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Exponential Dichotomy and Mild Solutions of Nonautonomous Equations in Banach Spaces

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We prove that the exponential dichotomy of a strongly continuous evolution family on a Banach space is equivalent to the existence and uniqueness of continuous bounded mild solutions of the corresponding inhomogeneous equation. This result addresses nonautonomous abstract Cauchy problems with unbounded coefficients. The technique used involves evolution semigroups. Some applications are given to evolution families on scales of Banach spaces arising in center manifolds theory.

KEY WORDS: Exponential dichotomy; mild solutions; central manifolds.

1991 MATHEMATICS SUBJECT CLASSIFICATIONS: 47D99, 34D05, 34C35.

0. INTRODUCTION

In this paper we prove that a strongly continuous exponentially bounded evolution family $\{U(t,s)\}_{t \geqslant s}$ on a Banach space X has exponential dichotomy if (and only if) for each bounded continuous X-valued function

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$f \in C_b(\mathbb{R}, X)$ there exists a unique solution $u \in C_b(\mathbb{R}, X)$ to the following inhomogeneous equation:

$$u(t) = U(t, s)\, u(s) + \int_s^t U(t, \tau)\, f(\tau)\, d\tau, \qquad t \geqslant s \tag{0.1}$$

Our work continues the line of research that characterizes dichotomy in terms of "Perron-type" theorems.

As an example, consider a nonautonomous abstract Cauchy problem,

$$x'(t) = A(t)\, x(t), \quad x(s) = x_s, \quad x_s \in D(A(s)), \quad t \geqslant s, \quad t, s \in \mathbb{R} \tag{0.2}$$

on the Banach space X. Assume, for a moment, that (0.2) is well-posed in the sense that there exists an evolution (solving) family of operators $\{U(t, s)\}_{t \geqslant s}$ which gives a differentiable solution $x(\cdot)$. This means that $x(\cdot)\colon t \mapsto U(t, s)\, x(s)$, $t \geqslant s$, is differentiable for any given initial conditions $x(s) = x_s \in D(A(s))$, $x(t) \in D(A(t))$, and (0.2) holds. Now let f be a locally integrable X-valued function on $\mathbb{R}$ and consider the inhomogeneous equation:

$$x'(t) = A(t)\, x(t) + f(t), \qquad t \in \mathbb{R} \tag{0.3}$$

A function $u(\cdot)$ is called a *mild* solution of (0.3) if $u(\cdot)$ satisfies (0.1). Thus, our result shows, in particular, that the exponential dichotomic behavior of solutions to a nonautonomous abstract Cauchy problem (0.2) is equivalent to the existence and uniqueness of a bounded continuous *mild* solution to the *inhomogeneous* equation (0.3) for any $f \in C_b(\mathbb{R}, X)$. Let R denote the operator that recovers the solution $u = Rf$ to (0.1) for a given f. We note that if f is sufficiently smooth, then the function $u = Rf$ defined by (0.1) will be a differentiable solution to (0.3) (cf. Pazy (1983), Sections 5.5 and 5.7).

The characterization of dichotomy for (0.2) in terms of the solutions of the inhomogeneous equation (0.3) has a fairly long history that goes back to Perron (1930). Classical theorems of this type concerning differential equations with bounded operators $A(t)$ can be found in Daleckij and Krein (1974) (see also historical comments there) and Massera and Schaeffer (1966). For unbounded $A(t)$ a result of this type concerning *classical* solutions of (0.3), obtained by a completely different method, and applications of this result are contained in the book by Levitan and Zhikov (1982), Chapters 10, 11. Recent results for the finite dimensional case are obtained in Palmer (1988) and Ben-Artzi and Gohberg (1992) (see also the literature cited therein). "L_p-theorems" of this type can be found in Dore (1993). Results of this type for nonautonomous equations on the semiaxis $\mathbb{R}_+$ are considered (under some additional assumptions) in Rodrigues and

Ruas-Filho (1995), and a certain class of nonautonomous parabolic equations on the semiaxis is considered in Zhang (1995).

In the autonomous case, when $A(t) \equiv A$ is an unbounded operator which generates a strongly continuous semigroup on X, this result is proven in Prüss (1984), Theorem 4; our paper is a direct generalization of his result. In the current paper, we address the general nonautonomous abstract Cauchy problem (0.2) on $\mathbb{R}$ where the operators $A(t)$ (with domains $D(A(t))$) are time-dependent and unbounded.

Our main result is not restricted to well-posed abstract Cauchy problems but addresses, more generally, any problem (0.2) whose solution is associated with a strongly continuous evolution family $\{U(t,s)\}_{t \geqslant s}$. It is often the case that an equation (0.2) does not have a differentiable solution yet does have a solution, in a mild sense, which is associated with an evolution family. For example, consider $A(t) := A_0 + A_1(t)$ where A_0 generates a strongly continuous semigroup, $\{e^{tA_0}\}_{t \geqslant 0}$, and $t \mapsto A_1(t)$ is continuous from $\mathbb{R}$ to $B(X)$, the set of bounded linear operators on X. One can show that there exists an evolution family $\{U(t,s)\}_{t \geqslant s}$ which gives a mild solution, $x(t) = U(t,s)\, x(s)$, for $x(s) = x_s \in D(A(s)) = D(A_0)$, in the sense that it satisfies the integral equation

$$x(t) = e^{(t-s)A_0} x_s + \int_s^t e^{(t,\tau)A_0} A_1(\tau)\, x(\tau)\, d\tau$$

However, it is not necessarily the case that $U(t,s)\, x$ is differentiable in t for all $x \in D(A)$; see, for example, Curtain and Pritchard (1978), Chapters 2 and 9.

Therefore, we do not study only well-posed abstract Cauchy problems (0.2). Instead, we take as our starting point any strongly continuous evolution family. We assume, in addition, that $\{U(t,s)\}_{t \geqslant s}$ is exponentially bounded. This replaces in a natural way the classical assumption of bounded growth (see, e.g., Zhang (1995), Lemma 5).

We prove that an evolution family $\{U(t,s)\}_{t \geqslant s}$ has exponential dichotomy if and only if the following property holds:

(M) *For every $f \in C_b(\mathbb{R}, X)$ there exists a unique solution $u \in C_b(\mathbb{R}, X)$ to* (0.1).

This result is applied in Section 2 to show that a standing hypothesis in the paper Vanderbauwhede and Iooss (1992) on central manifold theory is, in fact, equivalent to the existence of the exponential dichotomy for $\{U(t,s)\}_{t \geqslant s}$.

The main result is proven in Section 1 using the relatively new technique of so-called evolution semigroups; see Howland (1974), Latushkin and

Montgomery-Smith (1994, 1995), Latushkin *et al.* (1996), Latushkin and Randolph (1995), Nagel (1995), Räbiger and Schnaubelt (1994, 1996), Rau (1994a, 1994b), Nguyen Van Minh (1994), van Neerven (1996), and also Johnson (1980) and Chicone and Swanson (1981) in a different context. For a given evolution family $\{U(t,s)\}_{t\geqslant s}$ the evolution semigroup $\{T^t\}_{t\geqslant 0}$ is defined on a "super-space" E of functions $f\colon \mathbb{R}\to X$ by the rule

$$(T^t f)(\tau)=U(\tau,\tau-t)\,f(\tau-t),\qquad \tau\in\mathbb{R},\qquad t\geqslant 0 \tag{0.4}$$

If $E=L_p(\mathbb{R},X)$, $1\leqslant p<\infty$, or $E=C_0(\mathbb{R},X)$, the space of continuous functions vanishing at infinity, then $\{T^t\}_{t\geqslant 0}$ is a strongly continuous semigroup; we will denote its generator by Γ.

The role of the evolution semigroup arises from the following facts. If A generates a strongly continuous semigroup, it is well known (see, e.g., Nagel (1984)) that the asymptotic behavior of solutions to the differential equation $x'(t)=Ax(t)$ is determined in many important cases by the location of the spectrum of A. There are, however, examples for which this property fails to hold, and for nonautonomous Cauchy problems (0.2), even for finite-dimensional X, the asymptotic behavior of a solution is most likely not determined by the spectra of the operators $A(t)$. However, *it is determined* by the spectrum of T^t or Γ. Namely, the following facts hold (see Latushkin and Montgomery-Smith (1994, 1995), Latushkin *et al.* (1996), Latushkin and Randolph (1995), Räbiger and Schnaubelt (1994, 1996), Rau (1994a, 1994b)). A strongly continuous evolution family $\{U(t,s)\}_{t\geqslant s}$ has exponential dichotomy on X if and only if the spectrum, $\sigma(T^t)$, $t>0$, does not intersect the unit circle or, equivalently, the operator Γ^{-1} is bounded on $C_0(\mathbb{R},X)$ or $L_p(\mathbb{R},X)$. Moreover, one can associate with $T=T^1$ on $C_0(\mathbb{R},X)$ a family of weighted shift operators, $\pi_s(T)$, $s\in\mathbb{R}$, acting on $c_0(\mathbb{Z},X)$, the space of X-valued sequences vanishing at infinity. The exponential dichotomy of $\{U(t,s)\}_{t\geqslant s}$ is equivalent to the fact that the operators $(I-\pi_s(T))^{-1}$ are bounded uniformly for $s\in\mathbb{R}$. In the main step of our proof we show that condition (M) implies this fact.

In Section 2 we discuss several modifications of the condition (M). First, we replace the space $C_b(\mathbb{R},X)$ in (M) by $C_0(\mathbb{R},X)$ and show that this new condition (M_{C_0}) is also equivalent to the dichotomy of the evolution family $\{U(t,s)\}_{t\geqslant s}$. Since $\{T^t\}_{t\geqslant 0}$ is a strongly continuous semigroup on $C_0(\mathbb{R},X)$ with the generator Γ, we are able to prove that the operator R, defined by (M_{C_0}) as $u=Rf$, is, in fact, equal to $-\Gamma^{-1}$. Thus, (M_{C_0}) is a "mild version" of the above-mentioned condition, $\Gamma^{-1}\in B(C_0(\mathbb{R},X))$, for the dichotomy of $\{U(t,s)\}_{t\geqslant s}$. Similar remarks hold when $C_0(\mathbb{R},X)$ is replaced by $L_p(\mathbb{R},X)$.