

Exponential rate of convergence to equilibrium for a model describing fiber lay-down processes

Jean Dolbeault^{a*}, Axel Klar^b, Clément Mouhot^c, Christian Schmeiser^d

^a*Ceremade (UMR CNRS 7534), Université Paris-Dauphine, Place de Lattre de Tassigny, F-75775 Paris Cédex 16, France.*

E-mail: dolbeaul@ceremade.dauphine.fr,

^b*Technische Universität Kaiserslautern, Fachbereich Mathematik, E. Schrödinger Straße, D-67663 Kaiserslautern, Germany.*

E-mail: klar@itwm.fhg.de,

^c*University of Cambridge, DAMTP, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WA, UK.*

E-mail: Clement.Mouhot@ens.fr,

^d*Fakultät für Mathematik, Universität Wien, Nordbergstraße 15, 1090 Wien, Austria. E-mail: Christian.Schmeiser@univie.ac.at*

Abstract: This paper is devoted to the adaptation of the method developed in (4; 3) to a Fokker-Planck equation for fiber lay-down which has been studied in (1; 5). Exponential convergence towards a unique stationary state is proved in a norm which is equivalent to a weighted L^2 norm. The method is based on a micro / macro decomposition which is well adapted to the diffusion limit regime.

KEY WORDS kinetic equations; stochastic differential equations; Fokker-Planck equation; fiber dynamics; hypocoercivity; spectral gap; Poincaré inequality; hypoelliptic operators; degenerate diffusion; transport operator; large time behavior; convergence to equilibrium; exponential rate of convergence

July 1, 2012

1 Introduction

The understanding of the shapes generated by the lay-down of flexible fibers onto a conveyor belt is of great interest in the production process of nonwovens textiles that find their applications, e.g., in composite materials like filters, textile and hygiene industry. In (5) a stochastic model for the fiber lay-down process, *i.e.* for the generation of a fiber web on a conveyor belt has been presented. Taking into account the fiber motion under the influence of turbulence, the process can be described by a system of stochastic differential equations. We shall focus on a very simple situation, which does not take into account the movement of the belt, and gives rise to a Fokker-Planck equation. Some numerical results will also be considered at the end of this paper.

An important criterion for the quality of the web and the resulting nonwoven material is how the solution converges to equilibrium. In particular, the speed of convergence to the stationary solution is important. The faster this convergence is, the more uniform the produced textile will be. From a technological point of view, process parameters should be adjusted such that the speed of convergence to equilibrium is optimal.

The trend to equilibrium for solutions of kinetic equations has been investigated in many papers using entropy methods; see for example (2; 13). A simplified approach has been suggested in (4; 3). This trend to equilibrium for the Fokker-Planck equation for fiber lay-down under consideration in this paper has already been investigated using Dirichlet forms and operator semi-group techniques in (8); an ergodic theorem and explicit rates of convergence have been established. In the present paper we prove the convergence at an exponential rate towards a unique stationary state in a weighted L^2 norm by adapting the method developed in (4; 3) to the setting of non-moving belts.

2 The model and main results

In the melt-spinning process of nonwoven textiles, hundreds of individual endless fibers obtained by the continuous extrusion through nozzles of a melted polymer are stretched and entangled by highly turbulent air flows to finally form a web on a conveyor belt (see (5) for more details). We describe the motion of an individual fiber, neglecting interactions with the others.

*Correspondence to: Jean Dolbeault: Ceremade (UMR CNRS 7534), Université Paris-Dauphine, Place de Lattre de Tassigny, F-75775 Paris Cédex 16, France. E-mail: dolbeaul@ceremade.dauphine.fr

An arclength parametrization of the laid down fiber in a coordinate system following the conveyor belt is given by $x_0(t) \in \mathbb{R}^2$, $t \geq 0$. The tangent vector is denoted by $dx_0(t)/dt = \tau(\alpha(t))$ with $\tau(\alpha) = (\cos \alpha, \sin \alpha)$, $\alpha \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Since the lay-down process is assumed to happen at the constant normalized speed 1 (equal to the spinning speed), $x_0(t)$ can also be interpreted as the position of the lay-down point at time t . If the conveyor belt moves with velocity κe_1 , the history of the lay-down point in the laboratory frame (as opposed to the conveyor belt frame) is given by $x(t) = x_0(t) + t \kappa e_1$, *i.e.*,

$$\frac{dx}{dt} = \tau(\alpha) + \kappa e_1. \quad (1)$$

The practical device for producing fiber lay-down (and how it can be modeled from a mathematical point of view) has been described with details in (10). It is a natural restriction that the speed of the conveyor belt cannot exceed the lay-down speed: $0 \leq \kappa \leq 1$, since otherwise a stationary lay-down point would be impossible. The lay-down process can now be determined by prescribing the dynamics of the angle $\alpha(t)$, described as a stochastic process. It is driven by a deterministic force trying to move the lay-down point towards the equilibrium position $x = 0$ and by a Brownian motion modeling the effect of the turbulent air flow:

$$d\alpha = -\tau^\perp(\alpha) \cdot \nabla V(x) dt + A dW, \quad (2)$$

where W denotes a one-dimensional Wiener process, $A > 0$ measures its strength relative to the deterministic forcing, $\tau^\perp = d\tau/d\alpha = (-\sin \alpha, \cos \alpha)$, and $V(x)$ is a potential such that e^{-V} is integrable with the normalisation $\int_{\mathbb{R}^2} e^{-V} dx = 1$ and $\nabla V(0) = 0$.

The system (1)–(2) defines a stochastic process on $\mathbb{R}^2 \times S^1$. The corresponding probability density $f(t, x, \alpha)$ satisfies the Fokker-Planck equation

$$\partial_t f + (\tau + \kappa e_1) \cdot \nabla_x f - \partial_\alpha(\tau^\perp \cdot \nabla_x V f) + D \partial_\alpha^2 f = 0,$$

(with the diffusivity $D = A^2/2$) which will be the object of our study. The analysis of the long time behaviour is considerably simplified in the case of a nonmoving conveyor belt:

$$\kappa = 0$$

because the stationary solution is explicit. Dealing with the moving belt requires new tools. We shall assume that the assumption $\kappa = 0$ holds true from now on. In this case the Fokker-Planck equation is written as an abstract ODE

$$\partial_t f + \mathbb{T}f = D \mathbb{L}f, \quad (3)$$

with $\mathbb{T}f = \tau \cdot \nabla_x f - \partial_\alpha(\tau^\perp \cdot \nabla_x V f)$ and $\mathbb{L}f = \partial_\alpha^2 f$.

It is easily seen that $F(x, \alpha) = e^{-V(x)}$ is an equilibrium solution of (3), lying in the intersection of the null spaces of \mathbb{T} and \mathbb{L} : $\mathbb{T}F = \mathbb{L}F = 0$.

A convenient functional analytic setting is introduced by the scalar product

$$\langle f, g \rangle := \int_{\mathbb{R}^2 \times S^1} fg d\mu, \quad d\mu(x, \alpha) := \frac{dx d\nu(\alpha)}{F(x, \alpha)}, \quad d\nu(\alpha) := \frac{d\alpha}{2\pi},$$

and by the associated norm $\|f\|^2 = \langle f, f \rangle$. On the space $L^2(\mathbb{R}^2 \times S^1, d\mu)$, the operator \mathbb{T} is skew symmetric, and the operator \mathbb{L} is symmetric and negative semi-definite. Thus, we have

$$\frac{d}{dt} \frac{\|f - F\|^2}{2} = D \langle \mathbb{L}f, f \rangle = -D \|\partial_\alpha f\|^2. \quad (4)$$

This identity reveals the main difficulty in proving convergence to equilibrium. The decay to equilibrium seems to stop, as soon as f is in the null space of \mathbb{L} consisting of all α -independent distributions. On the other hand, the decay equation (4) does not make use of the action of the operator \mathbb{T} and, in particular, of the fact that the equilibrium F is the unique probability density in $\mathcal{N}(\mathbb{T}) \cap \mathcal{N}(\mathbb{L})$. Any α -independent distribution function f is indeed unstable under the action of \mathbb{T} , unless $f = F$. Hence, convergence to the equilibrium can be expected and will be proven to be exponential. This is a so-called *hypocoercivity* result as defined in (13). A recently developed approach (4; 3) for proving hypocoercivity in the abstract setting (3) will be applied with a special emphasis on the behaviour of the decay rate as $D \rightarrow 0$ and $D \rightarrow \infty$. It requires assumptions on the potential V , which have already been used in (4):

(H1) *Regularity*: $V \in W_{\text{loc}}^{2,\infty}(\mathbb{R}^2)$.

(H2) *Normalization*: $\int_{\mathbb{R}^2} e^{-V} dx = 1$.

(H3) *Spectral gap condition*: there exists a positive constant Λ such that

$$\int_{\mathbb{R}^2} |\nabla_x u|^2 e^{-V} dx \geq \Lambda \int_{\mathbb{R}^2} u^2 e^{-V} dx$$

for any $u \in H^1(e^{-V} dx)$ such that $\int_{\mathbb{R}^2} u e^{-V} dx = 0$.

(H4) *Pointwise condition*: there exists $c_1 > 0$ such that

$$|\nabla_x^2 V(x)| \leq c_1 (1 + |\nabla_x V(x)|) \text{ for any } x \in \mathbb{R}^2.$$

Roughly speaking, (H2) and (H3) require a sufficiently strong growth of $V(x)$ as $|x| \rightarrow \infty$, whereas (H4) puts a limitation on the growth behavior. This leaves room, however, for a large class of confining potentials including $V(x) = (1 + |x|^2)^\beta$, $\beta \geq 1/2$.

In (4), the additional pointwise condition $\Delta_x V(x) \leq \frac{\theta}{2} |\nabla_x V(x)|^2 + c_0$ with $c_0 > 0$ and $\theta \in (0, 1)$ has been required. This is, however, a consequence of (H4) by $\Delta_x V(x) \leq \sqrt{2} |\nabla_x^2 V(x)|$ and the Young inequality $\sqrt{2} c_1 |\nabla_x V| \leq |\nabla_x V|^2/4 + 2c_1^2$, with $\theta = 1/2$ and $c_0 = c_1 + 2c_1^2$.

Theorem 1. *Let $f_0 \in L^2(\mathbb{R}^2 \times S^1, d\mu)$ and let (H1)–(H4) hold. Then, for every $\eta > 0$, the solution of (3) subject to the initial condition $f(t=0) = f_0$ satisfies*

$$\|f(t) - F\| \leq (1 + \eta) \|f_0 - F\| e^{-\lambda t} \quad \text{with} \quad \lambda = \frac{\eta}{1 + \eta} \frac{C_1 D}{1 + C_2 D^2},$$

where C_1 and C_2 are two positive constants which depend only on the potential V .

As a consequence, we have

$$\lambda = O(D) \quad \text{as } D \rightarrow 0 \quad \text{and} \quad \lambda = O(D^{-1}) \quad \text{as } D \rightarrow \infty.$$

Both results are sharp, as can be seen from the toy problem in (4). As $D \rightarrow 0$, dissipation is provided by the $O(D)$ right hand side of (3), which dominates λ . On the other hand, the dynamics for large D can be described by a macroscopic limit: see (1). When $D \rightarrow \infty$, the correct time scale is $t = O(D)$ and corresponds to a parabolic scaling, and therefore $\lambda = O(D^{-1})$ had to be expected.

However our method is not sharp in the sense that we cannot expect to obtain the optimal coefficients in the limiting cases above. Again this can be seen from the toy problem in (4), where the spectral gap can be explicitly computed.

3 Proof of Theorem 1

3.1 The modified entropy

We introduce the deviation $g := f - F$, satisfying (3) subject to $g(t=0) = f_0 - F$. Following (4), we denote the orthogonal projection to $\mathcal{N}(\mathbf{L})$ by

$$\Pi g := \rho_g = \int_{S^1} g d\nu.$$

Note that $\int_{\mathbb{R}^2} \rho_g dx = 0$ holds. In the following, we shall also need

$$\mathbf{T}\Pi g = \tau \cdot (\nabla_x \rho_g + \rho_g \nabla_x V) = \tau \cdot e^{-V} \nabla_x (e^V \rho_g), \quad (5)$$

with the consequence

$$\mathbf{T}\Pi \mathbf{T}\Pi = 0, \quad (6)$$

which is essential for the applicability of the method of (4). It implies that the macroscopic limit (corresponding to $D \rightarrow \infty$) in (3) is diffusive: see (1).

If $(\mathbf{T}\Pi)^*$ denotes the adjoint of $\mathbf{T}\Pi$ with respect to the scalar product $\langle \cdot, \cdot \rangle$, then, with the help of the operator

$$\mathbf{A} = (1 + (\mathbf{T}\Pi)^* \mathbf{T}\Pi)^{-1} (\mathbf{T}\Pi)^*$$

and an appropriately chosen $\varepsilon > 0$, the modified entropy functional is defined by

$$\mathbf{H}[g] := \frac{1}{2} \|g\|^2 + \varepsilon \langle \mathbf{A}g, g \rangle.$$

By (4, Lemma 1), (6) implies

$$\|Ag\| \leq \frac{1}{2} \|(1 - \Pi)g\|.$$

Hence the modified entropy functional is bounded from above and below by the square of the norm for any $\varepsilon \in (0, 1)$. More precisely we have:

$$\frac{1 - \varepsilon}{2} \|g\|^2 \leq H[g] \leq \frac{1 + \varepsilon}{2} \|g\|^2.$$

A straightforward computation gives

$$\frac{d}{dt} H[g] = -D[g], \quad (7)$$

with the entropy dissipation functional

$$D[g] = -D \langle Lg, g \rangle + \varepsilon \langle AT\Pi g, g \rangle + \varepsilon \langle AT(1 - \Pi)g, g \rangle - \varepsilon \langle TA g, g \rangle - \varepsilon D \langle ALg, g \rangle. \quad (8)$$

Notice indeed that the range of the operator A is contained in the space of α -independent functions, so that $\langle Ag, Lg \rangle = 0$ for any function $g \in L^2(\mathbb{R}^2 \times S^1, d\mu)$ which is in the domain of A .

3.2 Microscopic and macroscopic coercivity

The first term on the right hand side of (8) has already been computed in (4). With $d\nu = d\alpha/(2\pi)$, the Poincaré inequality on S^1 ,

$$\int_{S^1} |\partial_\alpha g|^2 d\nu \geq \int_{S^1} (g - \int_{S^1} g d\nu)^2 d\nu$$

implies the *microscopic coercivity* property

$$-\langle Lg, g \rangle \geq \|(1 - \Pi)g\|^2. \quad (9)$$

The operator $AT\Pi = (1 + (\Pi\Pi)^* \Pi\Pi)^{-1} (\Pi\Pi)^* \Pi\Pi$ shares its spectral decomposition with $(\Pi\Pi)^* \Pi\Pi$. For the latter we have, using (5)

$$\langle (\Pi\Pi)^* \Pi\Pi g, g \rangle = \|\Pi\Pi g\|^2 = \frac{1}{2} \int_{\mathbb{R}^2 \times S^1} e^{-V} |\nabla_x u_g|^2 dx d\nu,$$

with $u_g = e^V \rho_g$. The spectral gap condition (H3) implies the *macroscopic coercivity* property

$$\langle (\Pi\Pi)^* \Pi\Pi g, g \rangle \geq \frac{\Lambda}{2} \|\rho_g\|^2,$$

leading to

$$\langle AT\Pi g, g \rangle \geq \frac{\Lambda}{2 + \Lambda} \|\Pi g\|^2. \quad (10)$$

By (9) and (10), the sum of the first two terms in the entropy dissipation (8) is coercive. This will also be sufficient for controlling the remaining three terms, if the operators AT , TA , and AL are bounded, for $\varepsilon > 0$, small enough.

3.3 Boundedness of auxiliary operators

By (4, Lemma 1), we know that

$$\|TA g\| \leq \|(1 - \Pi)g\|. \quad (11)$$

The computation

$$(\Pi\Pi)^* Lg = -\Pi T Lg = -\nabla_x \cdot \Pi(\tau \partial_\alpha^2 g) = \nabla_x \cdot \Pi(\tau g) = -(\Pi\Pi)^* g$$

shows that $AL = -A$ and, thus,

$$\|ALg\| \leq \frac{1}{2} \|(1 - \Pi)g\|. \quad (12)$$

The most elaborate part of the analysis is to prove the boundedness of AT . Following the approach of (4), we consider its adjoint

$$(AT)^* = -T^2 \Pi (1 + (\Pi\Pi)^* \Pi\Pi)^{-1}.$$

For a given $g \in L^2(\mathbb{R}^2 \times S^1, d\mu)$, we introduce $h = (1 + (\Pi\Pi)^* \Pi\Pi)^{-1} g$ which, after solving for g and applying Π , becomes

$$\rho_g = \rho_h - \Pi T^2 \rho_h = e^{-V} u_h - \frac{1}{2} \nabla_x \cdot (e^{-V} \nabla_x u_h) \quad (13)$$

with $u_h = e^V \rho_h$. A straightforward computation gives

$$(\text{AT})^* g = -\mathbb{T}^2 \rho_h = e^{-V} [(\tau \cdot \nabla_x)^2 u_h - (\tau^\perp \cdot \nabla_x V) (\tau^\perp \cdot \nabla_x u_h)]$$

and, as a consequence,

$$\|(\text{AT})^* g\| \leq \|\nabla_x^{\otimes 2} u_h\|_{L^2(\mathbb{R}^2, e^{-V} dx)} + \|\nabla_x V\| \|\nabla_x u_h\|_{L^2(\mathbb{R}^2, e^{-V} dx)}.$$

Therefore, in order to prove the boundedness of $(\text{AT})^*$ (and, thus, of AT), we need to prove the boundedness of the right hand side in terms of $\|\rho_g\|$ for the solution u_h of the elliptic equation (13). This $L^2 \rightarrow H^2$ (with weight e^{-V}) elliptic regularity result has been derived in (4) under the assumptions (H1)–(H4) (see Proposition 5 for the first term and Lemma 8 for the second, where Assumption (H4) is explicitly needed). Collecting these results gives $\|(\text{AT})^* g\| \leq C_V \|g\|$ and therefore

$$\|\text{AT}(1 - \Pi)g\| \leq C_V \|(1 - \Pi)g\|, \quad (14)$$

where C_V depends only on the potential.

3.4 Hypocoercivity

Inserting estimates (9)–(12) and (14) in (8) gives

$$\begin{aligned} \mathbb{D}[g] &\geq D \|(1 - \Pi)g\|^2 + \frac{\varepsilon \Lambda}{2 + \Lambda} \|\Pi g\|^2 - \varepsilon (C_V + 1 + \frac{D}{2}) \|(1 - \Pi)g\| \|g\| \\ &\geq (D - \varepsilon (C_V + 1 + \frac{D}{2})) \|(1 - \Pi)g\|^2 + \frac{\varepsilon \Lambda}{2 + \Lambda} \|\Pi g\|^2 - \varepsilon (C_V + 1 + \frac{D}{2}) \|(1 - \Pi)g\| \|\Pi g\| \\ &\geq (D - \varepsilon (C_V + 1 + \frac{D}{2}) (1 + \frac{1}{2\delta})) \|(1 - \Pi)g\|^2 + \varepsilon \left(\frac{\Lambda}{2 + \Lambda} - \frac{\delta}{2} (C_V + 1 + \frac{D}{2}) \right) \|\Pi g\|^2, \end{aligned} \quad (15)$$

for an arbitrary $\delta > 0$. This shows already that coercivity can be achieved by first choosing δ and then ε , both small enough. With the choice

$$\delta = \frac{\Lambda}{(2 + \Lambda)(C_V + 1 + D/2)},$$

the coefficients on the right hand side of (15) can be written as $D - \varepsilon r(D)$ and εs with

$$r(D) := \frac{1}{2\Lambda} (2\Lambda + (2 + \Lambda)(C_V + 1 + \frac{D}{2})) (C_V + 1 + \frac{D}{2}), \quad s := \frac{\Lambda}{2(2 + \Lambda)}.$$

Then the optimal choice of ε , considering the form of the coefficients, would be

$$\bar{\varepsilon}(D) := \frac{D}{r(D) + s}.$$

However, we also have to guarantee $\varepsilon < 1$ for the definiteness of $\mathbb{H}[g]$ and actually, even stronger, $\frac{1+\varepsilon}{1-\varepsilon} \leq (1 + \eta)^2$ will be needed below, which can be guaranteed by the requirement $\varepsilon \leq \frac{\eta}{1+\eta}$. Moreover the two conditions are equivalent at first order for $\eta > 0$, small. These considerations lead to the choice

$$\varepsilon = \frac{\eta}{1 + \eta} \frac{\bar{\varepsilon}(D)}{\bar{\varepsilon}_{max}}, \quad \text{with } \bar{\varepsilon}_{max} = \max \left\{ 1, \max_{D>0} \bar{\varepsilon}(D) \right\},$$

which is finite because of $\bar{\varepsilon}(0) = \bar{\varepsilon}(\infty) = 0$. With this choice,

$$D - \varepsilon r(D) \geq \varepsilon s \geq \frac{\eta}{1 + \eta} \frac{2C_1 D}{1 + C_2 D^2} =: 2\lambda,$$

with appropriately chosen constants $C_1, C_2 > 0$, depending only on Λ and C_V and, thus, only on the potential V . The estimate

$$\mathbb{D}[g] \geq 2\lambda \|g\|^2 \geq \frac{4\lambda}{1 + \varepsilon} \mathbb{H}[g] > 2\lambda \mathbb{H}[g]$$

follows. Using this in (7) and the Gronwall lemma imply

$$\mathbb{H}[f(t) - F] \leq \mathbb{H}[f_0 - F] e^{-2\lambda t}.$$

Finally we obtain for the norm

$$\begin{aligned} \|f(t) - F\|^2 &\leq \frac{2}{1 - \varepsilon} \mathbb{H}[f(t) - F] \leq \frac{2}{1 - \varepsilon} \mathbb{H}[f_0 - F] e^{-2\lambda t} \\ &\leq \frac{1 + \varepsilon}{1 - \varepsilon} \|f_0 - F\|^2 e^{-2\lambda t} \leq (1 + \eta)^2 \|f_0 - F\|^2 e^{-2\lambda t}, \end{aligned}$$

which completes the proof of Theorem 1.

4 Concluding remarks

4.1 Numerical investigations

It is interesting to compare the rates predicted by the above results, which are only upper bounds, with numerical rates of convergence. We use a classical Monte-Carlo method with an Euler-Maruyama discretization scheme for all computations. A numerical investigation of the equations using a semi-Lagrangian method can be found in (12).

The exponential decay of the L^2 -difference to the stationary solution is observed in Figure 1. In Figure 2 the decay rates λ have been obtained from the above simulations for various values of A using a least square fit. The rate given by Theorem 1, *i.e.* $\lambda \sim \frac{C_1 D}{1 + C_2 D^2}$, $D = A^2/2$, fits qualitatively very well the curve obtained in Figure 2 when A is away from 0. In particular, values of A with an optimal rate of convergence can be determined from the numerical as well as the analytical results.

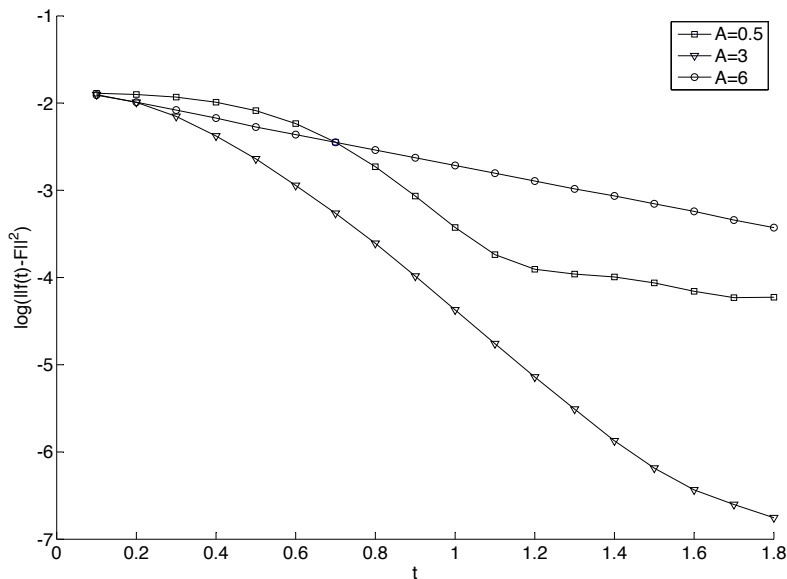


Figure 1: Plot of $t \mapsto \log \|f(t, \cdot, \cdot) - F\|^2$ for $A = 0.5, 3, 6$.

4.2 Perspectives

1. *Models where stationary solutions are not known explicitly.* The model considered in this paper can be extended in different directions, for instance by taking into account the movement of the belt, or by models where fibers have smoother trajectories than the ones considered in this paper, see (9; 1). In these cases the stationary solutions are not always known explicitly. The application of the entropy method presented above is then an open problem.
2. *3-D models.* To model fluid flow through a fiber web, the model has to be extended to three dimensional situations, see (11). For such a model exponential convergence to equilibrium, at least for the case $\kappa = 0$, can be proven with the same methods as in this paper.

Acknowledgements. This research project has been supported by the ANR project *CBDif-Fr* and *EVOL*, by the Excellence Center for Mathematical and Computational Modeling (CM)² and by Deutsche Forschungsgemeinschaft (DFG), KL 1105/18-1. The authors thank K. Fellner and P. Markowich for the organization of a conference on *Modern Topics in Nonlinear Kinetic Equations* (DAMTP, Cambridge, April 20-22, 2009) where this research project was initiated. They also thank an anonymous referee for his careful reading of the manuscript.

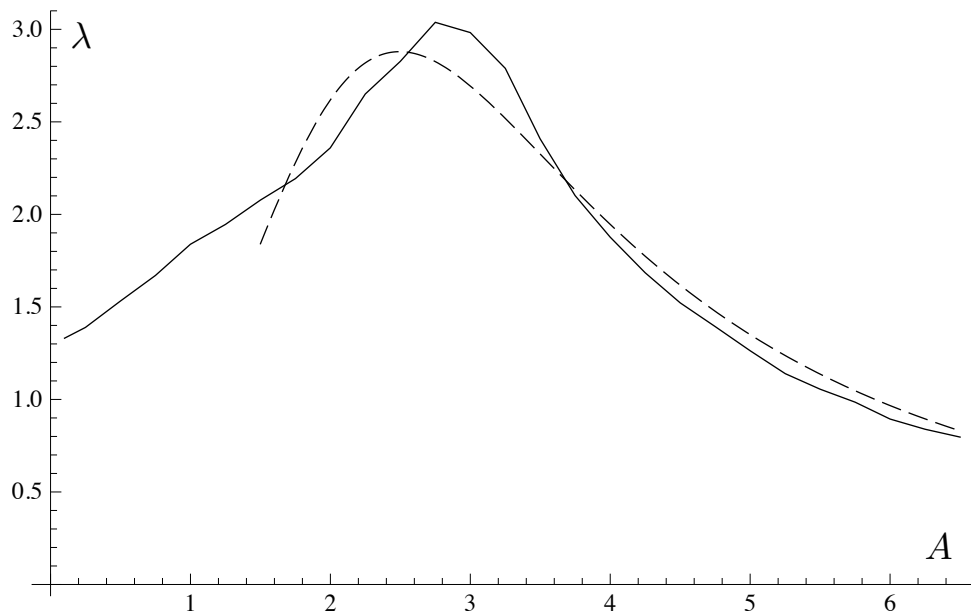


Figure 2: Plot of $\lambda = \lambda(A)$ as a function of A . Observed values fit correctly with the qualitative behavior predicted by our estimates, away from $A = 0$. Parameters C_1 and C_2 have been adjusted with a least square method for $A > 1.5$ (dashed curve).

References

- [1] L. L. BONILLA, T. GÖTZ, A. KLAR, N. MARHEINEKE, AND R. WEGENER, *Hydrodynamic limit of a Fokker-Planck equation describing fiber lay-down processes*, SIAM J. Appl. Math., 68 (2007/08), pp. 648–665.
- [2] L. DESVILLETES AND C. VILLANI, *On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: the linear Fokker-Planck equation*, Comm. Pure Appl. Math., 54 (2001), pp. 1–42.
- [3] J. DOLBEAULT, C. MOUHOT, AND C. SCHMEISER, *Hypocoercivity for kinetic equations with linear relaxation terms*, Comptes Rendus Mathématique, 347 (2009), pp. 511–516.
- [4] ———, *Hypocoercivity and stability for a class of kinetic models with mass conservation and a confining potential*. To appear in Transactions of the American Mathematical Society, 2011.
- [5] T. GÖTZ, A. KLAR, N. MARHEINEKE, AND R. WEGENER, *A stochastic model and associated Fokker-Planck equation for the fiber lay-down process in nonwoven production processes*, SIAM J. Appl. Math., 67 (2007), pp. 1704–1717.
- [6] T. GÖTZ, A. KLAR, AND A. UNTERREITER, *Asymptotics of fiber spinning equations*, in Progress in industrial mathematics at ECMI 2006, vol. 12 of Math. Ind., Springer, Berlin, 2008, pp. 697–702.
- [7] T. GÖTZ, A. KLAR, A. UNTERREITER, AND R. WEGENER, *Numerical evidence for the non-existence of stationary solutions of the equations describing rotational fiber spinning*, Math. Models Methods Appl. Sci., 18 (2008), pp. 1829–1844.
- [8] M. GROTHAUS AND A. KLAR, *Ergodicity and rate of convergence for a nonsectorial fiber lay-down process*, SIAM J. Math. Anal., 40 (2008), pp. 968–983.
- [9] M. HERTY, A. KLAR, S. MOTSCH, AND F. OLAWSKY, *A smooth model for fiber lay-down processes and its diffusion approximations*, Kinet. Relat. Models, 2 (2009), pp. 489–502.
- [10] A. KLAR, N. MARHEINEKE, AND R. WEGENER, *Hierarchy of mathematical models for production processes of technical textiles*, ZAMM Z. Angew. Math. Mech., 89 (2009), pp. 941–961.
- [11] A. KLAR, J. MARINGER, AND R. WEGENER, *A three dimensional model for the lay down of flexible fibers*, to appear in Mathematical Models and Methods in Applied Sciences, (2012).
- [12] A. KLAR, P. REUTERSWÄRD, AND M. SEAÏD, *A semi-Lagrangian method for a Fokker-Planck equation describing fiber dynamics*, J. Sci. Comput., 38 (2009), pp. 349–367.
- [13] C. VILLANI, *Hypocoercivity*, Mem. Amer. Math. Soc., 202 (2009), pp. iv+141.