Exponential martingales and changes of measure for counting processes

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Agenda

1. Modeling counting processes
2. Exponential martingales and changes of measure
3. Examples
4. Open problems

Modeling counting processes

A classical method for modeling discrete events in continuous time is through counting processes.

A statistical model for a counting process with intensity consists of:

- A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.
- A nonexplosive counting process N on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.
- A parametrized family $(\mu_\theta)_{\theta\in\Theta}$ of intensities.
- A corresponding family of probability measures P_θ such that under P_θ, N is a nonexplosive counting process with intensity μ_θ.

Problem. Given a family $(\mu_\theta)_{\theta\in\Theta}$, does there exist a statistical model corresponding to this family of candidate intensities? This is not a vacuous question, as many candidate intensities yield explosion.

Modeling counting processes

Solution approach on canonical spaces. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be the space of nonexplosive counting process trajectories endowed with the canonical σ-algebra and filtration, let $N : \Omega \to \Omega$ be the identity and let P be such that N is a homogeneous Poisson process.

Jacobsen (2005) gives sufficient criteria on μ_θ to ensure that there exists a probability measure P_θ equivalent to P such that under P_θ, N has intensity μ_θ.

This yields the existence of nonexplosive counting processes with intensity μ_θ and yields the existence of the statistical model.

Modeling counting processes

Benefits of the canonical setting:

- Precise expressions for the likelihood in terms of the conditional distributions of event times of the counting process with intensity μ_θ.
- Coupling arguments may be used to analyze non-explosion.

Drawbacks of the canonical setting:

- Only intensities depending on N are covered.
- Arguments are often based on very technical manipulations of the canonical space and various conditional distributions, instead of for example modern martingale theory.

Alternative approach. Consider a general filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ and formulate all issues in terms of martingales.

Modeling counting processes

A general problem statement. Assume given:

- A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.
- A positive, predictable and locally bounded intensity process λ.
- A counting process N with intensity λ.
- A parametrized family $(\mu_\theta)_{\theta\in\Theta}$ of intensities.

We seek: Sufficient criteria on μ_θ to ensure the existence of a probability measure P_θ equivalent to P such that under P_θ, N has intensity μ_θ.

As corollaries, we obtain: Explicit expressions for the likelihood, criteria for existence of counting processes with various intensities (corresponding to criteria for nonexplosion).

Exponential martingales and changes of measure

Definition. We say that a d-dimensional nonexplosive counting process N has intensity λ if $N_t^i - \int_0^t \lambda_s^i \, \mathrm{d}s$ is a local martingale, $i \leq d$.

From now on, assume given:

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ satisfying the usual conditions.
- Positive, predictable and locally bounded d-dimensional λ, μ.
- A d-dimensional counting process N with intensity λ.

For any semimartingale X with $\Delta X > -1$, we define

$$\mathcal{E}(X)_t = \exp\left(X_t - \frac{1}{2}[X^c]_t + \sum_{0<s\leq t} \log(1 + \Delta X_s) - \Delta X_s \right).$$

$\mathcal{E}(X)$ is the Doléans-Dade exponential of X. If X is a local martingale, so is $\mathcal{E}(X)$.

Exponential martingales and changes of measure

We define:

- $M_t^i = N_t^i - \int_0^t \lambda_s^i \,\mathrm{d}s$.
- $\gamma_t^i = \mu_t^i (\lambda^i)_t^{-1}$.
- $H_t^i = \gamma_t^i - 1$.
- $H \cdot M = \sum_{i=1}^d \int_0^t H_s^i \,\mathrm{d}M_s^i$.

Lemma. Assume that $\mathcal{E}(H \cdot M)$ is a martingale. Let $t \geq 0$. With Q_t being the measure with Radon-Nikodym derivative $\mathcal{E}(H \cdot M)_t$ with respect to P, N is a counting process under Q_t with intensity $1_{[0,t]}\mu + 1_{(t,\infty)}\lambda$.

Conclusion. In order to obtain the existence of the desired equivalent probability measures, we need criteria for the martingale property of $\mathcal{E}(H \cdot M)$.

Exponential martingales and changes of measure

Theorem. Assume that there is $\varepsilon > 0$ such that whenever $0 \leq u \leq t$ with $|t-u| \leq \varepsilon$, one of the following two conditions are satisfied:

$$E \exp\left(\sum_{i=1}^{d} \int_u^t (\gamma_s^i \log \gamma_s^i - (\gamma_s^i - 1))\lambda_s^i \,\mathrm{d}s\right) < \infty \quad \text{or}$$

$$E \exp\left(\sum_{i=1}^{d} \int_u^t \lambda_s^i \,\mathrm{d}s\right) < \infty \quad \text{and} \quad E \exp\left(\sum_{i=1}^{d} \int_u^t \log_+ \gamma_s^i \,\mathrm{d}N_s^i\right) < \infty,$$

where $\log_+ x = \max\{\log x, 0\}$. Then $\mathcal{E}(H \cdot M)$ is a martingale.

Exponential martingales and changes of measure

Corollary. Let $\lambda = 1$. Assume that there is $\varepsilon > 0$ such that whenever $0 \leq u \leq t$ with $|t - u| \leq \varepsilon$, one of the following two conditions are satisfied:

$$E \exp\left(\sum_{i=1}^{d} \int_u^t \mu_s^i \log_+ \mu_s^i \, \mathrm{d}s\right) < \infty \quad \text{or}$$
$$E \exp\left(\sum_{i=1}^{d} \int_u^t \log_+ \mu_s^i \, \mathrm{d}N_s^i\right) < \infty,$$

where $\log_+ x = \max\{\log x, 0\}$. Then $\mathcal{E}(H \cdot M)$ is a martingale.

Exponential martingales and changes of measure

Outline of proof:

1. Argue that it suffices to show that $\mathcal{E}((H \cdot M)^t - (H \cdot M)^u)$ is a martingale for $|t - u| \leq \varepsilon$.
2. Decompose μ into large and small parts and show a related decomposition for exponential martingales.
3. Apply two theorems of Lépingle & Mémin (1978) to the obtain the result.

Examples

Example 1. Let $\mu_t^i \leq \alpha + \beta \sum_{j=1}^d N_{t-}^j$. Then $\mathcal{E}(H \cdot M)$ is a martingale.

Example 1 shows that we may recover the classical affine criteria for non-explosion from the canonical case in the case of a general filtered space. This also extends the criterion from Gjessing et al. (2010) from an "$\mathcal{L}^p$"-criterion, $p > 1$, to an "$\mathcal{L}^p$"-criterion, $p \geq 1$.

Outline of proof: To use the first moment condition, use that $E \exp(\varepsilon X \log X)$ is finite when X is Poisson distributed and $0 < \varepsilon < 1$, choose $\varepsilon > 0$ such that $4\beta\varepsilon d < 1$. To use the second moment condition, use a Markov argument and that Poisson distributions have exponential moments of all orders, choose $\varepsilon > 0$ such that $\beta\varepsilon d < 1$.

Examples

Example 2. Consider $A : \mathbb{N}_0^d \times \mathbb{R}_+^d \to \mathbb{R}^d$, $B : \mathbb{N}_0^d \times \mathbb{R}_+^d \to \mathbb{M}(d,d)$ and $\sigma : \mathbb{N}_0^d \times \mathbb{R}_+^d \to \mathbb{M}(d,d)$. Assume that $A(\eta,\cdot)$, $B(\eta,\cdot)$ and $\sigma(\eta,\cdot)$ are continuous and bounded for $\eta \in \mathbb{N}_0^d$. Assume that σ is positive definite. Assume that for $\eta \in \mathbb{N}_0^d$, there is $\delta, c > 0$ such that

$$\sup_{t\geq 0} \|A(\eta,t)\|_2 \leq c\|\eta\|_1^{1-\delta}$$

$$\sup_{t\geq 0} \|\sigma(\eta,t)\|_2 \leq c\|\eta\|_1^{(1-\delta)/2}$$

$$\sup_{t\geq 0} \|B(\eta,t)\|_2 \leq c.$$

Examples

Example 2, contined. Let X be a solution to

$$\mathrm{d}X_t = (A(N_t, Z_t) + B(N_t, Z_t)X_t)\,\mathrm{d}t + \sigma(N_t, Z_t)\,\mathrm{d}W_t,$$

where W is a d-dimensional $(\mathcal{F}_t)$ Brownian motion and $Z_t^i = t - T^i_{N_t^i}$, where T_n^i is the n'th event time of N^i. Let $\phi : \mathbb{R}^d \to \mathbb{R}^d_+$ be Lipschitz. Assume that $\phi(x)^i \neq 0$ for $x_i \neq 0$. Put $\mu_t = \phi(X_t)$. Then $\mathcal{E}(H \cdot M)$ is a martingale.

Example 2 shows that we can use our results to construct counting processes where the intensity is driven by a SDE whose coefficients vary according to the history of the counting process.

Examples

Outline of proof: Note that conditionally on N, the intensity has the distribution of a Gaussian process. Apply bounds for $E \exp(c\|Z\|_2^{1+\varepsilon})$, with Z d-dimensionally Gaussian and $0 < \varepsilon < 1$, to obtain a bound for the conditional expectation

$$E\left(\exp\left(t\sum_{i=1}^{d}\mu_s^i \log_+ \mu_s^i\right)\middle| N\right).$$

Use this to obtain a bound of the unconditional expectation varying continuously in s, $0 \leq s \leq t$. Apply Jensen's inequality and further estimates to obtain the result.

Examples

Example 3. Let $\phi_i : \mathbb{R} \to (0, \infty)$ for $i \le d$ and $h_{ij} : \mathbb{R}_+ \to \mathbb{R}$ for $i, j \le d$. Define

$$\mu_t^i = \phi_i \left(\sum_{j=1}^{d} \int_0^{t-} h_{ij}(t-s) \,\mathrm{d}N_s^j \right).$$

Assume that ϕ^i is Borel measurable, that $\phi_i(x) \le |x|$ and that h_{ij} is bounded. Then $\mathcal{E}(H \cdot M)$ is a martingale.

Example 3 is an example of a sufficient criterion for non-explosion for multidimensional Hawkes processes.

Open problems

Questions not yet answered:

- Can the requirement that μ be positive be dropped if we require only absolute continuity instead of equivalence between measures?
- Can Example 2 be extended to the case where $\delta = 0$?
- Can the condition $E \exp(\sum_{i=1}^{d} \int_u^t \lambda_s^i \, \mathrm{d}s) < \infty$ in the second moment condition of the main theorem be removed?

Open problems

The final question inspires a more general conjecture:

Conjecture. If M is a purely discontinuous local martingale with $-1 < \Delta M \leq 0$, does it hold that $\mathcal{E}(M)$ is a martingale?

The rationale behind this conjecture: nonpositive jumps yield pointwisely smaller compensators, and therefore should not "cause explosion".

Open problems

The conjecture might lead to an extension of the Novikov condition in the following sense. By a theorem of Lépingle and Mémin, $\mathcal{E}(M)$ is a martingale if $E\exp(\Pi_p^* B_t)$ is finite for $t \geq 0$, where

$$B_t = \frac{1}{2}[M^c]_t + \sum_{0<s\leq t}(1+\Delta M_s)\log(1+\Delta M_s) - \Delta M_s.$$

Noting that $(1+x)\log(1+x) - x \leq x^2$ for $x > -1$, this yields that if $\langle M \rangle$ exists and $E\exp(\frac{1}{2}\langle M^c\rangle_t + \langle M^d\rangle_t)$ is finite for $t \geq 0$, then $\mathcal{E}(M)$ is a martingale.

However, for $x \geq 0$, it also holds that $(1+x)\log(1+x) - x \leq \frac{1}{2}x^2$. Therefore, if nonpositive jumps can be handled separately, the same constant $\frac{1}{2}$ as in the classical Novikov condition might be obtained.

Open problems

If the result should hold, however, it is essential that we are attempting to prove the martingale property and not the uniformly integrable martingale property: By an example in Protter & Shimbo (2006), it does not hold that finiteness of $E\exp(\alpha\langle M\rangle_\infty)$ implies that $\mathcal{E}(M)$ is a uniformly integrable martingale for M purely discontinuous and $\alpha < 1$.

The counterexample in Protter & Shimbo (2006) considers $M_t = -a(N_t - t)^T$ for N a standard Poisson process, a particular a with $0 < a < 1$ and a particular finite, but unbounded, stopping time T, and shows that $\mathcal{E}(M)$ is not a uniformly integrable martingale. However, $\mathcal{E}(M)$ is in fact a martingale, and so the conjecture we consider here is not disproved by this example.

Open problems

A method for proving the conjecture might be based on the following two results.

Lemma. Let M be a local martingale with $-1 < \Delta M$. Let (T_n) be a localising sequence such that $\mathcal{E}(M)^{T_n}$ is a martingale. $\mathcal{E}(M)$ is a martingale if and only if $\lim_n E\mathcal{E}(M)_{T_n} 1_{(T_n \le t)} = 0$.

Lemma. Let T be a stopping time, let $-1 < \xi \le 0$ be $\mathcal{F}_T$ measurable, let $A_t = \xi 1_{(t \ge T)}$ and let $M = A - \Pi_p^* A$. Then $\mathcal{E}(M)$ is a uniformly integrable martingale, and it holds that $E\mathcal{E}(M)_T 1_{(T \le t)} \le P(T \le t)$.

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