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# Exponential estimate for the asymptotics of Bergman kernels

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**Abstract** We prove an exponential estimate for the asymptotics of Bergman kernels of a positive line bundle under hypotheses of bounded geometry. Further, we give Bergman kernel proofs of complex geometry results, such as separation of points, existence of local coordinates and holomorphic convexity by sections of positive line bundles.

## 1 Introduction

Let $(X, \omega)$ be a symplectic manifold of real dimension $2n$. Assume that there exists a Hermitian line bundle $(L, h^L)$ over $X$ endowed with a Hermitian connection $\nabla^L$ with the property that

$$R^L = -2\pi\sqrt{-1}\,\omega, \tag{1.1}$$

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where $R^L = (\nabla^L)^2$ is the curvature of $(L, \nabla^L)$. Let $(E, h^E)$ be a Hermitian vector bundle on $X$ with Hermitian connection $\nabla^E$ and its curvature $R^E$.

Let $J$ be an almost complex structure which is compatible with $\omega$ (i.e., $\omega$ is $J$-invariant and $\omega(\cdot, J\cdot)$ defines a metric on $TX$). Let $g^{TX}$ be a $J$-invariant Riemannian metric on $X$. Let $d(x, y)$ be the Riemannian distance on $(X, g^{TX})$.

The spin$^c$ Dirac operator $D_p$ acts on $\Omega^{0,\bullet}(X, L^p \otimes E) = \bigoplus_{q=0}^n \Omega^{0,q}(X, L^p \otimes E)$, the direct sum of spaces of $(0, q)$–forms with values in $L^p \otimes E$.

We refer to the orthogonal projection $P_p$ from $L^2(X, E_p)$, the space of $L^2$-sections of $E_p := \Lambda^\bullet(T^{*(0,1)}X) \otimes L^p \otimes E$, onto $\mathrm{Ker}(D_p)$ as the *Bergman projection* of $D_p$. The Schwartz kernel $P_p(\cdot, \cdot)$ of $P_p$ with respect to the Riemannian volume form $dv_X(x')$ of $(X, g^{TX})$ is called the *Bergman kernel* of $D_p$.

**Theorem 1** *Suppose that $(X, g^{TX})$ is complete and $R^L$, $R^E$, $J$, $g^{TX}$ have bounded geometry (i.e., they and their derivatives of any order are uniformly bounded on $X$ in the norm induced by $g^{TX}$, $h^E$, and the injectivity radius of $(X, g^{TX})$ is positive). Assume also that there exists $\varepsilon > 0$ such that on $X$,*

$$\sqrt{-1}R^L(\cdot, J\cdot) > \varepsilon g^{TX}(\cdot, \cdot). \tag{1.2}$$

*Then there exist $\boldsymbol{c} > 0$, $\boldsymbol{p}_0 > 0$, which can be determined explicitly from the geometric data* [cf. (3.17)] *such that for any $k \in \mathbb{N}$, there exists $C_k > 0$ such that for any $p \geqslant \boldsymbol{p}_0$, $x, x' \in X$, we have*

$$\left|P_p(x, x')\right|_{\mathscr{C}^k} \leqslant C_k\, p^{n+\frac{k}{2}} \exp\!\left(-\boldsymbol{c}\sqrt{p}\, d(x, x')\right). \tag{1.3}$$

The pointwise $\mathscr{C}^k$-seminorm $\left|S(x, x')\right|_{\mathscr{C}^k}$ of a section $S \in \mathscr{C}^\infty(X \times X, E_p \boxtimes E_p^*)$ at a point $(x, x') \in X \times X$ is the sum of the norms induced by $h^L$, $h^E$ and $g^{TX}$ of the derivatives up to order $k$ of $S$ with respect to the connection induced by $\nabla^L$, $\nabla^E$ and the Levi-Civita connection $\nabla^{TX}$ evaluated at $(x, x')$.

Assume now $X = \mathbb{C}^n$ with the standard trivial metric, $E = \mathbb{C}$ with trivial metric. Assume also $L = \mathbb{C}$ and $h^L = e^{-\varphi}$ where $\varphi : X \to \mathbb{R}$ is a smooth plurisubharmonic potential such that (1.2) holds. Then the estimate (1.3) with $k = 0$ was basically obtained by [5] for $n = 1$, [8,12] for $n \geqslant 1$. In [7, Theorem 4.18] (cf. [17, Theorem 4.2.9]), a refined version of (1.3), i.e., the asymptotic expansion of $P_p(x, x')$ for $p \to +\infty$ with the exponential estimate was obtained.

When $(X, J, \omega)$ is a compact Kähler manifold, $E = \mathbb{C}$ with trivial metric, $g^{TX} = \omega(\cdot, J\cdot)$ and (1.1) holds, a better estimate than (1.3) with $k = 0$ and $d(x, x') > \delta > 0$ was obtained in [6].

Recently, Lu and Zelditch announced in [13, Theorem 2.1] the estimate (1.3) with $k = 0$ and $d(x, x') > \delta > 0$ when $(X, J, \omega)$ is a Kähler manifold, $E = \mathbb{C}$ with trivial metric, $g^{TX} = \omega(\cdot, J\cdot)$ and (1.1) holds (see also [1]).

Theorem 1 was known to the authors for several years, being an adaptation of [7, Theorem 4.18]. The recent papers [6,13] motivated us to publish our proof.

The next result describes the relation between the Bergman kernel on a Galois covering of a compact symplectic manifold and the Bergman kernel on the base, which is an interesting consequence of our proof of Theorem 1.

**Theorem 2** *Let $(X, \omega)$ be a compact symplectic manifold. Let $(L, \nabla^L, h^L)$, $(E, \nabla^E, h^E)$, $J$, $g^{TX}$ be given as above. Consider a Galois covering $\pi : \widetilde{X} \to X$ and let $\Gamma$ be the group of deck transformations. Let us decorate with tildes the preimages of objects living on the quotient, e.g., $\widetilde{L} = \pi^* L$, $\widetilde{\omega} = \pi^* \omega$ etc. Let $\widetilde{D}_p$ be the spin$^c$ Dirac operator acting on $\Omega^{0,\bullet}(\widetilde{X}, \widetilde{L}^p \otimes \widetilde{E})$ and $\widetilde{P}_p$ be the orthogonal projection from $L^2(\widetilde{X}, \widetilde{E}_p)$ onto $\operatorname{Ker}(\widetilde{D}_p)$. There exists $\boldsymbol{p}_1$ which depends only on the geometric data on $X$, such that for any $p > \boldsymbol{p}_1$ we have*

$$\sum_{\gamma \in \Gamma} \widetilde{P}_p(\gamma x, y) = P_p(\pi(x), \pi(y)), \quad \textit{for any } x, y \in \widetilde{X}. \tag{1.4}$$

As a particular case, (1.4) holds for a compact Kähler manifold $(X, J, \omega)$, a positive holomorphic Hermitian line bundle $(L, h^L)$, the projection $P_p$ on the space $H^0(X, L^p \otimes E)$ of holomorphic sections (which equals $\operatorname{Ker}(D_p)$ for $p$ large enough), and the projection $\widetilde{P}_p$ on the space $H^0_{(2)}(\widetilde{X}, \widetilde{L}^p \otimes \widetilde{E})$ of holomorphic $L^2$ sections (which equals $\operatorname{Ker}(\widetilde{D}_p)$ for $p$ large enough), see Remark 1, Theorem 5. In this context, for $g^{TX} = \omega(\cdot, J\cdot)$ and for $E = \mathbb{C}$ with trivial metric, (1.4) also appeared simultaneously in [13, Theorem 1] by a different method.

We also give proofs of some complex geometry results by using the expansion of the Bergman kernel.

**Theorem 3** *Let $(X, J, g^{TX})$ be a complete Hermitian manifold and $(L, h^L)$, $(E, h^E)$ be holomorphic Hermitian vector bundles over $X$, where $\operatorname{rk}(L) = 1$. Assume that $R^L$, $R^E$, $J$, $g^{TX}$ have bounded geometry and (1.2) holds. Then there exists $p_0 \in \mathbb{N}$ such that for all $p \geqslant p_0$, $X$ is holomorphically convex with respect to the bundle $L^p \otimes E$ and $H^0_{(2)}(X, L^p \otimes E)$ separates points and give local coordinates on $X$.*

The main tool here are the properties of peak sections at a point $x_0 \in X$, which are defined as sections in the orthogonal complement of the space of sections vanishing at $x_0$. We show for example in (4.21) that linearly independent peak sections at $x_0$ are asymptotically orthogonal in the neighborhood of $x_0$.

Theorem 3 implies immediately:

**Corollary 1** *Let $(X, J, \Theta)$ be a complete Hermitian manifold and let $\varphi$ be a smooth function on $X$. Assume that $\varphi$ is bounded from below, $\partial\overline{\partial}\varphi$, $J$, $g^{TX}$ have bounded geometry and there exists $\varepsilon > 0$ such that $\sqrt{-1}\partial\overline{\partial}\varphi \geqslant \varepsilon\Theta$ on $X$. Then $X$ is a Stein manifold.*

We refer to [15,19] for further applications of the off-diagonal expansion of the Bergman kernel to the Berezin-Toeplitz quantization, and [17] for a comprehensive study of the Bergman kernels and its applications.

This paper is organized as follows: In Sect. 2, we explain the spectral gap property of Dirac operators and the elliptic estimate. In Sect. 3, we establish Theorems 1, 2. In Sect. 4, we show what the result becomes in the complex case. We give some applications of Theorem 1 and of the diagonal expansion of the Bergman kernel, in particular, we establish Theorem 3 and Corollary 1.

## 2 Dirac operator and elliptic estimates

The almost complex structure $J$ induces a splitting $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$, where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of $J$ corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. Let $T^{*(1,0)}X$ and $T^{*(0,1)}X$ be the corresponding dual bundles. For any $v \in TX$ with decomposition $v = v_{1,0} + v_{0,1} \in T^{(1,0)}X \oplus T^{(0,1)}X$, let $v^*_{1,0} \in T^{*(0,1)}X$ be the metric dual of $v_{1,0}$. Then $c(v) = \sqrt{2}(v^*_{1,0} \wedge - i_{v_{0,1}})$ defines the Clifford action of $v$ on $\Lambda(T^{*(0,1)}X)$, where $\wedge$ and $i$ denote the exterior and interior product respectively. We denote

$$\Lambda^{0,\bullet} = \Lambda^{\bullet}(T^{*(0,1)}X), \quad E_p := \Lambda^{0,\bullet} \otimes L^p \otimes E. \tag{2.1}$$

Along the fibers of $E_p$, we consider the pointwise Hermitian product $\langle \cdot, \cdot \rangle$ induced by $g^{TX}$, $h^L$ and $h^E$. Let $dv_X$ be the Riemannian volume form of $(TX, g^{TX})$. The $L^2$–Hermitian product on the space $\Omega^{0,\bullet}_0(X, L^p \otimes E)$ of smooth compactly supported sections of $E_p$ is given by

$$\langle s_1, s_2 \rangle = \int_X \langle s_1(x), s_2(x) \rangle \, dv_X(x). \tag{2.2}$$

We denote the corresponding norm with $\|\cdot\|_{L^2}$ and with $L^2(X, E_p)$ the completion of $\Omega^{0,\bullet}_0(X, L^p \otimes E)$ with respect to this norm, and $\|B\|^{0,0}$ the norm of $B \in \mathscr{L}(L^2(X, E_p))$ with respect to $\|\cdot\|_{L^2}$.

Let $\nabla^{TX}$ be the Levi-Civita connection of the metric $g^{TX}$, and let $\nabla^{\det_1}$ be the connection on $\det(T^{(1,0)}X)$ induced by $\nabla^{TX}$ by projection. By [17, §1.3.1], $\nabla^{TX}$ (and $\nabla^{\det_1}$) induces canonically a Clifford connection $\nabla^{\text{Cliff}}$ on $\Lambda(T^{*(0,1)}X)$. Let $\nabla^{E_p}$ be the connection on $E_p = \Lambda(T^{*(0,1)}X) \otimes L^p \otimes E$ induced by $\nabla^{\text{Cliff}}$, $\nabla^L$ and $\nabla^E$.

We denote by $\text{Spec}(B)$ the spectrum of an operator $B$.

**Definition 1** The spin$^c$ Dirac operator $D_p$ is defined by

$$D_p = \sum_{j=1}^{2n} c(e_j) \nabla^{E_p}_{e_j} : \Omega^{0,\bullet}(X, L^p \otimes E) \longrightarrow \Omega^{0,\bullet}(X, L^p \otimes E), \tag{2.3}$$

with $\{e_i\}_i$ an orthonormal frame of $TX$.

Note that $D_p$ is a formally self-adjoint, first order elliptic differential operator. Since we are working on a complete manifold $(X, g^{TX})$, $D_p$ is essentially self-adjoint. This follows e.g., from an easy modification of the Andreotti-Vesentini lemma [17, Lemma 3.3.1] [where the particular case of a complex manifold and the Dirac operator (4.1) is considered]. Let us denote by $D_p$ the self-adjoint extension of $D_p$ defined initially on the space of smooth compactly supported forms, and by $\text{Dom}(D_p)$ its domain. By the proof of [16, Theorems 1.1, 2.5] (cf. [17, Theorems 1.5.7, 1.5.8] as the tensor $S$ therein is given by $S = -\frac{1}{2} J(\nabla^{TX} J)$), we have:

**Lemma 1** *If* $(X, g^{TX})$ *is complete and* $\nabla^{TX}J$, $R^{TX}$ *and* $R^E$ *are uniformly bounded on* $(X, g^{TX})$, *and* (1.2) *holds, then there exists* $C_L > 0$ *such that for any* $p \in \mathbb{N}$, *and any* $s \in \bigoplus_{q\geqslant 1}\Omega_0^{0,q}(X, L^p \otimes E)$, *we have*

$$\|D_p s\|_{L^2}^2 \geqslant \big(2p\mu_0 - C_L\big)\|s\|_{L^2}^2, \tag{2.4}$$

*with*

$$\mu_0 = \inf_{u\in T_x^{(1,0)}X,\, x\in X} R_x^L(u, \overline{u})/|u|_{g^{TX}}^2 > 0. \tag{2.5}$$

*Moreover*

$$\operatorname{Spec}(D_p^2) \subset \{0\} \cup [2p\mu_0 - C_L, +\infty[. \tag{2.6}$$

From now on, we assume that $(X, g^{TX})$ is complete and $R^L$, $R^E$, $J$, $g^{TX}$ have bounded geometry. The following elliptic estimate will be applied to get the kernel estimates. Notice that we do not use the condition that $\sqrt{-1}R^L(\cdot, J\cdot)$ is positive in the proof of Lemma 2.

**Lemma 2** *Given a sequence of smooth forms* $s_p \in \bigcap_{\ell\in\mathbb{N}} \operatorname{Dom}(D_p^\ell) \subset L^2(X, E_p)$ *and a sequence* $C_p' > 0$ $(p \in \mathbb{N})$, *assume that for any* $\ell \in \mathbb{N}$, *there exists* $C_\ell'' > 0$ *such that for any* $p \in \mathbb{N}^*$,

$$\left\|\left(\frac{1}{\sqrt{p}}D_p\right)^\ell s_p\right\|_{L^2} \leqslant C_\ell'' C_p'. \tag{2.7}$$

*Then for any* $k \in \mathbb{N}$, *there exists* $C_k > 0$ *such that for any* $p \in \mathbb{N}^*$ *and* $x \in X$ *the pointwise* $\mathscr{C}^k$*-seminorm satisfies*

$$\big|s_p(x)\big|_{\mathscr{C}^k} \leqslant C_k C_p' p^{\frac{n+k}{2}}. \tag{2.8}$$

*Proof* Let $\operatorname{inj}^X$ be the injectivity radius of $(X, g^{TX})$, and let $\varepsilon \in ]0, \operatorname{inj}^X[$. We denote by $B^X(x,\varepsilon)$ and $B^{T_xX}(0,\varepsilon)$ the open balls in $X$ and $T_xX$ with center $x \in X$ and radius $\varepsilon$, respectively. The exponential map $T_xX \ni Z \mapsto \exp_x^X(Z) \in X$ is a diffeomorphism from $B^{T_xX}(0,\varepsilon)$ on $B^X(x,\varepsilon)$. From now on, we identify $B^{T_xX}(0,\varepsilon)$ with $B^X(x,\varepsilon)$ for $\varepsilon < \operatorname{inj}^X$.

For $x_0 \in X$, we work in the normal coordinates on $B^X(x_0,\varepsilon)$. We identify $E_Z$, $L_Z$, $\Lambda(T_Z^{*(0,1)}X)$ to $E_{x_0}$, $L_{x_0}$, $\Lambda(T_{x_0}^{*(0,1)}X)$ by parallel transport with respect to the connections $\nabla^E$, $\nabla^L$, $\nabla^{\text{Cliff}}$ along the curve $[0,1] \ni u \mapsto uZ$. Thus on $B^{T_{x_0}X}(0,\varepsilon)$, $(E_p, h^{E_p})$ is identified to the trivial Hermitian bundle $(E_{p,x_0}, h^{E_{p,x_0}})$. Let $\{e_i\}_i$ be an orthonormal basis of $T_{x_0}X$. Denote by $\nabla_U$ the ordinary differentiation operator on $T_{x_0}X$ in the direction $U$. Let $\Gamma^E$, $\Gamma^L$, $\Gamma^{\Lambda^{0,\bullet}}$ be the corresponding connection forms of $\nabla^E$, $\nabla^L$ and $\nabla^{\text{Cliff}}$ with respect to any fixed frame for $E$, $L$, $\Lambda(T^{*(0,1)}X)$ which is parallel along the curve $[0,1] \ni u \mapsto uZ$ under the trivialization on $B^{T_{x_0}X}(0,\varepsilon)$.

Let $\{\widetilde{e}_i\}_i$ be an orthonormal frame on $TX$. On $B^{T_{x_0}X}(0,\varepsilon)$, we have

$$\begin{aligned}\nabla_{\cdot}^{E_{p,x_0}} &= \nabla_{\cdot} + p\,\Gamma^L(\cdot) + \Gamma^{\Lambda^{0,\bullet}}(\cdot) + \Gamma^E(\cdot),\\ D_p &= c(\widetilde{e}_j)\Big(\nabla_{\widetilde{e}_j} + p\Gamma^L(\widetilde{e}_j) + \Gamma^{\Lambda^{0,\bullet}}(\widetilde{e}_j) + \Gamma^E(\widetilde{e}_j)\Big).\end{aligned} \tag{2.9}$$

By [17, Lemma 1.2.4], for $\Gamma^{\bullet} = \Gamma^E, \Gamma^L, \Gamma^{\Lambda^{0,\bullet}}$, we have

$$\Gamma^{\bullet}_Z(e_j) = \frac{1}{2} R^{\bullet}_{x_0}(Z, e_j) + \mathscr{O}(|Z|^2). \tag{2.10}$$

Using an unit vector $S_L$ of $L_{x_0}$, we get an isometry

$$E_{p,x_0} \simeq \Big(\Lambda(T^{*(0,1)}X) \otimes E\Big)_{x_0} =: \boldsymbol{E}_{x_0}. \tag{2.11}$$

For $s \in \mathscr{C}^\infty(\mathbb{R}^{2n}, \boldsymbol{E}_{x_0})$, $Z \in \mathbb{R}^{2n}$ and $t = \frac{1}{\sqrt{p}}$, set

$$\begin{aligned}&(S_t s)(Z) = s(Z/t), \quad \nabla_t = S_t^{-1} t \nabla^{E_{p,x_0}} S_t,\\ &\boldsymbol{D}_t = S_t^{-1} t D_p S_t.\end{aligned} \tag{2.12}$$

Let $f : \mathbb{R} \to [0,1]$ be a smooth even function such that

$$f(v) = \begin{cases} 1 & \text{for } |v| \leqslant \varepsilon/2,\\ 0 & \text{for } |v| \geqslant \varepsilon' \text{ with } \varepsilon' < \varepsilon.\end{cases} \tag{2.13}$$

Set

$$\sigma_p(Z) = s_p(Z) f(|Z|/t), \quad \widetilde{\sigma}_p = S_t^{-1}\sigma_p \in \mathscr{C}_0^\infty(\mathbb{R}^{2n}, \boldsymbol{E}_{x_0}). \tag{2.14}$$

Then by (2.12) and (2.14), we have

$$\begin{aligned}&\widetilde{\sigma}_p(Z) = s_p(tZ) f(|Z|)\\ &(\boldsymbol{D}_t^l \widetilde{\sigma}_p)(Z) = \sum_{j=0}^{l} \binom{l}{j} \underbrace{\big[\boldsymbol{D}_t, \ldots, [\boldsymbol{D}_t}_{(l-j)\text{ times}}, f(|\cdot|)]\ldots\big](Z)\,(\boldsymbol{D}_t^j S_t^{-1} s_p)(Z).\end{aligned} \tag{2.15}$$

Now by (2.9), (2.10) and (2.12), we get

$$\nabla_t = \nabla_0 + \mathscr{O}(t), \quad \boldsymbol{D}_t = \boldsymbol{D}_0 + \mathscr{O}(t), \tag{2.16}$$

where $\boldsymbol{D}_0$ is an elliptic operator on $\mathbb{R}^{2n}$. By (2.16) and our assumption on bounded geometry, $\underbrace{\big[\boldsymbol{D}_t, \ldots, [\boldsymbol{D}_t}_{(l-j)\text{ times}}, f(|\cdot|)]\ldots\big]$ is a differential operator of order $l - j - 1$

for $l > j$, whose coefficients are uniformly bounded on $B^{T_{x_0}X}(0,\varepsilon)$, with respect to $x_0 \in X$, $p \in \mathbb{N}$, and $(\boldsymbol{D}_t^j S_t^{-1} s_p)(Z) = \left(\left(\frac{1}{\sqrt{p}} D_p\right)^j s_p\right)(tZ)$. Thus (2.7), (2.9) and (2.15) and elliptic estimates imply

$$\int\limits_{|Z|<\varepsilon} |\boldsymbol{D}_t^l \widetilde{\sigma}_p(Z)|^2 \, dZ \leqslant (l+1) \int\limits_{|Z|<\varepsilon} \Big\{ \Big| f(|\cdot|) \boldsymbol{D}_t^l S_t^{-1} s_p \Big|^2 (Z) + l^2 \Big| [\boldsymbol{D}_t, f(|\cdot|)] \boldsymbol{D}_t^{l-1} S_t^{-1} s_p \Big|^2 (Z) + \widetilde{C}_l \sum_{j=0}^{l-2} \Big| \boldsymbol{D}_t^{l-j-1} \varphi(|\cdot|) \boldsymbol{D}_t^j S_t^{-1} s_p \Big|^2 (Z) \Big\} dZ, \tag{2.17}$$

here $\varphi : \mathbb{R} \to [0,1]$ is a smooth even function such that $\varphi = 1$ on $\mathrm{supp}(f)$ and $\varphi(v) = 0$ for $|v| \geqslant \varepsilon''$ with $\varepsilon'' < \varepsilon$. By repeating the above argument, we get

$$\begin{aligned} \int\limits_{|Z|<\varepsilon} |\boldsymbol{D}_t^l \widetilde{\sigma}_p(Z)|^2 \, dZ &\leqslant C_0 p^n \int\limits_{|Z|<\varepsilon/\sqrt{p}} \sum_{j=0}^{l} \left| \left( \frac{1}{\sqrt{p}} D_p \right)^j s_p(Z) \right|^2 dZ \\ &\leqslant C_1 \sum_{j=0}^{l} (C''_j C'_p)^2 p^n. \end{aligned} \tag{2.18}$$

On $\mathbb{R}^{2n}$, we use the usual $\mathscr{C}^k$-seminorm, i.e., $|g(x)|_{\mathscr{C}^k} := \sum_{l \leqslant k} |\nabla_{e_{i_1}} \ldots \nabla_{e_{i_l}} g|(x)$. By the Sobolev embedding theorem (cf. [17, Theorem A.1.6]), (2.16), (2.18) and our assumption on bounded geometry, we obtain that for any $k \in \mathbb{N}$, there exists $C_k > 0$ such that for any $x_0 \in X$, $p \in \mathbb{N}^*$, we have $\left|\widetilde{\sigma}_p(0)\right|_{\mathscr{C}^k} \leqslant C_k C'_p p^{n/2}$. Going back to the original coordinates (before rescaling), we get

$$\left|\sigma_p(x_0)\right|_{\mathscr{C}^k} \leqslant C_k C'_p p^{\frac{n+k}{2}}. \tag{2.19}$$

The proof of Lemma 2 is completed. □

## 3 Proofs of Theorems 1 and 2

For $x, x' \in X$, let $\exp\left(-\frac{u}{p} D_p^2\right)(x,x')$, $\left(\frac{1}{p} D_p^2 \exp\left(-\frac{u}{p} D_p^2\right)\right)(x,x')$ be the smooth kernels of the operators $\exp\left(-\frac{u}{p} D_p^2\right)$, $\frac{1}{p} D_p^2 \exp\left(-\frac{u}{p} D_p^2\right)$ with respect to $dv_X(x')$.

**Theorem 4** *There exists $\boldsymbol{a} > 0$ such that for any $m \in \mathbb{N}$, $u_0 > 0$, there exists $C > 0$ such that for $u \geqslant u_0$, $p \in \mathbb{N}^*$, $x, x' \in X$, we have*

$$\left| \exp\left(-\frac{u}{p} D_p^2\right)(x,x') \right|_{\mathscr{C}^m} \leqslant C p^{n+\frac{m}{2}} \exp\left(\mu_0 u - \frac{\boldsymbol{a} p}{u} d(x,x')^2\right), \tag{3.1}$$

*and that for $u \geqslant u_0$, $p \geqslant 2C_L/\mu_0$, $x, x' \in X$, we have*

$$\Big|\Big(\frac{1}{p}\,D_p^2\exp\Big(-\frac{u}{p}D_p^2\Big)\Big)(x,x')\Big|_{\mathscr{C}^m} \leqslant Cp^{n+\frac{m}{2}}\exp\Big(-\frac{1}{2}\mu_0 u-\frac{\boldsymbol{a}p}{u}d(x,x')^2\Big). \tag{3.2}$$

*Proof* For any $u_0 > 0$, $k \in \mathbb{N}$, there exists $C_{u_0,k} > 0$ such that for $u \geqslant u_0$, $p \in \mathbb{N}^*$, we have

$$\Big\|\Big(\frac{1}{\sqrt{p}}\,D_p\Big)^k\exp\Big(-\frac{u}{p}D_p^2\Big)\Big\|^{0,0} \leqslant C_{u_0,k}, \tag{3.3}$$

and by Lemma 1, that for $u \geqslant u_0$, $p \geqslant 2C_L/\mu_0$, we have

$$\Big\|\Big(\frac{1}{\sqrt{p}}\,D_p\Big)^k\frac{1}{p}\,D_p^2\exp\Big(-\frac{u}{p}D_p^2\Big)\Big\|^{0,0} \leqslant C_{u_0,k}e^{-\mu_0 u}. \tag{3.4}$$

From Lemma 2, (3.3) and (3.4), for any $u_0 > 0$, $m \in \mathbb{N}$, there exists $C'_{u_0,m} > 0$ such that for $u \geqslant u_0$, $p \in \mathbb{N}^*$, $x, x' \in X$, we have

$$\Big|\exp\Big(-\frac{u}{p}D_p^2\Big)(x,x')\Big|_{\mathscr{C}^m} \leqslant C'_{u_0,m}\,p^{n+\frac{m}{2}}, \tag{3.5}$$

and that for $u \geqslant u_0$, $p \geqslant 2C_L/\mu_0$, $x, x' \in X$, we have

$$\Big|\Big(\frac{1}{p}\,D_p^2\exp\Big(-\frac{u}{p}D_p^2\Big)\Big)(x,x')\Big|_{\mathscr{C}^m} \leqslant C'_{u_0,m}\,p^{n+\frac{m}{2}}e^{-\mu_0 u}. \tag{3.6}$$

To obtain (3.1) and (3.2) in general, we proceed as in the proof of [7, Theorem 4.11] and [3, Theorem 11.14] (cf. [17, Theorem 4.2.5]). For $h > 1$ and $f$ from (2.13), put

$$\begin{aligned} K_{u,h}(a) &= \int_{-\infty}^{+\infty}\cos(v\sqrt{2u}a)\exp\Big(-\frac{v^2}{2}\Big)\Big(1-f\Big(\frac{1}{h}\sqrt{2u}v\Big)\Big)\frac{dv}{\sqrt{2\pi}} \\ H_{u,h}(a) &= \int_{-\infty}^{+\infty}\cos(v\sqrt{2u}a)\exp\Big(-\frac{v^2}{2}\Big)f\Big(\frac{1}{h}\sqrt{2u}v\Big)\frac{dv}{\sqrt{2\pi}}\cdot \end{aligned} \tag{3.7}$$

By (3.7), we infer

$$K_{u,h}\Big(\frac{1}{\sqrt{p}}D_p\Big)+H_{u,h}\Big(\frac{1}{\sqrt{p}}D_p\Big)=\exp\Big(-\frac{u}{p}D_p^2\Big). \tag{3.8}$$

Using finite propagation speed of solutions of hyperbolic equations [17, Theorem D.2.1], and (3.7), we find that

$$
\begin{aligned}
&\operatorname{supp} H_{u,h}\left(\frac{1}{\sqrt{p}}D_p\right)(x,\cdot) \subset B^X(x, \varepsilon h/\sqrt{p}), \text{ and} \\
&H_{u,h}\left(\frac{1}{\sqrt{p}}D_p\right)(x,\cdot) \text{ depends only on the restriction of } D_p \text{ to } B^X(x, \varepsilon h/\sqrt{p}).
\end{aligned}
\tag{3.9}
$$

Thus from (3.8) and (3.9), we get for $x, x' \in X$,

$$
K_{u,h}\left(\frac{1}{\sqrt{p}}D_p\right)(x,x') = \exp\Big(-\frac{u}{p}D_p^2\Big)(x,x'), \quad \text{if } \sqrt{p}\, d(x,x') \geqslant \varepsilon h. \tag{3.10}
$$

By (3.7), there exist $C', C_1 > 0$ such that for any $c > 0$, $m \in \mathbb{N}$, there is $C > 0$ such that for $u \geqslant u_0$, $h > 1$, $a \in \mathbb{C}$, $|\mathrm{Im}(a)| \leqslant c$, we have

$$
|a|^m |K_{u,h}(a)| \leqslant C \exp\left(C'c^2u - \frac{C_1}{u}h^2\right). \tag{3.11}
$$

Using Lemma 2 and (3.11) we find that for $\boldsymbol{K}(a) = K_{u,h}(a)$ or $a^2K_{u,h}(a)$,

$$
\left|\boldsymbol{K}\left(\frac{1}{\sqrt{p}}D_p\right)(x,x')\right|_{\mathscr{C}^m} \leqslant C_2 p^{n+\frac{m}{2}} \exp\left(C'c^2u - \frac{C_1}{u}h^2\right). \tag{3.12}
$$

Setting $h = \sqrt{p}\, d(x,x')/\varepsilon$ in (3.12), we get that for any $p \in \mathbb{N}^*$, $u \geqslant u_0$, $x, x' \in X$ such that $\sqrt{p}d(x,x') \geqslant \varepsilon$, we have

$$
\left|\boldsymbol{K}\left(\frac{1}{\sqrt{p}}D_p\right)(x,x')\right|_{\mathscr{C}^m} \leqslant C p^{n+\frac{m}{2}} \exp\left(C'c^2u - \frac{C_1}{\varepsilon^2 u}p\, d(x,x')^2\right). \tag{3.13}
$$

By (3.5), (3.10) and (3.13), we infer (3.1). By (3.6), (3.10) and (3.13), we infer (3.2). The proof of Theorem 4 is completed. □

*Proof of Theorem 1* Analogue to [7, (4.89)] (or [17, (4.2.22)]), we have

$$
\exp\Big(-\frac{u}{p}D_p^2\Big) - P_p = \int_u^{+\infty} \frac{1}{p} D_p^2 \exp\Big(-\frac{u_1}{p}D_p^2\Big)du_1. \tag{3.14}
$$

Note that $\frac{1}{4}\mu_0 u + \frac{\boldsymbol{a}}{u} p\, d(x,x')^2 \geqslant \sqrt{\boldsymbol{a}\mu_0 p}\, d(x,x')$, thus

$$\begin{aligned}\int_u^{+\infty} \exp\Big(-\frac{1}{2}\,\mu_0 u_1 - \frac{\boldsymbol{a}}{u_1} p\, d(x,x')^2\Big) du_1 \\ \leqslant \exp(-\sqrt{\boldsymbol{a}\mu_0 p}\, d(x,x')) \int_u^{+\infty} \exp\Big(-\frac{1}{4}\,\mu_0 u_1\Big) du_1 \\ = \frac{4}{\mu_0} \exp\Big(-\frac{1}{4}\mu_0 u - \sqrt{\boldsymbol{a}\mu_0 p}\, d(x,x')\Big).\end{aligned} \tag{3.15}$$

By (3.2), (3.14) and (3.15), there exists $C > 0$ such that for $u \geqslant u_0$, $p \geqslant 2C_L/\mu_0$, $x, x' \in X$, we have

$$\begin{aligned}\Big|\Big(\exp\Big(-\frac{u}{p}\, D_p^2\Big) - P_p\Big)(x,x')\Big|_{\mathscr{C}^m} \\ \leqslant C p^{n+\frac{m}{2}} \exp\Big(-\frac{1}{4}\mu_0 u - \sqrt{\boldsymbol{a}\mu_0 p}\, d(x,x')\Big).\end{aligned} \tag{3.16}$$

By (3.1) and (3.16), we get (1.3) with

$$\boldsymbol{p}_0 = 2C_L/\mu_0\,, \quad \boldsymbol{c} = \sqrt{\boldsymbol{a}\mu_0}. \tag{3.17}$$

The proof of Theorem 1 is completed. □

*Proof of Theorem 2* It is standard that for any $p \in \mathbb{N}^*$, $u > 0$, and $x, y \in \widetilde{X}$,

$$\exp\Big(-uD_p^2\Big)(\pi(x),\pi(y)) = \sum_{\gamma\in\Gamma} \exp\Big(-u\widetilde{D}_p^2\Big)(\gamma x, y). \tag{3.18}$$

Let us denote for $r > 0$ by $N(r) = \# B^{\widetilde{X}}(x,r) \cap \Gamma y$. Let $K > 0$ be such that the sectional curvature of $(X, g^{TX})$ is $\geqslant -K^2$. By [20], there exists $C > 0$ such that for any $r > 0$, $x, y \in \widetilde{X}$, we have

$$N(r) \leqslant C e^{(2n-1)Kr}. \tag{3.19}$$

[Note that in the proof of (3.1), we did not use Lemma 1. From (3.1) and (3.19), we know the right hand side of (3.18) is absolutely convergent with respect to $\mathscr{C}^k$-norm for any $k \in \mathbb{N}$, and verifies the heat equation on $X$, thus we also get a proof of (3.18)].

Take $\boldsymbol{p}_1 = \max\{\boldsymbol{p}_0, (2n-1)^2K^2/\boldsymbol{c}^2\}$. Then for any $p > \boldsymbol{p}_1$, by Theorem 4, (3.15) and (3.19), we know that

$$\begin{aligned}\sum_{\gamma\in\Gamma} \int_u^{+\infty} \Big(\frac{1}{p}\, \widetilde{D}_p^2 \exp\Big(-\frac{u_1}{p}\, \widetilde{D}_p^2\Big)\Big)(\gamma x, y) du_1 \\ = \int_u^{+\infty} \sum_{\gamma\in\Gamma} \Big(\frac{1}{p}\, \widetilde{D}_p^2 \exp\Big(-\frac{u_1}{p}\, \widetilde{D}_p^2\Big)\Big)(\gamma x, y) du_1.\end{aligned} \tag{3.20}$$

Moreover, (1.3) and (3.19) show that $\sum_{\gamma\in\Gamma}\widetilde{P}_p(\gamma x, y)$ is absolutely convergent with respect to $\mathscr{C}^k$-norm for any $k\in\mathbb{N}$, $p>\boldsymbol{p}_1$. From Theorem 4, (3.14) for $\widetilde{D}_p$, (3.18) and (3.20), we get

$$\exp\Big(-\frac{u}{p}D_p^2\Big)(\pi(x),\pi(y))-\sum_{\gamma\in\Gamma}\widetilde{P}_p(\gamma x, y)$$
$$=\int_u^{+\infty}\Big(\frac{1}{p}D_p^2\exp\Big(-\frac{u_1}{p}D_p^2\Big)\Big)(\pi(x),\pi(y))du_1. \qquad (3.21)$$

Now from (3.14) for $D_p$ and (3.21), we obtain (1.4). Note that the constants $C_L$, $\mu_0$ in Lemma 1, $\boldsymbol{a}$ in Theorem 4 depend only on the geometric data on $X$, not on the covering map $\pi:\widetilde{X}\to X$, thus $\boldsymbol{p}_1$ depends only on the geometric data on $X$. □

*Remark 1* Let $A\in\Omega^3(X)$. We assume $A$ and its derivatives are bounded on $X$. Set

$$^cA=\sum_{i<j<k}A(e_i,e_j,e_k)c(e_i)c(e_j)c(e_k),\quad D_p^A:=D_p+{}^cA. \qquad (3.22)$$

Then $D_p^A$ is a modified Dirac operator (cf. [17, §1.3.3], [2]). As we are in the bounded geometry context, Lemmas 1, 2 still hold if we replace $D_p$ by $D_p^A$ (cf. [17, Theorems 1.5.7, 1.5.8]). This implies that Theorem 4 holds for $D_p^A$, thus Theorems 1, 2 hold for the orthogonal projection from $L^2(X,E_p)$ onto $\mathrm{Ker}(D_p^A)$.

## 4 The holomorphic case

We discuss now the particular case of a complex manifold, cf. the situation of [17, §6.1.1]. Let $(X,J)$ be a complex manifold with complex structure $J$ and complex dimension $n$. Let $g^{TX}$ be a Riemannian metric on $TX$ compatible with $J$, and let $\Theta=g^{TX}(J\cdot,\cdot)$ be the (1, 1)-form associated to $g^{TX}$ and $J$. We call $(X,J,g^{TX})$ or $(X,J,\Theta)$ a Hermitian manifold. A Hermitian manifold $(X,J,g^{TX})$ is called complete if $g^{TX}$ is complete. Moreover let $(L,h^L)$, $(E,h^E)$ be holomorphic Hermitian vector bundles on $X$ and $\mathrm{rk}(L)=1$. Consider the holomorphic Hermitian (Chern) connections $\nabla^L$, $\nabla^E$ on $(L,h^L)$, $(E,h^E)$.

This section is organized as follows. In Sect. 4.1, we explain Theorem 1 in the holomorphic case. In Sect. 4.2, we give some Bergman kernel proofs of some known results about separation of points, existence of local coordinates and holomorphic convexity. The usual proofs use the $L^2$ estimates for the $\overline{\partial}$-equation introduced by Andreotti-Vesentini and Hörmander. For plenty of informations about holomorphic convexity of coverings (Shafarevich conjecture) and its role in algebraic geometry see [14].

### 4.1 Theorems 1, 2 in the holomorphic case

The space of holomorphic sections of $L^p \otimes E$ which are $L^2$ with respect to the norm given by (2.2) is denoted by $H^0_{(2)}(X, L^p \otimes E)$. Let $P_p(x, x')$, $(x, x' \in X)$ be the Schwartz kernel of the orthogonal projection $P_p$, from the space of $L^2$-sections of $L^p \otimes E$ onto $H^0_{(2)}(X, L^p \otimes E)$, with respect to the Riemannian volume form $dv_X(x')$ associated to $(X, g^{TX})$.

**Theorem 5** *Let $(X, J, g^{TX})$ be a complete Hermitian manifold. Assume that $R^L$, $R^E$, $J$, $g^{TX}$ have bounded geometry and (1.2) holds. Then the uniform exponential estimate (1.3) holds for the Bergman kernel $P_p(x, x')$ associated with $H^0_{(2)}(X, L^p \otimes E)$. Moreover, Theorem 2 still holds for this version of $P_p$.*

*Proof* Let $\overline{\partial}^{L^p\otimes E,*}$ be the formal adjoint of the Dolbeault operator $\overline{\partial}^{L^p\otimes E}$ with respect to the Hermitian product (2.2) on $\Omega^{0,\bullet}_0(X, L^p \otimes E)$. Set

$$D_p = \sqrt{2}\left(\overline{\partial}^{L^p\otimes E} + \overline{\partial}^{L^p\otimes E,*}\right). \tag{4.1}$$

Using the assumption of the bounded geometry, we have by [17, (6.1.8)] for $p$ large enough,

$$\operatorname{Ker}(D_p) = \operatorname{Ker}(D_p^2) = H^0_{(2)}(X, L^p \otimes E). \tag{4.2}$$

Observe that $D_p$ is a modified Dirac operator as in Remark 1 with $A = \frac{\sqrt{-1}}{4}(\partial - \overline{\partial})\Theta$, see [2, Theorem 2.2], cf. also [17, Theorem 1.4.5]. (In particular, if $(X, J, g^{TX})$ is a complete Kähler manifold, then the operator $D_p$ from (4.1) is a Dirac operator in the sense of Sect. 2.) Thus by Remark 1, Theorems 1, 2 still hold for the kernel $P_p(x, x')$ under the assumption of bounded geometry. □

Theorem 5 has the following consequence.

**Corollary 2** *Let $(X, \omega)$ be a complete Kähler manifold such that $\omega$ has bounded geometry. Let $(L, h^L)$ be a holomorphic Hermitian line bundle such that $R^L = -2\pi\sqrt{-1}\,\omega$. Then the estimate (1.3) holds for $P_p(x, x')$ associated with $H^0_{(2)}(X, L^p)$.*

### 4.2 Holomorphic convexity of manifolds with bounded geometry

We will identify the 2-form $R^L$ with the Hermitian matrix $\dot{R}^L \in \operatorname{End}(T^{(1,0)}X)$ such that for $W, Y \in T^{(1,0)}X$,

$$R^L(W, \overline{Y}) = \langle \dot{R}^L W, \overline{Y}\rangle. \tag{4.3}$$

Analogous to the result of Theorem 1 about the uniform off-diagonal decay of the Bergman kernel, a straightforward adaptation of the technique used in this paper yields the uniform diagonal expansion for manifolds and bundles with bounded geometry (cf. [17, Theorem 4.1.1] for the compact case, [17, Theorems 6.1.1, 6.1.4] for other cases of non-compact manifolds including the covering manifolds). This was already observed in [17, Problem 6.1]. The following theorem allows to construct uniform peak sections of the powers $L^p$ for $p$ sufficiently large.

**Theorem 6** *Let $(X, J, g^{TX})$ be a complete Hermitian manifold. Assume that $R^L$, $R^E$, $J$, $g^{TX}$ have bounded geometry and (1.2) holds. Then there exist smooth coefficients $\boldsymbol{b}_r(x) \in \mathrm{End}(E)_x$ which are polynomials in $R^{TX}$, $R^E$ (and $d\Theta$, $R^L$) and their derivatives with order $\leqslant 2r-2$ (resp. $2r-1$, $2r$) and reciprocals of linear combinations of eigenvalues of $\dot{R}^L$ at $x$, and $\boldsymbol{b}_0 = \det(\dot{R}^L/(2\pi))\mathrm{Id}_E$, such that for any $k, \ell \in \mathbb{N}$, there exists $C_{k,\ell} > 0$ such that for any $p \in \mathbb{N}^*$, $x \in X$,*

$$\left| P_p(x,x) - \sum_{r=0}^{k} \boldsymbol{b}_r(x) p^{n-r} \right|_{\mathscr{C}^\ell} \leqslant C_{k,\ell}\, p^{n-k-1}. \tag{4.4}$$

*Moreover, the expansion is uniform in the following sense: for any fixed $k, \ell \in \mathbb{N}$, assume that the derivatives of $g^{TX}$, $h^L$, $h^E$ with order $\leqslant 2n+2k+\ell+6$ run over a set bounded in the $\mathscr{C}^\ell$–norm taken with respect to the parameter $x \in X$ and, moreover, $g^{TX}$ runs over a set bounded below, then the constant $C_{k,\ell}$ is independent of $g^{TX}$; and the $\mathscr{C}^\ell$-norm in (4.4) includes also the derivatives with respect to the parameters.*

We will actually make use in the following only of the case $\ell = 0$ from Theorem 6.

**Theorem 7** *Let $(X, J, g^{TX})$ be a complete Hermitian manifold. Assume that $R^L$, $R^E$, $J$, $g^{TX}$ have bounded geometry and (1.2) holds. Then there exists $p_0 \in \mathbb{N}$ such that for all $p \geqslant p_0$, the bundle $L^p \otimes E$ is generated by its global holomorphic $L^2$-sections.*

*Proof* By Theorem 6,

$$P_p(x,x) = \boldsymbol{b}_0(x) p^n + \mathscr{O}(p^{n-1}), \quad \text{uniformly on } X, \tag{4.5}$$

where $\boldsymbol{b}_0 = \det(\dot{R}^L/2\pi)\mathrm{Id}_E$. Due to (1.2), the function $\det(\dot{R}^L/2\pi)$ is bounded below by a positive constant. Hence (4.5) implies that there exists $p_0 \in \mathbb{N}$ such that for all $p \geqslant p_0$ and all $x \in X$ the endomorphism $P_p(x,x) \in \mathrm{End}(E_x)$ is invertible. In particular, for any $v \in L^p_x \otimes E_x$ there exists $S = S(x,v) \in H^0_{(2)}(X, L^p \otimes E)$ with $S(x) = v$. □

We can apply Theorem 7 to the following situation. Let $h_p$ be a Hermitian metric on $L^p$. Let $\pi : \widetilde{X} \to X$ be a Galois covering and consider

$$\widetilde{\Theta} = \pi^*\Theta,\ \ dv_{\widetilde{X}} = \widetilde{\Theta}^n/n!, (\widetilde{L}^p, \widetilde{h}_p) = (\pi^* L^p, \pi^* h_p), (\widetilde{E}, h^{\widetilde{E}}) = (\pi^* E, \pi^* h^E), \tag{4.6}$$

and let $L^2(\widetilde{X}, \widetilde{L}^p \otimes \widetilde{E})$ be the $L^2$-space of sections of $\widetilde{L}^p \otimes \widetilde{E}$ with respect to $\widetilde{h}_p$, $h^{\widetilde{E}}$, $dv_{\widetilde{X}}$.

**Corollary 3** *Let $(X, \Theta)$ be a compact Hermitian manifold, L be a positive line bundle over $X$. Then there exists $p_0 \in \mathbb{N}$ such that for all Galois covering $\pi : \widetilde{X} \to X$, for all $p \geqslant p_0$, and all Hermitian metrics $h_p$, $h^E$ on $L^p$, $E$, the bundle $\widetilde{L}^p \otimes \widetilde{E}$ is generated by its global holomorphic $L^2$-sections.*

*Proof* Indeed, $(\widetilde{X}, g^{T\widetilde{X}})$ is complete and $R^{\widetilde{L}}$, $R^{\widetilde{E}}$, $\widetilde{J}$, $g^{T\widetilde{X}}$ have bounded geometry. Thus the conclusion follows immediately from Theorem 7 for metrics $h_p$ of the form $(h^L)^p$, where $h^L$ is a positively curved metric on $L$. That $p_0$ is independent of the covering $\pi : \widetilde{X} \to X$ follows from the dependency conditions of $p_0$ in Theorem 7. Observe finally that the $L^2$ condition is independent of the Hermitian metric $h_p$, $h^E$ chosen on $L^p$, $E$ over the compact manifold $X$. □

Note that instead of using Theorem 7 we could have also concluded by using [17, Theorem 6.1.4] or [18, Theorem 3.14]. The latter shows that, roughly speaking, the asymptotics of the Bergman kernel on the base manifold and on the covering are the same. Note also that by Theorem 7 or [17, Theorem 6.1.4] we obtain an estimate from below of the von Neumann dimension of the space of $L^2$ holomorphic sections (cf. [17, Remark 6.1.5]):

$$\dim_\Gamma H^0_{(2)}(\widetilde{X}, \widetilde{L}^p \otimes \widetilde{E}) \geqslant \frac{p^n}{n!}\mathrm{rk}(E) \int_X \left(\frac{\sqrt{-1}}{2\pi} R^L\right)^n + o(p^n) \quad p \to \infty. \tag{4.7}$$

This was used in [23] and [17, §6.4] to obtain weak Lefschetz theorems, extending results from [22].

The following definition was introduced in [21, Definition 4.1] for line bundles.

**Definition 2** Suppose $X$ is a complex manifold, $(F, h^F) \to X$ is a holomorphic Hermitian vector bundle. The manifold $X$ is called *holomorphically convex with respect to* $(F, h^F)$ if, for every infinite subset $\Sigma$ without limit points in $X$, there is a holomorphic section $S$ of $F$ on $X$ such that $|S|_{h^F}$ is unbounded on $\Sigma$. The manifold $X$ is called holomorphically convex if it is holomorphically convex with respect to the trivial line bundle.

Since it suffices to consider any infinite subset of $\Sigma$, in order to prove the holomorphic convexity we may assume that $S$ is actually equal to a sequence of points $\{x_i\}_{i\in\mathbb{N}}$ without limit points in $X$.

**Theorem 8** *Let $(X, J, g^{TX})$ be a complete Hermitian manifold. Assume that $R^L$, $R^E$, $J$, $g^{TX}$ have bounded geometry and (1.2) holds. Then there exists $p_0 \in \mathbb{N}$ such that for all $p \geqslant p_0$, $X$ is holomorphically convex with respect to the bundle $L^p \otimes E$.*

*Proof* If $X$ is compact the assertion is trivial. We assume in the sequel that $X$ is non-compact. We use the following lemma.

**Lemma 3** *There exists $p_0 \in \mathbb{N}$ such that for all $p \geqslant p_0$, for any compact set $K \subset X$ and any $\varepsilon, M > 0$ there exists a compact set $K(\varepsilon, M) \subset X$ with the property that for any $x \in X\backslash K(\varepsilon, M)$ there exists $S \in H^0_{(2)}(X, L^p \otimes E)$ with $|S(x)| \geqslant M$ and $|S| \leqslant \varepsilon$ on $K$.*

*Proof* Let $p_0 \in \mathbb{N}$ be as in the conclusion of Theorem 7. For any $x \in X$, $w \in L^p_x \otimes E_x$, consider the section

$$S \in H^0_{(2)}(X, L^p \otimes E), \quad y \longmapsto P_p(y, x) \cdot w. \tag{4.8}$$

Since $P_p(x, x)$ is invertible, we can find for any given $v \in L^p_x \otimes E_x$ a section $S_{x,v}$ as in (4.8) such that $S_{x,v}(x) = v$. Thus for any $x \in X$ there exists $v(x) \in L^p_x \otimes E_x$ such that $|S_{x,v(x)}(x)| = 2M$. By Theorem 1, for any fixed $0 < r < \text{inj}^X$, $S_{x,v(x)}$ has exponential decay outside the ball $B(x, r)$, uniformly in $x \in X$ (cf. also (4.20)). We can now choose $\delta > 0$, such that for any $x \in X$ with $d(x, K) > \delta$ we have $|S_{x,v(x)}(y)| \leqslant \varepsilon$ for $y \in K$. We set finally $K(\varepsilon, M) = \{z \in X : d(z, K) \leqslant \delta\}$. The proof of Lemma 3 is completed. □

In order to finish the proof of Theorem 8, let us choose an exhaustion $\{K_i\}_{i\in\mathbb{N}}$ of $X$ with compact sets, i.e., $K_i \subset \mathring{K}_{i+1}$ and $X = \bigcup_{i\in\mathbb{N}} K_i$. Consider a sequence $\{x_i\}_{i\in\mathbb{N}}$ in $X$ without limit points. Using Lemma 3, we construct inductively a sequence of holomorphic sections $\{S_i\}_{i\in\mathbb{N}}$ and a subsequence $\{\nu(i)\}_{i\in\mathbb{N}}$ of $\mathbb{N}$ such that

$$|S_i| \leqslant 2^{-i} \text{ on } K_i \text{ and } |S_i(x_{\nu(i)})| \geqslant 2^i + \sum_{j<i} |S_j(x_{\nu(i)})|, \tag{4.9}$$

where $\nu(i)$ is the smallest index $j$ such that $x_j \in X \backslash K_i(2^{-i}, 2^i + \sum_{j<i} |S_j(x_{\nu(i)})|)$. Then $S = \sum_{i\in\mathbb{N}} S_i$ converges uniformly on any compact set of $X$, hence defines a holomorphic section of $L^p \otimes E$ on $X$, and satisfies $|S(x_{\nu(i)})| \to \infty$ as $i \to \infty$. □

*Remark 2* **(a)** Napier [21, Theorem 4.2] proves a similar result for a complete Kähler manifold with bounded geometry and for $(E, h^E)$ trivial. His notion of bounded geometry (cf. [21, Definition 3.1]) is that of Cheng-Yau [4]. It requires that there exists a covering of $X$ by coordinate Euclidean balls of a fixed radius in which the corresponding Euclidean metrics are uniformly (in $\mathscr{C}^0$-norm) comparable to the metric $g^{TX}$. This definition implies that the injectivity radius of $(X, g^{TX})$ is positive. On the other hand, we require in our definition of bounded geometry that also the derivatives of the various geometric data to be bounded. Working with a weaker notion than that used in the present paper, Napier first concludes the holomorphic convexity for the adjoint bundle $L^p \otimes K_X$ (twisting with $K_X$ is necessary for the application of the $L^2$ method for solving the $\overline{\partial}$-equation, due to Andreotti-Vesentini and Hörmander, see e.g., [9, Théorème 5.1], [17, Theorem B.4.6]). If the Ricci curvature of $g^{TX}$ (i.e., the curvature $R^{\det}$ of the holomorphic Hermitian connection $\nabla^{\det}$ on $K_X^* = \det(T^{(1,0)}X)$) is bounded from below, then [21, Theorem 4.2(ii)] shows that $X$ is holomorphically convex with respect to $L^p$, for $p$ sufficiently large. Note that our notion of bounded geometry implies that the Ricci curvature of $g^{TX}$ is bounded from below.

**(b)** In the conditions of Theorem 8 there exists $p_0 \in \mathbb{N}$ such that for all $p \geqslant p_0$ we have:

(i) $H^0_{(2)}(X, L^p \otimes E)$ separate points of $X$, i.e., for any $x, y \in X$, $x \neq y$, there exists $S \in H^0_{(2)}(X, L^p \otimes E)$ with $S(x) = 0$, $S(y) \neq 0$.

(ii) $H^0_{(2)}(X, L^p \otimes E)$ gives local coordinates on $X$, i.e., for any $x \in X$ the derivative

$$d_x : H^0_{(2)}(X, \mathscr{I}_x \otimes L^p \otimes E) \to T^{*(1,0)}_x X \otimes L^p_x \otimes E_x$$

is surjective, where $H^0_{(2)}(X, \mathscr{I}_x \otimes L^p \otimes E)$ is the space of holomorphic $L^2$ sections of $L^p \otimes E$ vanishing at $x$.

The items (i) and (ii) follow from the $L^2$ estimates for $\overline{\partial}$ for singular metrics by using similar arguments as in [9,21]. We show next how they also follow from the asymptotics of the Bergman kernel.

*Proof of Theorem 3* The holomorphic convexity follows from Theorem 8.

We show that $H^0_{(2)}(X, L^p \otimes E)$ gives local coordinates for $p$ large enough. Let us fix $x_0 \in X$. Consider normal coordinates centered at $x_0$ as in the proof of Lemma 2, and choose $\{w_j\}_{j=1}^n$ an orthonormal basis of $T^{(1,0)}_{x_0} X$ such that

$$\dot{R}^L(x_0) = \mathrm{diag}(a_1(x_0), \ldots, a_n(x_0)) \in \mathrm{End}(T^{(1,0)}_{x_0} X). \tag{4.10}$$

We fix an orthonormal basis of $T_{x_0}X$ given by $e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j)$ and $e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j)$. Then $z_i = Z_{2i-1} + \sqrt{-1} Z_{2i}$ is a complex coordinate of $Z \in \mathbb{R}^{2n} \simeq (T_{x_0}X, J)$. Let $e_L$, $\{u_\ell\}_\ell$ be frames of $L$, $E$ which are parallel with respect to $\nabla^L$, $\nabla^E$ along the curve $[0, 1] \ni u \to uZ$. Denote by $a_i = a_i(x_0)$ and

$$\begin{aligned} |Z|^2_a &:= \frac{1}{4} \sum_{i=1}^n a_i |z_i|^2, \\ \mathscr{P}_{x_0}(Z, Z') &= \frac{1}{(2\pi)^n} \prod_{i=1}^n a_i \ \exp\left( -\frac{1}{4} \sum_i a_i \left( |z_i|^2 + |z'_i|^2 - 2 z_i \overline{z}'_i \right) \right), \\ S^p_\ell(Z) &= f(|Z|) e^{-p|Z|^2_a} e_L^{\otimes p} \otimes u_\ell, \quad S^p_{j,\ell}(Z) = z_j S^p_\ell(Z), \end{aligned} \tag{4.11}$$

where $f$ is defined in (2.13). Then by [17, Theorem 4.2.1], there exist a smooth family relative to the parameter $x_0 \in X$, of odd degree polynomials $J_{1,x_0}(Z, Z') \in \mathrm{End}(E_{x_0})[Z, Z']$, and $J_{0,x_0}(Z, Z') = \mathrm{Id}_{E_{x_0}}$, such that for $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < \varepsilon$, $\alpha \in \mathbb{N}^{2n}$, we have

$$\begin{aligned} &\left| \frac{\partial^\alpha}{\partial Z^\alpha} \left( p^{-n} P_{p,x_0}(Z, Z') - \sum_{r=0}^1 (J_{r,x_0} \mathscr{P}_{x_0}) (\sqrt{p} Z, \sqrt{p} Z') p^{-\frac{r}{2}} \right) \right| \\ &\quad \leqslant C p^{(|\alpha|-2)/2} (1 + \sqrt{p}\,|Z| + \sqrt{p}\,|Z'|)^M e^{-C_0 \sqrt{p}\,|Z-Z'|} + \mathscr{O}(p^{-\infty}). \end{aligned} \tag{4.12}$$

For $r = 0, 1$, set

$$S^p_{j,\ell,r}(Z) = \int_{Z' \in \mathbb{C}^n} p^n (J_{r,x_0} \mathscr{P}_{x_0})(\sqrt{p} Z, \sqrt{p} Z') S^p_{j,\ell}(Z') dv_X(Z'). \tag{4.13}$$

Then by (2.13), (4.11) and (4.12), for $|Z| < \varepsilon/4$ and $\boldsymbol{B} \in \left\{1, \frac{\partial}{\partial z_j}\right\}$,

$$\begin{aligned}&\left|P_p(S_\ell^p)(Z) - e^{-p|Z|_a^2} e_L^{\otimes p} \otimes u_\ell\right| \\ &\quad \leqslant Cp^{-1/2}(1+\sqrt{p}\,|Z|)^M\, e^{-C_0'\sqrt{p}\,|Z|} + \mathcal{O}(p^{-\infty}) \\ &\left|\boldsymbol{B}\Big(P_p(S_{j,\ell}^p)(Z) - \sum_{r=0}^{1} S_{j,\ell,r}^p(Z) p^{-\frac{r}{2}}\Big)\right| \\ &\quad \leqslant Cp^{-1/2}(1+\sqrt{p}\,|Z|)^M\, e^{-C_0'\sqrt{p}\,|Z|} + \mathcal{O}(p^{-\infty}).\end{aligned} \tag{4.14}$$

By (4.13), we have for $|Z| < \varepsilon/4$ in the sense of the $\mathscr{C}^1$-seminorm,

$$\begin{aligned}S_{j,\ell,0}^p(Z) &= z_j e^{-p|Z|_a^2} e_L^{\otimes p} \otimes u_\ell + \mathcal{O}(p^{-\infty}),\\ S_{j,\ell,1}^p(Z) &= p^{-\frac{1}{2}} Q_{j,\ell,1}(\sqrt{p}Z) e^{-p|Z|_a^2} + \mathcal{O}(p^{-\infty}),\end{aligned} \tag{4.15}$$

and $Q_{j,\ell,1}(Z)$ are even polynomials on $Z$, thus $(dS_{j,\ell,1}^p)(0) = \mathscr{O}(p^{-\infty})$.

Now from (4.13), (4.14) and (4.15), there exists $p_0 \in \mathbb{N}$ such that uniformly for $x_0 \in X$, $p \geqslant p_0$, the differential

$$d_{x_0} : H^0_{(2)}(X, \mathscr{I}_{x_0} \otimes L^p \otimes E) \to (T^{*(1,0)}X \otimes L^p \otimes E)_{x_0}$$

is surjective, namely $d_{x_0} P_p(S_{j,l}^p) = dz_j \otimes e_L^{\otimes p} \otimes u_\ell$, for $1 \leqslant j \leqslant n$, $1 \leqslant l \leqslant \mathrm{rk}(E)$.

We prove now $H^0_{(2)}(X, L^p \otimes E)$ separates the points in $X$ for $p$ large enough. We will construct first peak sections at $x \in X$ (cf. [17, (5.1.24)] for $E = \mathbb{C}$). Set

$$\Phi_p(x) = \{s \in H^0_{(2)}(X, L^p \otimes E) : s(x) = 0\}. \tag{4.16}$$

Let $\psi_x : H^0_{(2)}(X, L^p \otimes E) \to (L^p \otimes E)_x$ be the evaluation map $\psi_x s = s(x)$. Let $\psi_x^*$ be its adjoint. By Theorem 6, there exists $p_0 \in \mathbb{N}$, such that $\Phi_p(x)$ has codimension $\mathrm{rk}(E)$ in $H^0_{(2)}(X, L^p \otimes E)$ for any $x \in X$ and $p \geqslant p_0$.

For $x_0 \in X$, let $\{S_{x_0,j}^p\}_{j=1}^{\mathrm{rk}(E)}$ be an orthonormal basis of the orthogonal complement $\Phi_p(x_0)^\perp$ of $\Phi_p(x_0)$, and we call $\{S_{x_0,j}^p\}_{j=1}^{\mathrm{rk}(E)}$ peak sections at $x_0$. Then $\psi_{x_0} : \mathbb{C}\{S_{x_0,j}^p\} \to (L^p \otimes E)_{x_0}$ is bijective for $p \geqslant p_0$ and

$$\begin{aligned}&\Big(S_{x_0,j}^p(x_0)^*, (\psi_x\psi_x^*)^{-1} S_{x_0,k}^p(x_0)\Big) = \Big\langle S_{x_0,j}^p(x_0), (\psi_x\psi_x^*)^{-1} S_{x_0,k}^p(x_0)\Big\rangle_{x_0}\\ &\quad = \Big\langle (\psi_{x_0})^{-1}(S_{x_0,j}^p(x_0)), (\psi_{x_0})^{-1}(S_{x_0,k}^p(x_0))\Big\rangle = \Big\langle S_{x_0,j}^p, S_{x_0,k}^p\Big\rangle = \delta_{jk}.\end{aligned} \tag{4.17}$$

We have also

$$P_p(x, x_0) = \sum_{j=1}^{\mathrm{rk}(E)} S_{x_0,j}^p(x) \otimes S_{x_0,j}^p(x_0)^*. \tag{4.18}$$

By [17, (5.1.46)], we know that for any $x \in X$,

$$P_p(x,x) = \psi_x \psi_x^* \in \mathrm{End}(E)_x. \tag{4.19}$$

From (4.17), (4.18) and (4.19), we get for any $p \geqslant p_0$,

$$S^p_{x_0,k}(x) = P_p(x,x_0)P_p(x_0,x_0)^{-1}S^p_{x_0,k}(x_0). \tag{4.20}$$

From (4.12), we deduce that for a sequence $\{r_p\}$ with $r_p \to 0$ and $r_p\sqrt{p} \to \infty$,

$$\int_{B^X(x_0,r_p)} \left\langle S^p_{x_0,j}, S^p_{x_0,k}\right\rangle(x)\, dv_X(x) = \delta_{jk} - o(1), \quad \text{for } p \to \infty. \tag{4.21}$$

Note that (4.21) holds for any orthonormal basis $\{S^p_{x_0,j}\}_{j=1}^{\mathrm{rk}(E)}$ of $\Phi_p(x_0)^\perp$.

Let us assume that there exists a sequence of distinct points $x_p \neq y_p$ such that $\Phi_p(x_p) = \Phi_p(y_p)$. Since $\Phi_p(x_p) = \Phi_p(y_p)$ we can construct as above the peak sections $S^p_{x_p,j} = S^p_{y_p,j}$ as an orthonormal basis of $\Phi_p(x_p)^\perp = \Phi_p(y_p)^\perp$. We fix in the sequel such sections which peaks at both $x_p$ and $y_p$.

We consider the distance $d(x_p,y_p)$ between the two points $x_p$ and $y_p$. By passing to a subsequence we have two possibilities: either $\sqrt{p}d(x_p,y_p) \to \infty$ as $p \to \infty$ or there exists a constant $C > 0$ such that $d(x_p,y_p) \leqslant C/\sqrt{p}$ for all $p$.

Assume that the first possibility is true. For large $p$, we learn from relation (4.21) that the mass of $S^p_{x_p,j} = S^p_{y_p,j}$ (which is 1) concentrates both in neighborhoods $B(x_p,r_p)$ and $B(y_p,r_p)$ with $r_p = d(x_p,y_p)/2$ and approaches therefore 2 if $p \to \infty$. This is a contradiction which rules out the first possibility.

Now we proceed as in [17, §8.3.5]. We identify as usual $B^X(x_p,\varepsilon)$ to $B^{T_{x_p}X}(0,\varepsilon)$ so the point $y_p$ gets identified to $Z_p/\sqrt{p}$ where $Z_p \in B^{T_{x_p}X}(0,C)$. We define then

$$f_p : [0,1] \longrightarrow \mathbb{R}\,, \quad f_p(t) = \frac{\sum_{j=1}^{\mathrm{rk}(E)} |S^p_{x_p,j}(tZ_p/\sqrt{p})|^2}{\mathrm{Tr}\, P_p(tZ_p/\sqrt{p}, tZ_p/\sqrt{p})}\,. \tag{4.22}$$

We have $f_p(0) = f_p(1) = 1$ (as $S^p_{x_p,j} = S^p_{y_p,j}$) and $f_p(t) \leqslant 1$. We deduce the existence of a point $t_p \in ]0,1[$ such that $f_p''(t_p) = 0$. Equations (4.12), (4.20), (4.22) imply the estimate

$$f_p(t) = e^{-\frac{t^2}{4}\sum_j a_j |z_{p,j}|^2}\big(1 + g_p(tZ_p)/\sqrt{p}\big) \tag{4.23}$$

and the $\mathscr{C}^2$ norm of $g_p$ over $B^{T_{x_p}X}(0,C)$ is uniformly bounded in $p$. From (4.23) and $f_p(1) = 1$, we infer that $|Z_p|_a^2 := \frac{1}{4}\sum_j a_j|z_{p,j}|^2 = \mathscr{O}(1/\sqrt{p})$. Using a limited expansion $e^x = 1 + x + x^2\varphi(x)$ for $x = t^2|Z_p|_a^2$ in (4.23) and taking derivatives, we obtain

$$f_p''(t) = -2|Z_p|_a^2 + \mathscr{O}(|Z_p|_a^4) + \mathscr{O}(|Z_p|_a^2/\sqrt{p}) = (-2 + \mathscr{O}(1/\sqrt{p}))|Z_p|_a^2. \tag{4.24}$$

Evaluating at $t_p$ we get $0 = f_p''(t_p) = (-2+\mathscr{O}(1/\sqrt{p}))|Z_p|_a^2$, which is a contradiction since by assumption $Z_p \neq 0$. □

*Remark 3* **(a)** It is possible to show the separation of points in Theorem 3 by proving a non-compact version of [11, Theorem 1.8], where the asymptotics of the Bergman kernel for space of sections of a positive line bundle twisted with the Nadel multiplier sheaf of a singular metric are obtained. The only thing needed for the non-compact extension is the spectral gap, which in present case is provided by Lemma 1.

**(b)** In the same vein we can show the separation of points and existence of local coordinates on a compact set under the hypotheses of [17, Theorem 6.1.1] (which are less restrictive than bounded geometry). Let $(X, J, g^{TX})$ be a complete Kähler manifold and $(L, h^L)$, $(E, h^E)$ be holomorphic Hermitian vector bundles on $X$ and $\mathrm{rk}(L) = 1$. Assume also that there exist $\epsilon, C > 0$ such that on $X$,

$$\sqrt{-1}R^L(\cdot, J\cdot) \geqslant \epsilon g^{TX}(\cdot,\cdot), \quad \sqrt{-1}(R^{\det} + R^E) > -C\Theta \mathrm{Id}_E, \tag{4.25}$$

with $\Theta = g^{TX}(J\cdot,\cdot)$. Then for any compact set $K \subset X$ there exists $p_0 = p_0(K) \in \mathbb{N}$ such that for all $p \geqslant p_0$, the sections of $H^0_{(2)}(X, L^p \otimes E)$ separate points and give local coordinates on $K$.

*Proof of Corollary 1* We apply Theorem 8 for the trivial line bundle $L = X \times \mathbb{C}$ endowed with the metric $h^L$ defined by $|1|^2_{h^L} = e^{-2\varphi}$. Then $R^L = 2\partial\overline{\partial}\varphi$. Thus $X$ is holomorphically convex with respect to $(L, e^{-2p\varphi})$ for $p$ sufficiently large. Since $\varphi$ is bounded from below this implies that $X$ is holomorphically convex with respect to the trivial line bundle endowed with the trivial metric. Moreover, Theorem 3 shows that global holomorphic functions on $X$ separate points and give local coordinates on $X$. Hence $X$ is Stein. □

**Corollary 4** *Let $(X, J, \Theta)$ be a compact Hermitian manifold and $(L, h^L)$ be a positive line bundle over $X$. Then there exists $p_0 \in \mathbb{N}$ such that for all Galois covering $\pi : \widetilde{X} \to X$, for all $p \geqslant p_0$, and all Hermitian metrics $h_p$, $h^E$ on $L^p$, $E$, $\widetilde{X}$ is holomorphically convex with respect to the bundle $\widetilde{L}^p \otimes \widetilde{E}$.*

*Proof* The conclusion follows immediately from Theorem 8 for metrics $h_p$ of the form $(h^L)^p$, where $h^L$ is a positively curved metric on $L$. Observe finally that the convexity is independent of the Hermitian metrics $h_p$, $h^E$ chosen on $L^p$, $E$ over the compact manifold $X$. □

*Remark 4* We can prove Corollary 4 also without the use of the off-diagonal decay of the Bergman kernel. What is actually needed is Corollary 3, which uses only the diagonal expansion of the Bergman kernel.

In order to carry out the proof we show that Lemma 3 follows from Corollary 3. By this corollary, for any $x \in \widetilde{X}$ there exists $S_x \in H^0_{(2)}(\widetilde{X}, \widetilde{L}^p \otimes \widetilde{E})$ such that $|S_x(x)| \geqslant M$. Since $S_x \in L^2(\widetilde{X}, \widetilde{L}^p \otimes \widetilde{E})$ is holomorphic, there exists a compact set $A(S_x, \varepsilon) \subset \widetilde{X}$ such that $|S_x| \leqslant \varepsilon$ on $\widetilde{X} \setminus A(S_x, \varepsilon)$.

To prove this, let $F \subset \widetilde{X}$ be a compact fundamental set and $U$ a relatively compact open neighborhood of $F$. Let $\Gamma$ be the group of deck transformations of $\pi : \widetilde{X} \to X$.

For any $\gamma \in \Gamma$ we have $\sup_{\gamma F} |S| \leq C\|S\|_{L^1(\gamma U)} \leq C'\|S\|_{L^2(\gamma U)}$, see [10, Theorem 2.2.3]. Due to the bounded geometry, the constant $C'$ does not depend on $\gamma$. Now choose a compact set $B(S_x, \varepsilon)$ such that the $L^2$ norm of $S_x$ over $\widetilde{X} \setminus B(S_x, \varepsilon)$ is smaller than $\varepsilon/C' > 0$. We set $A(S_x, \varepsilon) = \cup\{\gamma F : \gamma U \cap B(S_x, \varepsilon) \neq \emptyset\}$. Using what has been said, there exists a compact set $F(\varepsilon, M) \supset F$ and sections $S_1, \ldots, S_m \in H^0_{(2)}(\widetilde{X}, \widetilde{L}^p \otimes \widetilde{E})$ such that

$$\begin{aligned} &\max_{1\leqslant i\leqslant m} |S_i(x)| \geqslant M \quad \text{for all } x \in F \\ &\max_{1\leqslant i\leqslant m} |S_i(x)| \leqslant \varepsilon \quad \text{for all } x \in \widetilde{X}\setminus F(\varepsilon, M). \end{aligned} \tag{4.26}$$

Let $K \subset \widetilde{X}$ be a compact set. Define

$$K(\varepsilon, M) = \bigcup\{\gamma F : \gamma \in \Gamma,\ K \cap \gamma F(\varepsilon, M) \neq \emptyset\}.$$

Consider now $x_0 \in \widetilde{X}\setminus K(\varepsilon, M)$. Then there is $\gamma \in \Gamma$ such that $\gamma^{-1}x_0 \in F$. It follows that $K \cap \gamma F(\varepsilon, M) = \emptyset$ so there is $S_i$ such that $|\gamma S_i(x_0)| \geqslant M$ and $|\gamma S_i| \leqslant \varepsilon$ on $K$.

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