Exponential Decay of Solutions to Hyperbolic Equations in Bounded Domains

JEFFREY RAUCH & MICHAEL TAYLOR

Communicated by RALPH PHILLIPS

§1. Introduction. In this paper we consider the decay of solutions to hyperbolic equations in bounded domains. In contrast to equations in exterior domains where local decay takes place because signals radiate to infinity, in the bounded case there must be some direct dissipative mechanism. We discuss two types. In one kind the energy of a wave will decrease when it passes through some fixed subregion of the domain. The second sort of decay occurs at the boundary; energy is lost when a wave is reflected from a fixed subset of the boundary.

Iwasaki [4, §2.2] has shown under broad hypotheses that either sort of decay mechanism forces the energy of solutions to decrease to zero as $t \to +\infty$. Here, we investigate conditions which imply that the decay is exponential.

For all problems the basic hypothesis is that there is a time $T > 0$ such that a bicharacteristic ray starting at any point and suitably reflected at the boundary reaches the region where decay takes place in time less than T. The idea behind this assumption is that the energy of a solution to a hyperbolic equation $Lu = 0$ is largely carried along bicharacteristic rays of L, so such a hypothesis says that all such rays spend plenty of time in the region where energy loss occurs.

We treat problems on manifolds without boundary in section 2 and 3. In section 4 we study problems with boundary, but we can only treat the case of one space variable.

§2. First order systems on a manifold without boundary. Let $L = \partial/\partial t - G(x, D_x)$, where G is a first order system of differential operators with smooth coefficients on M. We assume M is a smooth compact manifold without boundary endowed with a volume element. Assume G satisfies the dissipative condition $G + G^* = -F(x)$ where F is a non-negative self-adjoint matrix function. L is called a dissipative symmetric hyperbolic system.

It follows that the initial value problem

$$Lu = 0$$

$$u(0, x) = \phi$$

has a unique solution, and if $\phi \in L^2(M)$, then $\|u(t)\|_{L^2(M)}{}^2 \leqq \|\phi\|_{L^2(M)}{}^2$ for all $t \geqq 0$.

We give here conditions which yield an exponential rate of decay $\|u(t)\|^2 \leqq ce^{-\alpha t} \|u(0)\|^2$, $t \geqq 0$, where c and α are independent of u. Let $g(x, \xi)$ be the principal symbol of G. The first condition is as follows.

(A) *$F(x) \geqq \eta I > 0$ on an open set $U \subset M$ with the property that there is a number T_0 such that any null bicharacteristic strip of* $\det(\tau - i\, g(x, \xi))$ *in $T^*(\mathbf{R} \times M) \backslash 0$ passes through $T^*((0, T_0) \times U)\backslash 0$.*

In order to prove our exponential decay result, we shall need the following result on propagation of singularities, which is a simple extension of proposition 3.3.1 of Hörmander [3].

Proposition 1. *Let $p(x, D)$ be a $k \times k$ system of (pseudo) differential operators, of order m, on a manifold Ω, and assume that $q(x, \xi) =$* $\det p(x, \xi)$ *has real principal part q_M, of order $M = km$. Let $p(x, D)u = f$ and let $\gamma: I \rightarrow T^*(\Omega)\backslash 0$ be a null bicharacteristic strip for q_M; $I = [t_0, t_1]$. If $f \in H^s$ on $\gamma(I)$ and if $u \in H^{s+m-1}$ at $\gamma(t_0)$, then $u \in H^{s+m-1}$ on $\gamma(I)$.*

Proof. If ${}^{co}p(x, \xi)$ is the cofactor matrix of $p(x, \xi)$, then ${}^{co}p(x, D)p(x, D) = q_M(x, D) + r(x, D)$ where r has order $M - 1$. Then $(q_M(x, D) + r(x, D))u = {}^{co}p(x, D)f \in H^{s+m(k-1)}$ on $\gamma(I)$. Since the principal part of $q_M + r$ is scalar, Hörmander's theorem yields $u \in H^{s+m-1}$ on $\gamma(I)$, as desired. (In [3] the result is only stated for operators acting on scalar valued functions, but the proof applies without change to this case.)

Let $\omega = (0, T) \times U$, $\Omega = (0, T) \times M$, and let $E = \{u \in H^{-1}(\Omega): u \in L^2(\omega)$ and $Lu \in L^2(\Omega)\}$. Then E is a Banach space with norm $\|u\|_{H^{-1}(\Omega)} + \|u\|_{L^2(\omega)} + \|Lu\|_{L^2(\Omega)}$. If $T > T_0$ then assumption A and proposition 1 yield the inclusion $E \subset L^2(\Omega)$.

To prove this we first extend u to $\mathbf{R} \times M$. Since $u \in E$ it follows that $u \in C((0, T): H^{-2}(M))$. Define $F \in L^2(\mathbf{R} \times M)$ by $F = Lu$ in Ω and $F = 0$ otherwise and let $\tilde{u}$ be the unique solution of $L\tilde{u} = F$, $\tilde{u}(T/2) = u(T/2)$. Uniqueness for the Cauchy problem implies that $u = \tilde{u}$ on Ω so it suffices to show that $\tilde{u} \in L^2$ at every $\Gamma = (t, x, \tau, \xi) \in T^*(\Omega)\backslash 0$. If $\det(\tau - ig(x, \xi)) \neq 0$ then $\tilde{u} \in H^1$ at Γ since $L\tilde{u}$ is square integrable. On the other hand if $\det(\tau - ig(x, \xi)) = 0$ then the null bicharacteristic through Γ passes through a point $(t_0, x_0, \tau_0, \xi_0) \in T^*(\omega)\backslash 0$. Proposition 1 implies that $\tilde{u} \in L^2$ at Γ. It is easily verified that $E \hookrightarrow L^2(\Omega)$ has closed graph, so it is continuous. This yields the inequality.

(1) $$\|u\|_{L^2(\Omega)}^2 \leqq c\,\|Lu\|_{L^2(\Omega)}^2 + c\,\|u\|_{L^2(\omega)}^2 + c\,\|u\|_{H^{-1}(\Omega)}^2.$$

This estimate is not good enough; we must get rid of the nuisance term $\|u\|_{H^{-1}(\Omega)}$. To do this we will have to make an additional assumption. Indeed, so far it is possible that there exist solutions to $Lu = 0$ which vanish on U for all $t \geq 0$, and such solutions would not decay. To prevent this possibility, we make the following assumption.

(B) *If T is taken large enough, then $Lu = 0$ on Ω, $u = 0$ on ω implies $u \equiv 0$ on Ω.*

If M is an analytic manifold and L has analytic coefficients, then assumption (B) is satisfied, by Holmgren's uniqueness theorem. We have the following additional result.

Proposition 2. *If (A) holds and if $G - \rho$ has the unique continuation property for every $\rho \in \mathbf{C}$ with $\operatorname{Re} \rho \leqq 0$, then hypothesis (B) is satisfied.*

Proof. Suppose assumption (A) is satisfied for all $T \geqq T_0$. If $V_T = \{u \in L^2((0, T) \times M): Lu = 0; u = 0 \text{ on } (0, T) \times U\}$ then inequality (1) implies that V_T is finite dimensional. As $T \to \infty$, the V_T form a decreasing family of finite dimensional spaces; hence they stabilize; $V_{T_1} = V_\infty$. We must show that $V_\infty = \{0\}$. If $u \in V_\infty$, then $Lu = 0$ on $(0, \infty) \times M$ and $u = 0$ on $(0, \infty) \times U$. If $i(\partial/\partial t) + G$ had the unique continuation property the results of [7, §3] would imply that $u \equiv 0$. Since we are considering a weaker hypothesis here, we argue as follows.

We may norm V_∞ by $v \to [\int_0^1 (\|v(t)\|_{L^2(M)})^2\,dt]^{1/2}$. Define a semigroup Q^t on V_∞ by $Q^t f(s, x) = f(s + t, x)$. Clearly Q^t is a continuous semigroup of contractions on V_∞. It follows that $Q^t = e^{t\Lambda}$ where Λ is a linear transformation on V_∞ with spectrum in the left half plane. Let $v \in V_\infty$ be an eigenvector of Λ with eigenvalue ρ. Then $v(t) = e^{\rho t} v(0)$, and $Lu = 0$ implies $(G - \rho)v(0) = 0$. Since $v \in V_\infty$ we have $v(0) = 0$ on ω and the unique continuation principle for $G - \rho$ implies $v(0) = 0$, so $v \equiv 0$, and we are done.

We now proceed to improve inequality (1). First, let $\mathcal{L}$ be the closed linear operator from $L^2(\Omega)$ to $L^2(\Omega) \oplus L^2(\omega)$ given by $\mathcal{L}u = (Lu, u|_\omega)$ with domain $D(\mathcal{L}) = \{u \in L^2(\Omega): Lu \in L^2(\Omega)\}$. If K is the imbedding of $L^2(\Omega)$ into $H^{-1}(\Omega)$ then K is compact and for $T \geqq T_0$ inequality (1) asserts that

$$\|u\|_{L^2(\Omega)}^2 \leqq c\,\|\mathcal{L}u\|_{L^2(\Omega)\oplus L^2(\omega)}^2 + c\,\|Ku\|_{H^{-1}(\Omega)}^2, \quad u \in D(\mathcal{L}).$$

The next lemma shows that the range of $\mathcal{L}$ is closed.

Lemma. *Let E, X, Y be Banach spaces and let $\mathcal{L}: D(\mathcal{L}) \subset E \to X$ be a closed linear operator and $K: E \to Y$ be compact. If $\|u\|_E \leqq c\,\|\mathcal{L}u\|_X + c\,\|Ku\|_Y$ for all $u \in D(\mathcal{L})$, then $\mathcal{L}$ has closed range.*

This is a standard Fredholm type result and is proved in Lions and Magenes [5], p. 171.

Now if assumption (B) is satisfied, $\mathcal{L}$ is one-to-one, so $\mathcal{L}^{-1}: R(\mathcal{L}) \to L^2(\Omega)$ is well defined and, by the closed graph theorem, continuous. This yields the inequality

$$\|u\|_{L^2(\Omega)}^2 \leqq c\,\|Lu\|_{L^2(\Omega)}^2 + c\,\|u\|_{L^2(\omega)}^2, \tag{2}$$

which is the desired improvement of inequality (1). We now have all the tools we need.

Theorem 1. *Let the dissipative symmetric operator* $L = \partial/\partial t - G(x, D_x)$ *satisfy* (A) *and* (B). *Then if* $Lu = 0$ *on* $(0, \infty) \times M$ *and* $u(0) \in L^2(M)$, *we have*

$$\|u(t)\|_{L^2(M)}^2 \leqq ce^{-\alpha t}\,\|u(0)\|_{L^2(M)}^2$$

for constants $c, \alpha > 0$ *independent of* u.

Proof.

$$\begin{aligned}(d/dt)\,\|u(t)\|_{L^2(M)}^2 &= 2\,\mathrm{Re}\,(Gu, u) = -2\int_M F(x)u\cdot u\,dx \\ &\leqq -2\eta\,\|u(t)\|_{L^2(\omega)}^2. \\ \therefore\ \|u(T)\|_{L^2(M)}^2 &\leqq \|u(0)\|^2 - 2\eta\,\|u\|_{L^2(\omega)}^2 \\ &\leqq \|u(0)\|^2 - c\,\|u\|_{L^2(\Omega)}^2\end{aligned}$$

by inequality (2). Since $\|u(T)\|_{L^2(M)} \leqq \|u(t)\|_{L^2(M)}$ for $0 \leqq t \leqq T$, we have

$$\|u\|_{L^2(\Omega)}^2 = \int_0^T \|u(t)\|_{L^2(M)}^2\,dt \geqq T\,\|u(T)\|_{L^2(M)}^2.$$

Therefore $\|u(T)\|_{L^2(M)} \leqq (1 + cT)^{-1/2}\,\|u(0)\|_{L^2(M)}$. From this inequality, exponential decay follows easily.

We should point out that the methods of Ralston [6] show that if L is strictly hyperbolic then (A) is a necessary condition for exponential decay. Most operators which satisfy this hypothesis are strictly hyperbolic.

§3. Second order operators. An analogous result can be obtained for solutions to second order hyperbolic equations of the form

$$L\phi = \frac{\partial^2}{\partial t^2}\phi - A\phi + b(x)\frac{\partial}{\partial t}\phi = 0.$$

We suppose that $A = a(x, D_x)$ in a second order elliptic operator an M and that $-A$ is strictly positive self-adjoint. We also suppose $\mathrm{Re}\, b \geqq 0$ on M. Physically, the term $b(\partial/\partial t)u$ represents a resistance or friction.

Let H be the Hilbert space $D((-A)^{1/2}) \oplus L^2(M)$ with norm

$$\|(u_1, u_2)\|_H^2 = \|(-A)^{1/2}u_1\|_{L^2(M)}^2 + \|u_2\|_{L^2(M)}^2.$$

Define a closed linear operator G on H by

$$G = \begin{pmatrix} 0 & I \\ A & -b \end{pmatrix}, \qquad D(G) = D(A) \oplus D((-A)^{1/2}).$$

It follows that G is a bounded perturbation of the skew-adjoint operator

$$G_0 = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix},$$

so G generates a one parameter group P^t on H.

Note that if we set $u_1 = \phi$, $u_2 = (\partial\phi/\partial t)$, $u = (u_1, u_2)$, the initial value problem $L\phi = 0$, $\phi(0) = \psi_1$, $(\partial\phi/\partial t)(0) = \psi_2$ is equivalent to $\partial u/\partial t = Gu$, $u(0) = (\psi_1, \psi_2)$.

The basic identity expressing dissipation of energy is

$$\frac{d}{dt} ||u(t)||_H^2 = 2 \operatorname{Re} (Gu, u)_H = -2 \int_M (\operatorname{Re} b) \, |u_2|^2 \, dx. \tag{3}$$

In particular P^t is a semigroup of contractions on H. We make the following assumption which is analogous to (A).

(A′) *The principal symbol of $a(x, D_x)$ is a scalar symbol $a(x, \xi)I$, and* $\operatorname{Re} b \geqq \eta I > 0$ *on a subset U of M with the property that there is a number T_0 such that every null bicharacteristic strip of $\partial^2/\partial t^2 - a(x, D_x)$ in $T^*(\mathbf{R} \times M)\backslash 0$ passes through $T^*((0, T_0) \times U)\backslash 0$.*

The appropriate analogue of (B) will then automatically hold, since second order elliptic operators with real (scalar) symbols always have the unique continuation property. (See [1] or [2].)

To show that there is a substantial amount of energy dissipation we must get a lower bound for $||u_2||_{L^2(\omega)}$ where $\omega = (0, T) \times U$ as in §2. Since $u_2 = \partial\phi/\partial t$ and $L\phi = 0$, we have $Lu_2 = 0$ and assumption (A′) shows that if $T \geqq T_0$ and $u_2 \in L^2(\omega)$, then $u_2 \in L^2(\Omega)$, and

$$||u_2||_{L^2(\Omega)}^2 \leqq c \, ||u_2||_{L^2(\omega)}^2 + c \, ||u_2||_{H^{-1}(\Omega)}^2.$$

Using the unique continuation property we may reason as in §2 to show that if T is taken sufficiently large, then

$$||u_2||_{L^2(\Omega)}^2 \leqq c \, ||u_2||_{L^2(\omega)}^2. \tag{4}$$

Combining inequalities (3) and (4) yields

$$||u(T)||_H^2 \leqq ||u(0)||_H^2 - c \, ||u_2||_{L^2(\Omega)}^2. \tag{5}$$

This inequality does not lead directly to energy decay. Fortunately, we can estimate $||u(t)||_H$ in terms of $||u_2(t)||_{L^2(M)}$ as follows. Suppose $T > 2$ and let $g \in C_0^\infty(0, T)$ be a fixed positive function, equal to 1 on $(1, T - 1)$. Then, with norms and inner products those of $L^2(M)$, we have

$$\int_1^{T-1} ||(-A)^{1/2}\phi(t)||^2\,dt \leqq \int_0^T ||g(t)(-A)^{1/2}\phi(t)||^2\,dt$$

$$= \int_0^T g(t)(-A\phi(t), \phi(t))\,dt.$$

Since $A\phi = \partial^2\phi/\partial t^2 + b(\partial\phi/\partial t)$, we can insert this identity into the integral above and integrate by parts with respect to t, obtaining

$$\int_0^T \{g'(\phi', \phi) + g\,||\phi'||^2 - g(b\phi', \phi)\}\,dt.$$

Estimating this crudely we have, for all $\epsilon > 0$,

$$\int_1^{T-1} ||(-A)^{1/2}\phi(t)||^2\,dt \leqq c(1 + 1/\epsilon)\int_0^T \left|\left|\frac{\partial}{\partial t}\phi\right|\right|^2 dt + c\epsilon\int_0^T ||\phi||^2\,dt.$$

It follows that for ϵ small

$$||u_2||_{L^2(\Omega)}{}^2 \geqq c_1\epsilon\int_1^{T-1} ||u(t)||_H{}^2\,dt - c_2\epsilon^2\int_0^T ||u(t)||_H{}^2\,dt$$

$$\geqq c_1\epsilon(T-2)\,||u(T)||_H{}^2 - c_2\epsilon^2 T\,||u(0)||_H{}^2,$$

where we have used the fact that $||u(t)||_H$ is a decreasing function of t. This estimate together with (5) yields

$$||u(T)||_H{}^2 \leqq \frac{1 + c_2\epsilon^2 T}{1 + c_1\epsilon(T-2)}\,||u(0)||_H{}^2. \tag{6}$$

We now choose ϵ so small that the factor in (6) is less than 1, and we have the desired estimate. To sum up, we have proved:

Theorem 2. *If assumption* (A′) *is satisfied, then if*

$$L\phi = 0, \qquad u_1 = \phi, \qquad u_2 = \partial\phi/\partial t, \qquad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

we have

$$||u(t)||_H{}^2 \leq ce^{-\alpha t}\,||u(0)||^2$$

for constants c; $\alpha > 0$ *independent of* ϕ, t.

§4. One space variable. We consider the system

$$0 = \partial_t u - A(x)\partial_x u - B(x)u \equiv (\partial_t - G)u \tag{7}$$

where $A(x)$ is a smooth $k \times k$ Hermitian symmetric matrix valued function on $I = [a, b]$. We suppose that G is dissipative, that is, $B + B^* - \partial_x A \leqq 0$. In this case $0 \geqq \text{Re}\,(G\phi, \phi)_{L^2(I)}$ for all $\phi \in C_0^\infty(I)$. In addition, we suppose that $A(x)$ is nonsingular for all x in I. It follows that the number, ℓ, of positive eigenvalues (counting multiplicity) of A is independent of x. The differential equation is supplemented by homogeneous boundary conditions

(8) $u \in N_a$ when $x = a$ and $u \in N_b$ when $x = b$ where N_a and N_b are subspace of $\mathbf{C}^k$. These boundary spaces are assumed *maximal dissipative*, that is $\dim N_a = \ell$, $\dim N_b = k - \ell$ and

$$\langle A(a)v, v\rangle \geqq 0 \quad \text{for} \quad v \in N_a\,, \quad \langle A(b)v, v\rangle \leqq 0 \quad \text{for} \quad v \in N_b$$

where $\langle\ ,\ \rangle$ is the scalar product in $\mathbf{C}^k$.

With these hypotheses it follows that for any $f \in L^2(I)$ there is a unique $u \in C(\mathbf{R}_+ : L^2(I))$ which is a weak solution of (7), (8) in $\mathbf{R}_+ \times I$ with $u(0) = f$. In addition, $||u(t)||$ is a decreasing function of t.

To ensure exponential decay we must consider the rays. For convenience we assume that $0 < \ell < k$, since if all the eigenvalues of A have the same sign a similar argument can be given. For $x \in I$ let $\lambda_{\min}(x)$ (resp. $\lambda_{\max}(x)$) be the smallest (resp. largest) eigenvalue of $A(x)$. $\lambda_{\max}$ and $\lambda_{\min}$ are Lipshitz continuous on I. An integral curve of the vector field $\partial_t - \lambda_{\max}(x)\partial_x$ (resp. $\lambda_{\min}$) is called a slow characteristic moving to the left (resp. right). These curves represent paths of signals traveling as slowly as possible. Let T_{left} be the time it takes as slow left characteristic starting at the right hand endpoint of I to reach $x = a$. Similarly T_{right} is the time required for a slow right characteristic to cross I. Let $T = T_{\text{left}} + T_{\text{right}}$. Intuitively a signal starting at any point in I will have reached every other point, after suitable reflection at the boundary, in time T.

We say that N_a *is strictly dissipative* if $\langle A(a)v, v\rangle \geqq c\langle v, v\rangle$ for all $v \in N_a$. A similar definition applies to N_b. We say that G *is strictly dissipative at* $x \in I$ if $B(x) + B^*(x) - \partial_x A(x) < 0$.

Theorem 3. *Suppose N_a or N_b is strictly dissipative or G is strictly dissipative at some $x \in I$. Then if u is a solution of (7), (8), with $u(0) \in L^2(I)$ then $||u(t)|| \leqq ce^{-\alpha t}\,||u(0)||$ with c and α positive constants independent of u. In fact for any $t > T$, $||u(t)|| < K\,||u(0)||$ for some $K < 1$ which depends only on t.*

Proof. We treat the case where N_a is strictly dissipative. The other possibilities are handled in a similar fashion. The basic identity reflecting energy decay is

$$\frac{d}{dt}\,||u(t)||_{L^2(I)}{}^2 = \int_I \langle(B + B^* - \partial_x A)u, u\rangle\,dx$$
$$+ \langle A(b)u(b), u(b)\rangle - \langle A(a)u(a), u(a)\rangle.$$

This implies that

$$||u(t)||_{L^2(I)}{}^2 \leqq ||u(0)||_{L^2(I)}{}^2 - c\int_0^t ||u(s, a)||^2\,ds$$

where c is the constant from the dissipativity of N_a. To prove Theorem 3 we show that if $t > T$ then

$$\int_a^t \|u(s, a)\|^2 \, ds \geqq \text{const} \, \|u(t)\|_{L^2(I)}^2. \tag{9}$$

Let Γ_u(u for upper) be the integral curve of $\partial_t - \lambda_{\max}(x)\partial_x$ which passes through (t, a) and Γ_ℓ the integral curve of $\partial_t - \lambda_{\min}(x)\partial_x$ through $(0, a)$. These curves intersect the boundary $x = b$ at times t_u and t_ℓ with $t_u - t_\ell = t - T > 0$. The equation $\partial_x u = A^{-1}\partial_t u - A^{-1}Bu$ is viewed as a symmetric hyperbolic equation with the direction of increasing x as time-like. The curves Γ_u and Γ_ℓ were chosen so that the domain of dependence of the slab $[t_\ell, t_u] \times I$ is contained in the initial segment $[0, t] \times \{a\}$. Standard energy methods imply that

$$\int_{t_u}^{t_\ell} \|u(s)\|_{L^2(I)}^2 \, ds \leqq \text{const} \int_0^t \|u(s, a)\|^2 \, ds \tag{10}$$

using the fact that $\|u(s)\|_{L^2(I)}$ is a decreasing function of s we see that the left hand side of (10) dominates $(t_u - t_\ell) \, \|u(t)\|^2$ which completes the proof of (9).

References

1. N. Aronszajn, *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of the second order*, J. Math. Pure Appl. **36** (1957), 235–249.
2. H. Cordes, *Uber die eindeutige bestimmtheit der losungen elliptischen differentialgleichungen durch anfangsvorgaben*, Nachr. Akad. Wiss. Göttingen Math.-Phys. K1, IIa, No. 11 (1956), 239–258.
3. L. Hormander, *On the existence and the regularity of solutions of linear pseudo-differential equations*, L'Enseignment Math. **17** (1971), 99–163.
4. N. Iwasaki, *Local decay of solutions for symmetric hyperbolic systems with dissipative and coercive boundary conditions in exterior domains*, Publ. RIMS Kyoto U. **5** (1969), 193–218.
5. J. Lions & E. Magenes, *Problèmes aux limites non-homogeneous et applications Vol. 1*, Dunod, Paris 1968.
6. J. Ralston, *Local decay of solutions of conservative first order hyperbolic systems in odd dimensional space*, Trans. Amer. Math Soc. (to appear).
7. J. Rauch & M. Taylor, *Penetration into shadow regions and unique continuation properties in hyperbolic mixed problems.* Indiana Math. J. **22** (1972), 277–285.

The authors were partially supported by National Science Foundation Grant GP 34160.

University of Michigan

Date communicated: September 10, 1973