Gamma \times]0,T[$ into the following subsets

$$\begin{cases} Q = Q_1 \cup Q_2 \text{ with } Q_1 = \{v_1 \neq 0\}, \; Q_2 = \{v_1 = 0\}, \\ \Sigma = \Sigma_1 \cup \Sigma_2 \text{ with } \Sigma_1 = \{v_2 \neq 0\}, \; \Sigma_2 = \{v_2 = 0\}. \end{cases}$$

From (i) and from (32), we easily deduce, by Lebesgue's theorem, that

$$\begin{aligned} f'(\lambda_n v_{1n}) &\to f'(-\infty)_{\chi\{v_1 < 0\}} + f'(+\infty)_{\chi\{v_1 > 0\}} = q_1(x,t), \text{ strongly in } L^2(Q_1), \\ g'(\lambda_n v_{2n}) &\to g'(-\infty)_{\chi\{v_2 < 0\}} + g'(+\infty)_{\chi\{v_2 > 0\}} = q_2(x,t), \text{ strongly in } L^2(\Sigma_1). \end{aligned} \tag{44}$$

(We note by χ_A the characteristic function of A). Therefore

$$(z_1, z_2) = (q_1 w_1, q_2 w_2) \text{ in } Q_1 \times \Sigma_1. \tag{45}$$

On the other hand, the fact that (by definition)

$$v = (0,0) \qquad \text{a.e.} \qquad \text{in} \qquad Q_2 \times \Sigma_2$$

and

$$\begin{aligned} & v \in H^1(Q) \times H^1(\Sigma), \\ & (v_1)_{tt} - \Delta v_1 \in L^2(Q), \\ & (v_2)_{tt} + \partial_\nu v_1 - \Delta_T v_2 \in L^2(\Sigma), \end{aligned}$$

imply

$$\begin{cases} (v_1)_{tt} - \Delta v_1 & = 0 \quad \text{a.e. in } Q_2, \\ (v_2)_{tt} + \partial_\nu v_1 - \Delta_T v_2 & = 0 \quad \text{a.e. on } \Sigma_2. \end{cases}$$

But clearly , $z \in L^2(Q) \times L^2(\Sigma)$ satisfies

$$\begin{cases} z_1 = -\frac{d}{dt}((v_1)_{tt} - \Delta v_1) & \text{in } Q, \\ z_2 = -\frac{d}{dt}((v_2)_{tt} + \partial_\nu v_1 - \Delta_T v_2) & \text{in } \Sigma, \end{cases}$$

and therefore

$$z = (0,0) \quad \text{a.e.} \quad Q_2 \times \Sigma_2. \tag{46}$$

From (43), (45) and (46), we conclude that, in addition to (38), w satisfies

$$\begin{cases} (w_1)_{tt} - \Delta w_1 + \widetilde{q_1}(x,t)w_1 = 0 \ \text{in}\, Q, \\ (w_2)_{tt} + \partial_\nu w_1 - \Delta_T w_2 + \widetilde{q_2}(x,t)w_2 = 0 \ \text{in}\, \Sigma, \\ w_{1_\Sigma} = w_2, \end{cases}$$

with

$$\widetilde{q_1}(x,t) \in L^\infty(Q), \ \ \widetilde{q_1} = q_1 \text{ in } Q_1, \quad \widetilde{q_1} = 0 \text{ in } Q_2,$$
$$\widetilde{q_2}(x,t) \in L^\infty(\Sigma), \ \ \widetilde{q_2} = q_2 \text{ in } \Sigma_1, \quad \widetilde{q_2} = 0 \text{ in } \Sigma_2.$$

Recapitulating, we see that, in each of these three possible situations **a**), **b**) and **c**), the function $w \in L^2(Q) \times L^2(\Sigma)$ satisfies (38) and a Cauchy-Ventcel problem of type

$$\begin{cases} (w_1)_{tt} - \Delta w_1 + b_1(x,t)w_1 = 0 & \text{in } Q, \\ (w_2)_{tt} + \partial_\nu w_1 - \Delta_T w_2 + b_2(x,t)w_2 = 0 & \text{on } \Sigma, \\ w_{1_\Sigma} = w_2, \end{cases} \tag{47}$$

for some potential $b \in L^\infty_+(Q) \times L^\infty_+(\Sigma)$.

In order to apply (13) we must prove that $w \in H^1(Q) \times H^1(\Sigma)$. This can be done by proving (by the same perturbation argument that we have used above) and estimate of type (21) for system (47). Applying (13) we deduce $w \equiv 0$ and therefore $v = v(x)$.

Taking into account that $v = v(x)$ is a stationary solution of (35), we deduce that

$$\begin{cases} -\Delta v_1 + p_1(x,t)v_1 = 0 & \text{in } \ \Omega, \ \forall t \in]0,T[\,, \\ \partial_\nu v_1 - \Delta_T v_2 + p_2(x,t)v_2 = 0 & \text{on } \ \Gamma, \ \ \forall t \in]0,T[\,, \\ v_{1_\Sigma} = v_2, \end{cases}$$

and since $p_1 \geq 0$, $p_2 \geq 0$, then $v \equiv 0$.This clearly contradicts (33) and the proof of the theorem is now completed.

3. The superlinear case.

In this section we study the exponential decay of solutions of (1)-(7) in the case where the nonlinearities f and $g \in C^1(\mathbb{R})$, in addition to (3), satisfies (5), i.e. f and g are superlinear.

The proofs of our estimations are based on multiplier techniques.

The main result of this section concerns the simplest case where ω is a neighbourhood of the wole boundary Γ.

Theorem 3.1: *Assume that hypotheses (2)-(4) are satisfied . Assume also that (the superlinear case), there exist some constants δ_1 and $\delta_2 > 0$ such that*

$$\begin{cases} \exists\, \delta_1 > 0 : \ f(s)s \geq (2+\delta_1)F(s), \\ \exists\, \delta_2 > 0 : g(s)s \geq (2+\delta_2)G(s), \end{cases} \qquad \forall\, s \in \mathbb{R},$$

Then there exits some constants $C \geq 1$, $\gamma > 0$ such that the estimate (10) holds for every solution $u = u(x,t)$ of (1)-(7) with initial data $\left(u^0, u^1\right) \in \mathbf{V} \times \mathbf{H}$.

Proof: As in the proof of Theorem 2.1, we note that it is sufficient to prove an estimate of type (15) to obtain (10) with C and $\gamma > 0$ given by (17).
In order to prove (15) we proceed in several steps.
Step 1
Lemma 3.1: *Let Ω be a bounded, open, connected set in $\mathbb{R}^n$ $(n \geq 2)$ having a boundary $\Gamma = \partial\Omega$ of class C^2 and $q \in (W^{1,\infty}(\overline{\Omega}))^n$. Then, for every weak solution u of (1) we have the following identity*

$$\begin{aligned}
&\tfrac{1}{2}\int_\Sigma (q.\nu)\left[|u_2'|^2 - |\bigtriangledown_T u_2|^2 + |\partial_\nu u_1|^2\right] d\Sigma \\
&= (u_1', q.\bigtriangledown u_1)_\Omega \,|_0^T + (u_2', q_T.\bigtriangledown_T u_2)_\Omega \,|_0^T \\
&+\tfrac{1}{2}\int_Q (divq)\left[|u_1'|^2 - |\bigtriangledown u_1|^2\right] dxdt + \tfrac{1}{2}\int_\Sigma (divq_T)\left[|u_2'|^2 - |\bigtriangledown_T u_2|^2\right] d\Sigma \\
&+ \int_Q \nabla u_1.\nabla q.\nabla u_1 dxdt + \int_\Sigma \nabla_T u_2.\nabla_T q_T.\nabla_T u_2 d\Sigma \\
&- \int_Q (divq)F(u_1)dxdt - \int_\Sigma (divq_T)G(u_2)d\Sigma + \\
&+ \int_Q a u_1' q.\bigtriangledown u_1 dxdt + \int_\Sigma b u_2' q_T.\bigtriangledown_T u_2 d\Sigma,
\end{aligned} \tag{48}$$

where q_T is the tangential component of q and $\bigtriangledown_T q_T$ are *the components of tangential gradient .*
Proof: We proceed as in J. L. Lions [7]. In a first step we take the data $u^0 \in D_A$ and $u^1 \in \mathbf{V}$. We multiply the equation $(1)_1$and $(1)_2$ respectively by $q.\bigtriangledown u_1$ and $q_T.\bigtriangledown_T u_2$, we integrate over Q and Σ and do sommation. Integrating by parts, we obtain the identity for regular data. In the general case of a weak solution for $u^0 \in \mathbf{V}$and $u^1 \in \mathbf{H}$, we approximate by regular data and use the proposition 2 of K. Lemrabet, D. E. Teniou [6] and pass to the limit.
Lemma 3.2: *We have the following identity*

$$\begin{aligned}
&(u_1', m.\bigtriangledown u_1)_\Omega \,|_0^T + (u_2', m.\bigtriangledown_T u_2)_\Sigma \,|_0^T \\
&+\tfrac{n}{2}\int_Q \left[|u_1'|^2 - |\bigtriangledown u_1|^2\right] dxdt + \int_Q |\bigtriangledown u_1|^2 dxdt \\
&+\tfrac{1}{2}\int_\Sigma (n-1)\left[|u_2'|^2 - |\bigtriangledown_T u_2|^2\right] d\Sigma + \int_\Sigma |\bigtriangledown_T u_2|^2 d\Sigma \\
&-n\int_Q F(u_1)dxdt - \int_\Sigma (n-1)\,G(u_2)d\Sigma + \int_Q a u_1' m.\bigtriangledown u_1 dxdt \\
&= \int_\Sigma \tfrac{(m.\nu)TrB}{2}[|u_2'|^2 - |\bigtriangledown_T u_2|^2]d\Sigma - \int_\Sigma m.\nu TrB.G(u_2)d\Sigma \\
&+ \int_\Sigma \tfrac{(m.\nu)}{2}\{|u_2'|^2 - |\bigtriangledown_T u_2|^2 + \left|\tfrac{\partial u_1}{\partial\nu}\right|^2 + 2B.(\bigtriangledown_T u_2, \bigtriangledown_T u_2)\}d\Sigma
\end{aligned} \tag{49}$$

where B is the second fundamental form of Γ (the curvature operator) and T_r is the trace.
Proof: To obtain (49) , we apply (48) with $q(x) = x - x^0 = m(x)$ for $x^0 \in \mathbb{R}^n$ and remark that

$$\nabla.m = n, \quad \nabla_T.m_T = n - 1 - m.\nu TrB$$

Lemma 3.3: *We have the following identity*

$$\begin{aligned}
&\int_\Gamma \xi u_2(u_2'+\tfrac{bu_2}{2})|_0^T+\int_\Omega \xi u_1(u_1'+\tfrac{au_1}{2})\ |_0^T\\
&=\int_Q \xi\left[|u_1'|^2-|\nabla u_1|^2\right]dxdt\\
&+\int_Q u_1\nabla\xi.\nabla u_1dxdt-\int_Q \xi f(u_1)u_1dxdt\\
&+\int_\Sigma \xi\left[|u_2'|^2-|\nabla_T u_2|^2\right]d\Sigma\\
&+\int_\Sigma u_2\nabla_T\xi.\nabla_T u_2d\Sigma-\int_\Sigma \xi g(u_2)u_2d\Sigma
\end{aligned}\tag{50}$$

Proof: We multiply the equations $(1)_1$ and $(1)_2$ by $\xi(x)u_1$ and $\xi(x)u_2$ respectively, with $\xi\in C^1(\overline{\Omega})$. Integrating by parts we obtain (50).

Lemma 3.4: *We have the following identity*

$$\begin{aligned}
&\int_\Gamma u_2(u_2'+\tfrac{bu_2}{2}|_0^T+\int_\Omega u_1(u_1'+\tfrac{au_1}{2})\ |_0^T=\\
&\int_Q\left[|u_1'|^2-|\nabla u_1|^2\right]dxdt-\int_Q f(u_1)u_1dxdt\\
&+\int_\Sigma\left[|u_2'|^2-|\nabla_T u_2|^2\right]d\Sigma-\int_\Sigma g(u_2)u_2d\Sigma
\end{aligned}$$

Proof: We apply (50) with $\xi=1$.
Combining (49) and (51) we obtain the following Lemma

Lemma 3.5: *The following identity holds:*

$$\begin{aligned}
&(u_1',m.\nabla u_1+\alpha_1u_1(u_1'+\tfrac{au_1}{2}))_\Omega\ |_0^T+(u_2',m_T.\nabla_T u_2+\alpha_2u_2(u_2'+\tfrac{bu_2}{2}))_\Gamma\ |_0^T\\
&+(\tfrac{n}{2}-\alpha_1)\int_Q|u_1'|^2dxdt+(1+\alpha_1-\tfrac{n}{2})\int_Q|\nabla u_1|^2dxdt\\
&+\alpha_1\int_Q f(u_1)u_1dxdt-n\int_\Omega F(u_1)dx+(\tfrac{n-1}{2}-\alpha_2)\int_\Sigma|u_2'|^2d\Sigma\\
&+(1+\alpha_2-\tfrac{n-1}{2})\int_\Sigma|\nabla_T u_2|^2d\Sigma+\alpha_2\int_\Sigma g(u_2)u_2d\Sigma-\int_\Sigma[n-1]G(u_2)d\Sigma\\
&+\int_Q au_1'm.\nabla u_1dxdt+\int_\Sigma bu_2'm_T.\nabla_T u_2d\Sigma\\
&=\int_\Sigma\tfrac{(m.\nu)TrB}{2}[|u_2'|^2-|\nabla_T u_2|^2]d\Sigma-\int_\Sigma m.\nu TrB.G(u_2)d\Sigma\\
&+\int_\Sigma\tfrac{(m.\nu)}{2}\{|u_2'|^2-|\nabla_T u_2|^2+|\partial_\nu u_1|^2+2B.(\nabla_T u_2,\nabla_T u_2)\}d\Sigma,\ \ \forall\alpha\in\mathbb{R}
\end{aligned}\tag{52}$$

We note that (5) implies the existence of some constant $(\alpha_1,\alpha_2)\in(\frac{n-3}{2},\frac{n-1}{2})\times(\frac{n-3}{2},\frac{n-1}{2})$,such that

$$\begin{aligned}
&f(s)s\geq\tfrac{n+\gamma_1}{\alpha_1}F(s),\\
&g(s)s\geq\tfrac{n+\gamma_2}{\alpha_2}G(s)\quad\forall s\in\mathbb{R},
\end{aligned}$$

where γ_1 and γ_2 are two constants >0. With this choice of (α_1,α_2), from (52) we deduce, for some $C>0$, the estimate

$$\begin{aligned}
&C\int_0^T E(t)dt\leq\tfrac{1}{2}\int_\Sigma(m.\nu)\left[|u_2'|^2-|\nabla_T u_2|^2+|\partial_\nu u_1|^2\right]d\Sigma\\
&+\left|\int_Q au_1'm.\nabla u_1dxdt\right|+\left|\int_\Sigma bu_2'm_T.\nabla_T u_2d\Sigma\right|+X+Z
\end{aligned}\tag{53}$$

where

$$\begin{aligned}
&X=\big|(u_1',m.\nabla u_1+\alpha_1u_1(u_1'+\tfrac{au_1}{2}))_\Omega\ |_0^T\\
&+(u_2',m_T.\nabla_T u_2+\alpha_2u_2(u_2'+\tfrac{bu_2}{2}))_\Gamma\ |_0^T\big|,\\
&Z=\Big|\int_\Sigma\tfrac{(m.\nu)TrB}{2}[|u_2'|^2-|\nabla_T u_2|^2]d\Sigma-\int_\Sigma m.\nu TrB.G(u_2)d\Sigma\\
&+\int_\Sigma\tfrac{(m.\nu)}{2}\{2B\,(\nabla_T u_2,\nabla_T u_2)\}d\Sigma\Big|.
\end{aligned}\tag{54}$$

We have

$$\begin{aligned}&\left|\int_Q a u_1' m.\nabla u_1 dxdt\right| \leq \varepsilon \|m\|^2_{L^2(\Omega)} \int_Q |\nabla u_1|^2 dxdt \\ &+\frac{1}{2\varepsilon}\|a\|_{L^2(\Omega)} \int_Q a\,|u_1'|^2 dxdt\end{aligned} \tag{55}$$

$$\begin{aligned}&\left|\int_\Sigma b u_2' m_T.\nabla u_2 d\Sigma\right| \leq \varepsilon \|m_T\|^2_{L^2(\Gamma)} \int_\Sigma |\nabla_T u_2|^2 d\Sigma \\ &+\frac{1}{2\varepsilon}\|b\|_{L^2(\Gamma)} \int_\Sigma b\,|u_2'|^2 d\Sigma\end{aligned} \tag{55'}$$

for any ε.

Combining (53) with (55) and (55') where $\varepsilon > 0$ is taken small enough, we deduce

$$\begin{aligned}&\int_0^T E(t)dt \leq C\left\{\int_\Sigma (m.\nu)\left[|u_2'|^2 - |\nabla_T u_2|^2 + |\partial_\nu u_1|^2\right] d\Sigma\right. \\ &\left.+\int_Q a\,|u_1'|^2 dxdt + \int_\Sigma b\,|u_2'|^2 d\Sigma + X + Z\right\}\end{aligned} \tag{56}$$

Step 2

We now estimate the quantity

$$\int_\Sigma (m.\nu)\left[|u_2'|^2 - |\nabla_T u_2|^2 + |\partial_\nu u_1|^2\right] d\Sigma$$

in term of

$$\int_Q a\,|u_1'|^2 dxdt + \int_\Sigma b(x)\,|u_2'|^2 \Sigma.$$

Following the method of proof of Lemma 2.3, Chap. VII in J. L. Lions [7], we construct a neighbourhood $\widehat{\omega}$ of $\overline{\Gamma(x^0)}$ such that

$$\overline{\widehat{\omega}} \cap \Omega \subset \omega$$

and a vector field $h \in (W^{1,\infty}(\overline{\Omega}))^n$ such that

$$h = \nu \text{ on } \Gamma(x^0), \quad h.\nu \geq 0 \quad \text{a.e.} \quad \text{in} \quad \Gamma. \tag{57}$$

and

$$h = 0 \qquad \text{on } \Omega \setminus \widehat{\omega} \tag{58}$$

(See Remark 3.2 Chap.I of J. L. Lions [7] for the construction of this vector field). Applying identity (48) with $q = h$, we easily deduce the existence of some constant $C > 0$ such that

$$\begin{aligned}&\int_{\Sigma(x^0)}\left[|u_2'|^2 - |\nabla_T u_2|^2 + |\partial_\nu u_1|^2\right] d\Sigma \\ &\leq \int_\Sigma (h.\nu)\left[|u_2'|^2 - |\nabla_T u_2|^2 + |\partial_\nu u_1|^2\right] d\Sigma \\ &\leq C\left\{\int_0^T \int_{\widehat{\omega}}\left[|u_1'|^2 + |\nabla u_1|^2 + F(u_1)\right] dxdt\right. \\ &\left.+\int_\Sigma\left[|u_2'|^2 + |\nabla_T u_2|^2 + G(u_2)\right] d\Sigma\right\} + \\ &2\left(\int_\Omega u_1' h.\nabla u_1 dx\right)\Big|_0^T + 2\left(\int_\Gamma u_2' h_T.\nabla_T u_2 dx\right)\Big|_0^T\end{aligned} \tag{59}$$

We construct then a function $\eta \in W^{1,\infty}(\Omega)$ satisfying

$$\begin{array}{ll} 0 \leq \eta \leq 1 & \text{a.e. in } \Omega, \\ \eta = 1 & \text{a.e. in } \widehat{\omega}, \\ \eta = 0 & \text{a.e. in } \Omega \setminus \omega, \end{array} \tag{60}$$

and

$$\frac{|\nabla \eta|^2}{\eta} \in L^\infty(\omega). \tag{62}$$

(See Lemma 2.4 Chap.VII in J. L. Lions [7] for the construction of this function). Applying identity (50) with $\xi = \eta$ we deduce

$$\begin{array}{l} \int_Q \eta \left[|\nabla u_1|^2 + f(u_1)u_1 \right] dxdt - \int_Q u_1 \bigtriangledown \eta . \bigtriangledown u_1 dxdt \\ + \int_\Sigma \eta \left[|\nabla_T u_2|^2 + g(u_2)u_2 \right] d\Sigma - \int_\Sigma u_2 \nabla_T \eta . \bigtriangledown_T u_2 d\Sigma \\ \leq C \left\{ \int_0^T \int_\omega |u_1'|^2 dxdt + \int_\Sigma |u_2'|^2 d\Sigma + Y \right\} \end{array} \tag{63}$$

where

$$Y = \left| \int_\Gamma \eta_T . u_2 (u_2' + \frac{bu_2}{2}) \mid_0^T + \int_\Omega \eta u_1 (u_1' + \frac{au_1}{2}) \mid_0^T \right|. \tag{64}$$

On the other hand we have

$$\begin{array}{l} \left| \int_Q u_1 \bigtriangledown \eta . \bigtriangledown u_1 dxdt \right| \leq \varepsilon \int_Q \eta \left| \bigtriangledown u_1 \right|^2 dxdt + \frac{1}{2\varepsilon} \int_Q \frac{|\nabla \eta|^2}{\eta} |u_1|^2 dxdt, \\ \left| \int_\Sigma u_2 \bigtriangledown_T \eta . \bigtriangledown_T u_2 d\Sigma \right| \leq \varepsilon \int_\Sigma \eta \left| \bigtriangledown_T u_2 \right|^2 d\Sigma + \frac{1}{2\varepsilon} \int_\Sigma \frac{|\nabla \eta_T|^2}{\eta} |u_2|^2 dxdt. \end{array} \tag{65}$$

Combining (63) with (65) for $\varepsilon \in]0, 1[$, we deduce

$$\begin{array}{l} \int_0^T \int_\omega \left[|\nabla u_1|^2 + F(u_1) \right] dxdt + \int_\Sigma \left[|\nabla u_2|^2 + G(u_2) \right] d\Sigma \\ \leq \int_Q \eta \left[|\nabla u_1|^2 + F(u_1) \right] dxdt + \int_\Sigma \eta \left[|\nabla u_2|^2 + G(u_2) \right] d\Sigma \\ \leq \int_Q \eta \left[|\nabla u_1|^2 + f(u_1)u_1 \right] dxdt + \int_\Sigma \eta \left[|\nabla u_2|^2 + g(u_2)u_2 \right] d\Sigma \\ \leq C \left\{ \int_0^T \int_\omega |u_1'|^2 dxdt + \int_\Sigma |u_2'|^2 \Sigma + \int_Q |u_1|^2 dxdt + \int_\Sigma |u_2|^2 d\Sigma + Y \right\} \end{array} \tag{66}$$

From (59) and (66), we obtain

$$\begin{array}{l} \int_{\Sigma(x^0)} \left[|u_2'|^2 - |\bigtriangledown_T u_2|^2 + |\partial_\nu u_1|^2 \right] d\Sigma \leq \\ C \left\{ \int_Q a |u_1'|^2 dxdt + \int_\Sigma b |u_2'|^2 \Sigma + \int_Q |u_1|^2 dxdt + \int_\Sigma |u_2|^2 d\Sigma \right. \\ \left. + \int_\Omega u_1' h . \bigtriangledown u_1 dx \mid_0^T + \int_\Gamma u_2' h_T . \bigtriangledown_T u_2 dx \mid_0^T + Y \right\} \end{array} \tag{67}$$

Combining (56) and (67) we get

$$\begin{array}{l} TE(T) \leq \int_0^T E(t) dt \leq \\ C \left\{ \int_Q a |u_1'|^2 dxdt + + \int_\Sigma b |u_2'|^2 d\Sigma + \int_Q |u_1|^2 dxdt + \int_\Sigma |u_2|^2 d\Sigma \right\} \\ + \int_\Omega u_1' h . \bigtriangledown u_1 dx \mid_0^T + \int_\Gamma u_2' h_T . \bigtriangledown_T u_2 dx \mid_0^T + Y + X + Z. \end{array} \tag{68}$$

We now remark that

$$
\begin{aligned}
&X+Y+\left|\int_\Omega u_1' h.\bigtriangledown u_1 dx\right|_0^T+\left|\int_\Gamma u_2' h_T.\bigtriangledown_T u_2 dx\right|_0^T\\
&\le C\left\{E(0)+E(T)\right\}\le C\left\{2E(T)+\int_Q a\left|u_1'\right|^2dxdt+\int_\Sigma b(x)\left|u_2'\right|^2\Sigma\right\},
\end{aligned}
\tag{69}
$$

and that

$$
Z\le\int_\Sigma\left|u_2'\right|^2d\Sigma++\int_\Sigma\left|\nabla_T u_2\right|^2d\Sigma+\int_\Sigma\left|u_2\right|^2d\Sigma.
$$

Combining (67) with (68) for $T>0$ large enough, we obtain

$$
E(T)\le C\left\{\int_Q a\left|u_1'\right|^2dxdt+\int_\Sigma b(x)\left|u_2'\right|^2\Sigma+\int_0^T|u|_{\mathbf{H}}^2\,dt\right\}. \tag{70}
$$

Step 3

As in the proof of Theorem 2.1, we must obtain the following estimate

$$
\int_0^T|u|_{\mathbf{H}}^2\,dt\le C\left\{\int_Q a\left|u_1'\right|^2dxdt+\int_\Sigma b(x)\left|u_2'\right|^2d\Sigma\right\}. \tag{71}
$$

We will argue by contradiction. If (71) were not satisfied for some constant $C>0$, there will exist a sequence $\{u_n\}$ of solutions of $(1)_1$ and $(1)_2$ verifying (23). We define $\{\lambda_n\}$ and $\{v_n\}$ by (24) and (26). The functions $\{v_n\}$ satisfy (26) with the nonlinearities f_n and g_n given by (27). On the other hand the sequence $\{v_n\}$ satisfies also (28)-(29).

We now remark that the constant $C>0$ in the estimate (70) depends on the nonlinearities f and g but only in terms of the constants δ_1 and δ_2 of hypothesis (5). But those constants $\delta_i,\ i=1,2$, are uniform with respect to the rescaled family of nonlinearities (27). Therefore, the constant C of (70) is uniform on $\{f_n\}$ and $\{g_n\}$.

Thus, from (28)-(29) we deduce that $\{v_n\}$ is bounded in $H^1(Q)\times H^1(\Sigma)$. Therefore, we may extract some sequences verifying (31) and (32). The limit $v\in H^1(Q)\times H^1(\Sigma)$ will satisfy (33) and (34). In order to contradict (33), we want to apply again a uniqueness argument showing that $v\equiv 0$. But the the question is now simpler than in Theorem 2.1.

We remark that the sequence $\{\lambda_n\}$ is necessarily bounded. Indeed, assume that there is a subsequence (still denoted by $\{\lambda_n\}$) such that

$$
\lambda_n\to\infty. \tag{72}
$$

From the uniform estimate (70) we know that

$$
\begin{aligned}
&F_n(z)=\int_0^T f_n(z)dz=\tfrac{1}{\lambda_n^2}F_n(\lambda_n z)\text{ is uniformly bounded in }L^1(Q),\\
&G_n(z)=\int_0^T g_n(z)dz=\tfrac{1}{\lambda_n^2}G_n(\lambda_n z)\text{ is uniformly bounded in }L^1(\Sigma).
\end{aligned}
\tag{73}
$$

But from (5) one easily deduce that

$$\begin{aligned} &F(s) \geq c_1 \left|s\right|^{2+\delta_1}, \\ &G(s) \geq c_2 \left|s\right|^{2+\delta_2}, \quad \forall \left|s\right| \geq 1 \end{aligned} \tag{74}$$

with $c_1 = \min\{F(1), F(-1)\}$, $c_2 = \min\{G(1), G(-1)\}$.

Combining (73) et (74) we conclude that

$$\begin{aligned} &\lambda_n^{\delta_1} \int_0^T \int_{\left\{v_{1n} \geq \lambda_n^{-1}\right\}} \left|v_{1n}\right|^{2+\delta_1} + \int_0^T \int_{\left\{v_{1n} \geq \lambda_n^{-1}\right\}} F_n(\lambda_n v_{1n}) \leq C_1 \\ &\lambda_n^{\delta_2} \int_0^T \int_{\left\{v_{2n} \geq \lambda_n^{-1}\right\}} \left|v_{2n}\right|^{2+\delta_2} + \int_0^T \int_{\left\{v_n \geq \lambda_n^{-1}\right\}} G_n(\lambda_n v_{2n}) \leq C_2 \end{aligned}$$

which imply

$$\int\int \left|v_{1n}\right|^{2+\delta_1} \to 0, \quad \int\int \left|v_{2n}\right|^{2+\delta_2} \to 0, \quad n \to \infty,$$

and therefore a contradiction of (33).
The sequence $\{\lambda_n\}$ being bounded we must only consider the situations (**a**) and (**b**)of step 3 of the proof of Theorem 2.1.
In the situation **(a)** we proceed as follows. Since v_n is bounded in

$$(L^\infty(0,T;H^1(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega))) \cap (L^\infty(0,T;H^1(\Gamma)) \cap W^{1,\infty}(0,T;L^2(\Gamma))),$$

then v_n is relatively compact in $(L^\infty(0,T;H^{1-\varepsilon}(\Omega)) \cap (L^\infty(0,T;H^{1-\varepsilon}(\Gamma))$ for every $\varepsilon \geq 0$. Taking into account that p satisfies $(n-2)p \leq 2n$, we deduce that the sequence $f_n(v_{1n})$ (resp. $g_n(v_{2n})$) converges strongly to $\lambda^{-1} f(\lambda v_1)$ in $L^\infty(0,T;L^r(\Omega)$ (resp. to $\lambda^{-1} g(\lambda v_1)$ in $L^\infty(0,T;L^r(\Sigma))$ for every $r \in \left[2, \frac{2n}{p(n-2)}\right]$ (when $n=2$ the convergence holds in $L^\infty(0,T;L^r(\Omega)$ (resp. ins $L^\infty(0,T;L^r(\Gamma))$ for every $r \geq 1$). Therefore we may go to the limit in (26) and we deduce that $w = v_t$ (where v is the limit of v_n) satisfies (37)-(38). On the other hand , since $(n-2)p \leq 2n$ we deduce that

$$b = (b_1, b_2) = (f'(\lambda), g(\lambda)) \in L_+^\infty(0,T;L^n(\Omega)) \times L_+^\infty(0,T;L^n(\Gamma))$$

(when $n=2$, $b \in L_+^\infty(0,T;L^r(\Omega)) \times L_+^\infty(0,T;L^r(\Gamma))$ for every $r \geq 1$).
In the situation **(b)** we proceed as in the proof of Theorem 2.1.and we deduce that w satisfies (38)-(40). In both cases we see that $w \in L^2(Q) \times L^2(\Sigma)$ solves an equation of type (37) with

$$b = (b_1, b_2) \in L_+^\infty(0,T;L^r(\Omega)) \times L_+^\infty(0,T;L^r(\Gamma))$$

and verifies (38). Applying the unique continuation result of A. Ruiz [9] we deduce, $T >$diameter of Ω, $w \equiv 0$ which contradicts (33).
The proof of the theorem is now complete.

4. Appendix

For the proof of Lemma 2.1 and Lemma 2.2, we assume that the domain Ω has a boundary Γ of class C^2 . For $x^0 \in \mathbb{R}^n$ arbitrary, we consider, for $\varepsilon > 0$ the sets

$$O_\varepsilon = \underset{x \in \Gamma(x^0)}{\cup} B(x, \varepsilon); \quad \omega_\varepsilon = O_\varepsilon \cap \Omega,$$

where $B(x, \varepsilon)$ is the ball of centre x and of radius ε in $\mathbb{R}^n$.

Proof of Lemma 2.1.

The first estimation is the following: if $T > 2\mathbb{R}(x^0)$,

$$\left(T - 2\mathbb{R}(x^0)\right) E_0 \leq C \left[\int_\Sigma \left(|\varphi_2'|^2 - |\nabla_T \varphi_2|^2 \right) d\Sigma + \int_\Sigma |\partial_\nu \varphi_1|^2 \, d\Sigma \right] \tag{75}$$

for any solution φ of (11) which corresponds to the data $\left(\varphi^0, \varphi^1\right) \in \mathbf{V} \times \mathbf{H}$. This estimation is proved in K. Lemrabet, D. E. Teniou [5].

Let $\alpha > 0$ be such that $T - 2\alpha > 2R(x^0)$. From (75) and for $C > 0$ large enough,

$$\begin{aligned} E_0 \leq C \Big[& \textstyle\int_\alpha^{T-\alpha} \int_\Gamma \left(|\varphi_2'|^2 - |\nabla_T \varphi_2|^2 \right) d\Sigma \\ & + \textstyle\int_\alpha^{T-\alpha} \int_{\Gamma(x^0)} |\partial_\nu \varphi_1|^2 \, d\Sigma \Big], \; \forall \left(\varphi^0, \varphi^1\right) \in \mathbf{V} \times \mathbf{H}. \end{aligned} \tag{76}$$

It remains to prove the following estimation

$$\begin{aligned} & \textstyle\int_\alpha^{T-\alpha} \int_\Gamma \left(|\varphi_2'|^2 - |\nabla_T \varphi_2|^2 \right) d\Sigma + \int_\alpha^{T-\alpha} \int_{\Gamma(x^0)} |\partial_\nu \varphi_1|^2 \, d\Sigma \\ & \leq C \Big\{ \textstyle\int_0^T \int_\omega |\varphi_1'(x,t)|^2 \, dx dt + \int_0^T \int_\Gamma |\varphi_2'(x,t)|^2 \, d\Sigma \\ & + \textstyle\int_0^T |\nabla \varphi_1|^2_{L^2(\Omega)} \, dt + \int_0^T |\nabla_T \varphi_2|^2_{L^2(\Gamma)} \, dt \Big\}. \end{aligned} \tag{77}$$

We use the multiplier method and we apply the identity (48) with

$$\sigma(x,t) = t \, (T - t) \, h(x), \tag{78}$$

where $h \in \left[C^1\left(\overline{\Omega}\right)\right]^n$ is the vector field introduced in Remark 3.2 of Chap. I of J. L. Lions [7] , which satifies

$$h = \nu \;\text{ on } \Gamma(x^0), \;\; h.\nu \geq 0 \;\; \text{a.e.} \;\; \text{on} \;\; \Gamma, \; Supp\,\{h\} \subset \omega. \tag{79}$$

We obtain

$$\begin{aligned} & \textstyle\int_\alpha^{T-\alpha} \int_\Gamma \left(|\varphi_2'|^2 - |\nabla_T \varphi_2|^2 \right) d\Sigma + \int_\alpha^{T-\alpha} \int_{\Gamma(x^0)} |\partial_\nu \varphi_1|^2 \, d\Sigma \\ & \leq \textstyle\int_\Sigma \sigma.\nu \left(|\varphi_2'|^2 - |\nabla_T \varphi_2|^2 \right) d\Sigma + \int_0^T \int_{\Gamma(x^0)} \sigma.\nu \, |\partial_\nu \varphi_1|^2 \, d\Sigma \\ & \leq C\,(\omega) \Big\{ \textstyle\int_0^T \int_\omega |\varphi_1'(x,t)|^2 \, dx dt + \int_\Sigma |\varphi_2'(x,t)|^2 \, d\Sigma \\ & + \textstyle\int_0^T |\nabla \varphi_1|^2_{L^2(\Omega)} \, dt + \int_0^T |\nabla_T \varphi_2|^2_{L^2(\Gamma)} \, dt \Big\}. \end{aligned} \tag{80}$$

which ends the proof.

Proof of Lemma 2.3. Consider the case where $\omega = \omega_\varepsilon$ with $\varepsilon > 0$. Let $\widehat{\omega} = \omega_{\varepsilon/2}$ be.We fixe $\alpha > 0$ such that $T - 2\alpha > 2\mathbb{R}(x^0)$. From Lemma 2.1 and for $C > 0$ large enough,

$$\begin{aligned} E_0 \leq C \Big\{ & \textstyle\int_\alpha^{T-\alpha} \int_{\widehat{\omega}} |\varphi_1'(x,t)|^2 \, dxdt + \int_\alpha^{T-\alpha} \int_\Gamma |\varphi_2'(x,t)|^2 \, d\Sigma \\ & + \textstyle\int_\alpha^{T-\alpha} |\nabla \varphi_1|^2_{L^2(\Omega)} \, dt + \int_\alpha^{T-\alpha} |\nabla_T \varphi_2|^2_{L^2(\Gamma)} \, dt \Big\} . \end{aligned} \tag{81}$$

(We recall $C = C(1/\varepsilon)$)
We consider a sequence of functions

$$\eta_\varepsilon \in W^{1,\infty}(O_\varepsilon)$$

which we denote by η and such that

$$0 \leq \eta \leq 1 \text{ in } O_\varepsilon, \ \eta = 1 \text{ in } \widehat{\omega} = \omega_{\varepsilon/2}. \tag{82}$$

It is clear that we can take η such that

$$\left\| \frac{|\nabla \eta(x)|^2}{\eta(x)} \right\|_{L^2(O_\varepsilon)} = O\left(1/\varepsilon^2\right) \tag{83}$$

Indeed, we can take for example

$$\eta = \begin{cases} 1 & \text{in } O_{\varepsilon/2} \\ \frac{1}{\varepsilon^2}\left(\varepsilon - d\left(x, \partial\omega_{\varepsilon/2}\right)\right)^2 & \text{in } O_\varepsilon \backslash O_{\varepsilon/2} \end{cases} \tag{84}$$

where $d\left(x, \partial\omega_{\varepsilon/2}\right)$ is the distance from x to the boundary $\partial\omega_{\varepsilon/2}$.
We use the multiplier method of Lemma 2.1, here with

$$\xi(x,t) = t(T-t)\,\eta\,(\varphi_1, \varphi_2). \tag{85}$$

We denote by $\zeta(t) = t(T-t)$.
Multiplying (11) by ξ, integrating in Q and on Σ and doing sommation, we obtain

$$\begin{aligned} & \textstyle\int_Q \left[\zeta(t)\,\eta(x)\,|\varphi_1'|^2 + \zeta'(t)\,\eta(x)\,\varphi_1\varphi_1'\right] dxdt \\ & + \textstyle\int_\Sigma \left[\zeta(t)\,\eta(x)\,|\varphi_2'|^2 + \zeta'(t)\,\eta(x)\,\varphi_2\varphi_2'\right] d\Gamma dt \\ & = \textstyle\int_Q \left[\zeta(t)\,\eta(x)\,|\nabla\varphi_1|^2 + \zeta'(t)\,\varphi_1 \nabla\eta.\nabla\varphi_1\right] dxdt \\ & + \textstyle\int_\Sigma \left[\zeta(t)\,\eta(x)\,|\nabla_T\varphi_2|^2 + \zeta'(t)\,\varphi_2 \nabla_T\eta.\nabla_T\varphi_2\right] d\Gamma dt. \end{aligned} \tag{86}$$

On the other hand, for every $\varepsilon > 0$, we have

$$\begin{aligned} & \left| \textstyle\int_Q \zeta'(t)\,\varphi_1 \nabla\eta.\nabla\varphi_1 dxdt + \int_\Sigma \zeta'(t)\,\varphi_2 \nabla_T\eta.\nabla_T\varphi_2 d\Sigma \right| \\ & \leq \varepsilon \textstyle\int_Q \zeta(t)\,\eta(x)\,|\nabla\varphi_1|^2 \, dxdt + \frac{1}{4\varepsilon} \int_Q \zeta(t) \frac{|\nabla\eta(x)|^2}{\eta(x)} |\nabla\varphi_1|^2 \, dxdt \\ & + \varepsilon \textstyle\int_\Sigma \zeta(t)\,\eta(x)\,|\nabla_T\varphi_2|^2 \, d\Sigma + + \frac{1}{4\varepsilon} \int_\Sigma \zeta(t) \frac{|\nabla_T\eta(x)|^2}{\eta(x)} |\nabla_T\varphi_2|^2 \, d\Sigma. \end{aligned} \tag{87}$$

Combining (81), (86) and (87) for $\varepsilon < 1$, we obtain, for $C > 0$ large enough, the estimation (19) where the order of C is $C = O(1/\varepsilon^3)$.

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