EXPONENTIAL MIXING FOR THE TEICHMÜLLER FLOW

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ABSTRACT

We study the dynamics of the Teichmüller flow in the moduli space of Abelian differentials (and more generally, its restriction to any connected component of a stratum). We show that the (Masur-Veech) absolutely continuous invariant probability measure is exponentially mixing for the class of Hölder observables. A geometric consequence is that the $SL(2, \mathbf{R})$ action in the moduli space has a spectral gap.

1. Introduction

Let $\mathcal{M}_g$ be the moduli space of non-zero Abelian differentials on a compact Riemann surface of genus $g \geq 1$. Alternatively, $\mathcal{M}_g$ can be seen as the moduli space of translation surfaces of genus g: outside the zero set of an Abelian differential ω there are preferred local charts where $\omega = dz$, and the coordinate changes of those charts are translations. Let $\mathcal{M}_g^{(1)} \subset \mathcal{M}_g$ denote the subspace of surfaces with normalized area $\int |\omega|^2 = 1$.

By postcomposing the preferred charts with an element of $GL(2, \mathbf{R})$ one obtains another translation structure: this gives a natural $GL(2, \mathbf{R})$ action on $\mathcal{M}_g$. The $SL(2, \mathbf{R})$ action preserves $\mathcal{M}_g^{(1)}$. The Teichmüller flow on $\mathcal{M}_g$ is defined as the diagonal action of $SL(2, \mathbf{R})$: $\mathcal{TF}_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : \mathcal{M}_g \to \mathcal{M}_g$.

The space $\mathcal{M}_g$ is naturally stratified: given a list $\kappa = (\kappa_1, \ldots, \kappa_s)$ of positive integers with $\sum(\kappa_i - 1) = 2g - 2$, we let $\mathcal{M}_{g,\kappa}$ be the space of Abelian differentials whose zeroes have order $\kappa_1 - 1, \ldots, \kappa_s - 1$. The strata are obviously invariant by the $GL(2, \mathbf{R})$ action.

The strata $\mathcal{M}_{g,\kappa}$ are not necessarily connected (a classification of connected components is given in [KZ]). Let $\mathscr{C}$ be a connected component of some stratum $\mathcal{M}_{g,\kappa}$, and let $\mathscr{C}^{(1)} = \mathscr{C} \cap \mathcal{M}_g^{(1)}$. It has a natural structure of an analytic variety, and hence a natural Lebesgue measure class. By the fundamental work of Masur [Ma] and Veech [Ve1], there exists a unique probability measure $\nu_{\mathscr{C}^{(1)}}$ on $\mathscr{C}^{(1)}$ which is equivalent to Lebesgue measure, invariant by the Teichmüller flow, and ergodic. Veech later showed in [Ve2] that $\nu_{\mathscr{C}^{(1)}}$ is actually mixing, meaning

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that for any observables $f, g \in \mathrm{L}^2(\nu_{\mathscr{C}^{(1)}})$ one has

$$\textbf{(1.1)} \qquad \int_{\mathscr{C}^{(1)}} f \circ \mathscr{TF}_t(x) g(x)\, \mathrm{d}\nu_{\mathscr{C}^{(1)}}(x) - \int_{\mathscr{C}^{(1)}} f\, \mathrm{d}\nu_{\mathscr{C}^{(1)}} \int_{\mathscr{C}^{(1)}} g\, \mathrm{d}\nu_{\mathscr{C}^{(1)}} \to 0.$$

In this paper we are concerned with the speed of mixing of the Teichmüller flow, that is, the rate of the convergence in (1.1), for a suitable class of observables.

Main Theorem. — *The Teichmüller flow (restricted to any connected component of any stratum of the moduli space of Abelian differentials) is exponentially mixing for Hölder observables.*

The complete formulation of this result, specifying in particular what is understood by a Hölder observable in this non-compact setting, is given in Theorem 2.14.

Previously it had been shown by Bufetov [Bu] that the Central Limit Theorem holds for the Teichmüller flow (for suitable classes of observables). Though he did not obtain rates of mixing for the Teichmüller flow itself, he did obtain *stretched exponential* estimates for a related discrete time transformation (the Zorich renormalization algorithm for interval exchange transformations). In this paper we will also work with a discrete time transformation, though not directly with the Zorich renormalization.

This paper has two main parts: first we obtain exponential recurrence estimates, and then, using ideas first introduced by Dolgopyat [Do] and developed in [BV], we obtain exponential mixing. The proof of exponential recurrence uses an "induction on the complexity" scheme. Intuitively, the dynamics at "infinity" of the Teichmüller flow can be partially described by the dynamics in simpler (lower dimensional) connected components of strata, and we obtain estimates by induction all the way from the simplest of the cases. A simpler version of this scheme was used to show some combinatorial richness of the Teichmüller flow in the proof of the Zorich–Kontsevich conjecture [AV]. The recurrence estimates thus obtained are close to optimal.

It should be noted that our work does not use the $\mathrm{SL}(2, \mathbf{R})$ action for the estimates, and can be used to obtain new proofs of some previously known results which used to depend on the $\mathrm{SL}(2, \mathbf{R})$ action. In the other direction, however, it was pointed out to us by Bufetov that our main theorem has an important new corollary for the $\mathrm{SL}(2, \mathbf{R})$ action. It regards the nature of the corresponding unitary representation of $\mathrm{SL}(2, \mathbf{R})$ on the space $\mathrm{L}^2_0(\nu_{\mathscr{C}^{(1)}})$ of L^2 zero-average functions.

Corollary **1.1.** — *The action of* $\mathrm{SL}(2, \mathbf{R})$ *on* $\mathrm{L}^2_0(\nu_{\mathscr{C}^{(1)}})$ *has a spectral gap.*

The notion of spectral gap and the derivation of the corollary from the Main Theorem are discussed in Appendix B.

Remark. — Using Corollary 1.1 and the results of Ratner [Rt], it is possible to extend the main theorem to a larger class of functions, namely, the functions which are Hölder continuous in the direction of the $SO(2, \mathbf{R})$ action. This point of view even makes unnecessary the discussion of the metric on the Teichmüller space in Section 2.2.2, or the smoothing arguments in Lemma 4.7. However, we nevertheless include these arguments to keep the presentation completely independent of the $SL(2, \mathbf{R})$ action.

Remark. — Exponential recurrence estimates for the Teichmüller flow were first obtained by Athreya [At], who used the $SL(2, \mathbf{R})$ action to prove them for some large compact sets (which are, in particular, $SO(2, \mathbf{R})$ invariant). Our work allows us to obtain exponential recurrence for certain smaller compact sets, for which the first return map has good hyperbolic and distortion properties. Bufetov has independently obtained a proof of exponential recurrence estimates for such small compact sets, using the method of [Bu]. Those estimates, while non-optimal, are enough to obtain exponential mixing using the remainder of our argument.

We should also point out that recurrence estimates are often useful in statistical arguments in a very practical sense. For instance, the proof of typical weak mixing in [AF] can be made more transparent using such estimates.

2. Statements of the results

2.1. *Exponential mixing for excellent hyperbolic semiflows.* — To prove exponential decay of correlations for the Teichmüller flow, we will show that this flow can be reduced to an abstract flow with good hyperbolic properties. In this paragraph, we describe some assumptions under which such a flow is exponentially mixing.

By definition, a Finsler manifold is a smooth manifold endowed with a norm on each tangent space, which varies continuously with the base point.

Definition **2.1.** — *A* John domain Δ *is a finite dimensional connected Finsler manifold, together with a measure* Leb *on* Δ, *with the following properties:*

1. *For* $x, x' \in \Delta$, *let* $d(x, x')$ *be the infimum of the length of a* C^1 *path contained in* Δ *and joining* x *and* x'. *For this distance,* Δ *is bounded and there exist constants* C_0 *and* ε_0 *such that, for all* $\varepsilon < \varepsilon_0$, *for all* $x \in \Delta$, *there exists* $x' \in \Delta$ *such that* $d(x, x') \leq C_0\varepsilon$ *and such that the ball* $B(x', \varepsilon)$ *is compactly contained in* Δ.
2. *The measure* Leb *is a fully supported finite measure on* Δ, *satisfying the following inequality: for all* $C > 1$, *there exists* $A > 1$ *such that, whenever a ball* $B(x, r)$ *is compactly contained in* Δ, $\mathrm{Leb}(B(x, Cr)) \leq A\,\mathrm{Leb}(B(x, r))$.

For example, if Δ is an open subset of a larger manifold, with compact closure, whose boundary is a finite union of smooth hypersurfaces in general position, and Leb is obtained by restricting to Δ a smooth measure defined in a neighborhood of $\overline{\Delta}$, then (Δ, Leb) is a John domain.

Definition **2.2.** — *Let* L *be a finite or countable set, let* Δ *be a John domain, and let* $\{\Delta^{(l)}\}_{l\in L}$ *be a partition into open sets of a full measure subset of* Δ. *A map* $T : \bigcup_l \Delta^{(l)} \to \Delta$ *is a* uniformly expanding Markov map *if*

1. *For each* l, T *is a* C^1 *diffeomorphism between* $\Delta^{(l)}$ *and* Δ, *and there exist constants* $\kappa > 1$ *(independent of* l*) and* $C_{(l)}$ *such that, for all* $x \in \Delta^{(l)}$ *and all* $v \in T_x\Delta$, $\kappa\|v\| \leq \|DT(x)\cdot v\| \leq C_{(l)}\|v\|$.
2. *Let* $J(x)$ *be the inverse of the Jacobian of* T *with respect to* Leb. *Denote by* $\mathscr{H}$ *the set of inverse branches of* T. *The function* $\log J$ *is* C^1 *on each set* $\Delta^{(l)}$ *and there exists* $C > 0$ *such that, for all* $h \in \mathscr{H}$, $\|D((\log J)\circ h)\|_{C^0(\Delta)} \leq C$.

Such a map T preserves a unique absolutely continuous measure μ. Its density is bounded from above and from below and is C^1. This measure is ergodic and even mixing (see e.g. [Aar]). Notice that Leb is not assumed to be absolutely continuous with respect to Lebesgue measure. Although this will be the case in most applications, this definition covers also e.g. the case of maximum entropy measures when L is finite (in which case $\log J$ is constant, which yields $D((\log J)\circ h) = 0$).

Definition **2.3.** — *Let* $T : \bigcup_l \Delta^{(l)} \to \Delta$ *be a uniformly expanding Markov map on a John domain. A function* $r : \bigcup \Delta^{(l)} \to \mathbf{R}_+$ *is a* good roof function *if*

1. *There exists* $\varepsilon_1 > 0$ *such that* $r \geq \varepsilon_1$.
2. *There exists* $C > 0$ *such that, for all* $h \in \mathscr{H}$, $\|D(r\circ h)\|_{C^0} \leq C$.
3. *It is not possible to write* $r = \psi + \phi\circ T - \phi$ *on* $\bigcup \Delta^{(l)}$, *where* $\psi : \Delta \to \mathbf{R}$ *is constant on each set* $\Delta^{(l)}$ *and* $\phi : \Delta \to \mathbf{R}$ *is* C^1.

If r is a good roof function for T, we will write $r^{(n)}(x) = \sum_{k=0}^{n-1} r(T^k x)$.

Definition **2.4.** — *A good roof function* r *as above has* exponential tails *if there exists* $\sigma_0 > 0$ *such that* $\int_\Delta e^{\sigma_0 r}\,d\text{Leb} < \infty$.

If $\widehat{\Delta}$ is a Finsler manifold, we will denote by $C^1(\widehat{\Delta})$ the set of functions $u : \widehat{\Delta} \to \mathbf{R}$ which are bounded, continuously differentiable, and such that $\sup_{x\in\widehat{\Delta}} \|Du(x)\| < \infty$. Let

$$\|u\|_{C^1(\widehat{\Delta})} = \sup_{x\in\widehat{\Delta}} |u(x)| + \sup_{x\in\widehat{\Delta}} \|Du(x)\| \tag{2.1}$$

be the corresponding norm.

Definition **2.5.** — *Let* $T : \bigcup_l \Delta^{(l)} \to \Delta$ *be a uniformly expanding Markov map, preserving an absolutely continuous measure* μ. *An* hyperbolic skew-product *over* T *is a map* $\widehat{T}$ *from a dense open subset of a bounded connected Finsler manifold* $\widehat{\Delta}$, *to* $\widehat{\Delta}$, *satisfying the following properties:*

1. *There exists a continuous map* $\pi : \widehat{\Delta} \to \Delta$ *such that* $T \circ \pi = \pi \circ \widehat{T}$ *whenever both members of this equality are defined.*
2. *There exists a probability measure* ν *on* $\widehat{\Delta}$, *giving full mass to the domain of definition of* $\widehat{T}$, *which is invariant under* $\widehat{T}$.
3. *There exists a family of probability measures* $\{\nu_x\}_{x\in\Delta}$ *on* $\widehat{\Delta}$ *which is a disintegration of* ν *over* μ *in the following sense:* $x \mapsto \nu_x$ *is measurable,* ν_x *is supported on* $\pi^{-1}(x)$ *and, for any measurable set* $A \subset \widehat{\Delta}$, $\nu(A) = \int \nu_x(A)\,d\mu(x)$.
   Moreover, this disintegration satisfies the following property: there exists a constant $C > 0$ *such that, for any open subset* $O \subset \bigcup \Delta^{(l)}$, *for any* $u \in C^1(\pi^{-1}(O))$, *the function* $\bar{u} : O \to \mathbf{R}$ *given by* $\bar{u}(x) = \int u(y)\,d\nu_x(y)$ *belongs to* $C^1(O)$ *and satisfies the inequality*

$$\sup_{x\in O} \|D\bar{u}(x)\| \le C \sup_{y\in\pi^{-1}(O)} \|Du(y)\|. \qquad (\mathbf{2.2})$$

4. *There exists* $\kappa > 1$ *such that, for all* $y_1, y_2 \in \widehat{\Delta}$ *with* $\pi(y_1) = \pi(y_2)$, *holds*

$$d(\widehat{T}y_1, \widehat{T}y_2) \le \kappa^{-1} d(y_1, y_2). \qquad (\mathbf{2.3})$$

Let $\widehat{T}$ be an hyperbolic skew-product over a uniformly expanding Markov map T. Let r be a good roof function for T, with exponential tails. It is then possible to define a space $\widehat{\Delta}_r$ and a semiflow $\widehat{T}_t$ over $\widehat{T}$ on $\widehat{\Delta}$, using the roof function $r \circ \pi$, in the following way. Let $\widehat{\Delta}_r = \{(y, s) : y \in \pi^{-1}\left(\bigcup_l \widehat{\Delta}_l\right), 0 \le s < r(\pi y)\}$. For almost all $y \in \widehat{\Delta}$, all $0 \le s < r(\pi y)$ and all $t \ge 0$, there exists a unique $n \in \mathbf{N}$ such that $r^{(n)}(\pi y) \le t + s < r^{(n+1)}(\pi y)$. Set $\widehat{T}_t(y, s) = (\widehat{T}^n y, s + t - r^{(n)}(\pi y))$. This is a semiflow defined almost everywhere on $\widehat{\Delta}_r$. It preserves the probability measure $\nu_r = \nu \otimes \mathrm{Leb}/(\nu \otimes \mathrm{Leb})(\widehat{\Delta}_r)$. Using the canonical Finsler metric on $\widehat{\Delta}_r$, namely the product metric given by $\|(u, v)\| := \|u\| + \|v\|$, we define the space $C^1(\widehat{\Delta}_r)$ as in (2.1). Notice that $\widehat{\Delta}_r$ is not connected, and the distance between points in different connected components is infinite.

Definition **2.6.** — *A semiflow* $\widehat{T}_t$ *as above is called an* excellent hyperbolic semiflow.

The main motivations for this definition are that the Teichmüller flow is isomorphic to an excellent hyperbolic semiflow – the proof of this isomorphism will take a large part of this article – and that such a flow has exponential decay of correlations:

Theorem **2.7.** — *Let* $\widehat{T}_t$ *be an excellent hyperbolic semi-flow on a space* $\widehat{\Delta}_r$, *preserving the probability measure* ν_r. *There exist constants* $C > 0$ *and* $\delta > 0$ *such that, for all functions* $U, V \in C^1(\widehat{\Delta}_r)$, *for all* $t \geq 0$,

$$\left| \int U \cdot V \circ \widehat{T}_t \, d\nu_r - \left(\int U \, d\nu_r \right) \left(\int V \, d\nu_r \right) \right| \leq C \|U\|_{C^1} \|V\|_{C^1} e^{-\delta t}. \tag{2.4}$$

We will see the consequences of this theorem in the next sections. The proof of Theorem 2.7 will be deferred to Sections 7 and 8.

2.2. *The Teichmüller flow*

2.2.1. *Teichmüller space, moduli space and the Teichmüller flow.* — Let $g \in \mathbf{N}^*$ and $s \in \mathbf{N}^*$ be positive integers. Take M a compact orientable C^∞ surface of genus g, and let $\Sigma = \{A_1, \ldots, A_s\}$ be a subset of M. Let $\kappa = (\kappa_1, \ldots, \kappa_s) \in (\mathbf{N}^*)^s$ be such that $\sum(\kappa_i - 1) = 2g - 2$.

A *translation structure* on (M, Σ) with singularities type κ is an atlas on $M \backslash \Sigma$ for which the coordinate changes are translations, and such that each singularity A_i has a neighborhood which is isomorphic to the κ_i-fold covering of a neighborhood of 0 in $\mathbf{R}^2 \backslash \{0\}$. The Teichmüller space $\mathscr{Q}_{g,\kappa} = \mathscr{Q}(M, \Sigma, \kappa)$ is the set of such structures modulo isotopy rel. Σ. It has a canonical structure of manifold.

Let us describe this manifold structure by introducing charts through the *period map* Θ. Let ξ be a translation structure on (M, Σ). If $\gamma \in C^0([0, T], M)$ is a path on M, then it is possible to lift it in $\mathbf{R}^2$, starting from 0: this lifting is possible locally outside of the singularities, and the local form of the translation structure close to the singularities implies that this lifting is also possible at the singularities. Taking the value of the lifting at T, we get a *developing map*

$$D_\xi : C^0([0, T], M) \to \mathbf{R}^2. \tag{2.5}$$

This yields a linear map $H_1(M, \Sigma; \mathbf{Z}) \to \mathbf{R}^2$, i.e., an element of $H^1(M, \Sigma; \mathbf{R}^2)$. It is invariant under isotopy rel. Σ. Hence, it defines a map $\Theta : \mathscr{Q}(M, \Sigma, \kappa) \to H^1(M, \Sigma; \mathbf{R}^2)$.

This map is a local diffeomorphism for the canonical manifold structure of $\mathscr{Q}(M, \Sigma, \kappa)$, and gives in particular local coordinates. It even endows $\mathscr{Q}(M, \Sigma, \kappa)$ with a complex affine manifold structure.

A translation structure on (M, Σ) defines a volume form on $M \backslash \Sigma$ (the pull-back of the standard volume form on $\mathbf{R}^2$ by any translation chart). The manifold M has finite area for this volume form. Let $\mathscr{Q}^{(1)}(M, \Sigma, \kappa)$ be the smooth hypersurface of $\mathscr{Q}(M, \Sigma, \kappa)$ given by area 1 translation structures.

The space $H^1(M, \Sigma; \mathbf{R}^2)$ has a standard volume form (the Lebesgue form giving covolume 1 to the integer lattice). Pulling it back locally with Θ, we obtain a smooth measure μ on $\mathcal{Q}(M, \Sigma, \kappa)$. It induces a smooth measure $\mu^{(1)}$ on the hypersurface $\mathcal{Q}^{(1)}(M, \Sigma, \kappa)$.

The group $SL(2, \mathbf{R})$ acts on $\mathcal{Q}(M, \Sigma, \kappa)$ by postcomposition in the charts. It preserves the hypersurface $\mathcal{Q}^{(1)}(M, \Sigma, \kappa)$ and leaves invariant the measures μ and $\mu^{(1)}$. In particular, the action of $\mathcal{TF}_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ is a measure preserving flow, called the *Teichmüller flow*.

The modular group of (M, Σ) is the group of diffeomorphisms of M fixing Σ, modulo isotopy rel. Σ. It acts on the Teichmüller space $\mathcal{Q}(M, \Sigma, \kappa)$. The quotient is denoted by $\mathcal{M}_{g,\kappa} = \mathcal{M}(M, \Sigma, \kappa)$ and is called the *moduli space*. The action of the modular group on $\mathcal{Q}(M, \Sigma, \kappa)$ is proper and faithful, but it is not free. Hence, $\mathcal{M}(M, \Sigma, \kappa)$ has a complex affine orbifold structure.

Since the action of the modular group preserves the measure μ and the hypersurface $\mathcal{Q}^{(1)}$, we also obtain a measure ν on the moduli space, as well as a codimension 1 hypersurface $\mathcal{M}^{(1)}(M, \Sigma, \kappa)$ of area 1 translation structures, and a measure $\nu^{(1)}$ on it. Moreover, the action of $SL(2, \mathbf{R})$ commutes with the action of the modular group, whence $SL(2, \mathbf{R})$ still acts on $\mathcal{M}(M, \Sigma, \kappa)$ and $\mathcal{M}^{(1)}(M, \Sigma, \kappa)$, preserving respectively ν and $\nu^{(1)}$. In particular, the action of $\mathcal{TF}_t$ defines a flow on $\mathcal{M}(M, \Sigma, \kappa)$, that we still call the Teichmüller flow.

Theorem **2.8** *(Masur, Veech). — The measure $\nu^{(1)}$ has finite mass. Moreover, on each connected component of $\mathcal{M}^{(1)}(M, \Sigma, \kappa)$, the Teichmüller flow is ergodic, and even mixing.*

Our goal in this paper is to estimate the speed of mixing of the Teichmüller flow. Our estimates will in particular give a new proof of Theorem 2.8.

2.2.2. *A Finsler metric on the Teichmüller space.* — For a general dynamical system, the exponential decay of correlations usually only holds at best for sufficiently regular functions. In our case, "regular" will mean Hölder continuous, for some natural metric. This metric will be a Finsler metric on the Teichmüller space, invariant under the action of the modular group.

Let ξ be a translation structure on (M, Σ) with singularities type κ. The *saddle connections* of ξ are the unit speed geodesic paths $\gamma : [0, T] \to M$ such that $\gamma^{-1}(\Sigma) = \{0, T\}$. Equivalently, these are straight lines (for the translation structure) connecting two singularities, and without singularity in their interiors. If γ is a saddle connection, then $D_\xi(\gamma)$ is a complex number measuring the holonomy of the translation structure along γ. If $[\gamma]$ is the class of γ in $H_1(M, \Sigma; \mathbf{Z})$, then $D_\xi(\gamma) = \Theta(\xi)([\gamma])$ by definition of Θ.

The saddle connections define in particular elements of $H_1(M, \Sigma; \mathbf{Z})$. They are invariant under isotopy, and depend only on the class of ξ in $\mathcal{Q}(M, \Sigma, \kappa)$. The following lemma is well known (see e.g. [EM]).

Lemma **2.9.** — *Any translation surface* ξ *admits a triangulation whose vertices are the singularities* Σ *and whose edges are saddle connections. In particular, the saddle connections generate the homology* $H_1(M, \Sigma; \mathbf{R})$.

Proposition **2.10.** — *Let* $q \in \mathcal{Q}(M, \Sigma, \kappa)$, *and let* ξ *be a translation surface representing* q. *Let* $\{\gamma_n\}$ *be the set of its saddle connections. Define a function* $\|\cdot\|_q$ *on* $H^1(M, \Sigma; \mathbf{C})$ *by*

$$\|\omega\|_q = \sup_{n\in\mathbf{N}} \left| \frac{\omega([\gamma_n])}{\Theta(q)([\gamma_n])} \right|. \tag{2.6}$$

This function defines a norm on $H^1(M, \Sigma; \mathbf{C})$.

Proof. — Let $\|\cdot\|$ be any norm on $H_1(M, \Sigma; \mathbf{R})$. We will prove the existence of $C > 0$ such that, for any saddle connection γ, $C^{-1}\|[\gamma]\| \le |\Theta(q)([\gamma])| \le C\|[\gamma]\|$. Since the saddle connections generate the homology, this will easily imply the result of the proposition.

Since $\gamma \mapsto \Theta(q)([\gamma])$ is linear, the inequality $|\Theta(q)([\gamma])| \le C\|[\gamma]\|$ is trivial. For the converse inequality, let $L > 0$ be such that any point of M can be joined to a point of Σ by a path of length at most L. The inequality $C^{-1}\|\gamma\| \le |\Theta(q)([\gamma])|$ is trivial for the (finite number of) saddle connections of length $\le L$. Consider now a saddle connection γ with length $\ge L$, and let $n \ge 2$ be such that $(n/2)L \le |\Theta(q)([\gamma])| \le nL$. We can subdivide γ in n segments $[x_i, x_{i+1}]$ of length at most L. Joining each x_i to a singularity, we obtain a decomposition in homology $[\gamma] = \sum_{i=1}^n [\gamma_i]$, where γ_i is a path of length at most 3L. There exists a constant C such that any such path γ_i satisfies $\|[\gamma_i]\| \le C$, and we obtain $\|[\gamma]\| \le nC \le \frac{2C}{L}|\Theta(q)([\gamma])|$. □

Proposition **2.11.** — *The map from* $\mathcal{Q}(M, \Sigma, \kappa)$ *to the set of norms on* $H^1(M, \Sigma; \mathbf{C})$ *given by* $q \mapsto \|\cdot\|_q$ *is continuous.*

Proof. — Let $\varepsilon > 0$. By compactness of the unit ball, there exists a finite number of saddle connections $\gamma_1, \dots, \gamma_N$ such that, for any $\omega \in H^1(M, \Sigma; \mathbf{C})$,

$$\|\omega\|_q \le (1+\varepsilon) \sup_{1\le n\le N} \left| \frac{\omega([\gamma_n])}{\Theta(q)([\gamma_n])} \right|. \tag{2.7}$$

If q' is close enough to q, the saddle connections γ_i survive in q', and we get

$$\|\omega\|_{q'} \ge \sup_{1\le n\le N} \left| \frac{\omega([\gamma_n])}{\Theta(q')([\gamma_n])} \right| \ge (1-\varepsilon) \sup_{1\le n\le N} \left| \frac{\omega([\gamma_n])}{\Theta(q)([\gamma_n])} \right| \ge \frac{1-\varepsilon}{1+\varepsilon}\|\omega\|_q. \tag{2.8}$$

For the converse inequality, we have to prove that the new saddle connections appearing in q' do not increase the norm too much. Let ξ be a translation surface representing q. By Lemma 2.9, ξ is obtained by gluing a finite number of triangles along some parallel edges. A translation surface ξ' close to ξ is obtained by modifying slightly the sides of these triangles in $\mathbf{R}^2$ and then gluing them along the same pattern. Hence, we get a map $\phi_{\xi\xi'} : \xi \to \xi'$ which is affine in each triangle of the triangulation. Moreover, if ξ' is close enough to ξ, the differential of $\phi_{\xi\xi'}$ is ε-close to the identity

Let γ' be a saddle connection in ξ'. The path $\phi_{\xi\xi'}^{-1}(\gamma')$ is a union of a finite number of segments in ξ, and its length is at most $(1+\varepsilon)|D_{\xi'}(\gamma')|$. It is homotopic to a unique geodesic path γ in ξ. This path is a union of a finite number of saddle connections $\gamma_1, \ldots, \gamma_N$, with $\sum |D_\xi(\gamma_i)| \leq (1+\varepsilon)|D_{\xi'}(\gamma')|$. For $\omega \in H^1(M, \Sigma; \mathbf{C})$, we get

$$\left|\frac{\omega([\gamma'])}{\Theta(q')([\gamma'])}\right| = \frac{\left|\sum_{i=1}^N \omega([\gamma_i])\right|}{|D_{\xi'}(\gamma')|} \leq (1+\varepsilon)\frac{\sum_{i=1}^N |\omega([\gamma_i])|}{\sum_{i=1}^N |D_\xi(\gamma_i)|}$$
$$\leq (1+\varepsilon) \sup_{1\leq i\leq N} \frac{|\omega([\gamma_i])|}{|D_\xi(\gamma_i)|} \leq (1+\varepsilon)\|\omega\|_q.$$

Hence, we obtain $\|\omega\|_{q'} \leq (1+\varepsilon)\|\omega\|_q$. □

Since the tangent space of $\mathcal{Q}(M, \Sigma, \kappa)$ is everywhere identified through Θ with $H^1(M, \Sigma; \mathbf{C})$, the norm $\|\cdot\|_q$ gives a Finsler metric on $\mathcal{Q}(M, \Sigma, \kappa)$. It defines a distance (which is infinite for points in different connected components) on $\mathcal{Q}(M, \Sigma, \kappa)$ as follows: the distance between two points x, x' is the infimum of the length (measured with the Finsler metric) of a C^1 path joining x and x'.

Let $\operatorname{sys} : \mathcal{Q}(M, \Sigma, \kappa) \to \mathbf{R}_+$ be the systole function, i.e., the shortest length of a saddle connection. It is bounded on $\mathcal{Q}^{(1)}(M, \Sigma, \kappa)$.

Lemma **2.12.** — *The function* $q \mapsto \log(\operatorname{sys}(q))$ *is* 1*-Lipschitz on the space* $\mathcal{Q}(M, \Sigma, \kappa)$.

Proof. — We will prove that, for any C^1 path $\rho : (-1, 1) \to \mathcal{Q}(M, \Sigma, \kappa)$ with $\rho(0) = q$ and $\rho'(0) = \omega \in H^1(M, \Sigma; \mathbf{C})$ holds

$$\limsup_{t\to 0} \frac{|\log \operatorname{sys}(\rho(t)) - \log \operatorname{sys}(q)|}{|t|} \leq \|\omega\|_q. \tag{2.9}$$

This will easily imply the result.

In a translation surface representing q, there is a finite number of saddle connections $\gamma_1, \ldots, \gamma_N$ with minimal length. For small enough t, $\operatorname{sys}(\rho(t)) =$

$\min_{1\le i\le N}|\Theta(\rho(t))([\gamma_i])|$. Moreover,

$$\log|\Theta(\rho(t))([\gamma_i])| - \log(\mathrm{sys}(q)) = \log\left|\frac{\Theta(q)([\gamma_i]) + t\omega([\gamma_i]) + o(t)}{\Theta(q)([\gamma_i])}\right|$$
$$= t\Re\left(\frac{\omega([\gamma_i])}{\Theta(q)([\gamma_i])}\right) + o(t).$$

Hence,

$$|\log\mathrm{sys}(\rho(t)) - \log\mathrm{sys}(q)| \le |t|\max_{1\le i\le N}\left|\frac{\omega([\gamma_i])}{\Theta(q)([\gamma_i])}\right| + o(t)$$
$$\le |t|\|\omega\|_q + o(t).$$

□

By construction, the norm $\|\cdot\|_q$ is invariant under the action of the modular group. As a consequence, the modular group acts by isometries on $\mathscr{Q}(M, \Sigma, \kappa)$. Hence, the distance on $\mathscr{Q}(M, \Sigma, \kappa)$ induces a distance on the quotient $\mathscr{M}(M, \Sigma, \kappa)$. It is Finsler outside of the singularities of this orbifold. Notice that the systole is also invariant under the modular group, and passes to the quotient. We will still denote by sys this new function. The function $\log\circ\mathrm{sys}$ is still 1-Lipschitz on $\mathscr{M}(M, \Sigma, \kappa)$.

The systole plays an important role in the topology of $\mathscr{M}^{(1)}(M, \Sigma, \kappa)$ since, for all $\varepsilon > 0$, the set $\{q \in \mathscr{M}^{(1)}(M, \Sigma, \kappa) : \mathrm{sys}(q) \ge \varepsilon\}$ is compact. To say it differently, a sequence $q_n \in \mathscr{M}^{(1)}(M, \Sigma, \kappa)$ diverges to infinity if and only if $\mathrm{sys}(q_n) \to 0$.

Corollary **2.13.** — *The distance on $\mathscr{Q}^{(1)}(M, \Sigma, \kappa)$ is complete.*

Proof. — It is sufficient to prove the same statement in the quotient $\mathscr{M}^{(1)}(M, \Sigma, \kappa)$. If q_n is a Cauchy sequence in $\mathscr{M}^{(1)}(M, \Sigma, \kappa)$, the sequence $\log\mathrm{sys}(q_n)$ is also Cauchy by Lemma 2.12. Hence, $\mathrm{sys}(q_n)$ is bounded away from 0. In particular, the sequence q_n remains in a compact subset of the moduli space $\mathscr{M}^{(1)}(M, \Sigma, \kappa)$, and converges to any of its cluster values. □

Any element $\omega \in H^1(M, \Sigma; \mathbf{C})$ can be written uniquely as $\omega = a + ib$ where $a, b \in H^1(M, \Sigma; \mathbf{R})$. Let $\overline{\omega} = a - ib$. In this notation, the differential of the action of the Teichmüller flow is given by

$$\left.\frac{d\mathscr{TF}_t(q)}{dt}\right|_{t=0} = \overline{\Theta(q)}. \tag{2.10}$$

Hence, $\left\|\left.\frac{d\mathscr{TF}_t(q)}{dt}\right|_{t=0}\right\|_q \le 1$. In particular, the Teichmüller flow satisfies

$$d(\mathscr{TF}_t(q), q) \le |t|. \tag{2.11}$$

The same inequality holds in the quotient space $\mathscr{M}(M, \Sigma, \kappa)$.

If $q \in \mathscr{Q}(\mathrm{M}, \Sigma, \kappa)$ and $\omega = a + ib \in \mathrm{H}^1(\mathrm{M}, \Sigma; \mathbf{C})$ (identified through Θ with the tangent space of $\mathscr{Q}(\mathrm{M}, \Sigma, \kappa)$ at q), then the differential of the Teichmüller flow is given by

$$(\mathbf{2.12}) \qquad \mathrm{D}\mathscr{T\!F}_t(q) \cdot \omega = e^t a + i e^{-t} b.$$

This implies the inequality

$$(\mathbf{2.13}) \qquad e^{-2|t|} \|\omega\|_q \leq \|\mathrm{D}\mathscr{T\!F}_t(q) \cdot \omega\|_{\mathscr{T\!F}_t(q)} \leq e^{2|t|} \|\omega\|_q,$$

which corresponds to the classical fact that the extreme Lyapunov exponents of the Teichmüller flow are -2 and 2.

2.2.3. *Exponential decay of correlations.* — Let $\mathscr{C}^{(1)}$ be a connected component of $\mathscr{M}^{(1)}(\mathrm{M}, \Sigma, \kappa)$. It is an orbifold, and is endowed with a finite mass measure $\nu_{\mathscr{C}^{(1)}}$ (which we will assume to be normalized so that it is a probability measure), and a distance $d_{\mathscr{C}^{(1)}}$. The Teichmüller diagonal flow $\mathscr{T\!F}_t$ acts ergodically on $\mathscr{C}^{(1)}$ and preserves the measure $\nu_{\mathscr{C}^{(1)}}$.

For $0 < \alpha \leq 1$ and $f : \mathscr{C}^{(1)} \to \mathbf{R}$, we will denote by $\omega_\alpha(f, x)$ the local Hölder constant of f at x, i.e.

$$(\mathbf{2.14}) \qquad \omega_\alpha(f, x) = \sup_{\substack{y \in \mathrm{B}(x,1) \\ y \neq x}} \frac{|f(y) - f(x)|}{d_{\mathscr{C}^{(1)}}(y, x)^\alpha}.$$

For $k \in \mathbf{N}$ and $0 < \alpha \leq 1$, let $\mathscr{D}_{k,\alpha}$ be the set of functions $f : \mathscr{C}^{(1)} \to \mathbf{R}$ such that the norm

$$(\mathbf{2.15}) \qquad \|f\|_{\mathscr{D}_{k,\alpha}} := \sup_{x \in \mathscr{C}^{(1)}} |f(x)| \operatorname{sys}(x)^k + \sup_{x \in \mathscr{C}^{(1)}} \omega_\alpha(f, x) \operatorname{sys}(x)^k$$

is finite. This is the set of functions which are locally α-Hölder at each point and do not behave worse than $\operatorname{sys}(x)^{-k}$ at infinity. When f is compactly supported, this condition reduces to the fact that f is α-Hölder, but it is much more permissive in general.

For example, if a function $f : \mathscr{C}^{(1)} \to \mathbf{R}$ is compactly supported and C^1 (meaning that its lift to the manifold $\mathscr{Q}^{(1)}(\mathrm{M}, \Sigma, \kappa)$ is C^1), then it belongs to all spaces $\mathscr{D}_{k,\alpha}$.

The main result of this article is the following theorem:

Theorem **2.14.** — *Let $k \in \mathbf{N}$ and $0 < \alpha \leq 1$. Let $p, q \in \mathbf{R}_+$ be such that $1/p + 1/q < 1$. Then there exist constants $\delta > 0$ and $\mathrm{C} > 0$ (depending on k, α, p, q) such* *that, for all functions* $f : \mathscr{C}^{(1)} \to \mathbf{R}$ *belonging to* $\mathscr{D}_{k,\alpha} \cap \mathrm{L}^p(\nu_{\mathscr{C}^{(1)}})$ *and* $g : \mathscr{C}^{(1)} \to \mathbf{R}$ *belonging to* $\mathscr{D}_{k,\alpha} \cap \mathrm{L}^q(\nu_{\mathscr{C}^{(1)}})$*, for all* $t \geq 0$*, holds*

$$\left| \int f \cdot g \circ \mathscr{TF}_t \, \mathrm{d}\nu_{\mathscr{C}^{(1)}} - \left(\int f \, \mathrm{d}\nu_{\mathscr{C}^{(1)}} \right) \left(\int g \, \mathrm{d}\nu_{\mathscr{C}^{(1)}} \right) \right| \leq \mathrm{C}\big(\|f\|_{\mathscr{D}_{k,\alpha}} + \|f\|_{\mathrm{L}^p}\big)\big(\|g\|_{\mathscr{D}_{k,\alpha}} + \|g\|_{\mathrm{L}^q}\big) e^{-\delta t}.$$

An important ingredient in the course of the proof will be recurrence estimates to a given compact set. We give here a consequence of these estimates, which is of independent interest:

Theorem **2.15.** — *Let* $\delta > 0$. *Then there exist a compact set* $\mathrm{K} \subset \mathscr{C}^{(1)}$ *and a constant* $\mathrm{C} > 0$ *such that, for all* $t \geq 0$*,*

$$\nu_{\mathscr{C}^{(1)}}\{x \in \mathscr{C}^{(1)} \,:\, \forall s \in [0, t],\ \mathscr{TF}_s(x) \notin \mathrm{K}\} \leq \mathrm{C} e^{-(1-\delta)t}. \tag{2.16}$$

This result easily implies the following corollary:

Corollary **2.16.** — *For all* $\delta > 0$*, there exists* $\mathrm{C} > 0$ *such that,* $\forall \varepsilon \geq 0$*,*

$$\nu_{\mathscr{C}^{(1)}}\{x \in \mathscr{C}^{(1)} \,:\, \mathrm{sys}(x) < \varepsilon\} \leq \mathrm{C}\varepsilon^{2-\delta}. \tag{2.17}$$

Proof. — Let K be a compact subset as in Theorem 2.15. On K, the systole is larger than a constant ε_0. If $\mathrm{sys}(x) < \varepsilon < \varepsilon_0$, then $\mathscr{TF}_t x \notin \mathrm{K}$ for $|t| \leq \log(\varepsilon_0/\varepsilon)$ since $\log \circ \mathrm{sys}$ is 1-Lipschitz and $d(\mathscr{TF}_t x, x) \leq |t|$. Hence,

$$\begin{aligned} &\nu_{\mathscr{C}^{(1)}}\big\{x \in \mathscr{C}^{(1)} \,:\, \mathrm{sys}(x) < \varepsilon\big\} \\ &\leq \nu_{\mathscr{C}^{(1)}}\big\{x \in \mathscr{C}^{(1)} \,:\, \forall s \in [-\log(\varepsilon_0/\varepsilon), \log(\varepsilon_0/\varepsilon)],\ \mathscr{TF}_s(x) \notin \mathrm{K}\big\} \\ &= \nu_{\mathscr{C}^{(1)}}\big\{x \in \mathscr{C}^{(1)} \,:\, \forall s \in [0, 2\log(\varepsilon_0/\varepsilon)],\ \mathscr{TF}_s(x) \notin \mathrm{K}\big\} \leq \mathrm{C}\left(\frac{\varepsilon}{\varepsilon_0}\right)^{2(1-\delta)}. \end{aligned}$$

□

This estimate is known not to be optimal: by the Siegel–Veech formula (see e.g. [EM]), there exists a constant $\mathrm{C} > 0$ such that

$$\nu_{\mathscr{C}^{(1)}}\{x \in \mathscr{C}^{(1)} \,:\, \mathrm{sys}(x) < \varepsilon\} \sim \mathrm{C}\varepsilon^{2}. \tag{2.18}$$

Notice however that the proof of this result relies heavily on the $\mathrm{SL}(2, \mathbf{R})$ action, while our estimates involve only the Teichmüller flow. Since the loss between (2.18) and (2.17) is arbitrarily small, Theorem 2.15 is quite sharp. In particular, the combinatorial estimates we will develop in Section 5 for the proofs of Theorems 2.14 and 2.15 are quasi-optimal.

Remark. — As a consequence of Corollary 2.16 (or of Equation (2.18)), the function $\phi : x \mapsto 1/\mathrm{sys}(x)$ belongs to L^p for all $p < 2$. Moreover, Lemma 2.12 shows that $\phi \in \mathscr{D}_{1,1}$.

3. The Veech flow

In this section we introduce the Veech flow, and discuss its basic combinatorics, related to interval exchange transformations. The Veech flow is a finite cover of the Teichmüller flow, and it will be shown in the next section that our results for the Teichmüller flow follow from corresponding results for the Veech flow.

We follow the presentation of [MMY].

3.1. *Rauzy classes and interval exchange transformations*

3.1.1. *Interval exchange transformations.* — An interval exchange transformation is defined as follows. Let $\mathscr{A}$ be some fixed alphabet on $d \geq 2$ letters.

1. Take an interval $I \subset \mathbf{R}$ (all intervals will be assumed to be closed at the left and open at the right),
2. Break it into $d \geq 2$ intervals $\{I_\alpha\}_{\alpha \in \mathscr{A}}$,
3. Rearrange the intervals in a new order (via translations) inside I.

Modulo translations, we may always assume that the left endpoint of I is 0. Thus the interval exchange transformation is entirely defined by the following data:

1. The lengths of the intervals $\{I_\alpha\}_{\alpha \in \mathscr{A}}$,
2. Their orders before and after rearranging.

The first are called length data, and are given by a vector $\lambda \in \mathbf{R}_+^{\mathscr{A}}$ (here and henceforth $\mathbf{R}_+ = (0, \infty)$). The second are called combinatorial data, and are given by a pair of bijections $\pi = (\pi_t, \pi_b)$ from $\mathscr{A}$ to $\{1, \dots, d\}$ (we will sometimes call such a pair of bijections a permutation). We denote the set of all such pairs of bijections by $\mathfrak{S}(\mathscr{A})$. The bijections $\pi_\varepsilon : \mathscr{A} \to \{1, \dots, d\}$ can be viewed as rows where the elements of $\mathscr{A}$ are displayed in the order $(\pi_\varepsilon^{-1}(1), \dots, \pi_\varepsilon^{-1}(d))$. Thus we can see an element of $\mathfrak{S}(\mathscr{A})$ as a pair of rows, the top (corresponding to π_t) and the bottom (corresponding to π_b) of π. The interval exchange transformation associated to these data will be denoted $f = f(\lambda, \pi)$.

Notice that if the combinatorial data are such that the set of the first k elements in the top and bottom of π coincide for some $1 \leq k < d$ then, irrespective of the length data, the interval exchange transformation splits into two simpler transformations. We are mostly interested in combinatorial data for which this does not happen, which we will call *irreducible*. Let $\mathfrak{S}^0(\mathscr{A}) \subset \mathfrak{S}(\mathscr{A})$ be the set of irreducible combinatorial data.

3.1.2. *Rauzy classes.* — A *diagram* (or directed graph) consists of two kinds of objects, vertices and (oriented) arrows joining two vertices. Thus, an arrow has a start and an end. A *path* of length $m \geq 0$ in the diagram is a finite sequence $v_0, \ldots, v_m$ of vertices and a sequence of arrows $a_1, \ldots, a_m$ such that a_i starts at v_{i-1} and ends in v_i. A path is said to start at v_0, end in v_m, and pass through $v_1, \ldots, v_{m-1}$. If γ_1 and γ_2 are paths such that the end of γ_1 is the start of γ_2, their concatenation is also a path, denoted by $\gamma_1\gamma_2$. We can identify paths of length zero with vertices and paths of length one with arrows. Paths of length zero are called trivial. We introduce a partial order on paths: $\gamma_s \leq \gamma$ if and only if γ starts by γ_s.

Given $\pi \in \mathfrak{S}^0(\mathscr{A})$ we consider two operations. Let α and β be the last elements of the top and bottom rows. The *top* operation keeps the top row unchanged, and it changes the bottom row by moving β to the position immediately to the right of the position occupied by α. When applying this operation to π, we will say that α *wins* and β *loses*. The *bottom* operation is defined in a similar way, just interchanging the words top and bottom, and the roles of α and β. In this case we say that β wins and α loses. Notice that both operations preserve the first elements of both the top and the bottom row.

It is easy to see that each of these operations gives a bijection of $\mathfrak{S}^0(\mathscr{A})$. A Rauzy class $\mathfrak{R}$ is a minimal non-empty subset of $\mathfrak{S}^0(\mathscr{A})$ which is invariant under the top and bottom operations. Given a Rauzy class $\mathfrak{R}$, we define a diagram, called *Rauzy diagram*. Its vertices are the elements of $\mathfrak{R}$ and for each vertex $\pi \in \mathfrak{R}$ and each of the operations considered above, we define an arrow joining π to the image of π by the corresponding operation. Notice that every vertex is the start and end of two arrows, one top and one bottom. Every arrow has a start, an end, a type (top or bottom), a winner and a loser. The set of all paths is denoted by $\Pi(\mathfrak{R})$.

3.1.3. *Linear action.* — Let $\mathfrak{R} \subset \mathfrak{S}^0(\mathscr{A})$ be a Rauzy class. To each path $\gamma \in \Pi(\mathfrak{R})$, we associate a linear map $\mathrm{B}_\gamma \in \mathrm{SL}(\mathscr{A}, \mathbf{Z})$ as follows. If γ is trivial, then $\mathrm{B}_\gamma = \mathrm{id}$. If γ is an arrow with winner α and loser β then $\mathrm{B}_\gamma \cdot e_\xi = e_\xi$ for $\xi \in \mathscr{A} \setminus \{\alpha\}$ and $\mathrm{B}_\gamma \cdot e_\alpha = e_\alpha + e_\beta$, where $\{e_\xi\}_{\xi \in \mathscr{A}}$ is the canonical basis of $\mathbf{R}^{\mathscr{A}}$. We extend the definition to paths so that $\mathrm{B}_{\gamma_1\gamma_2} = \mathrm{B}_{\gamma_2} \cdot \mathrm{B}_{\gamma_1}$.

3.2. *Rauzy induction.* — Let $\mathfrak{R} \subset \mathfrak{S}^0(\mathscr{A})$ be a Rauzy class, and define $\Delta^0_{\mathfrak{R}} = \mathbf{R}^{\mathscr{A}}_+ \times \mathfrak{R}$. Given (λ, π) in $\Delta^0_{\mathfrak{R}}$, we say that we can apply Rauzy induction to (λ, π) if $\lambda_\alpha \neq \lambda_\beta$, where $\alpha, \beta \in \mathscr{A}$ are the last elements of the top and bottom rows of π, respectively. Then we define (λ', π') as follows:

1. Let $\gamma = \gamma(\lambda, \pi)$ be a top or bottom arrow on the Rauzy diagram starting at π, according to whether $\lambda_\alpha > \lambda_\beta$ or $\lambda_\beta > \lambda_\alpha$.
2. Let $\lambda'_\xi = \lambda_\xi$ if ξ is not the winner of γ, and $\lambda'_\xi = |\lambda_\alpha - \lambda_\beta|$ if ξ is the winner of γ.
3. Let π' be the end of γ.

We say that (λ', π') is obtained from (λ, π) by applying Rauzy induction, of type top or bottom depending on whether the type of γ is top or bottom. We have that $\pi' \in \mathfrak{R}$ and $\lambda' \in \mathbf{R}_+^{\mathscr{A}}$. The interval exchange transformations $f : \mathrm{I} \to \mathrm{I}$ and $f' : \mathrm{I}' \to \mathrm{I}'$ specified by the data (λ, π) and (λ', π') are related as follows. The map f' is the first return map of f to a subinterval of I, obtained by cutting from I a subinterval with the same right endpoint and of length λ_ξ, where ξ is the loser of γ. The map $\mathrm{Q} : (\lambda, \pi) \mapsto (\lambda', \pi')$ is called *Rauzy induction map*. Its domain of definition, the set of all $(\lambda, \pi) \in \Delta_{\mathfrak{R}}^0$ such that $\lambda_\alpha \neq \lambda_\beta$ (where α and β are the last letters in the top and bottom rows of π), will be denoted by $\Delta_{\mathfrak{R}}^1$.

The connected components $\Delta_\pi = \mathbf{R}_+^{\mathscr{A}} \times \{\pi\}$ of $\Delta_{\mathfrak{R}}^0$ are naturally labeled by the elements of $\mathfrak{R}$, or equivalently, by paths in $\Pi(\mathfrak{R})$ of length 0. The connected components Δ_γ of $\Delta_{\mathfrak{R}}^1$ are naturally labeled by arrows, that is, paths in $\Pi(\mathfrak{R})$ of length 1. One easily checks that each connected component of $\Delta_{\mathfrak{R}}^1$ is mapped homeomorphically to some connected component of $\Delta_{\mathfrak{R}}^0$.

Let $\Delta_{\mathfrak{R}}^n$ be the domain of Q^n, $n \geq 2$. The connected components of $\Delta_{\mathfrak{R}}^n$ are naturally labeled by paths in $\Pi(\mathfrak{R})$ of length n: if γ is obtained by following a sequence of arrows $\gamma_1, \ldots, \gamma_n$, then $\Delta_\gamma = \{x \in \Delta_{\mathfrak{R}}^0 : \mathrm{Q}^{k-1}(x) \in \Delta_{\gamma_k},\ 1 \leq k \leq n\}$. If γ starts at π and ends at π', then for any $x = (\lambda, \pi) \in \Delta_\gamma$,

$$\mathrm{Q}^n(x) = \big((\mathrm{B}_\gamma^*)^{-1}\lambda, \pi'\big) \tag{3.1}$$

(here and in the following we will use A^* to denote the transpose of a matrix A). Indeed for arrows this follows from the definitions, and the extension to paths is then immediate. Moreover, $\Delta_\gamma = (\mathrm{B}_\gamma^* \cdot \mathbf{R}_+^{\mathscr{A}}) \times \{\pi\}$.

If γ is a path in $\Pi(\mathfrak{R})$ of length n ending at $\pi' \in \mathfrak{R}$, let

$$\mathrm{Q}^\gamma = \mathrm{Q}^n : \Delta_\gamma \to \Delta_{\pi'}. \tag{3.2}$$

This map is a homeomorphism.

Let $\Delta_{\mathfrak{R}}^\infty = \bigcap_{n \geq 0} \Delta_{\mathfrak{R}}^n$. A sufficient condition for (λ, π) to belong to $\Delta_{\mathfrak{R}}^\infty$ is for the coordinates of λ to be independent over $\mathbf{Q}$.

3.2.1. *Complete and positive paths*

Definition **3.1.** — *Let $\mathfrak{R} \subset \mathfrak{S}^0(\mathscr{A})$ be a Rauzy class. A path $\gamma \in \Pi(\mathfrak{R})$ is called* complete *if every $\alpha \in \mathscr{A}$ is the winner of some arrow composing γ.*

Lemma **3.2** *([MMY], §1.2.3, Proposition). — Let $(\lambda, \pi) \in \Delta_{\mathfrak{R}}^\infty$, and let $\Delta_{\gamma(n)}$ be the connected component of (λ, π) in $\Delta_{\mathfrak{R}}^n$. Then $\gamma(n)$ is complete for all n large enough.*

In particular any Rauzy diagram contains complete paths.

We say that $\gamma \in \Pi(\mathfrak{R})$ is k-complete if it is a concatenation of k complete paths. We say that $\gamma \in \Pi(\mathfrak{R})$ is *positive* if B_γ is given, in the canonical basis of $\mathbf{R}_+^{\mathscr{A}}$, by a matrix with all entries positive.

Lemma **3.3** *([MMY], §1.2.4, Lemma).* — *If γ is a k-complete path with $k \geq 2\#\mathscr{A} - 3$, then γ is positive.*

3.3. *Zippered rectangles.* — Let $\mathfrak{R} \subset \mathfrak{S}^0(\mathscr{A})$ be a Rauzy class. Let $\pi = (\pi_t, \pi_b) \in \mathfrak{R}$. Let $\Theta_\pi \subset \mathbf{R}^{\mathscr{A}}$ be the set of all τ such that

$$\sum_{\pi_t(\xi)\leq k} \tau_\xi > 0 \quad \text{and} \quad \sum_{\pi_b(\xi)\leq k} \tau_\xi < 0 \quad \text{for all } 1 \leq k \leq d-1. \tag{3.3}$$

Notice that Θ_π is an open convex polyhedral cone. It is non-empty, since the vector τ with coordinates $\tau_\xi = \pi_b(\xi) - \pi_t(\xi)$ belongs to Θ_π.

From the data (λ, π, τ), it is possible to define a marked translation surface $S = S(\lambda, \pi, \tau)$ in some $\mathscr{Q}_{g,\kappa}$, where g and κ depend only on π (see [MMY], §3.2). It is obtained (in the *zippered rectangles* construction) by gluing rectangles of horizontal sides λ_α and vertical sides h_α, where the height vector $h \in \mathbf{R}_+^{\mathscr{A}}$ is given by $h = -\Omega(\pi)\cdot\tau$, and $\Omega(\pi)$ is the linear operator on $\mathbf{R}^{\mathscr{A}}$,

$$\langle \Omega(\pi)\cdot e_x, e_y\rangle = \begin{cases} 1, & \pi_t(x) > \pi_t(y), \pi_b(x) < \pi_b(y), \\ -1, & \pi_t(x) < \pi_t(y), \pi_b(x) > \pi_b(y), \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

In particular, the area of the translation surface S is $A(\lambda, \pi, \tau) = -\langle \lambda, \Omega\cdot\tau\rangle$.

3.3.1. *Extension of induction to the space of zippered rectangles.* — If $\gamma \in \Pi(\mathfrak{R})$ is a path starting at π, let $\Theta_\gamma \subset \mathbf{R}^{\mathscr{A}}$ be defined by the condition

$$B_\gamma^* \cdot \Theta_\gamma = \Theta_\pi. \tag{3.5}$$

If γ is a top arrow ending at π', then Θ_γ is the set of all $\tau \in \Theta_{\pi'}$ such that $\sum_{x\in\mathscr{A}} \tau_x < 0$, and if γ is a bottom arrow ending at π', then Θ_γ is the set of all $\tau \in \Theta_{\pi'}$ such that $\sum_{x\in\mathscr{A}} \tau_x > 0$. Thus, the map

$$\widehat{Q}^\gamma : \Delta_\gamma \times \Theta_\pi \to \Delta_{\pi'} \times \Theta_\gamma, \quad \widehat{Q}^\gamma(\lambda, \pi, \tau) = \big(Q(\lambda, \pi), (B_\gamma^*)^{-1}\cdot\tau\big) \tag{3.6}$$

is invertible. Now we can define an invertible map by putting together the $\widehat{Q}^\gamma$ for every arrow γ. This is a map from $\bigcup \Delta_\gamma \times \Theta_\pi$ (where the union is taken over all $\pi \in \mathfrak{R}$ and all arrows γ starting at π) to $\bigcup \Delta_{\pi'} \times \Theta_\gamma$ (where the union is taken over all $\pi' \in \mathfrak{R}$ and all arrows ending at π'). We let $\widehat{\Delta}_{\mathfrak{R}} = \bigcup_{\pi\in\mathfrak{R}} \Delta_\pi \times \Theta_\pi$. The map $\widehat{Q}$ is a skew-product over Q: $\widehat{Q}(\lambda, \pi, \tau) = (Q(\lambda, \pi), \tau')$ where τ' depends on (λ, π, τ).

The translation surfaces S and S′ corresponding respectively to (λ, π, τ) and $\widehat{Q}(\lambda, \pi, \tau)$ are obtained by appropriate cutting and pasting, so they correspond to the same element in the moduli space $\mathcal{M}_{g,\kappa}$ (the marking on the homology is however not preserved), see [MMY], §4.1. We have thus a well defined map $\mathrm{proj} : \widehat{\Delta}_{\mathfrak{R}} \to \mathcal{C}$ satisfying

$$\mathrm{proj} \circ \widehat{Q} = \mathrm{proj}, \tag{3.7}$$

where $\mathcal{C} = \mathcal{C}(\mathfrak{R})$ is a connected component of $\mathcal{M}_{g,\kappa}$ (the connectivity of the image of proj is due to the relation (3.7)). In particular g and κ only depend on $\mathfrak{R}$.

Theorem **3.4** *(Veech). — If $\mathcal{C}$ is a connected component of $\mathcal{M}_{g,\kappa}$ then there exists a Rauzy class $\mathfrak{R}$ such that $\mathcal{C} = \mathcal{C}(\mathfrak{R})$.*

Theorem **3.5** *(Veech). — The image of* $\mathrm{proj} : \widehat{\Delta}_{\mathfrak{R}} \to \mathcal{C}$ *has full Lebesgue measure in $\mathcal{C}$.*

The action of $\widehat{Q}$ on $\widehat{\Delta}_{\mathfrak{R}}$ admits a nice fundamental domain. Let

$$\phi(\lambda, \pi, \tau) = \|\lambda\| = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha. \tag{3.8}$$

Let $\mho_{\mathfrak{R}} \subset \widehat{\Delta}_{\mathfrak{R}}$ be the set of all x such that either

1. $\widehat{Q}(x)$ is defined and $\phi(\widehat{Q}(x)) < 1 \leq \phi(x)$,
2. $\widehat{Q}(x)$ is not defined and $\phi(x) \geq 1$,
3. $\widehat{Q}^{-1}(x)$ is not defined and $\phi(x) < 1$.

It is a fundamental domain for the action of $\widehat{Q}$: each orbit of $\widehat{Q}$ intersects $\mho_{\mathfrak{R}}$ in exactly one point. The fibers of the map $\mathrm{proj} : \mho_{\mathfrak{R}} \to \mathcal{C}$ are almost everywhere finite (with constant cardinality). The projection of the standard Lebesgue measure on $\mho_{\mathfrak{R}}$ is (up to scaling) the standard volume form on $\mathcal{C}$.

3.3.2. *The Veech flow.* — There is a natural flow $\mathcal{TV}_t : \widehat{\Delta}_{\mathfrak{R}} \to \widehat{\Delta}_{\mathfrak{R}}$, $\mathcal{TV}_t(\lambda, \pi, \tau) = (e^t\lambda, \pi, e^{-t}\tau)$, which lifts the Teichmüller flow in $\mathcal{M}_{g,\kappa}$. This flow commutes with $\widehat{Q}$. The *Veech flow* $\mathcal{VT}_t : \mho_{\mathfrak{R}} \to \mho_{\mathfrak{R}}$ is defined by $\mathcal{VT}_t(x) = \widehat{Q}^n(\mathcal{TV}_t(x))$ where n is the unique value such that $\widehat{Q}^n(\mathcal{TV}_t(x)) \in \mho_{\mathfrak{R}}$. It lifts the Teichmüller flow on $\mathcal{C}$:

$$\mathrm{proj} \circ \mathcal{VT}_t = \mathcal{TF}_t \circ \mathrm{proj}. \tag{3.9}$$

Since both the flow $\mathcal{TV}_t$ and the map $\widehat{Q}$ trivially preserve the standard Lebesgue measure on $\widehat{\Delta}_{\mathfrak{R}}$, the Veech flow $\mathcal{VT}_t$ preserves the standard Lebesgue measure on $\mho_{\mathfrak{R}}$.

Let $\mho_{\mathfrak{R}}^{(1)} = \mathrm{proj}^{-1}(\mathcal{C}^{(1)})$ be the set of all (λ, π, τ) such that $A(\lambda, \pi, \tau) = 1$. The Veech flow leaves invariant $\mho_{\mathfrak{R}}^{(1)}$. It follows that its restriction $\mathcal{VT}_t : \mho_{\mathfrak{R}}^{(1)} \to \mho_{\mathfrak{R}}^{(1)}$ leaves invariant a smooth volume form $d\omega$ (such that $d\omega \wedge dA = d\mathrm{Leb}$), whose projection is, up to scaling, the standard volume form on $\mathcal{C}^{(1)}$.

Remark. — Veech's proof of the fact that the standard volume form on $\mathscr{C}^{(1)}$ is finite actually first establishes finiteness of the lift measure on $\mho^{(1)}_{\mathfrak{R}}$. A different proof of finiteness follows from our recurrence estimates.

Remark. — Finiteness is a crucial step in Veech's proof of conservativity of an absolutely continuous invariant measure for the Rauzy renormalization (which is itself the center of Veech's proof of unique ergodicity for typical interval exchange transformations [Ve1]). A different proof of conservativity for the Rauzy renormalization follows immediately from our recurrence estimates (the proof of which does not depend on the zippered rectangle construction).

4. Reduction to recurrence estimates

4.1. *Measurable models*

4.1.1. *The Veech flow as suspension over the Rauzy renormalization.* — Let $\widehat{\Upsilon}_{\mathfrak{R}} \subset \mho_{\mathfrak{R}}$ be the set of all (λ, π, τ) with $\phi(\lambda, \pi, \tau) = \|\lambda\| = \sum_{\alpha\in\mathscr{A}} \lambda_\alpha = 1$. The connected components of $\widehat{\Upsilon}_{\mathfrak{R}}$ are naturally denoted $\widehat{\Upsilon}_\pi$. Let $\widehat{\Upsilon}^{(1)}_{\mathfrak{R}} = \mho^{(1)}_{\mathfrak{R}} \cap \widehat{\Upsilon}_{\mathfrak{R}}$, $\widehat{\Upsilon}^{(1)}_{\pi} = \mho^{(1)}_{\mathfrak{R}} \cap \widehat{\Upsilon}_{\pi}$. Let $\Upsilon^n_{\mathfrak{R}} \subset \Delta^n_{\mathfrak{R}}$ be the set of (λ, π) with $\|\lambda\| = 1$. We let $\widehat{m}$ denote the induced Lebesgue measure to $\widehat{\Upsilon}^{(1)}_{\mathfrak{R}}$.

Notice that $\widehat{\Upsilon}^{(1)}_{\mathfrak{R}}$ is transverse to the Veech flow on $\mho^{(1)}_{\mathfrak{R}}$. We are interested in the first return map $\widehat{R}$ to $\widehat{\Upsilon}^{(1)}_{\mathfrak{R}}$. Its domain is the intersection of $\widehat{\Upsilon}^{(1)}_{\mathfrak{R}}$ with the domain of definition of $\widehat{Q}$, and we have

$$\widehat{R}(\lambda, \pi, \tau) = (e^r \lambda', \pi', e^{-r}\tau'), \tag{4.1}$$

where $(\lambda', \pi', \tau') = \widehat{Q}(\lambda, \pi, \tau)$ and $r = r(\lambda, \pi) = -\log\|\lambda'\| = -\log\phi(\lambda', \pi', \tau')$ is the first return time. The map $\widehat{R}$ is a skew-product: it can be written as $\widehat{R}(\lambda, \pi, \tau) = (R(\lambda, \pi), e^{-r}\tau')$. The map $R : \Upsilon^1_{\mathfrak{R}} \to \Upsilon^0_{\mathfrak{R}}$ is called the *Rauzy renormalization map*. The measure $\widehat{m}$ is invariant under $\widehat{R}$.

The Veech flow can thus be seen as a special suspension over the map $\widehat{R}$, which is itself an "invertible extension" of a non-invertible map R. This "suspension model" loses control of some orbits (the ones that do not return to $\widehat{\Upsilon}^{(1)}_{\mathfrak{R}}$), but those have zero Lebesgue measure, and will not affect further considerations.

4.1.2. *Precompact sections.* — In the above suspension model for the Veech flow, the underlying discrete transformation $\widehat{R}$ is only very weakly hyperbolic. This is related to the fact that the section $\widehat{\Upsilon}^{(1)}_{\mathfrak{R}}$ is too large (for instance, it has infinite area). Zorich [Z] has introduced an alternative section with finite area, but such a section is still somewhat too large, so that there is not a good control on distortion. In the following we will introduce a class of suitably small (precompact in $\widehat{\Upsilon}^{(1)}_{\mathfrak{R}}$) sections with good distortion estimates.

The section we will choose will be the intersection of $\widehat{\Upsilon}^{(1)}_{\mathfrak{R}}$ with (finite unions of) sets of the form $\Delta_\gamma \times \Theta_{\gamma'}$. Precompactness in the λ direction is equivalent to having $\mathrm{B}^*_\gamma \cdot (\overline{\mathbf{R}}^{\mathscr{A}}_+ \setminus \{0\}) \subset \mathbf{R}^{\mathscr{A}}_+$, which is equivalent to γ being a positive path. To take care of both the λ and the τ direction, we introduce the following notion.

Definition **4.1.** — *A path γ, starting in π_s and ending in π_e, is said to be strongly positive if it is positive and $(\mathrm{B}^*_\gamma)^{-1} \cdot (\overline{\Theta}_{\pi_s} \setminus \{0\}) \subset \Theta_{\pi_e}$.*

Lemma **4.2.** — *Let γ be a k-complete path with $k \geq 3\#\mathscr{A} - 4$. Then γ is strongly positive.*

Proof. — Let $d = \#\mathscr{A}$. Fix $\tau \in \overline{\Theta}_{\pi_s} \setminus \{0\}$. Write γ as a concatenation of arrows $\gamma = \gamma_1 \ldots \gamma_n$, and let π^{i-1} and π^i denote the start and the end of γ_i. Let $\tau^0 = \tau$, $\tau^i = (\mathrm{B}^*_{\gamma_i})^{-1} \cdot \tau^{i-1}$. We must show that $\tau^n \in \Theta_{\pi^n}$.

Let $h^i = -\Omega(\pi^i) \cdot \tau^i$. Notice that $\tau \in \overline{\Theta}_{\pi^0} \setminus \{0\}$ implies that $h^0 \in \overline{\mathbf{R}}^{\mathscr{A}}_+ \setminus \{0\}$. Indeed, since $\tau \in \overline{\Theta}_{\pi^0}$, for every $\xi \in \mathscr{A}$, we have $\sum_{\pi^0_t(\alpha)<\pi^0_t(\xi)} \tau_\alpha \geq 0$, $\sum_{\pi^0_b(\alpha)<\pi^0_b(\xi)} \tau_\alpha \leq 0$. Moreover, since $\tau \neq 0$, there exists $1 \leq k^t, k^b \leq d$ minimal such that $\tau_{(\pi^0_t)^{-1}(k^t)} \neq 0$ and $\tau_{(\pi^0_b)^{-1}(k^b)} \neq 0$. Since π^0 is irreducible, $\min\{k^t, k^b\} < d$. Noticing that

$$h^0_\xi = \sum_{\pi^0_t(\alpha)<\pi^0_t(\xi)} \tau_\alpha - \sum_{\pi^0_b(\alpha)<\pi^0_b(\xi)} \tau_\alpha, \tag{4.2}$$

we see that $h^0_\xi \geq 0$ for all ξ, and the inequality is strict if $\pi^0_t(\xi) = k^t + 1$ (if $k^t < d$) or if $\pi^0_b(\xi) = k^b + 1$ (if $k^b < d$).

Notice that $h^i = \mathrm{B}_{\gamma_i} \cdot h^{i-1}$, so if $\gamma_1 \ldots \gamma_i$ is a positive path then $h^i \in \mathbf{R}^{\mathscr{A}}_+$.

Let $0 \leq k^t_i, k^b_i \leq d-1$ be maximal such that

$$\sum_{\pi^i_t(\xi) \leq k} \tau^i_\xi > 0 \quad \text{for all } 1 \leq k \leq k^t_i, \tag{4.3}$$

$$\sum_{\pi^i_b(\xi) \leq k} \tau^i_\xi < 0 \quad \text{for all } 1 \leq k \leq k^b_i, \tag{4.4}$$

where π^i_t and π^i_b are the top and the bottom of π^i. We claim that

1. If $h^{i-1} \in \mathbf{R}^{\mathscr{A}}_+$ then $k^t_i \geq k^t_{i-1}$ and $k^b_i \geq k^b_{i-1}$,
2. If $h^{i-1} \in \mathbf{R}^{\mathscr{A}}_+$ and the winner of γ_i is one of the first $k^t_{i-1} + 1$ letters in the top of π^{i-1} then $k^t_i \geq \min\{d-1, k^t_{i-1} + 1\}$,
3. If $h^{i-1} \in \mathbf{R}^{\mathscr{A}}_+$ and the winner of γ_i is one of the first $k^b_{i-1} + 1$ letters in the bottom of π^{i-1} then $k^b_i \geq \min\{d-1, k^b_{i-1} + 1\}$.

Let us see that (1), (2) and (3) imply the result, which is equivalent to the statement that $k^t_n = d-1$ and $k^b_n = d-1$. We will show that $k^t_n = d-1$, the other estimate being analogous. Let us write $\gamma = \gamma_{(1)}\dots\gamma_{(3d-4)}$ where $\gamma_{(j)}$ is complete. Write $\gamma_{(j)} = \gamma_{s_j}\dots\gamma_{e_j}$. By Lemma 3.3, $h^k \in \mathbf{R}^{\mathscr{A}}_+$ for $k \geq e_{2d-3}$. From the definition of a complete path, for each $j > 2d-3$, there exists $e_{j-1} < i \leq e_j$ such that the winner of γ_i is one of the first $k^t_{e_{j-1}}+1$ letters in the top of π^{i-1}. It follows that $k^t_{e_j} \geq \min\{d-1, k^t_{e_{j-1}}+1\}$, and so $k^t_n = k^t_{e_{3d-4}} \geq \min\{d-1, k^t_{e_{2d-3}}+d-1\} = d-1$.

We now check (1), (2) and (3). Assume that $h^{i-1} \in \mathbf{R}^{\mathscr{A}}_+$, and that γ_i is a top, the other case being analogous. In this case $\pi^i_t = \pi^i_{t-1}$ and $\tau^i_\alpha = \tau^{i-1}_\alpha$ for $\pi^i_t(\alpha) < d$, hence $k^t_i \geq k^t_{i-1}$. This shows that the first claim of (1) holds. Moreover, (2) also holds since its hypothesis can only be satisfied if $k^t_{i-1} = d-1$.

If the winner of γ_i is not one of the $k^b_{i-1}+1$ first letters in the bottom of π^{i-1}, then for every $\alpha \in \mathscr{A}$ such that $1 \leq \pi^{i-1}_b(\alpha) \leq k^b_{i-1}$, we have $\pi^{i-1}_b(\alpha) = \pi^i_b(\alpha)$, $\tau^{i-1}_\alpha = \tau^i_\alpha$, so $k^b_i \geq k^b_{i-1}$.

If the winner β of γ_i appears in the k-th position in the bottom of π^{i-1} with $1 \leq k \leq k^b_{i-1}+1$, then

$$\sum_{\pi^i_b(\xi)\leq j} \tau^i_\xi = \sum_{\pi^{i-1}_b(\xi)\leq j} \tau^{i-1}_\xi < 0 \quad \text{for all } 1 \leq j \leq k-1, \tag{4.5}$$

$$\sum_{\pi^i_b(\xi)\leq j} \tau^i_\xi = \sum_{\pi^{i-1}_b(\xi)\leq j-1} \tau^{i-1}_\xi < 0 \quad \text{for all } k+1 \leq j \leq k^b_{i-1}+1, \tag{4.6}$$

$$\sum_{\pi^i_b(\xi)\leq k} \tau^i_\xi = \sum_{\pi^{i-1}_b(\xi)\leq d-1} \tau^{i-1}_\xi - h^{i-1}_\beta \leq -h^{i-1}_\beta < 0, \tag{4.7}$$

which implies that $k^b_i \geq \min\{d-1, k^b_{i-1}+1\}$.

This shows that both (3) and the second claim of (1) must hold. □

4.1.3. *A better model.* — We will now choose a specific precompact section, adapted for the problem of exponential mixing (Theorem 2.14). Our particular choice aims to simplify the combinatorial description of the first return map. We will later consider a different choice for the recurrence problem (Theorem 2.15).

Let $\gamma_* \in \Pi(\mathfrak{R})$ be a strongly positive path starting and ending in the same $\pi \in \mathfrak{R}$. Assume further that if $\gamma_* = \gamma_s\gamma = \gamma\gamma_e$ then either $\gamma = \gamma_*$ or γ is trivial.[1] We will say that γ_* is *neat*.

Let $\widehat{\Xi} = \widehat{\Upsilon}^{(1)}_{\mathfrak{R}} \cap (\Delta_{\gamma_*} \times \Theta_{\gamma_*})$, and let $\Xi = \Upsilon^0_{\mathfrak{R}} \cap \Delta_{\gamma_*}$. We are interested in the first return map $T_{\widehat{\Xi}}$ to $\widehat{\Xi}$ under the Veech flow. The connected components of its domain are given by $\widehat{\Upsilon}^{(1)}_{\mathfrak{R}} \cap (\Delta_{\gamma\gamma^*} \times \Theta_{\gamma_*})$, where γ is either γ_*, or a minimal

[1] Notice that if γ_* ends by a bottom arrow and starts by a sufficiently long (at least half the length of γ_*) sequence of top arrows then this last condition is automatically satisfied.

path of the form $\gamma_*\gamma_0\gamma_*$ not beginning by $\gamma_*\gamma_*$. The restriction of $T_{\widehat{\Xi}}$ to such a component is given by

$$(\mathbf{4.8}) \qquad T_{\widehat{\Xi}}(\lambda,\pi,\tau) = \left(\frac{(B_\gamma^*)^{-1}\cdot\lambda}{\|(B_\gamma^*)^{-1}\cdot\lambda\|}, \pi, \left\|(B_\gamma^*)^{-1}\cdot\lambda\right\|(B_\gamma^*)^{-1}\cdot\tau\right).$$

The return time function is just

$$(\mathbf{4.9}) \qquad r_{\widehat{\Xi}}(\lambda,\pi,\tau) = r_{\Xi}(\lambda,\pi) = -\log\left\|(B_\gamma^*)^{-1}\cdot\lambda\right\|.$$

The map $T_{\widehat{\Xi}}(\lambda,\pi,\tau) = (\lambda',\pi,\tau')$ is a skew-product over a non-invertible transformation $T_\Xi(\lambda,\pi) = (\lambda',\pi)$.

The Veech flow can be seen as a suspension over $T_{\widehat{\Xi}}$, with roof function $r_{\widehat{\Xi}}$. In this suspension model, many more orbits escape control (the ones that do not come back to $\widehat{\Xi}$). Still, due to ergodicity of the Veech flow, almost every orbit is captured by the suspension model.

4.2. *Hyperbolic properties.* — The transformation $T_{\widehat{\Xi}}$ turns out to have much better hyperbolic properties than $\widehat{R}$.

Lemma **4.3.** — $T_{\widehat{\Xi}}$ *is a hyperbolic skew-product over* T_Ξ.

Implicit in the above statement is the choice of probability measure ν and Finsler metric $\|\cdot\|_{\widehat{\Xi}}$ which are part of the Definition 2.5 of a hyperbolic skew-product. The choice of ν is clear (the normalized restriction of $\widehat{m}$ to $\widehat{\Xi}$) but there is some freedom in the choice of the Finsler metric. In order to enforce the hyperbolicity properties we want from $T_{\widehat{\Xi}}$, we will introduce a particular complete Finsler metric on $\widehat{\Upsilon}_\pi^{(1)}$, and then take $\|\cdot\|_{\widehat{\Xi}}$ as its restriction. By strong positivity of γ_*, $\widehat{\Xi}$ is a precompact open subset of $\widehat{\Upsilon}_{\mathfrak{R}}^{(1)}$, so $\widehat{\Xi}$ will have bounded diameter with respect to such metric.

4.2.1. *Hilbert metric.* — The Hilbert pseudo-metric on $\mathbf{R}_+^2$ is given by $\mathrm{dist}_{\mathbf{R}_+^2}(x,y) = \log\max_{1\le i,j\le 2}\frac{x_iy_j}{x_jy_i}$. One easily checks that if $B\in GL(2,\mathbf{R})$ is a linear map such that $B\cdot\mathbf{R}_+^2\subset\mathbf{R}_+^2$ then B contracts weakly the Hilbert pseudo-metric: $\mathrm{dist}_{\mathbf{R}_+^2}(B\cdot x, B\cdot y)\le\mathrm{dist}_{\mathbf{R}_+^2}(x,y)$. In particular, the Hilbert pseudo-metric is invariant under linear isomorphisms of $\mathbf{R}_+^2$.

More generally, if $C\subset\mathbf{R}^{\mathscr{A}}\setminus\{0\}$ is an open convex cone whose closure does not contain any one-dimensional subspace of $\mathbf{R}^{\mathscr{A}}$, one defines a Hilbert pseudo-metric on C as follows. If x and y are collinear then $\mathrm{dist}_C(x,y)=0$. Otherwise, C intersects the subspace generated by x and y in a cone isomorphic to $\mathbf{R}_+^2$. We let $\mathrm{dist}_C(x,y) = \mathrm{dist}_{\mathbf{R}_+^2}(\psi(x),\psi(y))$ where ψ is any such isomorphism. If $C=\mathbf{R}_+^{\mathscr{A}}$ then we have $\mathrm{dist}_C(x,y) = \max_{\alpha,\beta\in\mathscr{A}}\log\frac{x_\alpha y_\beta}{x_\beta y_\alpha}$.

If $C' \subset C$ is a smaller cone then the inclusion $C' \to C$ is a weak contraction of the respective Hilbert pseudo-metrics: $\mathbf{dist}_C(x,y) \le \mathbf{dist}_{C'}(x,y)$. Moreover, if the diameter of C' with respect to $\mathbf{dist}_C$ is bounded by some M then the contraction is definite: $\mathbf{dist}_C(x,y) \le \delta\,\mathbf{dist}_{C'}(x,y)$ where $\delta = \delta(M) < 1$.

We notice that the Hilbert pseudo-metric on a cone C induces the Hilbert metric on the space of rays $\{tx : t \in \mathbf{R}_+\}$ contained in C (which is a projective manifold). It is a complete Finsler metric.

4.2.2. *Uniform expansion and contraction.* — Recall that $\widehat{\Upsilon}^{(1)}_\pi$ is contained in $\Delta_\pi \times \Theta_\pi$, which is a product of two cones. In $\Delta_\pi \times \Theta_\pi$, we have the product Hilbert pseudo-metric $\mathbf{dist}((\lambda,\pi,\tau),(\lambda',\pi,\tau')) = \mathbf{dist}_{\Delta_\pi}((\lambda,\pi),(\lambda',\pi)) + \mathbf{dist}_{\Theta_\pi}(\tau,\tau')$. Each product of rays $\{(a\lambda,\pi,b\tau) : a,b \in \mathbf{R}_+\} \subset \Delta_\pi \times \Theta_\pi$ intersects transversely $\Upsilon^{(1)}_{\mathfrak{R}}$ in a unique point. It follows that the product Hilbert pseudo metric induces a metric $\mathbf{dist}$ on $\widehat{\Upsilon}^{(1)}_\pi$. It is a complete Finsler metric.

Proof of Lemma 4.3. — Let us first show that T_Ξ is a uniformly expanding Markov map (the underlying Finsler metric being the restriction of $\mathbf{dist}_{\Delta_\pi}$, and the underlying measure Leb being the induced Lebesgue measure). It is clear that Ξ is a John domain.

Condition (1) of Definition 2.2 is easily verified, except for the definite contraction of inverse branches. To check this property, we notice that an inverse branch can be written as $h(\lambda,\pi) = \left(\frac{B^*_\gamma\cdot\lambda}{\|B^*_\gamma\cdot\lambda\|},\pi\right)$. Since γ_* is neat, we can write $B^*_\gamma = B^*_{\gamma_*}B^*_{\gamma_0}$ for some γ_0. Thus h can be written as (the restriction of) the composition of two maps $\Delta_\pi \to \Delta_\pi$, $h = h_* \circ h_0$, where h_0 is weakly contracting and h_* is definitely contracting by precompactness of Ξ in Δ_π (which is a consequence of positivity of γ_*).

To check condition (2) of Definition 2.2, let $h(\lambda,\pi) = \left(\frac{B^*_\gamma\cdot\lambda}{\|B^*_\gamma\cdot\lambda\|},\pi\right)$ be an inverse branch of T_Ξ. The Jacobian of h at (λ,π) is $J\circ h(\lambda,\pi) = \left(\frac{1}{\|B^*_\gamma\cdot\lambda\|}\right)^d$, where $d = \#\mathscr{A}$. It follows that

$$(\mathbf{4.10}) \qquad \frac{J\circ h(\lambda,\pi)}{J\circ h(\lambda',\pi)} \le \sup_{\alpha\in\mathscr{A}} \left(\frac{\lambda_\alpha}{\lambda'_\alpha}\right)^d \le e^{d\,\mathrm{dist}_{\Delta_\pi}((\lambda,\pi),(\lambda',\pi))},$$

so that $\log J\circ h$ is d-Lipschitz with respect to $\mathbf{dist}_{\Delta_\pi}$.

To see that $T_{\widehat{\Xi}}$ is a hyperbolic skew-product over T_Ξ, one checks the conditions (1–4) of Definition 2.5. Condition (1) is obvious, and condition (4) follows from precompactness of $\widehat{\Xi}$ in $\Delta_\pi \times \Theta_\pi$ as before. Since $T_{\widehat{\Xi}}$ is a first return map, the restriction of $\widehat{m}$ to $\widehat{\Xi}$ is $T_{\widehat{\Xi}}$-invariant. Its normalization is the probability measure ν of condition (2). In order to check condition (3), it is convenient to trivialize $\widehat{\Xi}$ to a product (via the natural diffeomorphism $\widehat{\Xi} \to \Xi \times \mathbf{P}\Theta_{\gamma_*}$). Since ν has a smooth density with respect to the product of the Lebesgue measure on the factors, condition (3) follows by the Leibnitz rule. □

4.3. *Basic properties of the roof function.* — Let $\mathrm{H}(\pi) = \Omega(\pi) \cdot \mathbf{R}^{\mathcal{A}}$. Recall (from §3.3) that if $\tau \in \Theta_\pi$ then $-\Omega(\pi) \cdot \tau \in \mathbf{R}^{\mathcal{A}}_+$, and that Θ_π is non-empty, so $\mathrm{H}(\pi) \cap \mathbf{R}^{\mathcal{A}}_+ \neq \emptyset$.

Lemma **4.4.** — *Let* $\Gamma \subset \Pi(\mathfrak{R})$ *be the set of all* γ *such that* γ *is either* γ_*, *or a minimal path of the form* $\gamma_*\gamma_0\gamma_*$ *not beginning by* $\gamma_*\gamma_*$. *Let* $\mathrm{K} \subset \mathbf{P}\mathrm{H}(\pi)$ *be a closed set such that* $\mathrm{B}_\gamma \cdot \mathrm{K} = \mathrm{K}$ *for every* $\gamma \in \Gamma$. *Then either* $\mathrm{K} = \emptyset$ *or* $\mathrm{K} = \mathbf{P}\mathrm{H}(\pi)$.

Proof. — Let $\Pi(\pi) \subset \Pi(\mathfrak{R})$ be the set of all paths that start and end in π. Then any element of $\gamma_*\Pi(\pi)\gamma_*$ is a concatenation of elements of Γ. It follows that if K is invariant under all B_γ, $\gamma \in \Gamma$, then K is invariant under all B_γ, $\gamma \in \Pi(\pi)$: indeed $\mathrm{B}_\gamma \cdot \mathrm{K} = \mathrm{B}_{\gamma_*}^{-1} \cdot \mathrm{B}_{\gamma_*\gamma\gamma_*} \cdot \mathrm{B}_{\gamma_*}^{-1} \cdot \mathrm{K} = \mathrm{K}$, since γ_* and $\gamma_*\gamma\gamma_*$ are concatenation of elements of Γ. According to Corollary 3.6 of [AV], this implies that K is either empty or equal to $\mathbf{P}\mathrm{H}(\pi)$. □

Lemma **4.5.** — *The roof function* r_Ξ *is good (in the sense of Definition 2.3).*

Proof. — We check conditions (1–3) of Definition 2.3. Let $\Gamma \subset \Pi(\mathfrak{R})$ be the set defined in the previous lemma. Notice that Γ consists of positive paths.

The set $\mathscr{H}$ of inverse branches h of T_Ξ is in bijection with Γ, since each inverse branch is of the form $h(\lambda, \pi) = \left(\frac{\mathrm{B}^*_{\gamma_h} \cdot \lambda}{\|\mathrm{B}^*_{\gamma_h} \cdot \lambda\|}, \pi\right)$ for some $\gamma_h \in \Gamma$.

Let $h \in \mathscr{H}$. Then $r_\Xi(h(\lambda, \pi)) = \log \|\mathrm{B}^*_{\gamma_h} \cdot \lambda\|$. Since γ_h is positive, $r_\Xi \geq \log 2$, which implies condition (1). Notice that $r_\Xi \circ h = \frac{1}{d} \log \mathrm{J} \circ h$, where J is as in the condition (2) of Definition 2.2, so (2) follows (by the previous discussion, it even follows that $r_\Xi \circ h$ is 1-Lipschitz with respect to $\mathrm{dist}_{\Delta_\pi}$).

Let us check condition (3). We identify the tangent space to Ξ at a point $(\lambda, \pi) \in \Xi$ with $\mathrm{V} = \{\lambda \in \mathbf{R}^{\mathcal{A}} : \sum \lambda_\alpha = 0\}$. Assume that we can write $r_\Xi = \psi + \phi \circ \mathrm{T}_\Xi - \phi$ with $\phi \in \mathrm{C}^1$, ψ locally constant. Write $r^{(n)}(x) = \sum_{j=0}^{n-1} r_\Xi(\mathrm{T}^j_\Xi(\lambda, \pi))$. Then $\mathrm{D}(r^{(n)} \circ h^n) = \mathrm{D}\phi - \mathrm{D}(\phi \circ h^n)$, which can be rewritten

$$\frac{\|(\mathrm{B}^*_{\gamma_h})^n \cdot v\|}{\|(\mathrm{B}^*_{\gamma_h})^n \cdot \lambda\|} = \mathrm{D}\phi(\lambda, \pi) \cdot v - \mathrm{D}(\phi \circ h^n)(\lambda, \pi) \cdot v, \quad (\lambda, \pi) \in \Xi,\ v \in \mathrm{V},$$

or

$$\frac{\langle v, \mathrm{B}^n_{\gamma_h} \cdot (1, \dots, 1)\rangle}{\langle \lambda, \mathrm{B}^n_{\gamma_h} \cdot (1, \dots, 1)\rangle} = \mathrm{D}\phi(\lambda, \pi) \cdot v - \mathrm{D}(\phi \circ h^n)(\lambda, \pi) \cdot v, \quad (\lambda,\pi) \in \Xi,\ v \in \mathrm{V}.$$

Since $\mathrm{D}h^n \to 0$, we conclude that $[\mathrm{B}^n_{\gamma_h} \cdot (1, \dots, 1)] \in \mathbf{P}\mathbf{R}^{\mathcal{A}}$ converges to a limit $[w] \in \mathbf{P}\mathbf{R}^{\mathcal{A}}$ independent of h. This obviously implies that $[w]$ is invariant by all B_{γ_h},

$h \in \mathscr{H}$. Since w is a limit of positive vectors (vectors with positive coordinates), by the Perron–Frobenius Theorem, w is collinear with the (unique) positive eigenvector of B_{γ_h}, which also corresponds to the largest eigenvalue. Recalling that $H(\pi)$ is invariant under B_{γ_h}, and intersects $\mathbf{R}_+^{\mathscr{A}}$, it follows that $w \in H(\pi)$. According to the previous lemma, $K = \{[w]\} \subset \mathbf{P}H(\pi)$ should be either empty or equal to the whole $\mathbf{P}H(\pi)$, so $H(\pi)$ should be one-dimensional. This gives a contradiction since $H(\pi)$ is even dimensional (since $H(\pi)$ is the image of the antisymmetric operator $\Omega(\pi)$). □

4.4. *A recurrence estimate and exponential mixing.* — We will show later (in Section 6) the following recurrence estimate.

Theorem **4.6.** — *The roof function r_Ξ has exponential tails.*

We will now show how to conclude exponential mixing for the Teichmüller flow, Theorem 2.14, assuming the above recurrence estimate and the abstract result on exponential mixing for hyperbolic skew-product flows.

The map $T_{\widehat{\Xi}}$ and the roof function r_Ξ define together a flow $\widehat{T}_t$ on the space $\widehat{\Delta}_r = \{(x, y, s) : (x, y) \in \widehat{\Xi}, T_{\widehat{\Xi}}(x, y) \text{ is defined and } 0 \le s < r_\Xi(x)\}$. Since $T_{\widehat{\Xi}}$ is a hyperbolic skew-product (Lemma 4.3), and r_Ξ is a good roof function (Lemma 4.5) with exponential tails (Theorem 4.6), $\widehat{T}_t$ is an excellent hyperbolic semi-flow. By Theorem 2.7, we get exponential decay of correlations

$$(\mathbf{4.11}) \qquad C_t(\tilde{f}, \tilde{g}) = \int \tilde{f} \cdot \tilde{g} \circ \widehat{T}_t \, d\nu - \int \tilde{f} \, d\nu \int \tilde{g} \, d\nu,$$

for C^1 functions $\tilde{f}$, $\tilde{g}$, that is

$$(\mathbf{4.12}) \qquad |C_t(\tilde{f}, \tilde{g})| \le Ce^{-3\delta t} \|\tilde{f}\|_{C^1} \|\tilde{g}\|_{C^1},$$

for some $C > 0$, $\delta > 0$. This estimate holds for C^1 functions on $\widehat{\Delta}_r$, while Theorem 2.14 deals with Hölder functions on $\mathscr{C}^{(1)}$. Hence, one needs an additional lifting and smoothing argument, provided by the following technical lemma.

Let $P : \widehat{\Upsilon}_{\mathfrak{R}}^{(1)} \times \mathbf{R} \to \mathscr{C}^{(1)}$ be given by $P(z, s) = \mathscr{T}\mathscr{F}_s(\mathrm{proj}(z))$, where $\mathrm{proj} : \widehat{\Delta}_{\mathfrak{R}} \to \mathscr{C}$ is the natural projection.

Lemma **4.7.** — *For every $k \in \mathbf{N}$, $0 < \alpha \le 1$, $p > p' \ge 1$, $\delta > 0$, there exist $C > 0$, $\varepsilon_0 > 0$ with the following property. Let $f : \mathscr{C}^{(1)} \to \mathbf{R}$ be a function belonging to $\mathscr{D}_{k,\alpha} \cap L^p(\nu_{\mathscr{C}^{(1)}})$. For every $t > 0$, there exists a C^1 function $f^{(t)} : \widehat{\Delta}_r \to \mathbf{R}$, such that $\|f \circ P - f^{(t)}\|_{L^{p'}(\nu)} \le C(\|f\|_{\mathscr{D}_{k,\alpha}} + \|f\|_{L^p(\nu_{\mathscr{C}^{(1)}})})e^{-\varepsilon_0 t}$ and $\|f^{(t)}\|_{C^1(\widehat{\Delta}_r)} \le C\|f\|_{\mathscr{D}_{k,\alpha}} e^{\delta t}$.*

Proof. — We identify $\widehat{\Upsilon}_{\mathfrak{R}}^{(1)} \cap \Delta_\pi \times \Theta_\pi$ with a subset U of $\mathbf{R}^{2d-2}$ via a map $(\lambda, \pi, \tau) \mapsto (x, y)$, where $x, y \in \mathbf{R}^{d-1}$ are defined by $x_i = \frac{\lambda_{\pi_t^{-1}(i+1)}}{\lambda_{\pi_t^{-1}(1)}}$, $y_i = \frac{\tau_{\pi_t^{-1}(i+1)}}{\tau_{\pi_t^{-1}(1)}}$,

$1 \leq i \leq d-1$ (here π_t is the top of π). In this way, $\widehat{\Xi}$ becomes a precompact subset of $\mathbf{R}^{2d-2}$. Using this identification, we will write $r_\Xi(x)$ for $r_\Xi(\lambda, \pi)$.

This also provides an identification of $\widehat{\Delta}_r$ with a subset of $\mathrm{U}\times[0,\infty) \subset \mathbf{R}^{2d-1}$ via the map $(\lambda, \pi, \tau, s) \mapsto (x, y, s)$. We will use $\|\cdot\|$ to denote the usual norm in $\mathbf{R}^{2d-1}$, and $\mathbf{dist}$ for the corresponding distance.

Let $\|\cdot\|_{\mathrm{F}}$ be the Finsler metric on $\mathrm{U}\times\mathbf{R}$ obtained by pullback via P of the Finsler metric on $\mathscr{C}^{(1)}$ defined in §2.2.2. At a point $(x, y, s) \in \widehat{\Xi}\times\mathbf{R}$, we have the estimate $\mathrm{C}^{-1}e^{-2|s|}\|w\| \leq \|w\|_{\mathrm{F}} \leq \mathrm{C}e^{2|s|}\|w\|$ where w is a vector tangent to (x, y, s). This follows from precompactness of $\widehat{\Xi}$ when $s=0$, and the general case follows from this one by applying the Teichmüller flow, see (2.13). We let $\mathbf{dist}_{\mathrm{F}}$ be the metric in $\widehat{\Delta}_r$ corresponding to $\|\cdot\|_{\mathrm{F}}$. We recall that $\widehat{\Delta}_r$ is disconnected, so the $\mathbf{dist}_{\mathrm{F}}$ distance between two points of $\widehat{\Delta}_r$ is sometimes infinite.

There is another Finsler metric $\|\cdot\|_{\widehat{\Delta}_r}$ over $\widehat{\Delta}_r$, which is the product of $\|\cdot\|_{\widehat{\Xi}}$ (introduced in Section 4.2) in the (x, y) direction and the usual metric in the s direction. We recall that it is with respect to this metric that the $\mathrm{C}^1(\widehat{\Delta}_r)$ norm is defined. One easily checks that $\mathrm{C}^{-1}\|w\| \leq \|w\|_{\widehat{\Delta}_r} \leq \mathrm{C}\|w\|$.

We may assume that $\|f\|_{\mathscr{D}_{k,\alpha}} \leq 1$. This implies that for $z_0 = (x_0, y_0, s_0) \in \widehat{\Delta}_r$, $|f\circ \mathrm{P}(z_0)| \leq \mathrm{C}e^{ks_0}$ and if $\mathbf{dist}_{\mathrm{F}}(z, z_0) = r < 1$ then $|f\circ\mathrm{P}(z) - f\circ\mathrm{P}(z_0)| \leq \mathrm{C}e^{ks_0}r^\alpha$.

Let $\varepsilon > 0$. Let $\phi^{(t)} : \mathbf{R}^{2d-1} \to [0,\infty)$ be a C^∞ function supported in $\{z \in \mathbf{R}^{2d-1} : \|z\| < e^{-\varepsilon t}/10\}$, such that $\int_{\mathbf{R}^{2d-1}} \phi^{(t)}(z)\,\mathrm{d}z = 1$ and such that $\|\phi^{(t)}\|_{\mathrm{C}^1(\mathbf{R}^{2d-1})} \leq \mathrm{C}e^{2d\varepsilon t}$. Let $\psi^{(t)} : \mathbf{R}^{2d-1} \to \mathbf{R}$ be given by $\psi^{(t)}(x, y, s) = f\circ\mathrm{P}(x, y, s)$ if $(x, y, s) \in \widehat{\Delta}_r$ and $0 \leq s \leq \varepsilon t$, and $\psi^{(t)}(x, y, s) = 0$ otherwise. We will show that if ε is small enough then one can take $f^{(t)} = \phi^{(t)} * \psi^{(t)}|\widehat{\Delta}_r$, where $*$ denotes convolution.

Let us first check the assertion $\|f^{(t)}\|_{\mathrm{C}^1(\widehat{\Delta}_r)} \leq \mathrm{C}e^{\delta t}$. It is immediate to check that, by choosing $\varepsilon > 0$ small, we have indeed $\|f^{(t)}\|_{\mathrm{C}^1(\widehat{\Delta}_r)} \leq \mathrm{C}\|\psi^{(t)} * \phi^{(t)}\|_{\mathrm{C}^1(\mathbf{R}^{2d-1})} \leq \mathrm{C}\|\psi^{(t)}\|_{\mathrm{L}^1(\mathrm{d}z)}\|\phi^{(t)}\|_{\mathrm{C}^1(\mathbf{R}^{2d-1})} \leq \mathrm{C}e^{\delta t}$.

We will now check the other assertion $\|f\circ\mathrm{P} - f^{(t)}\|_{\mathrm{L}^{p'}(\nu)} \leq \mathrm{C}e^{-\varepsilon_0 t}$, assuming $\|f\|_{\mathscr{D}_{k,\alpha}} + \|f\|_{\mathrm{L}^p(\nu_{\mathscr{C}^{(1)}})} \leq 1$.

Choose $\mathrm{C}_0 > \max\{4, 2k/\alpha\}$. Let $\mathrm{Y} \subset \widehat{\Delta}_r$ be the union of the connected components of $\widehat{\Delta}_r$ which intersect $\{(x, y, s) \in \mathbf{R}^{2d-1} : s > \mathrm{C}_0^{-1}\varepsilon t\}$. Let $\mathrm{X} \subset \widehat{\Delta}_r \setminus \mathrm{Y}$ be the set of points with $\mathbf{dist}((x, y, s), \partial\widehat{\Delta}_r) \leq 4e^{-\varepsilon t}$. Thus $\widehat{\Delta}_r \setminus (\mathrm{X}\cup\mathrm{Y})$ consists of points well inside the connected components of $\widehat{\Delta}_r$ with not so long (maximal) return time.

Lemma **4.8.** — *We have* $\nu(\mathrm{X}\cup\mathrm{Y}) \leq \mathrm{C}e^{-\varepsilon t/\mathrm{C}}$.

Proof. — The function r_Ξ is a good roof function. Therefore, by condition (2) of Definition 2.3, we have $r_\Xi(x) > (\mathrm{C}\mathrm{C}_0)^{-1}\varepsilon t$ for every $(x, y, s) \in \mathrm{Y}$. By Theorem 4.6, $\nu(\mathrm{Y}) \leq \mathrm{C}e^{-\varepsilon t/\mathrm{C}}$.

The boundary of each connected component of $\widehat{\Delta}_r$ can be split in three parts: a floor (containing points (x, y, s) with $s = 0$), a roof (containing points (x, y, s) such that $r_\Xi(x) = s$) and a remaining lateral part.

Points $(x, y, s) \in X$ are at distance at most $4e^{-\varepsilon t}$ of either the floor, the roof, or the lateral part of the boundary of their connected component in $\widehat{\Delta}_r$: we can thus write $X = X_{\rm floor} \cup X_{\rm roof} \cup X_{\rm lat}$ (there is non-trivial intersection of $X_{\rm floor}$ and $X_{\rm roof}$ with $X_{\rm lat}$). We will now show that each of those three sets have ν-measure at most $Ce^{-\varepsilon t/C}$. Clearly $\nu(X_{\rm floor}) \leq Ce^{-\varepsilon t}$.

Using (2.13) and condition (2) of Definition 2.3, we see that if (x, y) is in the domain of $T_{\widehat{\Xi}}$ then $\|DT_{\widehat{\Xi}}(x, y)\|, \|DT_{\widehat{\Xi}}(x, y)^{-1}\| \leq Ce^{2r_\Xi(x)}$. Using condition (2) of Definition 2.3 again we get $\|Dr_\Xi(x)\| \leq C\|DT_{\widehat{\Xi}}(x, y)\| \leq Ce^{2r_\Xi(x)}$. Thus if $(x, y, s) \in X$ then r_Ξ is $Ce^{2\varepsilon t/C_0}$-Lipschitz restricted to the connected component of the domain of $T_{\widehat{\Xi}}$ containing (x, y), and we conclude that if $(x, y, s) \in X_{\rm roof}$ then $s \geq r_\Xi(x) - Ce^{-\varepsilon t/2}$, so $\nu(X_{\rm roof}) \leq Ce^{-\varepsilon t/2}$.

Projecting $X_{\rm lat}$ on (x, y), we obtain a set $Z \subset \widehat{\Xi}$. By Theorem 4.6, $\nu(X_{\rm lat}) \leq Ce^{-\varepsilon t/C}$ follows from $\widehat{m}(Z) \leq Ce^{-\varepsilon t/C}$. Let us show the latter estimate. Using that $T_{\widehat{\Xi}}$, restricted to a connected component of its domain intersecting Z, is $Ce^{2\varepsilon t/C_0}$-Lipschitz, we get that $T_{\widehat{\Xi}}(Z)$ is contained in a $Ce^{2\varepsilon t/C_0}e^{-\varepsilon t} \leq Ce^{-\varepsilon t/2}$ neighborhood (with respect to the metric dist) of the boundary of $\widehat{\Xi}$. Since $\widehat{m}$ is invariant and smooth, and the boundary of $\widehat{\Xi}$ is piecewise smooth, it follows that $\widehat{m}(Z) \leq Ce^{-\varepsilon t/2}$. □

Notice that $\log \frac{d\nu}{dz}$ is bounded over $\widehat{\Delta}_r$, so $\|f^{(t)}\|_{L^p(\nu)} \leq C\|f^{(t)}\|_{L^p(\,dz)} \leq C\|f \circ P\|_{L^p(\,dz)} \leq C\|f \circ P\|_{L^p(\nu)} = C\|f\|_{L^p(\nu_{\mathscr{C}(1)})} \leq C$. Hence $\|f \circ P - f^{(t)}\|_{L^p(\nu)} \leq C$ and using Lemma 4.8 we conclude that $\|\chi_{X\cup Y}(f \circ P - f^{(t)})\|_{L^{p'}(\nu)} \leq Ce^{-\varepsilon t/C}$, where $\chi_{X\cup Y}$ is the characteristic function of $X \cup Y$. On the other hand, if $z_0 \in \widehat{\Delta}_r \setminus (X \cup Y)$ and $\|z - z_0\| \leq e^{-\varepsilon t}/10$ then $\mathbf{dist}_F(z_0, z) \leq Ce^{2\varepsilon t/C_0}e^{-\varepsilon t} \leq Ce^{-\varepsilon t/2}$. It follows that $|\psi^{(t)}(z) - f \circ P(z_0)| = |f \circ P(z) - f \circ P(z_0)| < Ce^{-\alpha\varepsilon t/2}e^{k\varepsilon t/C_0}$. Thus $|f^{(t)}(z_0) - f \circ P(z_0)| \leq Ce^{-\alpha\varepsilon t/4}$. This implies that $\|\chi_{\widehat{\Delta}_r \setminus (X\cup Y)}(f \circ P - f^{(t)})\|_{L^\infty(\nu)} \leq Ce^{-\varepsilon t/C}$. The result follows. □

Let now k, α, p, q, f and g be as in Theorem 2.14. Let δ satisfy (4.12), and let ε_0 be given by Lemma 4.7. Choose $p > p' > 1$, $q > q' > 1$ such that $\frac{1}{p'} + \frac{1}{q'} = 1$. For $t > 0$, let $f^{(t)}$ and $g^{(t)}$ satisfy

$$\|f \circ P - f^{(t)}\|_{L^{p'}} \leq C(\|f\|_{\mathscr{D}_{k,\alpha}} + \|f\|_{L^p})e^{-\varepsilon_0 t}, \tag{4.13}$$

$$\|f^{(t)}\|_{C^1(\widehat{\Delta}_r)} \leq C(\|f\|_{\mathscr{D}_{k,\alpha}} + \|f\|_{L^p})e^{\delta t}, \tag{4.14}$$

$$\|g \circ P - g^{(t)}\|_{L^{q'}} \leq C(\|g\|_{\mathscr{D}_{k,\alpha}} + \|g\|_{L^q})e^{-\varepsilon_0 t}, \tag{4.15}$$

$$\|g^{(t)}\|_{C^1(\widehat{\Delta}_r)} \leq C(\|g\|_{\mathscr{D}_{k,\alpha}} + \|g\|_{L^q})e^{\delta t}. \tag{4.16}$$

Then (4.12), (4.14) and (4.16) imply

$$\begin{aligned}(\mathbf{4.17}) \qquad \left| \int f^{(t)} \cdot g^{(t)} \circ \widehat{\mathrm{T}}_t \, d\nu - \int f^{(t)} \, d\nu \int g^{(t)} \, d\nu \right| &\leq \mathrm{C} e^{-3\delta t} \| f^{(t)} \|_{\mathrm{C}^1} \| g^{(t)} \|_{\mathrm{C}^1} \\ &\leq \mathrm{C} e^{-\delta t} (\| f \|_{\mathscr{D}_{k,\alpha}} + \| f \|_{\mathrm{L}^p}) (\| g \|_{\mathscr{D}_{k,\alpha}} + \| g \|_{\mathrm{L}^q}).\end{aligned}$$

We have

$$\begin{aligned}&\int f \cdot g \circ \mathscr{T}\mathscr{F}_t \, d\nu_{\mathscr{C}^{(1)}} - \int f \, d\nu_{\mathscr{C}^{(1)}} \int g \, d\nu_{\mathscr{C}^{(1)}} \\ &\qquad = \int f \circ \mathrm{P} \cdot g \circ \mathrm{P} \circ \widehat{\mathrm{T}}_t \, d\nu - \int f \circ \mathrm{P} \, d\nu \int g \circ \mathrm{P} \, d\nu \\ &\qquad = \mathrm{C}_t(f \circ \mathrm{P}, g \circ \mathrm{P}).\end{aligned}$$

Using (4.13), (4.15) and (4.17) we get

$$\begin{aligned}|\mathrm{C}_t(f \circ \mathrm{P}, g \circ \mathrm{P})| &\leq \left| \mathrm{C}_t(f^{(t)}, g^{(t)}) \right| + \left| \mathrm{C}_t(f \circ \mathrm{P} - f^{(t)}, g \circ \mathrm{P}) \right| \\ &\quad + \left| \mathrm{C}_t(f \circ \mathrm{P}, g \circ \mathrm{P} - g^{(t)}) \right| + \left| \mathrm{C}_t(f \circ \mathrm{P} - f^{(t)}, g \circ \mathrm{P} - g^{(t)}) \right| \\ &\leq \left| \mathrm{C}_t(f^{(t)}, g^{(t)}) \right| + 2 \| f \circ \mathrm{P} - f^{(t)} \|_{\mathrm{L}^{p'}} \| g \|_{\mathrm{L}^{q'}} \\ &\quad + 2 \| f \|_{\mathrm{L}^{p'}} \| g \circ \mathrm{P} - g^{(t)} \|_{\mathrm{L}^{q'}} + 2 \| f \circ \mathrm{P} - f^{(t)} \|_{\mathrm{L}^{p'}} \| g \circ \mathrm{P} - g^{(t)} \|_{\mathrm{L}^{q'}} \\ &\leq \mathrm{C} e^{-\min(\delta, \varepsilon_0) t} (\| f \|_{\mathscr{D}_{k,\alpha}} + \| f \|_{\mathrm{L}^p}) (\| g \|_{\mathscr{D}_{k,\alpha}} + \| g \|_{\mathrm{L}^q}).\end{aligned}$$

This concludes the proof of Theorem 2.14, modulo Theorem 4.6 which will be proved in Sections 5 and 6, and Theorem 2.7 which will be proved in Sections 7 and 8. □

4.5. *A better recurrence estimate and the complement of large balls.* — In the formulation of Theorem 4.6, the particular recurrence estimate is not necessarily good because we were more concerned in obtaining not only a precompact transversal, but one for which the combinatorics of the first return map is particularly simple (it is in particular conjugate to a horseshoe on infinitely many symbols). By considering slightly more complicated combinatorics, one can get considerably better estimates:

Theorem **4.9.** — *For every $\delta > 0$, there exists a finite union $\widehat{\mathrm{Z}} = \bigcup \Delta_{\gamma_s} \times \Gamma_{\gamma_e}$ such that $\widehat{\mathrm{Z}}^{(1)} = \widehat{\mathrm{Z}} \cap \widehat{\Upsilon}^{(1)}_{\mathfrak{R}}$ is precompact in $\widehat{\Upsilon}^{(1)}_{\mathfrak{R}}$, and the first return time $r_{\widehat{\mathrm{Z}}}$ to $\widehat{\mathrm{Z}}$ under the Veech flow satisfies*

$$(\mathbf{4.18}) \qquad \int_{\widehat{\mathrm{Z}}} e^{(1-\delta) r_{\widehat{\mathrm{Z}}}} \, d\widehat{m} < \infty.$$

This result easily implies Theorem 2.15 (taking $\mathrm{K} = \mathrm{proj}(\widehat{\mathrm{Z}}^{(1)})$). It will be proved at the end of Section 6, by using a similar argument to the proof of Theorem 4.6.

5. A distortion estimate

The proof of the recurrence estimates is based on the analysis of the Rauzy renormalization map R. The key step involves a control on the measure of sets which present big distortion after some long (Teichmüller) time. In order to obtain nearly optimal estimates, we will need to carry on a more elaborate combinatorial analysis of Rauzy diagrams.

5.1. *Degeneration of Rauzy classes.* — Let $\mathfrak{R} \subset \mathfrak{S}^0(\mathscr{A})$ be a Rauzy class. Let $\mathscr{A}' \subset \mathscr{A}$ be a non-empty proper subset.

Definition **5.1.** — *An arrow is called $\mathscr{A}'$-colored if its winner belongs to $\mathscr{A}'$. A path $\gamma \in \mathfrak{R}(\pi)$ is $\mathscr{A}'$-colored if it is a concatenation of $\mathscr{A}'$-colored arrows.*

We call $\pi \in \mathfrak{R}$ *$\mathscr{A}'$-trivial* if the last letters on both the top and the bottom rows of π do not belong to $\mathscr{A}'$, *$\mathscr{A}'$-intermediate* if exactly one of those letters belong to $\mathscr{A}'$ and *$\mathscr{A}'$-essential* if both letters belong to $\mathscr{A}'$. Alternatively, $\pi \in \mathfrak{R}$ is trivial/intermediate/essential if it is the beginning (and ending) of exactly 0/1/2 $\mathscr{A}'$-colored arrows.

An *$\mathscr{A}'$-decorated Rauzy class* $\mathfrak{R}_* \subset \mathfrak{R}$ is a maximal subset whose elements can be joined by an $\mathscr{A}'$-colored path. We let $\Pi_*(\mathfrak{R}_*)$ be the set of all $\mathscr{A}'$-colored paths starting (and ending) at permutations in $\mathfrak{R}_*$. We will sometimes write Π_* for $\Pi_*(\mathfrak{R}_*)$.

A decorated Rauzy class is called trivial if it contains a trivial element π. In this case $\mathfrak{R}_* = \{\pi\}$ and $\Pi_*(\mathfrak{R}_*) = \{\pi\}$ (recall that vertices are identified with trivial (zero-length) paths).

A decorated Rauzy class is called essential if it contains an essential element.

Since $\Pi_*(\mathfrak{R}_*) \neq \Pi(\mathfrak{R})$ (for instance, $\Pi_*(\mathfrak{R}_*)$ does not contain complete paths), any essential decorated Rauzy class contains intermediate elements.

5.1.1. *Essential decorated Rauzy classes.* — Let $\mathfrak{R}_*$ be an essential decorated Rauzy class. Let $\mathfrak{R}_*^{\mathrm{ess}} \subset \mathfrak{R}_*$ be the set of essential elements of $\mathfrak{R}_*$. Let $\Pi_*^{\mathrm{ess}}(\mathfrak{R}_*) \subset \Pi_*(\mathfrak{R}_*)$ be the set of paths which start and end at an element of $\mathfrak{R}_*^{\mathrm{ess}}$.

An *arc* $\gamma \in \Pi_*(\mathfrak{R}_*)$ is a minimal non-trivial path in Π_*^{ess}. All arrows in an arc are of the same type and have the same winner, so the type and winner of an arc are well defined. Any element of $\mathfrak{R}_*^{\mathrm{ess}}$ is thus the start (and end) of one top arc and one bottom arc. The losers in an arc are all distinct, moreover the first loser is in $\mathscr{A}'$ (and the others are not).

If $\gamma \in \Pi_*(\mathfrak{R}_*)$ is an arrow, then there exist unique paths $\gamma_s, \gamma_e \in \Pi_*$ such that $\gamma_s \gamma \gamma_e$ is an arc, called the *completion* of γ. If π is intermediate, there is a single arc passing through π, the completion of the arrow starting (or ending) at π.

If $\pi \in \mathfrak{R}_*$ we define $\pi^{\mathrm{ess}} \in \mathfrak{R}_*^{\mathrm{ess}}$ as follows. If π is essential then $\pi^{\mathrm{ess}} = \pi$. If π is intermediate, let π^{ess} be the end of the arc passing through π.

To $\gamma \in \Pi_*$ we associate an element $\gamma^{\mathrm{ess}} \in \Pi_*^{\mathrm{ess}}$ as follows. For a trivial path $\pi \in \mathfrak{R}_*$, we use the previous definition of π^{ess}. Assuming that γ is an arrow, we distinguish two cases:

1. If γ starts in an essential element, we let γ^{ess} be the completion of γ,
2. Otherwise, we let γ^{ess} be the endpoint of the completion of γ.

We extend the definition to paths $\gamma \in \Pi_*$ by concatenation. Notice that if $\gamma \in \Pi_*^{\mathrm{ess}}$ then $\gamma^{\mathrm{ess}} = \gamma$.

5.1.2. *Reduction.* — We will now generalize the notion of simple reduction of [AV]. We will need the following concept.

Definition **5.2.** — *Given $\pi \in \mathfrak{S}(\mathscr{A})$ whose top and bottom rows end with different letters, we obtain the admissible end of π by deleting as many letters from the beginning of the top and bottom rows of π as necessary to obtain an admissible permutation. The resulting permutation π' belongs then to $\mathfrak{S}^0(\mathscr{A}')$ for some $\mathscr{A}' \subset \mathscr{A}$.*

Let $\mathfrak{R}_*$ be an essential decorated Rauzy class, and let $\pi \in \mathfrak{R}_*^{\mathrm{ess}}$. Delete all letters not belonging to $\mathscr{A}'$ from the top and bottom rows of π. The resulting permutation $\pi' \in \mathfrak{S}(\mathscr{A}')$ is not necessary admissible, but since π is essential the letters in the end of the top and bottom rows of π' are distinct. Let π^{red} be the admissible end of π'. We call π^{red} the *reduction* of π.

We extend the operation $\pi \mapsto \pi^{\mathrm{red}}$ of reduction from $\mathfrak{R}_*^{\mathrm{ess}}$ to the whole $\mathfrak{R}_*$ by taking the reduction of an element $\pi \in \mathfrak{R}_*$ as the reduction of π^{ess}.

If $\gamma \in \Pi_*^{\mathrm{ess}}$ is an arc, starting at π_s and ending at π_e, then the reductions of π_s and π_e belong to the same Rauzy class, and are joined by an arrow γ^{red} (called the reduction of γ) of the same type, same winner, and whose loser is the first loser of the arc γ. Thus the set of reductions of all $\pi \in \mathfrak{R}_*$ is a Rauzy class $\mathfrak{R}_*^{\mathrm{red}} \subset \mathfrak{S}^0(\mathscr{A}'')$ for some $\mathscr{A}'' \subset \mathscr{A}'$.

We define the reduction of a path $\gamma \in \Pi_*$ as follows. If γ is a trivial path or an arc, it is defined as above. We extend the definition to the case $\gamma \in \Pi_*^{\mathrm{ess}}$ by concatenation. In general we let the reduction of γ to be equal to the reduction of γ^{ess}.

Notice that the reduction map $\mathfrak{R}_*^{\mathrm{ess}} \to \mathfrak{R}_*^{\mathrm{red}}$ is a bijection. The reduction map $\Pi_*^{\mathrm{ess}} \to \Pi(\mathfrak{R}_*^{\mathrm{red}})$ is a bijection compatible with concatenation.

5.2. *Further combinatorics.* — Let $\mathscr{A}' \subset \mathscr{A}$ be a non-empty proper subset.

5.2.1. *Drift in essential decorated Rauzy classes.* — Let $\mathfrak{R}_* \subset \mathfrak{R}$ be an essential $\mathscr{A}'$-decorated Rauzy class.

For $\pi \in \mathfrak{R}_*$, let $\alpha_t(\pi)$ (respectively, $\alpha_b(\pi)$) be the rightmost letter in the top (respectively, bottom) row of π that belongs to $\mathscr{A} \setminus \mathscr{A}'$. Let $d_t(\pi)$ (respectively, $d_b(\pi)$) be the position of $\alpha_t(\pi)$ (respectively, $\alpha_b(\pi)$) in the top (respectively, bottom) of π. Let $d(\pi) = d_t(\pi) + d_b(\pi)$.

An essential element of $\mathfrak{R}_*$ is thus some π such that $d_t(\pi), d_b(\pi) < d$. If π_s is an essential element of $\mathfrak{R}_*$ and $\gamma \in \Pi_*(\mathfrak{R}_*)$ is an arrow starting at π_s and ending at π_e, then

1. $d_t(\pi_e) = d_t(\pi_s)$ or $d_t(\pi_e) = d_t(\pi_s) + 1$, the second possibility happening if and only if γ is a bottom whose winner precedes $\alpha_t(\pi_s)$ in the top of π_s.
2. $d_b(\pi_e) = d_b(\pi_s)$ or $d_b(\pi_e) = d_b(\pi_s) + 1$, the second possibility happening if and only if γ is a top whose winner precedes $\alpha_b(\pi_s)$ in the bottom of π_s.

In particular $d(\pi_e) = d(\pi_s)$ or $d(\pi_e) = d(\pi_s) + 1$. In the second case, we say that γ is *drifting*.

Let $\mathfrak{R}_*^{\mathrm{red}}$ be the reduction of $\mathfrak{R}_*$, so that $\mathfrak{R}_*^{\mathrm{red}} \subset \mathfrak{S}^0(\mathscr{A}'')$ for some $\mathscr{A}'' \subset \mathscr{A}'$. If $\pi \in \mathfrak{R}_*$ is essential then there exists $\alpha \in \mathscr{A}''$ that either precedes $\alpha_t(\pi)$ in the top of π or precedes $\alpha_b(\pi)$ in the bottom of π (we call such an α *good* for π). Indeed, if $\gamma \in \Pi_*(\mathfrak{R}_*)$ is a path starting at π, ending with a drifting arrow and minimal with this property then the winner of the last arrow of γ belongs to $\mathscr{A}''$ and either precedes $\alpha_t(\pi)$ in the top of π (if the drifting arrow is a bottom) or precedes $\alpha_b(\pi)$ in the bottom of π (if the drifting arrow is a top).

Notice that if $\gamma \in \Pi_*$ is an arrow starting and ending at essential elements π_s, π_e, then a good letter for π_s is also a good letter for π_e. Moreover, if γ is not drifting then the winner of γ is not a good letter for π_s.

5.2.2. *Standard decomposition of separated paths*

Definition **5.3.** — *An arrow is called $\mathscr{A} \setminus \mathscr{A}'$-separated if both its winner and loser belong to $\mathscr{A}'$. A path $\gamma \in \mathfrak{R}$ is $\mathscr{A} \setminus \mathscr{A}'$-separated if it is a concatenation of $\mathscr{A} \setminus \mathscr{A}'$-separated arrows.*

If $\gamma \in \Pi(\mathfrak{R})$ is a non-trivial maximal $\mathscr{A} \setminus \mathscr{A}'$-separated path, then there exists an essential $\mathscr{A}'$-decorated Rauzy class $\mathfrak{R}_* \subset \mathfrak{R}$ such that $\gamma \in \Pi_*(\mathfrak{R}_*)$. Moreover, if $\gamma = \gamma_1 \ldots \gamma_n$, then each γ_i starts at an essential element $\pi_i \in \mathfrak{R}_*$ (and γ_n ends at an intermediate element of $\mathfrak{R}_*$ by maximality).

Let $r = d(\pi_n) - d(\pi_1)$. Let $\gamma = \gamma^{(1)}\gamma^1 \ldots \gamma^{(r)}\gamma^r$ where the γ^i are drifting arrows and $\gamma^{(i)}$ are (possibly trivial) concatenations of non-drifting arrows. If α is a good letter for π_1, then it follows that α is not the winner of any arrow in any $\gamma^{(i)}$. The reduction of the $\gamma^{(i)}$ are thus non-complete paths in $\Pi(\mathfrak{R}_*^{\mathrm{red}})$, according to Definition 3.1.

5.3. *The distortion estimate.* — Let $\mathfrak{R} \subset \mathfrak{S}^0(\mathscr{A})$ be a Rauzy class. Let $\gamma \in \Pi(\mathfrak{R})$, and let π denotes its start. The domain of definition Δ_γ of Q^γ can be written as $\Delta'_\gamma \times \{\pi\}$, where $\Delta'_\gamma = B^*_\gamma \cdot \mathbf{R}^{\mathscr{A}}_+ \subset \mathbf{R}^{\mathscr{A}}_+$.

The distortion argument will involve not only the study of Lebesgue measure, but also of its forward images under the renormalization map. Technically, this is most conveniently done by introducing a class of measures which is invariant as a whole. For $q \in \mathbf{R}^{\mathscr{A}}_+$, we define a measure ν_q on the σ-algebra of subsets A of $\mathbf{R}^{\mathscr{A}}_+$ which are positively invariant (i.e., such that $\mathbf{R}_+ A = A$) by

$$(\mathbf{5.1}) \qquad \nu_q(A) = (\#\mathscr{A})!\,\mathrm{Leb}\big(A \cap \big\{\lambda \in \mathbf{R}^{\mathscr{A}}_+ \,:\, \langle \lambda, q\rangle < 1\big\}\big).$$

Equivalently, ν_q can be considered as a measure on the projective space $\mathbf{PR}^{\mathscr{A}}_+$. These measures satisfy $\nu_q(\mathbf{R}^{\mathscr{A}}_+) = \frac{1}{\prod_{\alpha\in\mathscr{A}} q_\alpha}$, and $\nu_q(B^*_\gamma \cdot A) = \nu_{B_\gamma \cdot q}(A)$.

If $q \in \mathbf{R}^{\mathscr{A}}_+$ and $\gamma \in \Pi(\mathfrak{R})$, these formulas translate in the following algorithm to compute $\nu_q(\Delta'_\gamma)$: start from $q^{(0)} = q$, let then $q^{(1)}$ be equal to $q^{(0)}$, except for the component $q^{(1)}_\beta$ of the loser β of the first arrow of γ. If α is its winner, set instead $q^{(1)}_\beta = q^{(0)}_\alpha + q^{(0)}_\beta$. Define then $q^{(2)}$ by the same process (but starting from $q^{(1)}$ and considering the second arrow of γ) and so on. If γ has length n, then $\nu_q(\Delta'_\gamma) = \frac{1}{\prod_{\alpha\in\mathscr{A}} q^{(n)}_\alpha}$. This holds since $q^{(n)} = B_\gamma \cdot q$ by construction.

In fact, we will not really study the measures ν_q, rather the quantities $\nu_q(\Delta'_\gamma)$ for $q \in \mathbf{R}^{\mathscr{A}}_+$ and $\gamma \in \Pi(\mathfrak{R})$. To deal with sets of paths instead of sets of simplices, we will introduce a more convenient formalism, in which conditioning is more or less transparent.

Given $\Gamma \subset \Pi(\mathfrak{R})$, $\gamma_s \in \Pi(\mathfrak{R})$, let $\Gamma_{\gamma_s} \subset \Gamma$ be the set of paths starting by γ_s, and let Γ^{γ_s} be the collection of ends γ_e of paths $\gamma = \gamma_s\gamma_e \in \Gamma$. Let $P_q(\Gamma \mid \gamma_s) = \frac{\nu_q(\bigcup_{\gamma\in\Gamma_{\gamma_s}} \Delta'_\gamma)}{\nu_q(\Delta'_{\gamma_s})}$. If π is the end of γ_s, we have $P_q(\Gamma \mid \gamma_s) = P_{B_{\gamma_s}\cdot q}(\Gamma^{\gamma_s} \mid \pi)$. If γ is an arrow starting at π with winner α and loser β, we have $P_q(\gamma \mid \pi) = \frac{q_\beta}{q_\alpha + q_\beta}$. More generally, for $\mathscr{A}' \subset \mathscr{A}$ and $q \in \mathbf{R}^{\mathscr{A}}_+$, let $N_{\mathscr{A}'}(q) = \prod_{\alpha\in\mathscr{A}'} q_\alpha$. Let also $N(q) = N_{\mathscr{A}}(q)$. Then, if $\gamma \in \Pi(\mathfrak{R})$ starts at π,

$$(\mathbf{5.2}) \qquad P_q(\gamma \mid \pi) = \frac{N(q)}{N(B_\gamma \cdot q)}.$$

A family $\Gamma_s \subset \Pi(\mathfrak{R})$ is called disjoint if no two elements are comparable (for the partial order defined in §3.1.2). If Γ_s is disjoint and $\Gamma \subset \Pi(\mathfrak{R})$ is a family such that any $\gamma \in \Gamma$ starts by some element $\gamma_s \in \Gamma_s$, then for every $\pi \in \mathfrak{R}$

$$(\mathbf{5.3}) \qquad P_q(\Gamma \mid \pi) = \sum_{\gamma_s\in\Gamma_s} P_q(\Gamma \mid \gamma_s) P_q(\gamma_s \mid \pi) \le P_q(\Gamma_s \mid \pi) \sup_{\gamma_s\in\Gamma_s} P_q(\Gamma \mid \gamma_s).$$

For $\mathscr{A}' \subset \mathscr{A}$ non-empty, let $M_{\mathscr{A}'}(q) = \max_{\alpha\in\mathscr{A}'} q_\alpha$. Let $M(q) = M_{\mathscr{A}}(q)$. The key distortion estimate is the following.

Theorem **5.4.** — *There exist* $C > 0$, $\theta > 0$, *depending only on* $\#\mathscr{A}$ *with the following property. Let* $\mathscr{A}' \subset \mathscr{A}$ *be a non-empty proper subset,* $0 \leq m \leq M$ *be integers,* $q \in \mathbf{R}_+^{\mathscr{A}}$. *Then for every* $\pi \in \mathfrak{R}$,

$$P_q\big(\gamma \in \Pi(\mathfrak{R}),\ M(B_\gamma \cdot q) > 2^M M(q) \ \textit{and}\ M_{\mathscr{A}'}(B_\gamma \cdot q) < 2^{M-m} M(q) \,\big|\, \pi\big) \\ \leq C(m+1)^\theta 2^{-m}.$$

A crucial feature of our arguments is that we obtain estimates which are *uniform in* q. To some extent, this will enable us to treat the process as if it were Markov: the past has indeed an influence since it changes the parameter q, replacing it by $B_\gamma \cdot q$, but since everything is uniform in q this does not matter.

The proof of Theorem 5.4 is based on induction on $\#\mathscr{A}$, and will take the remaining of this section.

5.4. *Reduction estimate.* — Let $\mathfrak{R}_*$ be an $\mathscr{A}'$-decorated Rauzy class, and let $\gamma \in \Pi_*(\mathfrak{R}_*)$ start at $\pi \in \mathfrak{R}_*$. If $\mathfrak{R}_*$ is essential, let $\mathfrak{R}_*^{\mathrm{red}} \subset \mathfrak{S}^0(\mathscr{A}'')$ be its reduction. Let q^{red} be the (canonical) projection of q on $\mathbf{R}^{\mathscr{A}''}$ (obtained by forgetting the coordinates in $\mathscr{A} \setminus \mathscr{A}''$). Then the projection of $B_\gamma \cdot q$ on $\mathbf{R}^{\mathscr{A}''}$ coincides with $B_{\gamma^{\mathrm{red}}} \cdot q^{\mathrm{red}}$. Notice also that the projections of q and $B_\gamma \cdot q$ on $\mathbf{R}^{\mathscr{A}' \setminus \mathscr{A}''}$ coincide. This gives the formula

$$\frac{P_q(\gamma \,|\, \pi)}{P_{q^{\mathrm{red}}}(\gamma^{\mathrm{red}} \,|\, \pi^{\mathrm{red}})} = \frac{N_{\mathscr{A} \setminus \mathscr{A}'}(q)}{N_{\mathscr{A} \setminus \mathscr{A}'}(B_\gamma \cdot q)}. \tag{5.4}$$

Proposition **5.5.** — *Let* $\mathfrak{R}_*$ *be an* $\mathscr{A}'$*-decorated Rauzy class, and let* $\Gamma \subset \Pi_*(\mathfrak{R}_*)$ *be a family of paths such that, for all* $\gamma \in \Gamma$, $N_{\mathscr{A} \setminus \mathscr{A}'}(B_\gamma \cdot q) \geq 2^M N_{\mathscr{A} \setminus \mathscr{A}'}(q)$. *Then for every* $\pi \in \mathfrak{R}_*$,

$$P_q(\Gamma \,|\, \pi) \leq 2^{-M}. \tag{5.5}$$

Proof. — We may assume that Γ is the collection of all minimal paths $\gamma \in \Pi_*(\mathfrak{R}_*)$ starting at π and satisfying $N_{\mathscr{A} \setminus \mathscr{A}'}(B_\gamma \cdot q) \geq 2^M N_{\mathscr{A} \setminus \mathscr{A}'}(q)$. If $\mathfrak{R}_*$ is trivial then either Γ is empty or $M = 0$ and the estimate is obvious. If $\mathfrak{R}_*$ is neither trivial nor essential, then Γ consists of a single path γ, and the result follows from the definition of $P_q(\gamma \,|\, \pi)$. If $\mathfrak{R}_*$ is essential, we notice that two distinct paths in Γ have disjoint reductions, so the estimate follows from (5.4). □

5.5. *The main induction scheme*

Definition **5.6.** — *A path* $\gamma \in \Pi(\mathfrak{R})$ *is called* $\mathscr{A}'$*-preferring if it is a concatenation of a* $\mathscr{A}'$*-separated path (first) and a* $\mathscr{A}'$*-colored path (second).*

A path is $\mathscr{A}'$-preferring if and only if it chooses a winner in $\mathscr{A}'$ whenever possible. In particular, if a path has no loser in $\mathscr{A}'$, then it is $\mathscr{A}'$-preferring. Notice that γ is $\mathscr{A}'$-preferring if and only if $\langle B_\gamma \cdot e_\alpha, e_\beta \rangle = 0$ for $\alpha \in \mathscr{A} \setminus \mathscr{A}'$, $\beta \in \mathscr{A}'$ (so B_γ is block-triangular). Notice also that the $\mathscr{A}'$-separated part or the $\mathscr{A}'$-colored part in an $\mathscr{A}'$-preferring path may very well be trivial.

Proposition **5.7.** — *There exist* $C > 0$, $\theta > 0$, *depending only on* $\#\mathscr{A}$ *with the following property. Let* $M \in \mathbf{N}$, $q \in \mathbf{R}_+^{\mathscr{A}}$. *Then for every* $\pi \in \mathfrak{R}$,

$$P_q\big(\gamma \textit{ is not complete and } M(B_\gamma \cdot q) > 2^M M(q) \,\big|\, \pi\big) \leq C(M+1)^\theta 2^{-M}.$$

Proposition **5.8.** — *There exist* $C > 0$, $\theta > 0$, *depending only on* $\#\mathscr{A}$ *with the following property. Let* $\mathscr{A}' \subset \mathscr{A}$ *be a non-empty proper subset,* $M \in \mathbf{N}$, $q \in \mathbf{R}_+^{\mathscr{A}}$. *Then for every* $\pi \in \mathfrak{R}$,

$$P_q\big(\gamma \textit{ is } \mathscr{A} \setminus \mathscr{A}'\textit{-separated and } M_{\mathscr{A}'}(B_\gamma \cdot q) > 2^M M(q) \,\big|\, \pi\big) \\ \leq C(M+1)^\theta 2^{-M}.$$

Proposition **5.9.** — *There exist* $C > 0$, $\theta > 0$, *depending only on* $\#\mathscr{A}$ *with the following property. Let* $\mathscr{A}' \subset \mathscr{A}$ *be a non-empty proper subset,* $M \in \mathbf{N}$, $q \in \mathbf{R}_+^{\mathscr{A}}$. *Then for every* $\pi \in \mathfrak{R}$,

$$P_q\big(\gamma \textit{ is } \mathscr{A}'\textit{-preferring and } M_{\mathscr{A}'}(B_\gamma \cdot q) \leq 2^M M(q) < M(B_\gamma \cdot q) \,\big|\, \pi\big) \\ \leq C(M+1)^\theta 2^{-M}.$$

The proof of Theorem 5.4 and Propositions 5.7, 5.8 and 5.9 will be carried out simultaneously in an induction argument on $d = \#\mathscr{A}$. For $d \geq 2$, consider the statements:

(A_d) Proposition 5.7 holds for $\#\mathscr{A} = d$,
(B_d) Proposition 5.8 holds for $\#\mathscr{A} = d$,
(C_d) Proposition 5.9 holds for $\#\mathscr{A} = d$,
(D_d) Theorem 5.4 holds for $\#\mathscr{A} = d$.

The induction step will be composed of four parts:

1. (A_j), $2 \leq j < d$, implies (B_d),
2. (B_d) implies (C_d),
3. (C_d) implies (D_d),
4. (D_j), $2 \leq j \leq d$, implies (A_d).

Notice that the start of the induction is trivial (for $d = 2$ the hypothesis in (1) is trivially satisfied).

In what follows, C and θ denote generic constants, whose actual value may vary during the course of the proof.

5.5.1. *Proof of (1).* — Let Γ be the set of all maximal $\mathscr{A}\setminus\mathscr{A}'$-separated γ starting at π such that $M_{\mathscr{A}'}(B_\gamma\cdot q) > 2^M M_{\mathscr{A}'}(q)$. By Lemma 3.2, it is sufficient to prove

$$(\mathbf{5.6})\qquad P_q(\Gamma\,|\,\pi) \le C(M+1)^\theta 2^{-M}.$$

If Γ is non-empty then π is essential (and if $\Gamma=\emptyset$ the statement is trivial). Let $\mathfrak{R}_*$ be the $\mathscr{A}'$-decorated class containing π. We have $\Gamma\subset\Pi_*(\mathfrak{R}_*)$. Decompose Γ into subsets $\Gamma_{\overline{M}}$, $\overline{M}\ge M$, containing the $\gamma\in\Gamma$ with $2^{\overline{M}+1}M_{\mathscr{A}'}(q)\ge M_{\mathscr{A}'}(B_\gamma\cdot q) > 2^{\overline{M}}M_{\mathscr{A}'}(q)$. Recall the decomposition of $\gamma\in\Gamma$, $\gamma=\gamma^{(1)}\gamma^1\ldots\gamma^{(r)}\gamma^r$ where $r=r(\gamma)<2d$. Let $\Gamma_{\overline{M},r}\subset\Gamma_{\overline{M}}$ collect the γ with $r(\gamma)=r$. Let $\gamma_{(i)}=\gamma^{(1)}\gamma^1\ldots\gamma^{(i)}\gamma^i$, $1\le i\le r$, and let $\gamma_{(0)}$ be the start of γ. To $\gamma\in\Gamma_{\overline{M},r}$ we associate $\underline{m}=(m_1,\ldots,m_r)$ where

$$(\mathbf{5.7})\qquad 2^{m_i}\le \frac{M_{\mathscr{A}'}(B_{\gamma_{(i)}}\cdot q)}{M_{\mathscr{A}'}(B_{\gamma_{(i-1)}}\cdot q)} < 2^{m_i+1}.$$

We have $2^{\sum m_i}M_{\mathscr{A}'}(q)\le M_{\mathscr{A}'}(B_\gamma\cdot q)\le 2^{2r+\sum m_i}M_{\mathscr{A}'}(q)$, so $\overline{M}+1\ge\sum m_i\ge\overline{M}-2r$. Let $\Gamma_{\overline{M},r,\underline{m}}$ collect the γ with the same $\underline{m}$. For $0\le i\le r$, let $\Gamma_{\overline{M},r,\underline{m},i}$ be the collection of all possible $\gamma_{(i)}$.

Let $\mathfrak{R}_*^{\mathrm{red}}\subset\mathfrak{S}^0(\mathscr{A}'')$ be the reduction of $\mathfrak{R}_*$. If $\gamma_s\in\Pi_*(\mathfrak{R}_*)$ is $\mathscr{A}\setminus\mathscr{A}'$-separated then

$$(\mathbf{5.8})\qquad P_q(\Gamma_{\overline{M},r,\underline{m},i}\,|\,\gamma_s) = P_{q^{\mathrm{red}}}\big(\Gamma^{\mathrm{red}}_{\overline{M},r,\underline{m},i}\,\big|\,\gamma_s^{\mathrm{red}}\big),$$

where q^{red} is the orthogonal projection of q on $\mathbf{R}^{\mathscr{A}''}$, $\Gamma^{\mathrm{red}}_{\overline{M},r,\underline{m},i}$ is the image of $\Gamma_{\overline{M},r,\underline{m},i}$ by the reduction map and γ_s^{red} is the reduction of γ_s. If $\gamma\in\Gamma_{\overline{M},r,\underline{m},i}$ starts by $\gamma_s\in\Gamma_{\overline{M},r,\underline{m},i-1}$ then we can write $\gamma=\gamma_s\gamma_a\gamma_b$, where γ_b is a drifting arrow, and γ_a is a concatenation of non-drifting arrows. Then γ_a^{red} is a non-complete path (in $\Pi(\mathfrak{R}_*^{\mathrm{red}})$) satisfying $M_{\mathscr{A}''}(B_{\gamma_a^{\mathrm{red}}}\cdot B_{\gamma_s^{\mathrm{red}}}\cdot q^{\mathrm{red}})\ge 2^{m_i-1}M_{\mathscr{A}''}(B_{\gamma_s^{\mathrm{red}}}\cdot q^{\mathrm{red}})$. By (A_j) with $j=\#\mathscr{A}''<d$,

$$(\mathbf{5.9})\qquad P_{q^{\mathrm{red}}}\big(\Gamma^{\mathrm{red}}_{\overline{M},r,\underline{m},i}\,\big|\,\gamma_s^{\mathrm{red}}\big)\le C(m_i+1)^\theta 2^{-m_i},\qquad \gamma_s\in\Gamma_{\overline{M},r,\underline{m},i-1}.$$

Each family $\Gamma_{\overline{M},r,\underline{m},i}$ is disjoint, so (5.8) and (5.9) imply

$$(\mathbf{5.10})\qquad P_q(\Gamma_{\overline{M},r,\underline{m},i}\,|\,\pi)\le C(m_i+1)^\theta 2^{-m_i}P_q(\Gamma_{\overline{M},r,\underline{m},i-1}\,|\,\pi),$$

which gives

$$(\mathbf{5.11})\qquad P_q(\Gamma_{\overline{M},r,\underline{m}}\,|\,\pi) = P_q(\Gamma_{\overline{M},r,\underline{m},r}\,|\,\pi)\le\prod_{i=1}^r C(m_i+1)^\theta 2^{-m_i}\le C(\overline{M}+1)^\theta 2^{-\overline{M}}.$$

Summing over the different $\underline{m}$ (with $\sum m_i\le\overline{M}+1$), $r<2d$, and $\overline{M}\ge M$, we get (5.6).

5.5.2. *Proof of (2).* — Let Γ be the set of all $\mathscr{A}'$-preferring γ such that $M_{\mathscr{A}'}(B_\gamma \cdot q) \leq 2^M M(q) < M(B_\gamma \cdot q)$, and which are minimal with those properties. Any $\gamma \in \Gamma$ is of the form $\gamma = \gamma_s \gamma_e$ where γ_s is $\mathscr{A}'$-separated and γ_e is $\mathscr{A}'$-colored. Let $\Gamma_s \subset \Pi(\mathfrak{R})$ collect all possible γ_s. Notice that Γ_s is disjoint.

Let $m = m(\gamma_s) \in [-1, M]$ be the smallest integer such that $M_{\mathscr{A}\setminus\mathscr{A}'}(B_{\gamma_s} \cdot q) \leq 2^{m+1} M(q)$. Notice that $M_{\mathscr{A}'}(B_{\gamma_s} \cdot q) = M_{\mathscr{A}'}(q) \leq M(q)$. Let $\Gamma_{s,m}$ collect all $\gamma_s \in \Gamma_s$ with $m(\gamma_s) = m$.

Let us show that for $\gamma_s \in \Gamma_{s,m}$

$$P_q(\Gamma \mid \gamma_s) \leq 2^{m+1-M}. \tag{5.12}$$

Let π_e be the ending of γ_s. Let Γ^{γ_s} be the set of all endings γ_e of paths $\gamma = \gamma_s \gamma_e \in \Gamma$ that begin with γ_s. Let $\mathfrak{R}_*$ be the $\mathscr{A}'$-decorated Rauzy class containing π_e. Then $\Gamma^{\gamma_s} \subset \Pi_*(\mathfrak{R}_*)$ is a collection of paths γ_e satisfying $M_{\mathscr{A}\setminus\mathscr{A}'}(B_{\gamma_e} \cdot B_{\gamma_s} \cdot q) > 2^M M(q) \geq 2^{M-1-m} M(B_{\gamma_s} \cdot q)$. In particular, $N_{\mathscr{A}\setminus\mathscr{A}'}(B_{\gamma_e} \cdot B_{\gamma_s} \cdot q) > 2^{M-1-m} N(B_{\gamma_s} \cdot q)$. Applying Proposition 5.5 to $B_{\gamma_s} \cdot q$, we obtain $P_q(\Gamma \mid \gamma_s) = P_{B_{\gamma_s} \cdot q}(\Gamma^{\gamma_s} \mid \pi_e) \leq 2^{m+1-M}$.

If $m \geq 0$ then $\Gamma_{s,m}$ consists of $\mathscr{A}'$-separated paths γ_s with $M_{\mathscr{A}\setminus\mathscr{A}'}(B_{\gamma_s} \cdot q) > 2^m M(q)$. By (B_d),

$$P_q(\Gamma_{s,m} \mid \pi) \leq C(m+2)^\theta 2^{-m}. \tag{5.13}$$

Notice that (5.13) is still satisfied (trivially) for $m = -1$. Putting together (5.13) and (5.12), and summing over m, we get

$$P_q(\Gamma \mid \pi) \leq C(M+1)^\theta 2^{-M}. \tag{5.14}$$

5.5.3. *Proof of (3).* — The proof is by descending recurrence on $\#\mathscr{A}'$. We may assume that $m > 0$ since the case $m = 0$ is trivial. Let $\Gamma \subset \Pi(\mathfrak{R})$ be the set of γ starting at π and such that $M(B_\gamma \cdot q) > 2^M M(q)$, $M_{\mathscr{A}'}(B_\gamma \cdot q) < 2^{M-m} M(q)$ and which are minimal with those properties. We want to estimate $P_q(\Gamma \mid \pi) \leq C(m+1)^\theta 2^{-m}$.

Let $\Gamma_D \subset \Gamma$ be the set of $\mathscr{A}'$-preferring paths. We have $P_q(\Gamma_D \mid \pi) \leq C(M+1)^\theta 2^{-M}$ by (C_d), so we just have to prove that $P_q(\Gamma \setminus \Gamma_D \mid \pi) \leq C(m+1)^\theta 2^{-m}$.

If $\gamma \in \Gamma \setminus \Gamma_D$, then at least one of the arrows composing γ has as winner an element of $\mathscr{A} \setminus \mathscr{A}'$, and as loser an element of $\mathscr{A}'$. Decompose $\gamma = \gamma_s \gamma_e$ with γ_s minimal such that no arrow composing γ_e has as winner an element of $\mathscr{A} \setminus \mathscr{A}'$, and as loser an element of $\mathscr{A}'$; let $n_0 = n_0(\gamma)$ be the length of γ_s. Let $\beta = \beta(\gamma) \in \mathscr{A} \setminus \mathscr{A}'$ be the winner of the last arrow of γ_s.

We can then write $\Gamma \setminus \Gamma_D$ as the union of Γ_β, $\beta \in \mathscr{A} \setminus \mathscr{A}'$, where Γ_β collects all γ with $\beta(\gamma) = \beta$. We only have to prove that $P_q(\Gamma_\beta \mid \pi) \leq C(m+1)^\theta 2^{-m}$ for any $\beta \in \mathscr{A} \setminus \mathscr{A}'$.

Let $\Gamma^*_\beta \subset \Gamma_\beta$ be the set of γ such that $\mathrm{M}(\mathrm{B}_{\gamma_s} \cdot q) \leq 2^{\mathrm{M}-m}\mathrm{M}(q)$. For $\gamma \in \Gamma^*_\beta$, write $\gamma = \gamma^*_s \gamma^*_e$ with γ^*_s minimal with $\mathrm{M}(\mathrm{B}_{\gamma^*_s} \cdot q) > 2^{\mathrm{M}-m}\mathrm{M}(q)$. In particular $\mathrm{M}(\mathrm{B}_{\gamma^*_s} \cdot q) \leq 2^{\mathrm{M}+1-m}\mathrm{M}(q)$. Let $n^*(\gamma)$ be the length of γ^*_s. Since $n^* > n_0$, γ^*_e is $\mathscr{A}'$-preferring. Notice that γ^*_e is also such that $\mathrm{M}(\mathrm{B}_{\gamma^*_e} \cdot \mathrm{B}_{\gamma^*_s} \cdot q) > 2^{m-1}\mathrm{M}(\mathrm{B}_{\gamma^*_s} \cdot q)$. By (C_d), it follows that $\mathrm{P}_q(\Gamma^*_\beta \,|\, \gamma^*_s) \leq \mathrm{C}m^\theta 2^{1-m}$. Since the collection of all possible γ^*_s is disjoint, we get

$$(\mathbf{5.15}) \qquad \mathrm{P}_q(\Gamma^*_\beta \,|\, \pi) \leq \mathrm{C}m^\theta 2^{1-m}.$$

Thus we only need to show that $\mathrm{P}_q(\Gamma_\beta \setminus \Gamma^*_\beta \,|\, \pi) \leq \mathrm{C}m^\theta 2^{-m}$.

Before continuing, let us notice that if $\#\mathscr{A}' = \#\mathscr{A} - 1$, then $\Gamma_\beta = \Gamma^*_\beta$. Indeed in this case $\mathscr{A} = \mathscr{A}' \cup \{\beta\}$, and since $\mathrm{M}_\beta(\mathrm{B}_{\gamma_s} \cdot q) \leq \mathrm{M}_{\mathscr{A}'}(\mathrm{B}_{\gamma_s} \cdot q)$, we have $\mathrm{M}(\mathrm{B}_{\gamma_s} \cdot q) \leq 2^{\mathrm{M}-m}\mathrm{M}(q)$. In particular, the previous argument is enough to establish (D_d) in the case $\#\mathscr{A}' = d - 1$, which allows us to start the reverse induction on $\#\mathscr{A}'$ used in the argument below.

For $\gamma \in \Gamma_\beta \setminus \Gamma^*_\beta$, there exists an integer $m_0 = m_0(\gamma) \in [0, m)$ such that $2^{\mathrm{M}-m_0}\mathrm{M}(q) \geq \mathrm{M}(\mathrm{B}_{\gamma_s} \cdot q) > 2^{\mathrm{M}-1-m_0}\mathrm{M}(q)$. We collect all γ with $m_0(\gamma) = m_0$ in Γ_{β,m_0}. It is enough to show that

$$(\mathbf{5.16}) \qquad \mathrm{P}_q(\Gamma_{\beta,m_0} \,|\, \pi) \leq \mathrm{C}(m+1)^\theta 2^{-m}.$$

Write $\gamma = \gamma^1_s \gamma^1_e = \gamma^2_s \gamma^2_e$ where γ^1_s, γ^2_s are minimal such that $\mathrm{M}(\mathrm{B}_{\gamma^1_s} \cdot q) > 2^{\mathrm{M}-m_0}\mathrm{M}(q)$, $\mathrm{M}(\mathrm{B}_{\gamma^2_s} \cdot q) > 2^{\mathrm{M}-1-m_0}\mathrm{M}(q)$. Let $n_1 = n_1(\gamma)$ and $n_2 = n_2(\gamma)$ be the lengths of γ^1_s and γ^2_s. We have $n_2 < n_0 < n_1$.[2]

Let $\Gamma^1_{\beta,m_0,s}$, $\Gamma^2_{\beta,m_0,s}$ collect all possible paths γ^1_s, γ^2_s as above. The families $\Gamma^1_{\beta,m_0,s}$, $\Gamma^2_{\beta,m_0,s}$ are disjoint. If $\gamma = \gamma^1_s \gamma^1_e \in \Gamma_{\beta,m_0,s}$ with $\gamma^1_s \in \Gamma^1_{\beta,m_0,s}$, the path γ^1_e is $\mathscr{A}'$-preferring and satisfies $\mathrm{M}(\mathrm{B}_{\gamma^1_e} \cdot \mathrm{B}_{\gamma^1_s} \cdot q) > 2^{m_0-1}\mathrm{M}(\mathrm{B}_{\gamma^1_s} \cdot q)$, $\mathrm{M}_{\mathscr{A}'}(\mathrm{B}_{\gamma^1_e} \cdot \mathrm{B}_{\gamma^1_s} \cdot q) < 2^{\mathrm{M}-m}\mathrm{M}(q) < \mathrm{M}(\mathrm{B}_{\gamma^1_s} \cdot q)$, so by (C_d) we have

$$(\mathbf{5.17}) \qquad \mathrm{P}_q\big(\Gamma_{\beta,m_0,s} \,\big|\, \gamma^1_s\big) \leq \mathrm{C}(m_0+1)^\theta 2^{-m_0}, \quad \gamma^1_s \in \Gamma^1_{\beta,m_0,s}.$$

On the other hand, $\mathrm{M}_\beta(\mathrm{B}_{\gamma^2_s} \cdot q) < \mathrm{M}_{\mathscr{A}'}(\mathrm{B}_{\gamma_s} \cdot q) < 2^{\mathrm{M}-m}\mathrm{M}(q)$ so that $\mathrm{M}_{\mathscr{A}' \cup \{\beta\}}(\mathrm{B}_{\gamma^2_s} \cdot q) < 2^{\mathrm{M}-m}\mathrm{M}(q)$. Then

$$(\mathbf{5.18}) \qquad \mathrm{P}_q\big(\Gamma^1_{\beta,m_0,s} \,\big|\, \pi\big) \leq \mathrm{P}_q\big(\Gamma^2_{\beta,m_0,s} \,\big|\, \pi\big) \leq \mathrm{C}(m-m_0)^\theta 2^{m_0+1-m},$$

where the first inequality is trivial and the second is by the reverse induction hypothesis (that is, (D_d) with $\mathscr{A}' \cup \{\beta\}$ instead of $\mathscr{A}'$, $\mathrm{M} - m_0 - 1$ instead of M and $m - m_0 - 1$ instead of m). Since $\Gamma^1_{\beta,m_0,s}$ is disjoint, (5.17) and (5.18) imply (5.16).

[2] Notice that we cannot have $n_2 = n_0$, since otherwise $\mathrm{M}_{\mathscr{A}'}(\mathrm{B}_{\gamma_s} \cdot q) = \mathrm{M}(\mathrm{B}_{\gamma_s} \cdot q) > 2^{\mathrm{M}-m}\mathrm{M}(q)$, so that $\gamma \notin \Gamma$.

5.5.4. *Proof of (4).* — Let $\gamma \in \Pi(\mathfrak{R})$ be a non-complete path starting at π. Let $\beta \in \mathscr{A}$ be a letter which is not winner of any arrow of γ, and let $\mathscr{A}' = \mathscr{A} \setminus \{\beta\}$. If $\mathfrak{R}_* \subset \mathfrak{R}$ is the $\mathscr{A}'$-decorated Rauzy class containing π then $\gamma \in \Pi_*(\mathfrak{R}_*)$. Let $\Gamma_\beta \subset \Pi_*(\mathfrak{R}_*)$ be the family of paths γ satisfying $\mathrm{M}(\mathrm{B}_\gamma \cdot q) > 2^{\mathrm{M}}\mathrm{M}(q)$ and minimal with this property. It is enough to show that

$$\text{(5.19)} \qquad \mathrm{P}_q(\Gamma_\beta \,|\, \pi) \leq \mathrm{C}(\mathrm{M}+1)^\theta 2^{-\mathrm{M}}$$

for an arbitrary choice of β and $\mathfrak{R}_*$.

First notice that $\mathfrak{R}_*$ cannot be a trivial decorated Rauzy class, since $\mathscr{A} \setminus \mathscr{A}'$ has a single element. If $\mathfrak{R}_*$ is neither trivial nor essential, then Γ_β contains a unique path γ starting at π. In this case $\mathrm{P}_q(\Gamma_\beta \,|\, \pi) = \mathrm{P}_q(\gamma \,|\, \pi) < 2^{-\mathrm{M}}$. It is enough then to consider the case where $\mathfrak{R}_*$ is essential.

Let $\Gamma^*_\beta \subset \Gamma_\beta$ be the set of all γ such that $\mathrm{M}_\beta(\mathrm{B}_\gamma \cdot q) \leq \mathrm{M}(q)$. By (D_d) applied to $\{\beta\}$, we have

$$\text{(5.20)} \qquad \mathrm{P}_q(\Gamma^*_\beta \,|\, \pi) \leq \mathrm{C}(\mathrm{M}+1)^\theta 2^{-\mathrm{M}}.$$

For $\gamma \in \Gamma_\beta \setminus \Gamma^*_\beta$, there is at least one arrow composing γ with β as loser. Let $\alpha = \alpha(\gamma)$ be the winner of the last such arrow. Let $m_0 = m_0(\gamma) \in [0, \mathrm{M}]$ be such that $2^{m_0}\mathrm{M}(q) < \mathrm{M}_\beta(\mathrm{B}_\gamma \cdot q) \leq 2^{m_0+1}\mathrm{M}(q)$. Write $\gamma = \gamma_s\gamma_e$ where γ_s is minimal with $\mathrm{M}_\beta(\mathrm{B}_{\gamma_s} \cdot q) > 2^{m_0}\mathrm{M}(q)$. Let $\mathrm{M}_0 = \mathrm{M}_0(\gamma) \in [m_0, \mathrm{M}]$ be such that $2^{\mathrm{M}_0}\mathrm{M}(q) < \mathrm{M}(\mathrm{B}_{\gamma_s} \cdot q) \leq 2^{\mathrm{M}_0+1}\mathrm{M}(q)$. Let $\Gamma \subset \Gamma_\beta \setminus \Gamma^*_\beta$ collect the γ with the same α, m_0 and M_0. It is enough to show that

$$\text{(5.21)} \qquad \mathrm{P}_q(\Gamma \,|\, \pi) \leq \mathrm{C}(\mathrm{M}+1)^\theta 2^{-\mathrm{M}}.$$

Let Γ_s be the family of possible γ_s for $\gamma \in \Gamma$. By (D_d) applied to $\{\beta\}$,

$$\text{(5.22)} \qquad \mathrm{P}_q(\Gamma \,|\, \gamma_s) \leq \mathrm{C}(\mathrm{M}+1-\mathrm{M}_0)^\theta 2^{\mathrm{M}_0-\mathrm{M}}, \quad \gamma_s \in \Gamma_s.$$

Let $\mathfrak{R}^{\mathrm{red}}_* \subset \mathfrak{S}^0(\mathscr{A}'')$ be the reduction of $\mathfrak{R}_*$. Notice that two distinct paths in Γ_s have disjoint reductions. Let $\Gamma^{\mathrm{red}}_s \subset \Pi(\mathfrak{R}^{\mathrm{red}}_*)$ be the image of Γ_s by the reduction map. Let q^{red} be the canonical projection of q on $\mathbf{R}^{\mathscr{A}''}$. Then by (5.4),

$$\text{(5.23)} \qquad \mathrm{P}_q(\Gamma_s \,|\, \pi) \leq \mathrm{P}_q\big(\Gamma^{\mathrm{red}}_s \,\big|\, \pi^{\mathrm{red}}\big) \sup_{\gamma_s \in \Gamma_s} \frac{\mathrm{N}_\beta(q)}{\mathrm{N}_\beta(\mathrm{B}_{\gamma_s} \cdot q)} \leq \mathrm{P}_q\big(\Gamma^{\mathrm{red}}_s \,\big|\, \pi^{\mathrm{red}}\big) 2^{-m_0}.$$

Notice that if $\gamma^{\mathrm{red}}_s \in \Gamma^{\mathrm{red}}_s$ then $\mathrm{M}_\alpha(\mathrm{B}_{\gamma^{\mathrm{red}}_s} \cdot q^{\mathrm{red}}) \leq 2^{m_0+1}\mathrm{M}(q)$, and if $\mathrm{M}_0 > m_0$ we also have $\mathrm{M}(\mathrm{B}_{\gamma^{\mathrm{red}}_s} \cdot q^{\mathrm{red}}) > 2^{\mathrm{M}_0}\mathrm{M}(q)$. Thus, if $\mathrm{M}_0 > m_0$, by (D_j) with $j = \#\mathscr{A}'' < d$,

$$\text{(5.24)} \qquad \mathrm{P}_q\big(\Gamma^{\mathrm{red}}_s \,\big|\, \pi^{\mathrm{red}}\big) \leq \mathrm{C}(\mathrm{M}_0+1-m_0)^\theta 2^{m_0-\mathrm{M}_0},$$

and we notice that (5.24) also holds, trivially, if $m_0 = \mathrm{M}_0$. Putting together (5.24), (5.23) and (5.22) we get (5.21).

6. Proof of the recurrence estimates

Lemma **6.1.** — *For every* $\widehat{\gamma} \in \Pi(\mathfrak{R})$, *there exist* $M \geq 0$, $\rho < 1$ *such that for every* $\pi \in \mathfrak{R}$, $q \in \mathbf{R}_+^{\mathscr{A}}$,

$$(\mathbf{6.1}) \qquad P_q\big(\gamma \text{ cannot be written as } \gamma_s\widehat{\gamma}\gamma_e \text{ and } M(B_\gamma \cdot q) > 2^M M(q) \,\big|\, \pi\big) \leq \rho.$$

Proof. — Fix $M_0 \geq 0$ large and let $M = 2M_0$. Let Γ be the set of all minimal paths γ starting at π which cannot be written as $\gamma_s\widehat{\gamma}\gamma_e$ and such that $M(B_\gamma \cdot q) > 2^M M(q)$. Any path $\gamma \in \Gamma$ can be written as $\gamma = \gamma_1\gamma_2$ where γ_1 is minimal with $M(B_\gamma \cdot q) > 2^{M_0}M(q)$. Let Γ_1 collect the possible γ_1. Then Γ_1 is disjoint. Let $\tilde{\Gamma}_1 \subset \Gamma_1$ be the set of all γ_1 such that $M_{\mathscr{A}'}(B_{\gamma_1} \cdot q) \geq M(q)$ for all $\mathscr{A}' \subset \mathscr{A}$ non-empty. By Theorem 5.4, if M_0 is sufficiently large we have

$$(\mathbf{6.2}) \qquad P_q(\Gamma_1 \setminus \tilde{\Gamma}_1 \,|\, \pi) < \frac{1}{2}.$$

For $\pi_e \in \mathfrak{R}$, let γ_{π_e} be a shortest possible path starting at π_e with $\gamma_{\pi_e} = \gamma_s\widehat{\gamma}$. If M_0 is sufficiently large then $\|B_{\gamma_{\pi_e}}\| < \frac{1}{d}2^{M_0-1}$. It follows that if $\gamma_1 \in \Gamma_1$ ends at π_e then

$$(\mathbf{6.3}) \qquad P_q(\Gamma \,|\, \gamma_1) \leq 1 - P_{B_{\gamma_1}\cdot q}(\gamma_{\pi_e} \,|\, \pi_e).$$

If furthermore $\gamma_1 \in \tilde{\Gamma}_1$ then

$$(\mathbf{6.4}) \qquad P_{B_{\gamma_1}\cdot q}(\gamma_{\pi_e} \,|\, \pi_e) = \frac{N(B_{\gamma_1} \cdot q)}{N(B_{\gamma_{\pi_e}} \cdot B_{\gamma_1} \cdot q)} \geq \frac{M(q)^d}{(2^{2M_0}M(q))^d} = 2^{-2dM_0}.$$

The result follows with $\rho = 1 - 2^{-2dM_0-1}$. □

Proposition **6.2.** — *For every* $\widehat{\gamma} \in \Pi(\mathfrak{R})$, *there exist* $\delta > 0$, $C > 0$ *such that for every* $\pi \in \mathfrak{R}$, $q \in \mathbf{R}_+^{\mathscr{A}}$ *and for every* $T > 1$

$$(\mathbf{6.5}) \qquad P_q(\gamma \text{ cannot be written as } \gamma_s\widehat{\gamma}\gamma_e \text{ and } M(B_\gamma \cdot q) > TM(q) \,|\, \pi) \leq CT^{-\delta}.$$

Proof. — Let M and ρ be as in the previous lemma. Let k be maximal with $T \geq 2^{k(M+1)}$. Let Γ be the set of minimal paths γ such that γ is not of the form $\gamma_s\widehat{\gamma}\gamma_e$ and $M(B_\gamma \cdot q) > 2^{k(M+1)}M(q)$. Any path $\gamma \in \Gamma$ can be written as $\gamma_1 \ldots \gamma_k$ where $\gamma_{(i)} = \gamma_1 \ldots \gamma_i$ is minimal with $M(B_{\gamma_{(i)}} \cdot q) > 2^{i(M+1)}M(q)$. Let $\Gamma_{(i)}$ collect the $\gamma_{(i)}$. Then the $\Gamma_{(i)}$ are disjoint. Moreover, by Lemma 6.1, for all $\gamma^{(i)} \in \Gamma_{(i)}$,

$$(\mathbf{6.6}) \qquad P_q\big(\Gamma_{(i+1)} \,\big|\, \gamma^{(i)}\big) \leq \rho.$$

This implies that $P_q(\Gamma \,|\, \pi) \leq \rho^k$. The result follows. □

Proof of Theorem 4.6. — Let π be the start of γ_*. The push-forward under radial projection of the measure ν_{q_0} onto $\Delta_\pi \cap \Upsilon^{(1)}_{\mathfrak{R}}$ yields a smooth measure $\tilde{\nu}$. It is enough to show that $\tilde{\nu}\{x \in \Xi : r_\Xi(x) \geq \log T\} \leq CT^{-\delta}$, for some $C > 0$, $\delta > 0$. A connected component of the domain of T_Ξ that intersects the set $\{x \in \Xi : r_\Xi(x) \geq \log T\}$ is of the form $\Delta_\gamma \cap \Upsilon^{(1)}_{\mathfrak{R}}$ where γ cannot be written as $\gamma_s \widehat{\gamma} \gamma_e$ with $\widehat{\gamma} = \gamma_* \gamma_* \gamma_* \gamma_*$ and $M(B_\gamma \cdot q_0) \geq C^{-1}T$, where $q_0 = (1, \ldots, 1)$ and C is a constant depending on γ_*. Thus

$$\begin{aligned}&\tilde{\nu}\{x \in \Xi : r_\Xi(x) \geq \log T\}\\ &\quad\leq P_{q_0}\big(\gamma \text{ can not be written as } \gamma_s \widehat{\gamma} \gamma_e \text{ and } M(B_\gamma \cdot q_0) \geq C^{-1}T \,\big|\, \pi\big).\end{aligned}$$

The result follows from the previous proposition. ◻

Lemma **6.3.** — *For every* $k_0 \geq 1$ *there exist* $C > 0$, $\theta > 0$, *depending only on* $\#\mathscr{A}$ *and* k_0 *with the following property. Let* $M \in \mathbf{N}$, $q \in \mathbf{R}^{\mathscr{A}}_+$. *Then for every* $\pi \in \mathfrak{R}$,

$$P_q\big(\gamma \textit{ is not } k_0\textit{-complete and } M(B_\gamma \cdot q) > 2^M M(q) \,\big|\, \pi\big) \leq C(M+1)^\theta 2^{-M}.$$

Proof. — The proof is by induction on k_0. For $k_0 = 1$, it is Proposition 5.7. Assume it holds for some $k_0 \geq 1$. Let Γ be the set of minimal paths which are not k_0+1-complete and such that $M(B_\gamma \cdot q) > 2^M M(q)$. Let $\Gamma_- \subset \Gamma$ be the set of paths which are not k_0-complete. Then $P_q(\Gamma_- \,|\, \pi) \leq C(M+1)^\theta 2^{-M}$ by the induction hypothesis. Every $\gamma \in \Gamma \setminus \Gamma_-$ can be written as $\gamma = \gamma_s \gamma_e$ with γ_s minimal k_0-complete. Let $m = m(\gamma_s) \in [0, M]$ be such that $2^m M(q) < M(B_{\gamma_s} \cdot q) \leq 2^{m+1} M(q)$. Let Γ_m collect the γ_s with $m(\gamma_s) = m$. Then Γ_m is disjoint. By the induction hypothesis $P_q(\Gamma_m \,|\, \pi) \leq C(m+1)^\theta 2^{-m}$ and by Proposition 5.7, $P_q(\Gamma \,|\, \gamma_s) \leq (M+1-m)^\theta 2^{m-M}$, $\gamma_s \in \Gamma_m$. The result follows by summing over m. ◻

Proposition **6.4.** — *For every* $k_0 \geq 2\#\mathscr{A} - 3$, $\delta > 0$, *there exist* $C > 0$ *and a finite disjoint set* $\Gamma_0 \subset \Pi(\mathfrak{R})$ *with the following properties:*

1. *If* $\gamma \in \Gamma_0$ *then* γ *is minimal* k_0*-complete,*
2. *For every* $\pi \in \mathfrak{R}$, $q \in \mathbf{R}^{\mathscr{A}}_+$, $T \geq 0$,

$$P_q(\gamma \textit{ cannot be written as } \gamma_s \gamma_0 \gamma_e \textit{ with } \gamma_0 \in \Gamma_0 \textit{ and } M(B_\gamma \cdot q) > TM(q) \,|\, \pi) \leq CT^{(\delta-1)}. \tag{6.7}$$

Proof. — Fix some $M \geq 0$. Let Γ_0 be the set of all minimal paths which are k_0-complete and such that $\|B_\gamma\| \leq 2^{M+2}$. Obviously Γ_0 satisfies condition (1). Let us show that if M is large then it also satisfies condition (2). It is sufficient to prove (6.7) for times T of the form $2^{k(M+1)}$.

For $k \geq 0$, let Γ be the set of paths γ such that γ is not of the form $\gamma_s\gamma_0\gamma_e$ with $\gamma_0 \in \Gamma_0$ and $M(B_\gamma \cdot q) > 2^{k(M+1)}M(q)$. Any path $\gamma \in \Gamma$ can be written as $\gamma_1 ... \gamma_k$ where $\gamma_{(i)} = \gamma_1 ... \gamma_i$ is minimal with $M(B_{\gamma_{(i)}} \cdot q) > 2^{i(M+1)}M(q)$. Let $\Gamma_{(i)}$ collect the $\gamma_{(i)}$. Then the $\Gamma_{(i)}$ are disjoint.

Notice that the γ_i are not $2k_0$-complete. Otherwise, $\gamma_i = \gamma_s\gamma_e$ with γ_s and γ_e k_0-complete. By Lemma 3.3, all coordinates of $B_{\gamma_s} \cdot B_{\gamma_{(i-1)}} \cdot q$ are larger than $M(B_{\gamma_{(i-1)}} \cdot q) > 2^{(i-1)(M+1)}$. It follows that $\|B_{\gamma_e}\| \leq 2^{M+2}$, so $\gamma_e \in \Gamma_0$, contradiction.

By the previous lemma, $P_q(\Gamma_{(i)} \mid \gamma_s) \leq C(M+1)^\theta 2^{-M}$, $\gamma_s \in \Gamma_{(i-1)}$. This implies that $P_q(\Gamma \mid \pi) \leq (C(M+1)^\theta 2^{-M})^k$. If M is large enough, this gives $P_q(\Gamma \mid \pi) \leq 2^{(\delta-1)k(M+1)}$. □

Proof of Theorem 4.9. — Let Γ_0 be as in the previous proposition, with $k_0 = 6\#\mathscr{A} - 8$. We let $\widehat{Z} = \bigcup \Delta_{\gamma_e} \times \Theta_{\gamma_s}$ where γ_s is minimal $4\#\mathscr{A} - 6$ complete, γ_e is minimal $2\#\mathscr{A} - 3$-complete and there exists $\gamma \in \Gamma_0$ that starts by $\gamma_s\gamma_e$. Its intersection with $\widehat{\Upsilon}^{(1)}$ is precompact by Lemmas 3.3 and 4.2.

Fix some component $\Delta_{\gamma_{e_0}} \times \Theta_{\gamma_{s_0}}$ of $\widehat{Z}$ and let us estimate $\widehat{m}\{x \in \Delta_{\gamma_{e_0}} \times \Theta_{\gamma_{s_0}} \cap \widehat{\Upsilon}^{(1)}_{\mathfrak{R}} : r_{\widehat{Z}}(x) > \log T\}$. Let π be the start of γ_{s_0}. If $\Delta_{\gamma_1} \times \Theta_{\gamma_2}$ is a component of the domain of the first return map to $\widehat{Z}$ that intersects $\{x \in \Delta_{\gamma_{e_0}} \times \Theta_{\gamma_{s_0}} : r_{\widehat{Z}}(x) > \log T\}$ then γ_1 cannot be written as $\gamma_s\gamma_0\gamma_e$ with $\gamma_0 \in \Gamma_0$. The projection of $\widehat{m}|\Delta_{\gamma_{e_0}} \times \Theta_{\gamma_{s_0}} \cap \widehat{\Upsilon}^{(1)}_{\mathfrak{R}}$ on $\Upsilon^{(1)}_{\mathfrak{R}}$ is absolutely continuous with a bounded density, so we conclude as in the proof of Theorem 4.6 that

$$\begin{aligned}\widehat{m}\Big\{x \in \Delta_{\gamma_{e_0}} \times \Theta_{\gamma_{s_0}} \cap \widehat{\Upsilon}^{(1)}_{\mathfrak{R}} : r_{\widehat{Z}}(x) > \log T\Big\} \\ \leq C P_{q_0}(\gamma \text{ cannot be written as } \gamma_s\gamma_0\gamma_e \text{ with } \gamma_0 \in \Gamma_0 \\ \text{and } M(B_\gamma \cdot q_0) > T \mid \pi),\end{aligned}$$

where $q_0 = (1, ..., 1)$. The result follows from the previous proposition. □

7. Exponential mixing for expanding semiflows

In this section and the next, our goal is to prove Theorem 2.7. As a first step, we will prove in this section an analogous result concerning expanding semiflows.

Let $T : \bigcup \Delta^{(l)} \to \Delta$ be a uniformly expanding Markov map on a John domain (Δ, Leb), with expansion constant $\kappa > 1$, and let $r : \Delta \to \mathbf{R}_+$ be a good roof function with exponential tails (as defined in Paragraph 2.1). Let $\Delta_r = \{(x, t) : x \in \Delta, 0 \leq t < r(x)\}$, we define a semi-flow $T_t : \Delta_r \to \Delta_r$, by $T_t(x, s) = (T^n x, s + t - r^{(n)}(x))$ where n is the unique integer satisfying $r^{(n)}(x) \leq t + s < r^{(n+1)}(x)$. Let μ be the absolutely continuous probability measure on Δ which is invariant under T, then the flow T_t preserves the probability measure $\mu_r = \mu \otimes \mathrm{Leb}/(\mu \otimes \mathrm{Leb})(\Delta_r)$.

We will also use the finite measure $\mathrm{Leb}_r = \mathrm{Leb} \otimes \mathrm{Leb}$ on Δ_r. In this section, we will be interested in the mixing properties of T_t. *Unless otherwise specified, all the integrals will be taken with respect to the measures* Leb *or* Leb_r.

Let us first define the class of functions for which we can prove exponential decay of correlations:

Definition **7.1.** — *A function* $U : \Delta_r \to \mathbf{R}$ *belongs to* $\mathscr{B}_0$ *if it is bounded, continuously differentiable on each set* $\Delta_r^{(l)} := \{(x,t) : x \in \Delta^{(l)},\ 0 < t < r(x)\}$, *and* $\sup_{(x,t)\in\bigcup\Delta_r^{(l)}} \|DU(x,t)\| < \infty$. *Write then*

$$\|U\|_{\mathscr{B}_0} = \sup_{(x,t)\in\bigcup\Delta_r^{(l)}} |U(x,t)| + \sup_{(x,t)\in\bigcup\Delta_r^{(l)}} \|DU(x,t)\|. \tag{7.1}$$

Notice that such a function is not necessarily continuous on the boundary of $\Delta_r^{(l)}$.

Definition **7.2.** — *A function* $U : \Delta_r \to \mathbf{R}$ *belongs to* $\mathscr{B}_1$ *if it is bounded and there exists a constant* $C > 0$ *such that, for all fixed* $x \in \bigcup_l \Delta^{(l)}$, *the function* $t \mapsto U(x,t)$ *is of bounded variation on the interval* $(0, r(x))$ *and its variation is bounded by* $Cr(x)$. *Let*

$$\|U\|_{\mathscr{B}_1} = \sup_{(x,t)\in\bigcup\Delta_r^{(l)}} |U(x,t)| + \sup_{x\in\bigcup\Delta^{(l)}} \frac{\mathrm{Var}_{(0,r(x))}(t\mapsto U(x,t))}{r(x)}. \tag{7.2}$$

This space $\mathscr{B}_1$ is very well suited for further extensions to the hyperbolic case. In this paper, the notation $C^1(X)$ for some space X always denotes the space of bounded continuous functions on X which are everywhere continuously differentiable and such that the norms of the differentials are bounded. Then the following inclusions hold:

$$C^1 \subset \mathscr{B}_0 \subset \mathscr{B}_1. \tag{7.3}$$

Theorem **7.3.** — *There exist constants* $C > 0$ *and* $\delta > 0$ *such that, for all functions* $U \in \mathscr{B}_0$ *and* $V \in \mathscr{B}_1$, *for all* $t \geq 0$,

$$\left|\int U\cdot V\circ T_t\,\mathrm{dLeb}_r - \left(\int U\,\mathrm{dLeb}_r\right)\left(\int V\,\mathrm{d}\mu_r\right)\right| \leq C\|U\|_{\mathscr{B}_0}\|V\|_{\mathscr{B}_1}e^{-\delta t}. \tag{7.4}$$

Remark. — Applying the previous theorem to the function $U(x,t)\cdot\frac{\mathrm{d}\mu}{\mathrm{dLeb}}(x)$, we also obtain

$$\left|\int U\cdot V\circ T_t\,\mathrm{d}\mu_r - \left(\int U\,\mathrm{d}\mu_r\right)\left(\int V\,\mathrm{d}\mu_r\right)\right| \leq C\|U\|_{\mathscr{B}_0}\|V\|_{\mathscr{B}_1}e^{-\delta t}. \tag{7.5}$$

Notation: when dealing with a uniformly expanding Markov map T, we will always denote by $\mathscr{H}_n$ the set of inverse branches of T^n.

The proof of Theorem 7.3 will take the rest of this section.

7.1. *Discussion of the aperiodicity condition.* — In this paragraph, we discuss several conditions on the return time r which turn out to be equivalent to the aperiodicity condition (3) in Definition 2.3.

Proposition **7.4.** — *Let* T *be a uniformly expanding Markov map for a partition* $\{\Delta^{(l)}\}$. *Let* $r : \Delta \to \mathbf{R}$ *be a function which is* C^1 *on each set* $\Delta^{(l)}$, *with* $\sup_{h\in\mathscr{H}} \|D(r\circ h)\|_{C^0} < \infty$. *Then the following conditions are equivalent:*

1. *There exists* $C > 0$ *such that there exists an arbitrarily large* n, *there exist* $h, k \in \mathscr{H}_n$, *there exists a continuous unitary vector field* $x \mapsto y(x)$ *such that, for all* $x \in \Delta$,

$$\left|D(r^{(n)} \circ h)(x) \cdot y(x) - D(r^{(n)} \circ k)(x) \cdot y(x)\right| > C. \tag{7.6}$$

2. *There exists* $C > 0$ *such that there exists an arbitrarily large* n, *there exist* $h, k \in \mathscr{H}_n$, *there exists* $x \in \Delta$ *and* $y \in T_x\Delta$ *with* $\|y\| = 1$ *such that*

$$\left|D(r^{(n)} \circ h)(x) \cdot y - D(r^{(n)} \circ k)(x) \cdot y\right| > C. \tag{7.7}$$

3. *It is not possible to write* $r = \psi + \phi \circ T - \phi$ *on* $\bigcup \Delta^{(l)}$, *where* $\psi : \Delta \to \mathbf{R}$ *is constant on each set* $\Delta^{(l)}$ *and* $\phi \in C^1(\Delta)$.
4. *It is not possible to write* $r = \psi + \phi \circ T - \phi$ *almost everywhere, where* $\psi : \Delta \to \mathbf{R}$ *is constant on each set* $\Delta^{(l)}$ *and* $\phi : \Delta \to \mathbf{R}$ *is measurable.*

The first condition is the (UNI) condition as given in [BV] in their one-dimensional setting.

Proof. — The implication (1) $\Rightarrow$ (2) is trivial. Let us prove (2) $\Rightarrow$ (1). Notice that there exists a constant c_0 such that, for any inverse branch $\ell \in \mathscr{H}_p$ of any iterate T^p of T, for any $x \in \Delta$ and any $y \in T_x\Delta$, $|D(r^{(p)} \circ \ell)(x) \cdot y| \leq c_0\|y\|$: for instance, take $c_0 = \frac{\sup_{h\in\mathscr{H}} \|D(r\circ h)\|_{C^0}}{1-\kappa^{-1}}$.

Let $C > 0$ be such that (7.7) is satisfied for infinitely many n. It is then possible to choose n large enough so that $c_0\kappa^{-n} \leq C/4$, $h, k \in \mathscr{H}_n$, $x_0 \in \Delta$ and $y_0 \in T_x\Delta$ such that (7.7) holds. Let $y_0(x)$ be a unitary vector field on a neighborhood U of x such that (7.7) still holds for $y_0(x)$. Fix a branch $l \in \mathscr{H}_m$ for some m such that $l(\Delta) \subset U$. Define a vector field y_1 on Δ by $y_1(x) = Dl(x)^{-1}y_0(lx)$. For any inverse branch $\ell \in \mathscr{H}_p$ for some $p \geq 1$, we have

$$\begin{aligned}\left|D(r^{(m+n+p)} \circ \ell \circ h \circ l)(x) \cdot y_1(x) - D(r^{(m+n)} \circ h \circ l)(x) \cdot y_1(x)\right|& \\ = \left|D(r^{(p)} \circ \ell)(hlx)Dh(lx) \cdot y_0(lx)\right|& \\ \leq c_0\|Dh(lx)\| \leq c_0\kappa^{-n} \leq C/4.&\end{aligned}$$

The same estimate applies to k. Since

$$\begin{aligned}&|D(r^{(m+n)} \circ h \circ l)(x) \cdot y_1(x) - D(r^{(m+n)} \circ k \circ l)(x) \cdot y_1(x)| \\ &\qquad = |D(r^{(n)} \circ h)(lx) \cdot y_0(lx) - D(r^{(n)} \circ k)(lx) \cdot y_0(lx)| \geq C,\end{aligned}$$

we get

$$\begin{aligned}&|D(r^{(m+n+p)} \circ \ell \circ h \circ l)(x) \cdot y_1(x) - D(r^{(m+n+p)} \circ \ell \circ k \circ l)(x) \cdot y_1(x)| \\ &\qquad \geq C/2.\end{aligned}$$

Finally, take $y(x) = y_1(x)/\|y_1(x)\|$. This proves (1).

The implication (2) $\Rightarrow$ (3) is easy: if it is possible to write $r = \psi + \phi \circ T - \phi$, then for all $h \in \mathscr{H}_n$, $r^{(n)} \circ h(x) = S_n\psi(h(x)) + \phi(x) - \phi(hx)$. Hence, if $\|y\| = 1$,

$$\begin{aligned}\left|D(r^{(n)} \circ h)(x) \cdot y - D(r^{(n)} \circ k)(x) \cdot y\right| &= \left|D(\phi \circ h)(x) \cdot y - D(\phi \circ k)(x) \cdot y\right| \\ &\leq 2\|\phi\|_{C^1}\kappa^{-n}.\end{aligned}$$

This quantity tends to 0 when $n \to \infty$, which is not compatible with (2).

Let us prove (3) $\Rightarrow$ (2). Assume that (2) does not hold, we will prove that r can be written as $\psi + \phi \circ T - \phi$. Let $\underline{h} = (h_1, h_2, \ldots)$ be a sequence of $\mathscr{H}$. Write $\underline{h}_n = h_n \circ \ldots \circ h_1$. Then

$$D\big(r^{(n)} \circ \underline{h}_n\big)(x) \cdot y = \sum_{k=1}^{n} D(r \circ h_k)(\underline{h}_{k-1}x) D\underline{h}_{k-1}(x) \cdot y. \tag{7.8}$$

The derivative of $r \circ h_k$ is uniformly bounded by assumption and $\|D\underline{h}_{k-1}(x)\| \leq \kappa^{-k+1}$. Therefore, this series is uniformly converging. Since (2) is not satisfied, its limit is independent of the sequence of inverse branches $\underline{h}$, and defines a continuous 1-form $\omega(x) \cdot y$ on Δ. It satisfies, for all $h \in \mathscr{H}$,

$$\omega(x) \cdot y = D(r \circ h)(x) \cdot y + \omega(hx) Dh(x) \cdot y. \tag{7.9}$$

Take a branch $h \in \mathscr{H}$, and let $\underline{h} = (h, h, \ldots)$. Let $x_0 \in \Delta$. The series of functions $\sum_{k=1}^{\infty}(r \circ \underline{h}_k - r \circ \underline{h}_k(x_0))$ is then summable in $C^1(\Delta)$, let us denote its sum by ϕ. By construction, $\omega(x) \cdot y = D\phi(x) \cdot y$ for all $x \in \Delta$ and $y \in T_x\Delta$. By (7.9), $D(r + \phi - \phi \circ T) = 0$. Hence, $r + \phi - \phi \circ T$ is constant on each $\Delta^{(l)}$, which concludes the proof.

The implication (4) $\Rightarrow$ (3) is trivial, we just have to prove (3) $\Rightarrow$ (4) to conclude. Assume that $r = \psi + \phi \circ T - \phi$ where ψ is constant on each set $\Delta^{(l)}$ and ϕ is measurable. We will prove that ϕ has a version which is C^1. Let $\mathscr{F}_n$ be the σ-algebra generated by the sets $h(\Delta)$ for $h \in \mathscr{H}_n$. It is an increasing sequence of σ-algebras. For almost all $x \in \Delta$, there exists a well defined sequence

$\overline{h} = (h_1, h_2, \ldots) \in \mathscr{H}^{\mathbf{N}}$ such that the element $F_n(x)$ of $\mathscr{F}_n$ containing x is given by $F_n(x) = h_1 \circ \ldots \circ h_n(\Delta)$. Equivalently, h_n is the unique element of $\mathscr{H}$ such that $T^{n-1}(x) \in h_n(\Delta)$. Since T is ergodic, almost every x is normal in the sense that, for any finite sequence $k_1, \ldots, k_p$ of elements of $\mathscr{H}$, there exist infinitely many n such that, for all $1 \leq i \leq p$, $h_{n+i} = k_i$.

The martingale convergence theorem shows that, for almost all $x \in \Delta$, for all $\varepsilon > 0$,

$$\textbf{(7.10)} \qquad \frac{\operatorname{Leb}\{x' \in F_n(x) \,:\, |\phi(x') - \phi(x)| > \varepsilon\}}{\operatorname{Leb}(F_n(x))} \to 0.$$

Take a point x_0 such that this convergence holds and which is normal. Replacing ϕ by $\phi - \phi(x_0)$, we can assume that $\phi(x_0) = 0$. Let $\overline{h} = (h_1, h_2, \ldots)$ be the corresponding sequence of $\mathscr{H}$ and write $\overline{h}_n = h_1 \circ \ldots \circ h_n$, so that $F_n(x_0) = \overline{h}_n(\Delta)$. Then (7.10) and distortion controls give, for all $\varepsilon > 0$,

$$\textbf{(7.11)} \qquad \operatorname{Leb}\left\{x \in \Delta \,:\, \left|\phi(\overline{h}_n x)\right| > \varepsilon\right\} \to 0.$$

Define a strictly increasing sequence m_k as follows: start from $m_1 = 1$. If m_k has been defined then, by normality of x_0, there exists $m_{k+1} > m_k$ such that $(h_1, \ldots, h_{m_{k+1}})$ finishes with $(h_1, \ldots, h_{m_k})$. By (7.11), we can choose a subsequence n_k of m_k such that

$$\textbf{(7.12)} \qquad \forall \varepsilon > 0, \ \sum_{k=1}^{\infty} \operatorname{Leb}\left\{x \in \Delta \,:\, \left|\phi(\overline{h}_{n_k} x)\right| > \varepsilon\right\} < \infty.$$

In particular, for almost all x, $\phi(\overline{h}_{n_k} x) \to 0$. Notice that $\phi(x) = \phi(\overline{h}_n x) + r^{(n)}(\overline{h}_n x) - S_n\psi(\overline{h}_n x)$. For almost all x, we get $\phi(x) = \lim_{k\to\infty} r^{(n_k)}(\overline{h}_{n_k} x) - S_{n_k}\psi(\overline{h}_{n_k} x)$. Moreover, the choice of m_k ensures that the sequence $D(r^{(n_k)} \circ \overline{h}_{n_k})$ is Cauchy. Hence, ϕ coincides almost everywhere with the C^1 function $\lim_{k\to\infty} r^{(n_k)} \circ \overline{h}_{n_k} - S_{n_k}\psi \circ \overline{h}_{n_k}$, which concludes the proof. □

7.2. *Existence of bump functions.* — The following technical lemma will prove useful later.

Lemma **7.5.** — *There exist constants* $C_1 > 1$ *and* $C_2 > 0$ *satisfying the following property: for any ball* $B(x, r)$ *compactly included in* Δ, *there exists a* C^1 *function* $\rho : \Delta \to [0, 1]$ *such that* $\rho = 0$ *on* $\Delta \backslash B(x, r)$, $\rho = 1$ *on* $B(x, r/C_1)$ *and* $\|\rho\|_{C^1} \leq C_2/r$.

Notice that this property is not true for any John domain, and uses the existence of the uniformly expanding Markov map T on Δ.

Proof. — Let $x_0 \in \Delta$ be in the domain of definition of all iterates of T. Let $\|\cdot\|'$ be a flat Riemannian metric on a neighborhood of x. By compactness, there exists a constant $K > 0$ such that, on a small neighborhood U of x_0, $K^{-1}\|\cdot\|' \leq \|\cdot\| \leq K\|\cdot\|'$.

For large enough n, the inverse branch $h \in \mathscr{H}_n$ such that $x_0 \in h(\Delta)$ satisfies $h(\Delta) \subset U$, since $\operatorname{diam}(h(\Delta)) \leq C\kappa^{-n}$. The set $h(\Delta)$ endowed with the distance given by $\|\cdot\|'$ is flat. Hence, there exists a constant $C > 0$ such that, given any ball $B' = B'(x, r)$ for this Euclidean distance, which is compactly included in $h(\Delta)$, there exists a C^1 function ρ supported in B', equal to 1 on $B'(x, r/2)$ and with $\|\rho\|_{C^1} \leq C/r$.

Since h and its inverse have uniformly bounded derivatives (with respect to $\|\cdot\|$ and $\|\cdot\|'$), this easily implies the lemma. □

The same compactness argument also implies the following lemma:

Lemma **7.6.** — *For all* $\varepsilon > 0$,

$$\sup\{k \in \mathbf{N} : \exists x_1, \dots, x_k \in \Delta \text{ with } d(x_i, x_j) \geq \varepsilon \text{ whenever } i \neq j\} < \infty.$$

7.3. *A Dolgopyat-like spectral estimate.* — The main step of the proof of Theorem 7.3 is the study of the spectral properties of weighted transfer operators L_s. Let $\sigma_0 > 0$ be such that $\int e^{\sigma_0 r}\,\mathrm{dLeb} < \infty$, which is possible since r has exponential tails. For $s \in \mathbf{C}$ with $\Re s > -\sigma_0$, define

$$L_s u(x) = \sum_{Ty=x} e^{-sr(y)} J(y) u(y). \tag{7.13}$$

For $s = \sigma + it$ with $\Re s > -\sigma_0$ and $t \in \mathbf{R}$, define a norm on $C^1(\Delta, \mathbf{C})$ by

$$\|u\|_{1,t} = \sup_{x\in\Delta} |u(x)| + \frac{1}{\max(1, |t|)} \sup_{x\in\Delta} \|Du(x)\|. \tag{7.14}$$

The main spectral estimate concerning the operators L_s is the following Dolgopyat-like estimate:

Proposition **7.7.** — *There exist* $\sigma_0' \leq \sigma_0$, $T_0 > 0$, $C > 0$ *and* $\beta < 1$ *such that, for all* $s = \sigma + it$ *with* $|\sigma| \leq \sigma_0'$ *and* $|t| \geq T_0$, *for all* $u \in C^1(\Delta)$, *for all* $k \in \mathbf{N}$,

$$\left\|L_s^k u\right\|_{L^2} \leq C\beta^k \|u\|_{1,t}. \tag{7.15}$$

This paragraph will be entirely devoted to the proof of Proposition 7.7. The proof will follow very closely the arguments in [BV], with small complications due to the general dimension.

For $s = 0$, L_s is the usual transfer operator. It acts on the space of C^1 functions, has a spectral gap, and a simple isolated eigenvalue at 1 (the corresponding eigenfunction will be denoted by f_0 and is the density of the invariant measure μ). For $\sigma \in \mathbf{R}$ close enough to 0, L_σ acting on $C^1(\Delta)$ is a continuous perturbation of L_0, by a straightforward computation. Hence, it has a unique eigenvalue λ_σ close to 1, and the corresponding eigenfunction f_σ (normalized so that $\int f_\sigma = 1$) is C^1, strictly positive, and tends to f_0 in the C^1 topology when $\sigma \to 0$.

Let $0 < \sigma_1 \leq \min(\sigma_0, 1)$ be such that f_σ is well defined and uniformly bounded from below for $\sigma \in [-\sigma_1, \sigma_1]$. For $s = \sigma + it$ with $|\sigma| \leq \sigma_1$ and $t \in \mathbf{R}$, define a modified transfer operator $\tilde{L}_s$ by

$$\tilde{L}_s(u) = \frac{L_s(f_\sigma u)}{\lambda_\sigma f_\sigma}. \tag{7.16}$$

It satisfies $\tilde{L}_\sigma 1 = 1$, and $|\tilde{L}_s u| \leq \tilde{L}_\sigma |u|$.

Lemma **7.8.** — *There exists a constant* C_3 *such that* $\forall n \geq 1$, $\forall s = \sigma + it$ *with* $\sigma \in [-\sigma_1, \sigma_1]$ *and* $t \in \mathbf{R}$, $\forall u \in C^1(\Delta)$, *holds for all* $x \in \Delta$

$$\left\| D\big(\tilde{L}_s^n u\big)(x) \right\| \leq C_3(|t| + 1)\tilde{L}_\sigma^n(|u|)(x) + \kappa^{-n}\tilde{L}_\sigma^n(\|Du\|)(x). \tag{7.17}$$

Proof. — We have

$$\tilde{L}_s^n u(x) = \sum_{h \in \mathcal{H}_n} \frac{(f_\sigma u)(hx) J^{(n)}(hx) e^{-sr^{(n)}(hx)}}{\lambda_\sigma^n f_\sigma(x)}, \tag{7.18}$$

where $r^{(n)}(x) = \sum_{k=0}^{n-1} r(T^k x)$ and $J^{(n)}(x) = \prod_{k=0}^{n-1} J(T^k x)$. Differentiating this expression, we obtain a sum of 5 terms: we can differentiate f_σ, or u, or $J^{(n)}$, or $r^{(n)}$, or $1/f_\sigma$.

Since f_σ is bounded in C^1 and uniformly bounded from below, and any inverse branch of T is contracting, there exists a constant $C > 0$ such that $\|D(f_\sigma \circ h)(x)\| \leq C f_\sigma(x)$. Hence, if we differentiate f_σ, the resulting term is bounded by $C\tilde{L}_\sigma^n(|u|)(x)$.

In the same way, distortion controls give $\|D(J^{(n)} \circ h)(x)\| \leq C J^{(n)} \circ h(x)$. We also have $\|D(1/f_\sigma)(x)\| \leq C/f_\sigma(x)$. Hence, the corresponding terms are also bounded by $C\tilde{L}_\sigma^n(|u|)(x)$.

Moreover, $D(e^{-sr^{(n)} \circ h})(x) = -sD(r^{(n)} \circ h)(x) e^{-sr^{(n)} \circ h(x)}$. The uniform contraction of h and the boundedness of the derivative of $r \circ \ell$ for $\ell \in \mathcal{H}$ show that this term is bounded by $C|s| e^{-\sigma r^{(n)} \circ h(x)}$. Hence, the resulting term is bounded by $C(|t| + 1)\tilde{L}_\sigma^n(|u|)(x)$.

Finally, $\|D(u \circ h)(x)\| \leq \kappa^{-n} \|Du(hx)\|$, which shows the required bound on the last term. □

From this point on, we will fix once and for all a constant $C_3 > 5$ satisfying the conclusion of Lemma 7.8. This lemma implies that the iterates of $\tilde{L}_s$ are bounded for the norm $\| \ \|_{1,t}$. More precisely, the following holds:

Lemma **7.9.** — *There exists a constant* $C > 1$ *such that, for all* $s = \sigma + it$ *with* $\sigma \in [-\sigma_1, \sigma_1]$ *and* $|t| \geq 10$, *for all* $k \in \mathbf{N}$, *for all* $u \in C^1(\Delta)$,

$$\left\| \tilde{L}_s^k u \right\|_{1,t} \leq C \|u\|_{C^0} + \frac{\kappa^{-k}}{|t|} \|Du\|_{C^0}. \tag{7.19}$$

In particular, $\left\| \tilde{L}_s^k u \right\|_{1,t} \leq C \|u\|_{1,t}$.

Proof. — The inequality $\left\| \tilde{L}_s^k u \right\|_{C^0} \leq \|u\|_{C^0}$ and Lemma 7.8 give

$$\begin{aligned} \left\| \tilde{L}_s^k u \right\|_{C^0} + \frac{\left\| D\left(\tilde{L}_s^k u\right) \right\|_{C^0}}{|t|} &\leq \|u\|_{C^0} + \frac{1}{|t|} \left[2C_3 |t| \|u\|_{C^0} + \kappa^{-k} \|Du\|_{C^0} \right] \\ &\leq C \|u\|_{C^0} + \frac{\kappa^{-k}}{|t|} \|Du\|_{C^0}. \end{aligned}$$

□

To prove Proposition 7.7, we need to get some contraction. This is easy to do if the derivative is large compared to the C^0 norm of the function:

Lemma **7.10.** — *There exists* $N_0 \in \mathbf{N}$ *such that any* $n \geq N_0$ *satisfies the following property. Let* $s = \sigma + it$ *with* $\sigma \in [-\sigma_1, \sigma_1]$ *and* $|t| \geq 10$. *Let* $v \in C^1(\Delta)$ *satisfy* $\sup \|Dv\| \geq 2C_3 |t| \sup |v|$. *Then*

$$\left\| \tilde{L}_s^n v \right\|_{1,t} \leq \frac{9}{10} \|v\|_{1,t}. \tag{7.20}$$

Proof. — We have

$$\left\| \tilde{L}_s^n v \right\|_{C^0} \leq \|v\|_{C^0} \leq \frac{1}{2C_3 |t|} \sup \|Dv(x)\| \leq \frac{1}{2C_3} \|v\|_{1,t}. \tag{7.21}$$

Moreover, for $x \in \Delta$

$$\begin{aligned} \left\| D\left(\tilde{L}_s^n v\right)(x) \right\| &\leq C_3 (1 + |t|) \tilde{L}_\sigma^n (|v|)(x) + \kappa^{-n} \tilde{L}_\sigma^n (\|Dv\|)(x) \\ &\leq C_3 (1 + |t|) \|v\|_{C^0} + \kappa^{-n} \|Dv\|_{C^0} \\ &\leq \left[\frac{1 + |t|}{2} + \kappa^{-n} |t| \right] \|v\|_{1,t}. \end{aligned}$$

Hence,

$$\|\tilde{L}_s^n v\|_{C^0} + \frac{1}{|t|}\|D(\tilde{L}_s^n v)\|_{C^0} \leq \left[\frac{1}{2C_3} + \frac{1+|t|}{2|t|} + \kappa^{-n}\right]\|v\|_{1,t}. \tag{7.22}$$

Since $C_3 \geq 5$ and $|t| \geq 10$, the conclusion of the lemma holds as soon as $\kappa^{-n} \leq \frac{1}{5}$. □

Hence, to prove Proposition 7.7, we will mainly have to deal with functions v satisfying $\sup\|Dv\| \leq 2C_3|t|\sup|v|$. For technical reasons, it is more convenient to introduce the following notation.

Definition **7.11.** — *For $t \in \mathbf{R}$, we will say that a pair (u, v) of functions on Δ belongs to $\mathscr{E}_t$ if $u : \Delta \to \mathbf{R}_+$ is C^1, $v : \Delta \to \mathbf{C}$ is C^1, $0 \leq |v| \leq u$ and*

$$\forall x \in \Delta, \quad \max(\|Du(x)\|, \|Dv(x)\|) \leq 2C_3|t|u(x). \tag{7.23}$$

Lemma **7.12.** — *There exists $N_1 \in \mathbf{N}$ such that any $n \geq N_1$ satisfies the following property. Let $s = \sigma + it$ with $\sigma \in [-\sigma_1, \sigma_1]$ and $|t| \geq 10$. Let $(u, v) \in \mathscr{E}_t$. Let $\chi \in C^1(\Delta)$ with $\|D\chi\| \leq |t|$ and $3/4 \leq \chi \leq 1$. Assume that*

$$\forall x \in \Delta, \quad \left|\tilde{L}_s^n v(x)\right| \leq \tilde{L}_\sigma^n(\chi u)(x). \tag{7.24}$$

Then $(\tilde{L}_\sigma^n(\chi u), \tilde{L}_s^n(v)) \in \mathscr{E}_t$.

Proof. — Let $(u, v) \in \mathscr{E}_t$ with $|t| \geq 10$. Let $n \in \mathbf{N}$. By Lemma 7.8, for $x \in \Delta$,

$$\left\|D\big(\tilde{L}_\sigma^n(\chi u)\big)(x)\right\| \leq C_3\tilde{L}_\sigma^n(\chi u)(x) + \kappa^{-n}\tilde{L}_\sigma^n(\|D(\chi u)\|)(x). \tag{7.25}$$

Since $(u, v) \in \mathscr{E}_t$ and $\|D\chi\| \leq |t|$,

$$\begin{aligned}\|D(\chi u)(x)\| &\leq |t|u(x) + \|Du(x)\| \leq (1 + 2C_3)|t|u(x) \\ &\leq \frac{4}{3}(1+2C_3)|t|(\chi u)(x).\end{aligned}$$

Hence,

$$\left\|D\big(\tilde{L}_\sigma^n(\chi u)\big)(x)\right\| \leq \left[C_3 + \kappa^{-n}\frac{4}{3}(1+2C_3)|t|\right]\tilde{L}_\sigma^n(\chi u)(x). \tag{7.26}$$

If n is large enough, the factor is $\leq 2C_3|t|$, and we get $\left\|D\big(\tilde{L}_\sigma^n(\chi u)\big)(x)\right\| \leq 2C_3|t|\tilde{L}_\sigma^n(\chi u)(x)$. This is half of what we have to prove.

Concerning v, Lemma 7.8 gives

$$\begin{aligned}\left\|D\big(\tilde{L}_s^n v\big)(x)\right\| &\leq C_3(1+|t|)\tilde{L}_\sigma^n(|v|)(x) + \kappa^{-n}\tilde{L}_\sigma^n(\|Dv\|)(x) \\ &\leq C_3(1+|t|)\frac{4}{3}\tilde{L}_\sigma^n(\chi u)(x) + \kappa^{-n}\frac{4}{3}2C_3|t|\tilde{L}_\sigma^n(\chi u)(x).\end{aligned}$$

If n is large enough, this quantity is again bounded by $2C_3|t|\tilde{L}_\sigma^n(\chi u)(x)$. □

If $h \in \mathcal{H}_n$, then $\|Dh(x) \cdot y\| \leq \kappa^{-n}\|y\|$. In particular, since r satisfies Condition (3) of Definition 2.3, the first condition of Proposition 7.4 gives $n \geq \max(N_0, N_1)$, two inverse branches $h, k \in \mathcal{H}_n$ and a continuous unitary vector field y_0 on Δ such that, for all $x \in \Delta$,

$$\left|D(r^{(n)} \circ h)(x) \cdot y_0(x) - D(r^{(n)} \circ k)(x) \cdot y_0(x)\right| \\ \geq 10C_3 \max(\|Dh(x) \cdot y_0(x)\|, \|Dk(x) \cdot y_0(x)\|).$$

Smoothing the vector field y_0, we get a smooth vector field y with $1 \leq \|y\| \leq 2$ such that, for all $x \in \Delta$,

$$\left|D(r^{(n)} \circ h)(x) \cdot y(x) - D(r^{(n)} \circ k)(x) \cdot y(x)\right| \\ \geq 9C_3 \max(\|Dh(x) \cdot y(x)\|, \|Dk(x) \cdot y(x)\|). \tag{7.27}$$

We fix n, h, k and y as above, until the end of the proof of Proposition 7.7.

Lemma **7.13.** — *There exist $\delta > 0$ and $\zeta > 0$ satisfying the following property. Let $s = \sigma + it$ with $\sigma \in [-\sigma_1, \sigma_1]$ and $|t| \geq 10$. Let $(u, v) \in \mathcal{E}_t$. For all $x_0 \in \Delta$ such that the ball $B(x_0, (\zeta + \delta)/|t|)$ is compactly included in Δ, there exists a point x_1 with $d(x_0, x_1) \leq \zeta/|t|$ such that one of the following possibilities holds:*

– Either, for all $x \in B(x_1, \delta/|t|)$,

$$\left|e^{-sr^{(n)}\circ h(x)}J(hx)(v \cdot f_\sigma)(hx) + e^{-sr^{(n)}\circ k(x)}J(kx)(v \cdot f_\sigma)(kx)\right| \\ \leq \frac{3}{4}e^{-\sigma r^{(n)}\circ h(x)}J(hx)(u \cdot f_\sigma)(hx) + e^{-\sigma r^{(n)}\circ k(x)}J(kx)(u \cdot f_\sigma)(kx).$$

– Or, for all $x \in B(x_1, \delta/|t|)$,

$$\left|e^{-sr^{(n)}\circ h(x)}J(hx)(v \cdot f_\sigma)(hx) + e^{-sr^{(n)}\circ k(x)}J(kx)(v \cdot f_\sigma)(kx)\right| \\ \leq e^{-\sigma r^{(n)}\circ h(x)}J(hx)(u \cdot f_\sigma)(hx) + \frac{3}{4}e^{-\sigma r^{(n)}\circ k(x)}J(kx)(u \cdot f_\sigma)(kx).$$

Proof. — Take some constants $\delta > 0$ and $\zeta > 0$. Let $t \in \mathbf{R}$ with $|t| \geq 10$. Take $(u, v) \in \mathcal{E}_t$. Consider $x_0 \in \Delta$ such that the ball $B(x_0, (\zeta + \delta)/|t|)$ is compactly included in Δ. If δ is small enough and ζ is large enough, we will find a point $x_1 \in B(x_0, \zeta/|t|)$ for which the conclusion of the lemma holds.

First case: Assume first that there exists $x_1 \in B(x_0, \zeta/|t|)$ such that $|v \circ h(x_1)| \leq u \circ h(x_1)/2$ or $|v \circ k(x_1)| \leq u \circ k(x_1)/2$. We will show that this point satisfies the required conclusion. The situation being symmetric, we can assume that $|v \circ h(x_1)| \leq u \circ h(x_1)/2$.

Since $(u, v) \in \mathcal{E}_t$, we have $\|Du(x)\| \leq 2C_3|t|u(x)$. As h is a contraction, this yields $\|D(u \circ h)(x)\| \leq 2C_3|t|u \circ h(x)$. We can integrate this inequality along an almost length-minimizing path between two points x, x': Gronwall's inequality gives $u(hx') \leq e^{2C_3|t|d(x,x')}u(hx)$.

For $x \in B(x_1, \delta/|t|)$, we get

$$\|D(v \circ h)(x)\| \leq 2C_3|t|u(hx) \leq 2C_3|t|e^{2C_3|t|\delta/|t|}u(hx_1). \tag{7.28}$$

Hence,

$$|v(hx) - v(hx_1)| \leq 2C_3|t|e^{2C_3\delta}u(hx_1)\delta/|t|. \tag{7.29}$$

Since $|v(hx_1)| \leq u(hx_1)/2$, we get

$$|v(hx)| \leq \left(\frac{1}{2} + 2C_3\delta e^{2C_3\delta}\right)u(hx_1) \leq \left(\frac{1}{2} + 2C_3\delta e^{2C_3\delta}\right)e^{2C_3\delta}u(hx). \tag{7.30}$$

If δ is small enough, we get $|v(hx)| \leq \frac{3}{4}u(hx)$ for all $x \in B(x_1, \delta/|t|)$. This concludes the proof.

Second case: Assume that, for all $x \in B(x_0, \zeta/|t|)$, holds $|v \circ h(x)| > u \circ h(x)/2$ and $|v \circ k(x)| > u \circ k(x)/2$.

Let $\phi : [0, \zeta/(2|t|)] \to \Delta$ be the solution of the equation $\phi'(\tau) = y(\phi(\tau))$ with $\phi(0) = x_0$. Write $x^\tau = \phi(\tau)$. We will first show that there exists $\tau \leq \zeta/(8|t|)$ for which $F(x^\tau) := e^{-sr^{(n)}\circ h(x^\tau)}J \circ h(x^\tau)(v \cdot f_\sigma)(hx^\tau)$ and $G(x^\tau) := e^{-sr^{(n)}\circ k(x^\tau)}J \circ k(x^\tau)(v \cdot f_\sigma)(kx^\tau)$ have opposite phases. Let $\gamma(\tau)$ be the difference of their phases.

On the set $h(B(x_0, \zeta/|t|)) \cup k(B(x_0, \zeta/|t|))$, the function v is non vanishing. Hence, it can locally be written as $v(x) = \rho(x)e^{i\theta(x)}$. Since $Dv(x) = D\rho(x)e^{i\theta(x)} + i\rho(x)e^{i\theta(x)}D\theta(x)$, the inequality $\|Dv(x)\| \leq 2C_3|t|u(x)$ yields

$$\|D\theta(x)\| \leq 2C_3|t|u(x)/\rho(x) \leq 4C_3|t|. \tag{7.31}$$

Since $\gamma(\tau) = -tr^{(n)}(hx^\tau) + \theta(hx^\tau) + tr^{(n)}(kx^\tau) - \theta(kx^\tau)$, we get

$$\begin{aligned}\gamma'(\tau) = t\left[D(r^{(n)} \circ k)(x^\tau) \cdot y(x^\tau) - D(r^{(n)} \circ h)(x^\tau) \cdot y(x^\tau)\right]\\ + D\theta(hx^\tau)Dh(x^\tau) \cdot y(x^\tau) - D\theta(kx^\tau)Dk(x^\tau) \cdot y(x^\tau).\end{aligned}$$

By (7.27) and (7.31), we get

$$\begin{aligned}|\gamma'(\tau)| &\geq 9C_3|t| \max(\|Dh(x^\tau) \cdot y(x^\tau)\|, \|Dk(x^\tau) \cdot y(x^\tau)\|)\\ &\quad - 4C_3|t|\|Dh(x^\tau) \cdot y(x^\tau)\| - 4C_3|t|\|Dk(x^\tau) \cdot y(x^\tau)\|\\ &\geq C_3|t| \max(\|Dh(x^\tau) \cdot y(x^\tau)\|, \|Dk(x^\tau) \cdot y(x^\tau)\|).\end{aligned}$$

There exists a constant $\gamma_0 > 0$ such that, for all $x \in \Delta$ and all $y \in T_x\Delta$ with $1 \leq \|y\| \leq 2$, $\|Dh(x) \cdot y\| \geq \gamma_0$ and $\|Dk(x) \cdot y\| \geq \gamma_0$. We get finally

$$|\gamma'(\tau)| \geq |t|C_3\gamma_0. \tag{7.32}$$

If $\zeta = 16\pi/(C_3\gamma_0)$, we obtain $\tau \in [0, \zeta/(8|t|)]$ for which $F(x^\tau)$ and $G(x^\tau)$ have opposite phases. Set $x_1 = x^\tau \in B(x_0, \zeta/(4|t|))$.

From the definition of F and the inequality $\|D(v \circ h)(x)\| \leq 4C_3|t||v(hx)|$ on the ball $B(x_0, \zeta/|t|)$, it is easy to check the existence of a constant C independent of δ such that, for all $x \in B(x_0, \zeta/|t|)$, $\|DF(x)\| \leq C|t||F(x)|$. If $x, x' \in B(x_0, \zeta/(3|t|))$, an almost length-minimizing path γ between x and x' is contained in $B(x_0, \zeta/|t|)$. Gronwall's inequality along this path yields $|F(x')| \leq e^{C|t|d(x,x')}|F(x)|$. Moreover, if Γ_F denotes the phase of $F(x)$, we have $\|D\Gamma_F(x)\| \leq C|t|$. On the ball $B(x_1, \delta/|t|)$ (which is included in $B(x_0, \zeta/(3|t|))$ as soon as $\delta \leq \zeta/12$), we get:

$$(\mathbf{7.33}) \qquad |\Gamma_F(x) - \Gamma_F(x_1)| \leq C\delta \text{ and } e^{-\delta C} \leq \frac{|F(x)|}{|F(x_1)|} \leq e^{\delta C}.$$

In the same way, if Γ_G denotes the phase of G, we have for all $x \in B(x_1, \delta/|t|)$

$$(\mathbf{7.34}) \qquad |\Gamma_G(x) - \Gamma_G(x_1)| \leq C\delta \text{ and } e^{-\delta C} \leq \frac{|G(x)|}{|G(x_1)|} \leq e^{\delta C}.$$

Assume for example that $|F(x_1)| \geq |G(x_1)|$ (the other case is symmetric). If δ is small enough, we get for all $x \in B(x_1, \delta/|t|)$

$$(\mathbf{7.35}) \qquad |\Gamma_F(x) - \Gamma_G(x) - \pi| \leq \pi/6 \text{ and } |F(x)| \geq |G(x)|/2.$$

We can then use the following elementary lemma:

Lemma **7.14.** — *Let $z = re^{i\theta}$ and $z' = r'e^{i\theta'}$ be complex numbers with $|\theta - \theta' - \pi| \leq \pi/6$ and $r' \leq 2r$. Then $|z + z'| \leq r + \frac{r'}{2}$.*

Proof. — We can assume that $\theta = 0$. Then

$$(\mathbf{7.36}) \qquad |z + z'|^2 = (r + r'\cos(\theta'))^2 + (r'\sin(\theta'))^2.$$

Since $\cos(\theta') \leq 0$ and $r' \leq 2r$, we have $r + r'\cos(\theta') \in [-r, r]$. Moreover, $|\sin(\theta')| \leq 1/2$. Hence,

$$(\mathbf{7.37}) \qquad |z + z'|^2 \leq r^2 + r'^2/4 \leq (r + r'/2)^2.$$

□

Together with (7.35), the lemma proves that, for all $x \in B(x_1, \delta/|t|)$,

$$(\mathbf{7.38}) \qquad |F(x) + G(x)| \leq |F(x)| + |G(x)|/2.$$

This proves that the second conclusion of Lemma 7.13 holds. □

From this point on, we fix the constants ζ and δ given by Lemma 7.13. Since Δ is a John domain, there exist constants C_0 and ε_0 such that, for all $\varepsilon < \varepsilon_0$, for all $x \in \Delta$, there exists $x' \in \Delta$ such that $d(x, x') \leq C_0\varepsilon$ and such that the ball $B(x', \varepsilon)$ is compactly contained in Δ. Choose $T_0 \geq 10$ such that $2(\zeta + \delta)/T_0 < \varepsilon_0$.

Lemma **7.15.** — *There exist* $\beta_0 < 1$ *and* $0 < \sigma_2 < \sigma_1$ *satisfying the following property. Let* $s = \sigma + it$ *with* $\sigma \in [-\sigma_2, \sigma_2]$ *and* $|t| \geq T_0$. *Let* $(u, v) \in \mathscr{E}_t$. *Then there exists* $\tilde{u} : \Delta \to \mathbf{R}$ *such that* $(\tilde{u}, \tilde{L}_s^n v) \in \mathscr{E}_t$ *and* $\int \tilde{u}^2 \, d\mu \leq \beta_0 \int u^2 \, d\mu$.

Proof. — Consider a maximal set of points $x_1, \dots, x_k \in \Delta$ such that the balls $B(x_i, 2(\zeta + \delta)/|t|)$ are compactly included in Δ, and two by two disjoint. By Lemma 7.6, this set is finite. The John domain condition on Δ ensures that Δ is covered by the balls $B(x_i, C_4/|t|)$ where $C_4 = (2 + C_0)2(\zeta + \delta)$.

In each ball $B(x_i, (\zeta + \delta)/|t|)$, there exists a ball $B_i' = B(x_i', \delta/|t|)$ on which the conclusion of Lemma 7.13 holds for the pair (u, v). We will write $\mathrm{type}(B_i') = h$ if the first conclusion of Lemma 7.13 holds, and $\mathrm{type}(B_i') = k$ otherwise. By Lemma 7.5, there exists a function ρ_i on Δ such that $\rho_i = 1$ on $B_i'' = B(x_i', \delta/(C_1|t|))$, $\rho_i = 0$ outside of B_i' and $\|\rho_i\|_{C^1} \leq C_2|t|/\delta$. We define a function ρ on Δ by

$$\rho = \left(\sum_{\mathrm{type}(B_i') = h} \rho_i \right) \circ T^n \tag{7.39}$$

on $h(\Delta)$,

$$\rho = \left(\sum_{\mathrm{type}(B_i') = k} \rho_i \right) \circ T^n \tag{7.40}$$

on $k(\Delta)$, and $\rho = 0$ on $\Delta \setminus (h(\Delta) \cup k(\Delta))$. This function satisfies $\|\rho\|_{C^1} \leq |t|/\eta_0$ for some constant η_0 independent of s, u, v, and we can assume $\eta_0 < 1/4$. Notice that η_0 depends on n, h, k and δ, which is not troublesome since these quantities are fixed once and for all. Define a new function $\chi = 1 - \eta_0\rho$. It takes its values in $[3/4, 1]$, with $\|D\chi\| \leq |t|$. Moreover, by construction,

$$\left| \tilde{L}_s^n v \right| \leq \tilde{L}_\sigma^n(\chi u). \tag{7.41}$$

We set $\tilde{u} = \tilde{L}_\sigma^n(\chi u)$. By (7.41) and Lemma 7.12, $(\tilde{u}, \tilde{L}_s^n v) \in \mathscr{E}_t$. We have to show that, for some constant $\beta_0 < 1$, $\int \tilde{u}^2 \, d\mu \leq \beta_0 \int u^2 \, d\mu$ as soon as σ is small enough.

The definition of $\tilde{\mathrm{L}}^n_\sigma$ gives

$$\begin{aligned}\lambda_\sigma^{2n} f_\sigma^2(x)\tilde{u}^2(x) &= \Big(\sum_{l\in\mathscr{H}_n} e^{-\sigma r^{(n)}(lx)}\mathrm{J}(lx)(\chi\cdot f_\sigma\cdot u)(lx)\Big)^2\\ &\le \Big(\sum_{l\in\mathscr{H}_n}\mathrm{J}(lx)\big(f_\sigma\cdot u^2\big)(lx)\Big)\Big(\sum_{l\in\mathscr{H}_n} e^{-2\sigma r^{(n)}(lx)}\mathrm{J}(lx)\big(f_\sigma\cdot\chi^2\big)(lx)\Big)\\ &\le \Big(\sup_\Delta \frac{f_\sigma}{f_0}\Big)\Big(\sum_{l\in\mathscr{H}_n}\mathrm{J}(lx)\big(f_0\cdot u^2\big)(lx)\Big)\\ &\quad\times\Big(\sup_\Delta\frac{f_\sigma}{f_{2\sigma}}\Big)\Big(\sum_{l\in\mathscr{H}_n} e^{-2\sigma r^{(n)}(lx)}\mathrm{J}(lx)\big(f_{2\sigma}\cdot\chi^2\big)(lx)\Big).\end{aligned}$$

If $x\in\mathrm{B}''_i$ with $\mathrm{type}(\mathrm{B}'_i)=h$, we have

$$\begin{aligned}\frac{1}{\lambda^n_{2\sigma}f_{2\sigma}(x)}\sum_{l\in\mathscr{H}_n} e^{-2\sigma r^{(n)}(lx)}\mathrm{J}(lx)\big(f_{2\sigma}\cdot\chi^2\big)(lx) &= \tilde{\mathrm{L}}^n_{2\sigma}(\chi^2)(x)\\ &= 1-\big(1-(1-\eta_0)^2\big)e^{-2\sigma r^{(n)}(hx)}\mathrm{J}(hx)\frac{f_{2\sigma}(hx)}{\lambda^n_{2\sigma}f_{2\sigma}(x)}.\end{aligned}$$

This is uniformly bounded by a constant $\eta_1<1$. The same inequality holds if $\mathrm{type}(\mathrm{B}'_i)=k$, with h replaced by k. Define a number

$$\xi(\sigma)=\Big(\sup_\Delta\frac{\lambda^n_{2\sigma}f_0(x)f_{2\sigma}(x)}{\lambda_\sigma^{2n}f_\sigma^2(x)}\Big)\Big(\sup_\Delta\frac{f_\sigma}{f_0}\Big)\Big(\sup_\Delta\frac{f_\sigma}{f_{2\sigma}}\Big). \tag{7.42}$$

Let $\mathrm{X}=\bigcup\mathrm{B}''_i$ and $\mathrm{Y}=\Delta\backslash\mathrm{X}$. We have proved that

$$\forall x\in\mathrm{X},\quad \tilde{u}^2(x)\le\eta_1\xi(\sigma)\tilde{\mathrm{L}}^n_0(u^2)(x). \tag{7.43}$$

If $x\notin\mathrm{X}$, there is no cancellation mechanism, and we simply have

$$\forall x\in\mathrm{Y},\quad \tilde{u}^2(x)\le\xi(\sigma)\tilde{\mathrm{L}}^n_0(u^2)(x). \tag{7.44}$$

The equations (7.43) and (7.44) are not sufficient by themselves to obtain an inequality $\int\tilde{u}^2\,\mathrm{d}\mu\le\beta_0\int u^2\,\mathrm{d}\mu$, one further argument is required.

Since $\|\mathrm{D}u\|\le 2\mathrm{C}_3|t|u$, $\|\mathrm{D}(u^2)\|\le 4\mathrm{C}_3|t|u^2$. Hence, $(u^2,u^2)\in\mathscr{E}_{2t}$. By Lemma 7.12, we obtain $(\tilde{\mathrm{L}}^n_0(u^2),\tilde{\mathrm{L}}^n_{2it}(u^2))\in\mathscr{E}_{2t}$. Hence, the function $w=\tilde{\mathrm{L}}^n_0(u^2)$ satisfies $\|\mathrm{D}w\|\le 4\mathrm{C}_3|t|w$. Gronwall's inequality then implies that, for all points $x,x'\in\Delta$, $w(x')\le w(x)e^{4\mathrm{C}_3|t|d(x,x')}$. In particular, there exists a constant C such that, for all points x,x' in a ball $\mathrm{B}(x_i,\mathrm{C}_4/|t|)$, $w(x')\le\mathrm{C}w(x)$. This yields

$$\frac{\int_{\mathrm{B}(x_i,\mathrm{C}_4/|t|)}w\,\mathrm{d}\mu}{\mu(\mathrm{B}(x_i,\mathrm{C}_4/|t|))}\le\mathrm{C}\frac{\int_{\mathrm{B}''_i}w\,\mathrm{d}\mu}{\mu(\mathrm{B}''_i)}. \tag{7.45}$$

Moreover, $\mathrm{Leb}(\mathrm{B}(x_i, \mathrm{C}_4/|t|))/\mathrm{Leb}(\mathrm{B}_i'')$ is uniformly bounded since (Δ, Leb) is a John domain, and the density of μ is bounded from above and below. We get another constant C' such that

$$\int_{\mathrm{B}(x_i,\mathrm{C}_4/|t|)} w \,\mathrm{d}\mu \le \mathrm{C}' \int_{\mathrm{B}_i''} w \,\mathrm{d}\mu. \tag{7.46}$$

Since the balls B_i'' are disjoint, we obtain

$$\int_{\mathrm{Y}} w \,\mathrm{d}\mu \le \mathrm{C}' \int_{\mathrm{X}} w \,\mathrm{d}\mu. \tag{7.47}$$

Consider finally a large constant A such that $(\mathrm{A}+1)\eta_1 + \mathrm{C}' \le \mathrm{A}$. With (7.43) and (7.44), we get

$$\begin{aligned}(\mathrm{A}+1)\int \tilde{u}^2 \,\mathrm{d}\mu &\le \xi(\sigma)\left[(\mathrm{A}+1)\int_{\mathrm{X}} \eta_1 w \,\mathrm{d}\mu + (\mathrm{A}+1)\int_{\mathrm{Y}} w \,\mathrm{d}\mu\right] \\ &\le \xi(\sigma)\left[(\mathrm{A}+1)\eta_1 \int_{\mathrm{X}} w \,\mathrm{d}\mu + \mathrm{A}\int_{\mathrm{Y}} w \,\mathrm{d}\mu + \mathrm{C}'\int_{\mathrm{X}} w \,\mathrm{d}\mu\right] \\ &\le \xi(\sigma)\mathrm{A}\int w \,\mathrm{d}\mu.\end{aligned}$$

Since $\int w \,\mathrm{d}\mu = \int \tilde{\mathrm{L}}_0^n(u^2) \,\mathrm{d}\mu = \int u^2 \,\mathrm{d}\mu$, we finally get

$$\int \tilde{u}^2 \,\mathrm{d}\mu \le \xi(\sigma)\frac{\mathrm{A}}{\mathrm{A}+1}\int u^2 \,\mathrm{d}\mu. \tag{7.48}$$

When $\sigma \to 0$, $\xi(\sigma)$ converges to 1. Hence, there exists $\sigma_2 > 0$ such that $\beta_0 = \sup_{|\sigma|\le\sigma_2} \xi(\sigma)\frac{\mathrm{A}}{\mathrm{A}+1}$ is < 1. □

Lemmas 7.10 and 7.15 easily imply Proposition 7.7:

Proof of Proposition 7.7. — Is is sufficient to prove that there exist $\beta < 1$ and $\mathrm{C} > 0$ such that, for all $m \in \mathbf{N}$, for all $s = \sigma + it$ with σ small enough and $|t| \ge \mathrm{T}_0$, for all $u \in \mathrm{C}^1(\Delta)$,

$$\left\|\tilde{\mathrm{L}}_s^{2mn} u\right\|_{\mathrm{L}^2(\mu)} \le \mathrm{C}\beta^m \|u\|_{1,t}. \tag{7.49}$$

Indeed, if (7.49) is proved, consider a general integer k and write it as $k = 2mn + r$ where $0 \le r \le 2n-1$. Then

$$\left\|\mathrm{L}_s^k u\right\|_{\mathrm{L}^2(\mathrm{Leb})} \le \mathrm{C}\lambda_\sigma^k \left\|\tilde{\mathrm{L}}_s^k u\right\|_{\mathrm{L}^2(\mu)} \le \mathrm{C}\lambda_\sigma^k \beta^m \left\|\tilde{\mathrm{L}}_s^r u\right\|_{1,t} \le \mathrm{C}\lambda_\sigma^k \beta^m \|u\|_{1,t}, \tag{7.50}$$

by Lemma 7.9. Choosing σ_0' small enough so that $\sup_{|\sigma|\le\sigma_0'} \lambda_\sigma \beta^{1/(2n)} < 1$, we obtain the full conclusion of Proposition 7.7.

Let us prove (7.49) for $u \in \mathrm{C}^1(\Delta)$. Suppose first that, for all $0 \le p < m$, $\|\mathrm{D}(\tilde{\mathrm{L}}_s^{pn} u)\|_{\mathrm{C}^0} \ge 2\mathrm{C}_3|t| \, \|\tilde{\mathrm{L}}_s^{pn} u\|_{\mathrm{C}^0}$. Then Lemma 7.10 gives

$$(\mathbf{7.51}) \qquad \|\tilde{\mathrm{L}}_s^{mn} u\|_{1,t} \le \left(\frac{9}{10}\right)^m \|u\|_{1,t}.$$

Since $\|\tilde{\mathrm{L}}_s^{2mn} u\|_{\mathrm{L}^2(\mu)} \le \|\tilde{\mathrm{L}}_s^{2mn} u\|_{1,t} \le \mathrm{C}\, \|\tilde{\mathrm{L}}_s^{mn} u\|_{1,t}$ by Lemma 7.9, (7.49) is satisfied.

Otherwise, let $p < m$ be the first time such that

$$(\mathbf{7.52}) \qquad \|\mathrm{D}(\tilde{\mathrm{L}}_s^{pn} u)\|_{\mathrm{C}^0} < 2\mathrm{C}_3|t| \, \|\tilde{\mathrm{L}}_s^{pn} u\|_{\mathrm{C}^0},$$

and let $v = \tilde{\mathrm{L}}_s^{pn} u$. Since $(\sup|v|, v) \in \mathscr{E}_t$, we can apply Lemma 7.15 and obtain a sequence of functions u_k with $u_0 = \sup|v|$, $\int u_k^2 \,\mathrm{d}\mu \le \beta_0^k \int u_0^2 \,\mathrm{d}\mu$, and $(u_k, \tilde{\mathrm{L}}_s^{kn} v) \in \mathscr{E}_t$. In particular,

$$\begin{aligned} \|\tilde{\mathrm{L}}_s^{2mn} u\|_{\mathrm{L}^2(\mu)} &= \|\tilde{\mathrm{L}}_s^{(2m-p)n} v\|_{\mathrm{L}^2(\mu)} \le \|u_{2m-p}\|_{\mathrm{L}^2(\mu)} \le \beta_0^{(2m-p)/2} \sup|v| \\ &\le \beta_0^{m/2} \|u\|_{\mathrm{C}^0}. \end{aligned}$$

This proves (7.49) and concludes the proof of Proposition 7.7. □

7.4. *A control in the norm $\|\cdot\|_{1,t}$.* — Although it will not be useful in this paper, it is worth mentioning that Proposition 7.7, which gives a control in the L^2 norm, easily implies an estimate in the stronger norm $\|\cdot\|_{1,t}$. This kind of estimate is especially useful for the study of zeta functions.

Proposition **7.16.** — *There exist $\sigma_0' \le \sigma_0$, $\mathrm{T}_0 > 0$, $\mathrm{C} > 0$ and $\beta < 1$ such that, for all $s = \sigma + it$ with $|\sigma| \le \sigma_0'$ and $|t| \ge \mathrm{T}_0$, for all $u \in \mathrm{C}^1(\Delta)$, for all $k \in \mathbf{N}$,*

$$(\mathbf{7.53}) \qquad \|\mathrm{L}_s^k u\|_{1,t} \le \mathrm{C}\lambda_\sigma^k \min(1, \beta^k |t|) \|u\|_{1,t}.$$

Proof. — It is sufficient to prove the existence of $\beta < 1$ such that

$$(\mathbf{7.54}) \qquad \|\tilde{\mathrm{L}}_s^{3k} u\|_{1,t} \le \mathrm{C}\beta^k |t| \|u\|_{1,t}$$

if $|\sigma|$ is small enough and $|t|$ is large enough. Indeed, together with Lemma 7.9, it implies the conclusion of the proposition.

Denote by $\mathrm{Lip}(\Delta)$ the set of Lipschitz functions on Δ, with its canonical norm

$$(\mathbf{7.55}) \qquad \|w\|_{\mathrm{Lip}} = \sup_{x\in\Delta} |w(x)| + \sup_{x\neq x'} \frac{|w(x) - w(x')|}{d(x, x')}.$$

We will use the following classical Lasota-Yorke inequality on the transfer operators $\tilde{L}_\sigma$, for small enough $|\sigma|$: there exist $C > 0$ and $\beta_1 < 1$ such that, for all $k \in \mathbf{N}$, for all $w \in \mathrm{Lip}(\Delta)$,

$$\left\| \tilde{L}_\sigma^k w \right\|_{\mathrm{Lip}} \le C\beta_1^k \|w\|_{\mathrm{Lip}} + C\|w\|_{L^1}. \tag{7.56}$$

Hence,

$$\left\| \tilde{L}_s^{2k} u \right\|_{C^0} \le \left\| \tilde{L}_\sigma^k \left(\left| \tilde{L}_s^k u \right| \right) \right\|_{C^0} \le C\beta_1^k \left\| \tilde{L}_s^k u \right\|_{\mathrm{Lip}} + C \left\| \tilde{L}_s^k u \right\|_{L^2}. \tag{7.57}$$

Moreover, $\left\| \tilde{L}_s^k u \right\|_{\mathrm{Lip}} \le |t| \left\| \tilde{L}_s^k u \right\|_{1,t} \le C|t| \|u\|_{1,t}$, and $\left\| \tilde{L}_s^k u \right\|_{L^2} \le \beta_2^k \|u\|_{1,t}$ for some $\beta_2 < 1$, by Proposition 7.7. Hence, there exists $\beta_3 < 1$ such that

$$\left\| \tilde{L}_s^{2k} u \right\|_{C^0} \le C|t|\beta_3^k \|u\|_{1,t}. \tag{7.58}$$

By Lemma 7.9, we get

$$\left\| \tilde{L}_s^{3k} u \right\|_{1,t} \le C \left\| \tilde{L}_s^{2k} u \right\|_{C^0} + \frac{\kappa^{-k}}{|t|} \left\| D\left(\tilde{L}_s^{2k} u \right) \right\|_{C^0}. \tag{7.59}$$

Notice that $\frac{\| D(\tilde{L}_s^{2k} u) \|_{C^0}}{|t|} \le \left\| \tilde{L}_s^{2k} u \right\|_{1,t} \le C\|u\|_{1,t}$. Together with (7.58), this implies (7.54) and concludes the proof of the proposition. ◻

7.5. *Proof of Theorem 7.3.* — Let $U \in \mathscr{B}_0$ and $V \in \mathscr{B}_1$ be such that $\int V \, d\mu_r = 0$. We will prove that there exist $\delta > 0$ independent of U, V, and $C > 0$ dependent of U, V such that

$$\forall t \ge 0, \quad \left| \int U \cdot V \circ T_t \right| \le C e^{-\delta t}. \tag{7.60}$$

By the closed graph theorem, this will imply Theorem 7.3.

For $t \ge 0$, let $A_t = \{(x, a) \in \Delta_r \,:\, a + t \ge r(x)\}$ and $B_t = \Delta_r \setminus A_t$. Then

$$\int U \cdot V \circ T_t = \int_{A_t} U \cdot V \circ T_t + \int_{B_t} U \cdot V \circ T_t =: \rho(t) + \bar{\rho}(t). \tag{7.61}$$

We have

$$\begin{aligned} |\bar{\rho}(t)| &\le C \int_{x \in \Delta} \max(r(x) - t, 0) \le C \int_{r(x) \ge t} r(x) \\ &\le C\|r\|_{L^2} (\mathrm{Leb}(x \,:\, r(x) > t))^{1/2}. \end{aligned} \tag{7.62}$$

Since r has exponentially small tails, this quantity decays exponentially. It is therefore sufficient to prove that $\rho(t)$ decays exponentially to conclude.

Since $\rho(t)$ is bounded, we can define, for $s \in \mathbf{C}$ with $\Re s > 0$,

$$\widehat{\rho}(s) = \int_0^\infty e^{-st} \rho(t) \, dt. \tag{7.63}$$

For $W : \Delta_r \to \mathbf{R}$ and $s \in \mathbf{C}$, set $\widehat{W}_s(x) = \int_0^{r(x)} W(x, a) e^{-sa} \, da$ when $x \in \Delta$.

Lemma **7.17.** — *Let* $s \in \mathbf{C}$ *with* $\Re s > 0$. *Then*

$$\widehat{\rho}(s) = \sum_{k=1}^{\infty} \int_{\Delta} \widehat{V}_s(x) \cdot \big(L_s^k \widehat{U}_{-s}\big)(x)\, dx. \tag{7.64}$$

Proof. — We compute

$$\begin{aligned}
\widehat{\rho}(s) &= \int_{x\in\Delta} \int_{a=0}^{r(x)} \int_{t+a\geq r(x)} e^{-st} U(x,a) V \circ T_t(x,a)\, dt\, da\, dx \\
&= \sum_{k=1}^{\infty} \int_{x\in\Delta} \int_{a=0}^{r(x)} \int_{b=0}^{r(T^k x)} U(x,a) V(T^k x, b) e^{-s(b+r^{(k)}(x)-a)}\, db\, da\, dx \\
&= \sum_{k=1}^{\infty} \int_{x\in\Delta} \widehat{U}_{-s}(x) e^{-sr^{(k)}(x)} \widehat{V}_s(T^k x)\, dx \\
&= \sum_{k=1}^{\infty} \int_{x\in\Delta} \widehat{V}_s(x) \big(L_s^k \widehat{U}_{-s}\big)(x)\, dx.
\end{aligned}$$

□

Lemma **7.18.** — *There exists* $C > 0$ *such that, for all* $s = \sigma + it$ *with* $|\sigma| \leq \sigma_0/4$ *and* $t \in \mathbf{R}$, *the function* $L_s \widehat{U}_{-s}$ *is* C^1 *on* Δ *and satisfies the inequality*

$$\big\| L_s \widehat{U}_{-s} \big\|_{1,t} \leq \frac{C}{\max(1, |t|)}. \tag{7.65}$$

Proof. — Let us first prove that there exists $C > 0$ such that, whenever $|\sigma| \leq \sigma_0/4$,

$$\forall x \in \Delta, \quad \big|\widehat{U}_{-s}(x)\big| \leq \frac{C}{\max(1,|t|)} e^{(\sigma_0/2) r(x)}. \tag{7.66}$$

Since $\widehat{U}_{-s}(x) = \int_{a=0}^{r(x)} U(x,a) e^{sa}\, da$, this is trivial if $|t| \leq 1$. If $|t| > 1$, an integration by parts gives

$$\widehat{U}_{-s}(x) = \int_{a=0}^{r(x)} U(x,a) e^{sa}\, da = \left[U(x,a) \frac{e^{sa}}{s} \right]_0^{r(x)} - \int_{a=0}^{r(x)} \partial_t U(x,a) \frac{e^{sa}}{s}\, da. \tag{7.67}$$

The boundary terms are bounded by $Ce^{(\sigma_0/4)r(x)}/|t|$, while the remaining term is at most

$$Cr(x) e^{(\sigma_0/4) r(x)}/|t| \leq C' e^{(\sigma_0/2) r(x)}/|t|. \tag{7.68}$$

This proves (7.66).

We can now compute

$$
\begin{aligned}
(\mathbf{7.69}) \qquad |\mathrm{L}_s\widehat{\mathrm{U}}_{-s}(x)| &= \left| \sum_{h\in\mathscr{H}} e^{-sr\circ h(x)} \mathrm{J}(hx)\widehat{\mathrm{U}}_{-s}(hx) \right| \\
&\leq \frac{\mathrm{C}}{\max(1,|t|)} \sum_{h\in\mathscr{H}} e^{(\sigma_0/4)r(hx)}\mathrm{J}(hx)e^{(\sigma_0/2)r(hx)}.
\end{aligned}
$$

This sum is bounded by $\frac{\mathrm{C}'}{\max(1,|t|)}$ since $\sigma_0/2+\sigma_0/4<\sigma_0$.

We have $\mathrm{L}_s\widehat{\mathrm{U}}_{-s}(x) = \sum_{h\in\mathscr{H}} e^{-sr\circ h(x)}\mathrm{J}(hx)\int_{a=0}^{r(hx)}\mathrm{U}(hx,a)e^{sa}\,\mathrm{d}a$. To obtain $\mathrm{D}(\mathrm{L}_s\widehat{\mathrm{U}}_{-s})(x)$, we can differentiate $e^{-sr\circ h(x)}$, or $\mathrm{J}(hx)$, or $\mathrm{U}(hx,a)$ in the integral, or the bound $r(hx)$ of the integral.

Since $\mathrm{D}(e^{-sr\circ h(x)}) = -s\mathrm{D}(r\circ h)(x)e^{-sr\circ h(x)}$, and $\|\mathrm{D}(r\circ h)\|$ is uniformly bounded, the corresponding term is bounded by $\mathrm{C}|s|\cdot \mathrm{C}/\max(1,|t|)$, by the computation done in (7.69). Since $\mathrm{D}(\mathrm{J}\circ h)(x)\leq \mathrm{C}\mathrm{J}(hx)$, the corresponding term is bounded by $\mathrm{C}/\max(1,|t|)$. If we differentiate $\mathrm{U}(hx,a)$ in the integral, the corresponding term is bounded by $\mathrm{C}\sum_{h\in\mathscr{H}} e^{(\sigma_0/4)r(hx)}\mathrm{J}(hx)e^{(\sigma_0/4)r(hx)}r(hx)$, which is still uniformly bounded. Finally, the last term satisfies a similar bound.

We have proved that $\|\mathrm{D}(\mathrm{L}_s\widehat{\mathrm{U}}_{-s})\|_{\mathrm{C}^0}\leq \mathrm{C}$ for some constant C. Together with the inequality $\|\mathrm{L}_s\widehat{\mathrm{U}}_{-s}\|_{\mathrm{C}^0}\leq \mathrm{C}/\max(1,|t|)$, it proves the lemma. □

Lemma **7.19.** — *There exists* $\mathrm{C}>0$ *such that, for* $s=\sigma+it$ *with* $|\sigma|\leq\sigma_0/4$ *and* $t\in\mathbf{R}$,

$$(\mathbf{7.70}) \qquad \left\|\widehat{\mathrm{V}}_s\right\|_{\mathrm{L}^2} \leq \frac{\mathrm{C}}{\max(1,|t|)}.$$

Proof. — The inequality (7.66) for $\widehat{\mathrm{V}}_s$ is trivial if $|t|\leq 1$, and can be proved by an integration by parts along the flow direction (using the bounded variation of $t\mapsto \mathrm{V}(x,t)$) if $|t|>1$. This concludes the proof since $\int_\Delta e^{\sigma_0 r}<\infty$. □

Corollary **7.20.** — *There exists* $\sigma_3>0$ *(independent of* U, V*) such that the function* $\widehat{\rho}$ *admits an analytic extension* ϕ *to the set* $\{s=\sigma+it : |\sigma|\leq\sigma_3, |t|\geq \mathrm{T}_0\}$. *This extension satisfies* $|\phi(s)|\leq \mathrm{C}/t^2$.

Proof. — For $s=\sigma+it$ with $|\sigma|\leq\sigma_0/4$ and $|t|\geq\mathrm{T}_0$, set $\phi(s)=\sum_{k=1}^\infty\int \widehat{\mathrm{V}}_s\cdot \mathrm{L}_s^k\widehat{\mathrm{U}}_{-s}$. By Lemma 7.17, it coincides with $\widehat{\rho}$ when $\Re s>0$.

We have to check that the series defining ϕ is summable, and that ϕ satisfies the bound $|\phi(s)|\leq \mathrm{C}/t^2$. By Proposition 7.7, Lemma 7.18 and Lemma 7.19, if $|\sigma|$ is small enough,

$$(\mathbf{7.71}) \qquad \left|\int \widehat{\mathrm{V}}_s\cdot \mathrm{L}_s^k\widehat{\mathrm{U}}_{-s}\right| \leq \left\|\widehat{\mathrm{V}}_s\right\|_{\mathrm{L}^2}\left\|\mathrm{L}_s^k\widehat{\mathrm{U}}_{-s}\right\|_{\mathrm{L}^2} \leq \left\|\widehat{\mathrm{V}}_s\right\|_{\mathrm{L}^2}\mathrm{C}\beta^{k-1}\left\|\mathrm{L}_s\widehat{\mathrm{U}}_{-s}\right\|_{1,t} \leq \frac{\mathrm{C}}{t^2}\beta^k.$$

This last term is summable and its sum is at most $\frac{\mathrm{C}}{(1-\beta)t^2}$. □

Lemma **7.21.** — *For all* $s = it \neq 0$, *there exists an open disk* O_s *with center* s *(independent of* U, V*) such that* $\widehat{\rho}$ *admits an analytic extension to* O_s.

Proof. — The operator L_s acting on C^1 satisfies a Lasota–Yorke inequality, by Lemma 7.8 and the compactness of the unit ball of $C^1(\Delta)$ in $C^0(\Delta)$. By Hennion's theorem [He], its spectral radius on C^1 is ≤ 1, and its essential spectral radius is < 1.

Let us prove that L_s has no eigenvalue of modulus 1. This is an easy consequence of the weak-mixing of the flow T_t, but we will rather derive it directly. Assume that there exists a nonzero C^1 function u and a complex number λ with $|\lambda| = 1$ such that $L_s u = \lambda u$. Then $|u| = |L_s u| \leq L_0|u|$. Since $\int |u| = \int L_0|u|$, we get $|u| = L_0|u|$. In particular, $|L_s u| = L_0|u|$, which means that all the complex numbers $e^{-itr(hx)}u(hx)$ have the same argument. Take $k \in \mathbf{N}$ such that $k|t| \geq T_0$. The complex numbers $e^{-itkr(hx)}u^k(hx)$ also have the same argument. Hence, $|L_{ks}(u^k)| = L_0|u^k|$. In the same way, for any $n \in \mathbf{N}$, $|L_{ks}^n(u^k)| = L_0^n|u^k|$. This is a contradiction, since $L_{ks}^n(u^k)$ tends to 0 in L^2 by Proposition 7.7, while $\int L_0^n|u^k| = \int |u^k|$ does not tend to 0 when $n \to \infty$.

We have proved that the spectral radius of L_s is < 1. Hence, there exists a disk O_s around s and constants $C > 0$, $r < 1$ such that, for all $s' \in O_s$ and for all $n \in \mathbf{N}$, $\left\|L_{s'}^n\right\|_{C^1} \leq Cr^n$. Since $L_{s'}\widehat{U}_{-s'}$ is uniformly bounded in C^1 by Lemma 7.18, the series $\sum_{k\geq 1}\int_\Delta \widehat{V}_{s'} \cdot L_{s'}^{k-1}(L_{s'}\widehat{U}_{-s'})$ is convergent on O_s. By Lemma 7.17, it coincides with $\widehat{\rho}(s')$ for $\Re s' > 0$. □

Lemma **7.22.** — *There exists an open disk* O_0 *with center* 0 *(independent of* U, V*) such that* $\widehat{\rho}$ *admits an analytic extension to* O_0.

Proof. — The transfer operator L_0 acting on C^1 has an isolated eigenvalue 1. For small s, L_s is an analytic perturbation of L_0. Hence, it admits an eigenvalue λ_s close to 1. Denote by P_s the corresponding spectral projection, and f_s the eigenfunction (normalized so that $\int f_s = 1$). On a disk O_0 centered in 0, it is possible to write $L_s = \lambda_s P_s + R_s$ where P_s and R_s commute, and $\left\|R_s^n\right\|_{C^1} \leq Cr^n$ for some uniform constants $C > 0$ and $r < 1$.

The function $s \mapsto \lambda_s$ is analytic in O_0, let us compute its derivative at 0. Since $\|L_s - L_0\|_{C^1} = O(s)$ and $\|f_s - f_0\|_{C^1} = O(s)$, we have

$$
\begin{aligned}
\lambda_s = \int L_s f_s &= \int (L_s - L_0)(f_s - f_0) + \int L_0(f_s - f_0) + \int L_s f_0 \\
&= O(s^2) + \int (f_s - f_0) + \int L_0\left(e^{-sr} f_0\right) = O(s^2) + 0 + \int e^{-sr}\,d\mu \\
&= 1 - s\int r\,d\mu + O(s^2).
\end{aligned}
$$

Hence, $\lambda'(0) = -\int r \, \mathrm{d}\mu \neq 0$. Shrinking O_0 if necessary, we can assume that λ_s is equal to 1 only for $s = 0$.

For $s \in O_0 \setminus \{0\}$, define a function

$$\phi(s) = \frac{1}{1-\lambda_s} \int_\Delta \widehat{V}_s \cdot P_s L_s \widehat{U}_{-s} + \sum_{k=0}^{\infty} \int_\Delta \widehat{V}_s \cdot R_s^k L_s \widehat{U}_{-s}, \tag{7.72}$$

where the last series is converging since $\left\| R_s^k \right\|_{C^1} \leq C r^k$ and $\|L_s \widehat{U}_{-s}\|_{C^1(\Delta)} \leq C$ by Lemma 7.18. It coincides with $\widehat{\rho}(s)$ when $\Re s > 0$. When $s \to 0$, the function $\frac{1}{1-\lambda_s}$ has a pole of order exactly one, since $\lambda'(0) \neq 0$. Let us show that $\int \widehat{V}_0 \cdot P_0 L_0 \widehat{U}_0 = 0$. This will conclude the proof, since the function ϕ, being bounded on a neighborhood of 0, can then be extended analytically to 0.

The function $P_0 L_0 \widehat{U}_0$ is proportional to f_0. Hence, it is sufficient to prove $\int \widehat{V}_0 f_0 = 0$. But

$$\int_\Delta \widehat{V}_0(x) f_0(x) \, \mathrm{dLeb}(x) = \int_{x\in\Delta} \int_{t=0}^{r(x)} V(x,t) \, \mathrm{d}t \, \mathrm{d}\mu(x) = \int V \, \mathrm{d}\mu_r = 0. \tag{7.73}$$

□

We will use the following classical Paley–Wiener theorem:

Theorem **7.23.** — *Let $\rho : \mathbf{R}_+ \to \mathbf{R}$ be a bounded measurable function. For $\Re s > 0$, define $\widehat{\rho}(s) = \int_{x=0}^{\infty} e^{-sx} \rho(x) \, \mathrm{d}x$. Suppose that $\widehat{\rho}$ can be analytically extended to a function ϕ on a strip $\{s = \sigma + it \,:\, |\sigma| < \varepsilon, t \in \mathbf{R}\}$ and that*

$$\int_{t=-\infty}^{\infty} \sup_{|z|<\varepsilon} |\phi(z+it)| \, \mathrm{d}t < \infty. \tag{7.74}$$

Then there exist a constant $C > 0$ and a full measure subset $A \subset \mathbf{R}_+$ such that, for all $x \in A$, $|\rho(x)| \leq C e^{-(\varepsilon/2)x}$.

Proof of Theorem 7.3. — We can summarize Corollary 7.20, Lemma 7.21 and Lemma 7.22 as follows: there exists $\sigma_4 > 0$ (independent of U, V) such that $\widehat{\rho}$ admits an analytic extension ϕ to the set $\{s = \sigma + it \,:\, |\sigma| \leq \sigma_4, t \in \mathbf{R}\}$. Moreover, there exists $C > 0$ such that this extension satisfies

$$|\phi(\sigma + it)| \leq C \min\left(1, \frac{1}{t^2}\right). \tag{7.75}$$

Together with Theorem 7.23, it implies that $\rho(t)$ decays exponentially on a subset of $\mathbf{R}_+$ of full measure. Hence, on a full measure subset of $\mathbf{R}_+$, $\left|\int U \cdot V \circ T_t\right| \leq C e^{-\delta t}$. Since $t \mapsto \int U \cdot V \circ T_t$ is continuous by dominated convergence, this inequality holds in fact everywhere. This concludes the proof of Theorem 7.3. □

8. Exponential mixing for hyperbolic semiflows

In this section, we will prove Theorem 2.7, using Theorem 7.3 and an approximation argument.

8.1. *Estimates on bad returns.* — In this paragraph, we will prove the following exponential estimate on the number of returns to the basis Δ:

Lemma **8.1.** — *Let* $\Psi_t(x,a)$ *be the number of returns to* Δ *of* (x,a) *before time* t, *i.e.,*

$$\Psi_t(x,a) = \sup\{n \in \mathbf{N} \,:\, a+t > r^{(n)}(x)\}. \tag{8.1}$$

For all $\kappa > 1$, *there exist* $C > 0$ *and* $\delta > 0$ *such that, for all* $t \geq 0$,

$$\int_{\Delta_r} \kappa^{-\Psi_t(x,a)}\, \mathrm{dLeb}_r \leq Ce^{-\delta t}. \tag{8.2}$$

Proof. — We have

$$\begin{aligned}\int_{\Delta_r} \kappa^{-\Psi_t(x,a)}\, \mathrm{dLeb}_r &= \sum_{n=0}^{\infty} \kappa^{-n} \mathrm{Leb}_r\{(x,a) \,:\, r^{(n)}(x) < a+t \leq r^{(n+1)}(x)\} \\ &\leq \sum_{n=0}^{\infty} \kappa^{-n} \mathrm{Leb}_r\{(x,a) \,:\, t \leq r^{(n+1)}(x)\}.\end{aligned}$$

Moreover, for $\sigma > 0$,

$$\begin{aligned}&\mathrm{Leb}_r\{(x,a) \,:\, t \leq r^{(n+1)}(x)\} \\ &\qquad = \int_{\Delta} r(x) 1_{r^{(n+1)}(x)\geq t} \leq \left(\int_{\Delta} r^2\right)^{1/2} \left(\int_{\Delta} 1_{r^{(n+1)}(x)\geq t}\right)^{1/2} \\ &\qquad \leq C\left(\int_{\Delta} e^{\sigma r^{(n+1)}(x)}/e^{\sigma t}\right)^{1/2}.\end{aligned}$$

If σ is small enough,

$$\int_{\Delta} e^{\sigma r^{(n+1)}(x)} = \int_{\Delta} L_{\sigma}^{n+1}(1) \leq C\lambda_{\sigma}^{n+1}. \tag{8.3}$$

Choosing σ small enough so that $\kappa^{-1}\sqrt{\lambda_\sigma} < 1$, we obtain $\int_{\Delta_r} \kappa^{-\Psi_t(x,a)}\, \mathrm{dLeb}_r \leq Ce^{-\sigma t/2}$. □

8.2. *Proof of Theorem 2.7.* — Let U, V be C^1 functions on $\widehat{\Delta}_r$, with $\int \mathrm{U}\, \mathrm{d}\nu_r = 0$. We will prove that $\int \mathrm{U} \cdot \mathrm{V} \circ \widehat{\mathrm{T}}_{2t}\, \mathrm{d}\nu_r$ decreases exponentially fast in t.

Define a function V_t on Δ_r by $\mathrm{V}_t(x, a) = \int_{y \in \pi^{-1}(x)} \mathrm{V} \circ \widehat{\mathrm{T}}_t(y, a)\, \mathrm{d}\nu_x(y)$. Let $\pi_r : \widehat{\Delta}_r \to \Delta_r$ be given by $\pi_r(y, a) = (\pi(x), a)$.

Lemma **8.2.** — *There exist* $\delta > 0$ *(independent of* U, V*) and* $\mathrm{C} > 0$ *such that, for all* $t \geq 0$,

$$\left| \int_{\widehat{\Delta}_r} \mathrm{U} \cdot \mathrm{V} \circ \widehat{\mathrm{T}}_{2t}\, \mathrm{d}\nu_r - \int_{\widehat{\Delta}_r} \mathrm{U} \cdot \mathrm{V}_t \circ \mathrm{T}_t \circ \pi_r\, \mathrm{d}\nu_r \right| \leq \mathrm{C} e^{-\delta t}. \tag{8.4}$$

Proof. — We have

$$\begin{aligned} &\left| \int \mathrm{U} \cdot \mathrm{V} \circ \widehat{\mathrm{T}}_{2t}\, \mathrm{d}\nu_r - \int \mathrm{U} \cdot \mathrm{V}_t \circ \mathrm{T}_t \circ \pi_r\, \mathrm{d}\nu_r \right| \\ &\quad = \left| \int \mathrm{U} \cdot (\mathrm{V} \circ \widehat{\mathrm{T}}_t - \mathrm{V}_t \circ \pi_r) \circ \widehat{\mathrm{T}}_t\, \mathrm{d}\nu_r \right| \\ &\quad \leq \mathrm{C} \int |\mathrm{V} \circ \widehat{\mathrm{T}}_t - \mathrm{V}_t \circ \pi_r| \circ \widehat{\mathrm{T}}_t\, \mathrm{d}\nu_r = \mathrm{C} \int |\mathrm{V} \circ \widehat{\mathrm{T}}_t - \mathrm{V}_t \circ \pi_r|\, \mathrm{d}\nu_r. \end{aligned}$$

Take $x \in \Delta$. If $\pi(y) = \pi(y') = x$, the contraction properties of $\widehat{\mathrm{T}}$ give $d(\widehat{\mathrm{T}}_t(y, a), \widehat{\mathrm{T}}_t(y', a)) \leq \kappa^{-\Psi_t(x,a)} d(y, y')$, where Ψ_t is defined in Lemma 8.1. Hence, $|\mathrm{V} \circ \widehat{\mathrm{T}}_t(y, a) - \mathrm{V} \circ \widehat{\mathrm{T}}_t(y', a)| \leq \mathrm{C}\kappa^{-\Psi_t(x,a)}$. Averaging over y', we obtain $|\mathrm{V} \circ \widehat{\mathrm{T}}_t(y, a) - \mathrm{V}_t(x, a)| \leq \mathrm{C}\kappa^{-\Psi_t(x,a)}$. Finally,

$$\int_{\widehat{\Delta}_r} |\mathrm{V} \circ \widehat{\mathrm{T}}_t - \mathrm{V}_t \circ \pi_r|\, \mathrm{d}\nu_r \leq \mathrm{C} \int_{\Delta_r} \kappa^{-\Psi_t(x,a)}\, \mathrm{d}\mu_r. \tag{8.5}$$

This quantity decays exponentially, by Lemma 8.1 (and since the density of μ is bounded). □

Lemma **8.3.** — *There exist* $\delta > 0$ *(independent of* U, V*) and* $\mathrm{C} > 0$ *such that, for all* $t \geq 0$,

$$\left| \int_{\widehat{\Delta}_r} \mathrm{U} \cdot \mathrm{V}_t \circ \mathrm{T}_t \circ \pi_r\, \mathrm{d}\nu_r \right| \leq \mathrm{C} e^{-\delta t}. \tag{8.6}$$

Proof. — Define a function $\bar{\mathrm{U}}$ on Δ_r by $\bar{\mathrm{U}}(x, a) = \int_{y \in \pi^{-1}(x)} \mathrm{U}(y, a)\, \mathrm{d}\nu_x(y)$. Since $\mathrm{U} \in \mathrm{C}^1(\widehat{\Delta}_r)$ and the measures ν_x satisfy the third property in the definition of hyperbolic skew-products, the function $\bar{\mathrm{U}}$ belongs to $\mathscr{B}_0$. Moreover, $\int_{\Delta_r} \bar{\mathrm{U}}\, \mathrm{d}\mu_r =$

$\int_{\widehat{\Delta}_r} \mathrm{U}\,\mathrm{d}\nu_r = 0$. Hence, Theorem 7.3 (or rather the remark following it) gives

$$(\mathbf{8.7})\qquad \left|\int_{\widehat{\Delta}_r} \mathrm{U}\cdot \mathrm{V}_t\circ \mathrm{T}_t\circ \pi_r\,\mathrm{d}\nu_r\right| = \left|\int_{\Delta_r} \bar{\mathrm{U}}\cdot \mathrm{V}_t\circ \mathrm{T}_t\,\mathrm{d}\mu_r\right| \leq \mathrm{C}e^{-\delta t}\|\bar{\mathrm{U}}\|_{\mathscr{B}_0}\|\mathrm{V}_t\|_{\mathscr{B}_1}.$$

To conclude the proof, it is thus sufficient to show that $\|\mathrm{V}_t\|_{\mathscr{B}_1}$ is uniformly bounded. First of all, since V is bounded, V_t is bounded.

Consider then $x \in \bigcup \Delta^{(l)}$. Take $0 < a < r(x)$. If $\mathrm{T}_t(x, a)$ is not of the form $(x', 0)$, then V_t is differentiable along the flow direction at (x, a). Its derivative is given by

$$(\mathbf{8.8})\qquad \int_{y\in\pi^{-1}(x)} (\partial_a \mathrm{V})(\widehat{\mathrm{T}}_t(y, a))\,\mathrm{d}\nu_x(y),$$

since the flow is an isometry in the flow direction. In particular, this derivative is bounded by $\|\mathrm{V}\|_{\mathrm{C}^1}$.

There is a finite number of points $0 < a_1 < \cdots < a_p < r(x)$ such that $\mathrm{T}_t(x, a_i)$ is of the form $(x', 0)$. Indeed, since r is uniformly bounded from below by a constant ε_1, there are at most $\frac{r(x)}{\varepsilon_1} + 1$ such points. At each of these points, V_t has a jump of at most $2\|\mathrm{V}\|_{\mathrm{C}^0}$. Finally, the variation of $a \mapsto \mathrm{V}_t(x, a)$ along the interval $(0, r(x))$ is at most

$$(\mathbf{8.9})\qquad \left(\frac{r(x)}{\varepsilon_1} + 1\right) 2\|\mathrm{V}\|_{\mathrm{C}^0} + r(x)\|\mathrm{V}\|_{\mathrm{C}^1} \leq \mathrm{C}r(x)\|\mathrm{V}\|_{\mathrm{C}^1}.$$

□

Lemmas 8.2 and 8.3 show that, for a uniform constant $\delta > 0$ and for some constant $\mathrm{C} > 0$ depending on U and V, for all $t \geq 0$,

$$(\mathbf{8.10})\qquad \left|\int \mathrm{U}\cdot \mathrm{V}\circ \widehat{\mathrm{T}}_{2t}\,\mathrm{d}\nu_r\right| \leq \mathrm{C}e^{-\delta t}.$$

By the closed graph theorem, the constant C can be chosen of the form $\mathrm{C}'\|\mathrm{U}\|_{\mathrm{C}^1}\|\mathrm{V}\|_{\mathrm{C}^1}$ for a uniform constant C'. This concludes the proof of Theorem 2.7.

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A. A simple distortion estimate

Here we present an alternative distortion estimate, Theorem A.2, which is far from optimal, but is enough to obtain exponential mixing, while being based on a much simpler argument. While much simpler, we have only noticed it after obtaining the nearly optimal estimate.

For $\mathscr{A}' \subset \mathscr{A}$ non-empty, let $\mathrm{m}_{\mathscr{A}'}(q) = \min_{\alpha\in\mathscr{A}'} q_\alpha$, and let $\mathrm{m}(q) = \mathrm{m}_{\mathscr{A}}(q)$. The other notations are those of §5.3.

Lemma **A.1** *(Kerckhoff, [K]). — For every* $\mathrm{T} > 0$, $q \in \mathbf{R}_+^{\mathscr{A}}$, $\alpha \in \mathscr{A}$, $\pi \in \mathfrak{R}$, *we have*

$$(\mathbf{A.1}) \qquad \mathrm{P}_q(\gamma \in \Gamma_\alpha(\pi),\ (\mathrm{B}_\gamma \cdot q)_\alpha > \mathrm{T} q_\alpha \mid \pi) < \mathrm{T}^{-1},$$

where $\Gamma_\alpha(\pi)$ *denotes the set of paths starting at* π *with no winner equal to* α.

Proof. — Let $\Gamma_\alpha^{(n)}(\pi) \subset \Gamma_\alpha(\pi)$ denote the set of paths of length at most n. We prove the inequality for $\Gamma_\alpha^{(n)}(\pi)$ by induction on n. The case $n = 0$ is clear. The case n follows immediately from the case $n-1$ when none of the rows of π end with α. Assume for instance that the top row of π ends with α and the bottom row with β. Then every path $\gamma \in \Gamma_\alpha^{(n)}(\pi)$ starts with the bottom arrow γ_s starting at π. Let $q' = \mathrm{B}_{\gamma_s} \cdot q$. We have $q'_\alpha = q_\alpha + q_\beta$ and $\mathrm{P}_q(\gamma_s \mid \pi) = \frac{q_\alpha}{q'_\alpha}$. The inequality follows by the induction hypothesis. □

Theorem **A.2.** — *There exists* $\mathrm{C} > 1$ *such that for every* $q \in \mathbf{R}_+^{\mathscr{A}}$, *if* $\pi \in \mathfrak{R}$

$$(\mathbf{A.2}) \qquad \mathrm{P}_q(\mathrm{M}(\mathrm{B}_\gamma \cdot q) < \mathrm{C} \min\{\mathrm{m}(\mathrm{B}_\gamma \cdot q), \mathrm{M}(q)\} \mid \pi) > \mathrm{C}^{-1}.$$

Proof. — For $1 \le k \le d$, let $\mathrm{m}_k(q) = \max \mathrm{m}_{\mathscr{A}'}(q)$ where the maximum is taken over all $\mathscr{A}' \subset \mathscr{A}$ such that $\#\mathscr{A}' = k$. In particular $\mathrm{m} = \mathrm{m}_d$. We will show that for $1 \le k \le d$ there exists $\mathrm{C} > 1$ such that

$$(\mathbf{A.3}) \qquad \mathrm{P}_q(\mathrm{M}(\mathrm{B}_\gamma \cdot q) < \mathrm{C} \min\{\mathrm{m}_k(\mathrm{B}_\gamma \cdot q), \mathrm{M}(q)\} \mid \pi) > \mathrm{C}^{-1}$$

(the case $k = d$ implying the desired statement). The proof is by induction on k. For $k = 1$ it is obvious. Assume that it is proved for some $1 \le k < d$ with $\mathrm{C} = \mathrm{C}_0$. Let Γ be the set of minimal paths γ starting at π with $\mathrm{M}(\mathrm{B}_\gamma \cdot q) < \mathrm{C}_0 \min\{\mathrm{m}_k(\mathrm{B}_\gamma \cdot q), \mathrm{M}(q)\}$. Then there exists $\Gamma_1 \subset \Gamma$ with $\mathrm{P}_q(\Gamma_1 \mid \pi) > \mathrm{C}_1^{-1}$ and $\mathscr{A}' \subset \mathscr{A}$ with $\#\mathscr{A}' = k$ such that if $\gamma \in \Gamma_1$ then $\mathrm{m}_k(\mathrm{B}_\gamma \cdot q) = \mathrm{m}_{\mathscr{A}'}(\mathrm{B}_\gamma \cdot q)$.

For $\gamma_s \in \Gamma_1$, choose a path $\gamma = \gamma_s \gamma_e$ with minimal length such that γ ends at a permutation π_e such that the top or the bottom row of π_e (and possibly both) ends by some element of $\mathscr{A} \setminus \mathscr{A}'$. Let Γ_2 be the collection of the $\gamma = \gamma_s \gamma_e$ thus obtained. Then $\mathrm{P}_q(\Gamma_2 \mid \pi) > \mathrm{C}_2^{-1}$ and $\mathrm{M}(\mathrm{B}_\gamma \cdot q) < \mathrm{C}_2 \mathrm{M}(\mathrm{B}_{\gamma_s} \cdot q)$ for $\gamma = \gamma_s \gamma_e \in \Gamma_2$.

Let Γ_3 be the set of paths $\gamma = \gamma_s\gamma_e$ such that $\gamma_s \in \Gamma_2$, the winner of the last arrow of γ_e belongs to $\mathscr{A}'$, the winners of the other arrows of γ_e belong to $\mathscr{A} \setminus \mathscr{A}'$, and we have $(B_\gamma \cdot q)_\alpha \leq 2d(B_{\gamma_s} \cdot q)_\alpha$ for all $\alpha \in \mathscr{A}'$. By Lemma A.1, $P_q(\Gamma_3 \,|\, \gamma_s) > \frac{1}{2}$, $\gamma_s \in \Gamma_2$, and $P_q(\Gamma_3 \,|\, \pi) > (2C_2)^{-1}$.

Let $\gamma = \gamma_s\gamma_e \in \Gamma_3$, $\gamma_s \in \Gamma_2$. If $M(B_\gamma \cdot q) > 2dM(B_{\gamma_s} \cdot q)$, we take γ_1 with $\gamma_s \leq \gamma_1 \leq \gamma$, of minimal length such that $M(B_{\gamma_1} \cdot q) > 2dM(B_{\gamma_s} \cdot q)$; there exists $\alpha \in \mathscr{A} \setminus \mathscr{A}'$ such that $M(B_{\gamma_1} \cdot q) = (B_{\gamma_1} \cdot q)_\alpha \leq 4dM(B_{\gamma_s} \cdot q)$. Moreover we have $m_{\mathscr{A}'}(B_{\gamma_1} \cdot q) > (C_0C_2 4d)^{-1}M(B_{\gamma_1} \cdot q)$ in this case. If $M(B_\gamma \cdot q) \leq 2dM(B_{\gamma_s} \cdot q)$, the loser α of the last arrow of γ (which belongs to $\mathscr{A} \setminus \mathscr{A}'$ by construction) satisfies $(B_\gamma \cdot q)_\alpha \geq (C_0C_2 2d)^{-1}M(B_\gamma \cdot q)$. This allows again to conclude: in any case there exists γ_1 with $\gamma_s \leq \gamma_1 \leq \gamma_e$ and $\mathscr{A}'_1$ with $\#\mathscr{A}'_1 = k+1$ such that $M(B_{\gamma_1} \cdot q) \leq 4dC_0C_2 \min\{m_{\mathscr{A}'_1}(B_{\gamma_1} \cdot q), M(q)\}$. Since the set Γ_4 of all γ_1 thus obtained satisfies $P_q(\Gamma_4 \,|\, \pi) \geq P_q(\Gamma_3 \,|\, \pi) > (2C_2)^{-1}$, (A.3) holds with $k+1$ instead of k. □

B. Spectral gap

This section is concerned with the natural action of $SL(2, \mathbf{R})$ on a connected component of a stratum $\mathscr{C}^{(1)}$. Though we have not used it elsewhere in this paper, this action is very important in several works on the Teichmüller flow, see for instance the work on Lyapunov exponents of [Fo].

We recall that the mere *existence* of this action has already important implications: for instance the action of non-compact one-parameter subgroups (which are conjugate either to the *Teichmüller flow* or the *horocycle flow*) is automatically mixing with respect to any ergodic invariant measure for the $SL(2, \mathbf{R})$ action. Thus, ergodicity of the Teichmüller flow ([Ma], [Ve1]) with respect to the absolutely continuous invariant measure on $\mathscr{C}^{(1)}$ implies mixing (which can be obtained also directly [Ve2]).

Here we will show how our analysis of the Teichmüller flow can be used to show that the $SL(2, \mathbf{R})$ action has a *spectral gap*. To put this concept in context, we recall some more general definitions.

Definition **B.1.** — *Let* G *be a (locally compact σ-compact) group. A (strongly continuous) unitary representation of* G *is said to have* almost invariant vectors *if for every* $\varepsilon > 0$ *and for every compact subset* $K \subset G$*, there exists a unit vector* v *such that* $\|g \cdot v - v\| < \varepsilon$ *for all* $g \in K$.

A unitary action which does not have almost invariant vectors is said to be isolated from the trivial representation.

If G *is a semi-simple Lie group (such as* $SL(2, \mathbf{R})$*), a representation which is isolated from the trivial representation is also said to have a* spectral gap.

Given a probability preserving action of $\mathrm{SL}(2, \mathbf{R})$, it thus makes sense to ask whether the corresponding unitary representation on L^2_0 (the space of zero-average L^2 functions) has a spectral gap. Ergodicity of the action is of course a necessary condition, being equivalent to the nonexistence of invariant unit vectors. It may happen for a group G that any unitary representation which has almost invariant vectors has indeed an invariant unit vector: this is one of the equivalent definitions of Kazhdan's property (T), and has several consequences. As it is well known, $\mathrm{SL}(2, \mathbf{R})$ does not have property (T), so the spectral gap is indeed a non-automatic property in this case.

The spectral gap for the $\mathrm{SL}(2, \mathbf{R})$ action on $\mathscr{C}^{(1)}$ can be also seen more geometrically as a statement about the foliated Laplacian on $\mathscr{C}^{(1)}/\mathrm{SO}(2, \mathbf{R})$,[3] or of the Casimir operator: the spectrum (for the action on L^2_0) does not contain 0.

The connection between the spectral gap for the $\mathrm{SL}(2, \mathbf{R})$ action and rates of mixing for non-compact one-parameter subgroups was used most notably by Ratner [Rt]. In her work, estimates on the rates of mixing are deduced from the spectral gap. That one could also go the other way around seems to be also understood (the argument is much easier than for the direction used by Ratner). It is possible however that this is the first time that it has been useful to consider this connection in the other direction.

The existence of a spectral gap has several ramifications. It is even interesting to just "go back" to rates of mixing using the work of Ratner. It implies *polynomial decay of correlations* for the horocycle flow. It even gives back extra information regarding the Teichmüller flow: it implies that exponential mixing holds for observables which are only Hölder along the $\mathrm{SO}(2, \mathbf{R})$ orbits (this notion of regularity is made precise in [Rt]). Further applications include exponential estimates for the Ball Averaging Problem, see [MNS].

The initial line of the arguments given here (reduction to a "reverse Ratner estimate") was explained to us by Nalini Anantharaman, Sasha Bufetov and Giovanni Forni. The proof of the "reverse Ratner estimate" was explained to us by Giovanni Forni.

Proposition **B.2.** — *Let us consider an ergodic action of* $\mathrm{SL}(2, \mathbf{R})$ *by measure-preserving automorphisms of a probability space. Let* ρ *be the corresponding representation on the space* H *of* L^2 *zero average functions. Assume that there exist* $\delta \in (0, 1)$ *and a dense subset of the subspace of* $\mathrm{SO}(2, \mathbf{R})$*-invariant functions* $\mathrm{H}' \subset \mathrm{H}$ *consisting of functions* ϕ *for which the correlations* $\langle \phi, \rho(g_t) \cdot \phi \rangle$, $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, *decay like* $\mathrm{O}(e^{-\delta t})$. *Then* ρ *is isolated from the trivial representation.*

[3] The space $\mathscr{C}^{(1)}/\mathrm{SO}(2, \mathbf{R})$ is foliated by quotients of $\mathrm{SL}(2, \mathbf{R})/\mathrm{SO}(2, \mathbf{R})$, which is a model for 2-dimensional hyperbolic space. In particular there is a natural leafwise metric of constant curvature -1, which allows us to define the foliated Laplacian, whose spectrum is contained in $[0, \infty)$.

Proof. — Let us decompose ρ into irreducible representations. Thus $\mathrm{H} = \int \mathrm{H}_\xi \, \mathrm{d}\mu(\xi)$, and there are irreducible actions ρ_ξ of $\mathrm{SL}(2, \mathbf{R})$ on each H_ξ which integrate to ρ.

Bargmann's classification (see [Rt] and the references therein) shows that all non-trivial irreducible representations fall into one of three series of representations: the principal, the complementary and the discrete series. Thus we have the corresponding decomposition $\mu = \mu_p + \mu_c + \mu_d$. We recall some basic facts that follow from this classification:

1. If ρ_ξ is in the complementary series, then there exists $s = s(\xi) \in (0, 1)$, such that ρ_ξ is isomorphic to the following representation ρ_s: the Hilbert space is

$$\mathscr{H}_s = \left\{ f : \mathbf{R} \to \mathbf{C} \, : \, \|f\|^2 = \int_{\mathbf{R}\times\mathbf{R}} \frac{f(x)\overline{f(y)}}{|x-y|^{1-s}} \, \mathrm{d}x \, \mathrm{d}y < \infty \right\}, \tag{B.1}$$

and the action is given by

$$\rho_s \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) = \frac{1}{(cx+d)^{1+s}} f\left(\frac{ax+b}{cx+d}\right). \tag{B.2}$$

2. The (integrated) representation ρ is isolated from the trivial representation if and only if there exists $\varepsilon > 0$ such that $s(\xi) < 1 - \varepsilon$ for μ_c-almost every ξ.
3. The space of $\mathrm{SO}(2, \mathbf{R})$ invariant vectors $\mathrm{H}'_\xi \subset \mathrm{H}_\xi$ is zero-dimensional (in the case of the discrete series) or one-dimensional (in the case of the principal and complementary series).

Let $\mathrm{H}' \subset \mathrm{H}$ be the set of $\mathrm{SO}(2, \mathbf{R})$ invariant functions. Then $\mathrm{H}' = \int \mathrm{H}'_\xi \, \mathrm{d}\mu(\xi)$. The point of the proof is the following lemma:

Lemma **B.3.** — *If ρ_ξ is in the complementary series and $\phi_\xi \in \mathrm{H}'_\xi$ is a non-zero vector, then $\langle \phi_\xi, \rho_\xi(g_t) \cdot \phi_\xi \rangle$ is* positive *and*

$$\lim_{t\to\infty} \frac{1}{t} \log \langle \phi_\xi, \rho_\xi(g_t) \cdot \phi_\xi \rangle = -1 + s(\xi). \tag{B.3}$$

Let us show how to conclude the proof using the lemma. Suppose by contradiction that ρ is not isolated from the trivial representation. There exists a function $\phi = \int \phi_\xi \, \mathrm{d}\mu(\xi) \in \mathrm{H}'$ whose correlations decay like $\mathrm{O}(e^{-\delta t})$ and such that

$$\mu_c\{\xi \, : \, \phi_\xi \neq 0 \text{ and } s(\xi) \in (1 - \delta/2, 1)\} > 0. \tag{B.4}$$

Write $\phi = \phi_p + \phi_c$ where ϕ_p is the part of ϕ corresponding to representations in the principal series, and ϕ_c corresponds to the complementary series (as discussed above, $\phi_\xi = 0$ for μ_d-almost every ξ). Then

$$\langle \phi, \rho(g_t) \cdot \phi \rangle = \langle \phi_p, \rho(g_t) \cdot \phi_p \rangle + \langle \phi_c, \rho(g_t) \cdot \phi_c \rangle. \tag{B.5}$$

By the results of Ratner [Rt], the correlations of ϕ_p decay at least as te^{-t}. Moreover, by (B.3), positivity, and (B.4), the second term is larger than $Ce^{-\delta t/2}$ for large t. This contradicts the speed of decay of correlations of ϕ. □

Proof of Lemma B.3. — A function $f \in \mathscr{H}_s$ is invariant under the $\mathrm{SO}(2, \mathbf{R})$ action if and only if it is smooth and satisfies the differential equation $(1+x^2)f'(x) + (1+s)xf(x) = 0$, i.e., $f(x) = \frac{c}{(1+x^2)^{(1+s)/2}}$.

For such a function f, the correlations are given by

$$\begin{aligned}\langle f, \rho_s(g_t)\cdot f\rangle &= |c|^2 e^{t(1+s)} \int_{\mathbf{R}\times\mathbf{R}} \frac{\mathrm{d}x\,\mathrm{d}y}{(1+x^2)^{(1+s)/2}(1+e^{4t}y^2)^{(1+s)/2}|x-y|^{1-s}} \\ &= |c|^2 e^{t(-1+s)} \int_{\mathbf{R}\times\mathbf{R}} \frac{\mathrm{d}x\,\mathrm{d}y}{(1+x^2)^{(1+s)/2}(1+y^2)^{(1+s)/2}|x-e^{-2t}y|^{1-s}}.\end{aligned} \tag{B.6}$$

This shows that the correlations are positive and that

$$\begin{aligned}&\liminf e^{t(1-s)}\langle f, \rho_s(g_t)\cdot f\rangle \\ &\quad \geq |c|^2 \int_{\mathbf{R}\times\mathbf{R}} \frac{\mathrm{d}x\,\mathrm{d}y}{(1+x^2)^{(1+s)/2}(1+y^2)^{(1+s)/2}|x|^{1-s}} > 0.\end{aligned} \tag{B.7}$$

Moreover, Ratner has proved the upper bound $\limsup e^{t(1-s)}\langle f, \rho_s(g_t)\cdot f\rangle < \infty$ in [Rt, Theorem 1] (the convergence of the last integral in (B.6) to the integral in (B.7) can also be verified directly). This concludes the proof of the lemma. □

Since our Main Theorem implies exponential decay of correlations for compactly supported smooth functions, the hypothesis of Proposition B.2 is satisfied. Corollary 1.1 follows.

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