EXPONENTIAL RUNGE-KUTTA METHODS FOR STIFF KINETIC EQUATIONS

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Abstract. We introduce a class of exponential Runge-Kutta integration methods for kinetic equations. The methods are based on a decomposition of the collision operator into an equilibrium and a non equilibrium part and are exact for relaxation operators of BGK type. For Boltzmann type kinetic equations they work uniformly for a wide range of relaxation times and avoid the solution of nonlinear systems of equations even in stiff regimes. We give sufficient conditions in order that such methods are unconditionally asymptotically stable and asymptotic preserving. Such stability properties are essential to guarantee the correct asymptotic behavior for small relaxation times. The methods also offer favorable properties such as nonnegativity of the solution and entropy inequality. For this reason, as we will show, the methods are suitable both for deterministic as well as probabilistic numerical techniques.

Keywords: Exponential integrators, Runge-Kutta methods, stiff equations, Boltzmann equation, fluid limits, asymptotic preserving schemes.

1. Introduction. The numerical solution of the Boltzmann equation close to fluid regimes represents a major computational challenge in rarefied gas dynamics. In such regimes, in fact, the mean free path becomes very small and standard computational approaches lose their efficiency due to the necessity of using very small time steps in deterministic schemes or, equivalently, a large number of collisions in probabilistic approaches. Several authors have tackled the problem in the past, and there is a large literature on the subject (see [2, 4, 10, 16, 14, 36] and the references therein). Most standard techniques are based on domain-decomposition strategies and/or model reduction asymptotic methods. The direct time discretization of the Boltzmann equation, in fact, represents a challenge in such stiff regimes due to the high dimensionality and the nonlinearity of the collision operator which makes unpractical the use of implicit solvers. Exponential methods, like the so-called time relaxed discretizations [16], combined with splitting strategies represent one possible way to overcome the problem. However, as discussed in various papers, the choice of the Maxwellian equilibrium truncation in the schemes was based more on physical considerations than on a direct mathematical derivation and it is an open problem to determine an optimal truncation criteria [5, 32]. Despite this, these discretizations have been applied with success both in the context of spectral methods as well as Monte Carlo methods [15, 31].

In this paper we propose a class of exponential integrators [21, 22] for the homogeneous Boltzmann equation and related kinetic equations which are based on explicit Runge-Kutta methods. The main advantage of the approach here proposed is that it works uniformly for a wide range of Knudsen numbers and avoids the solution of nonlinear systems of equations even in stiff regimes. Similarly to [16], we derive sufficient conditions such that the resulting schemes can be represented as well as convex combinations of density functions including a Maxwellian term. This property is essential to achieve nonnegativity, physical conservations and entropy inequality.

The starting point is the use of classical time spitting together with a decomposition of the gain term of the collision operator into an equilibrium and a non equilibrium part. Similar decompositions for the distribution function has been used in [2] to derive unconditionally stable schemes and in [30, 9, 13, 12] to develop hybrid Monte Carlo methods for kinetic equations. This decomposition of the Boltzmann integral has been introduced recently by Jin and Filbet [14] as a penalization method by a BGK-type relaxation operator to derive Implicit-Explicit Runge-Kutta schemes that overcome the stiffness of the full nonlinear collision operator. We recall that analogous penalization techniques based on linearized operators were previously used in the context of

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exponential methods for parabolic equations with applications to the Schrödinger equation [21, 22] and for kinetic equations [28, 16]. In particular, in the present paper, we will show that for space homogeneous equations the Maxwellian truncation criteria introduced in [16] is equivalent to the penalization method in [14]. Let us mention that a study of the accuracy property of splitting methods in stiff regimes is beyond the scopes of the present paper.

Even if we develop our schemes using the Boltzmann equation as a prototype model for its intrinsic difficulties and its relevance in applications, the methods applies to any large system of stiff ordinary differential equations of the form

$$Y' = \frac{1}{\varepsilon} R(Y), \quad Y(t_0) = Y_0, \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, $Y \in \mathbb{R}^N$ and the non-linear operator $R(Y)$ is a dissipative relaxation operator as in [7]. Such operator is endowed with a $n \times N$ matrix Q of rank $n < N$ such that $QR(Y) = 0, \forall Y$. This gives a vector of n conserved quantities $y = QY$. Solutions which belong to the kernel of the operator $R(Y) = 0$ are uniquely determined by the conserved quantities $Y = E(y)$ and characterize the manifold of local equilibria. Important examples of such dissipative relaxation operators arise in discrete kinetic theory, shallow water equations, granular gases, traffic flows and in general in finite difference/volume discretizations of several kinetic equations.

The methods here proposed use the following decomposition [14]

$$R(Y) = N(Y) + A(E(y) - Y), \tag{1.2}$$

where $N(Y)$ represents the non-dissipative non-linear part, $A > 0$ is an estimate of the Jacobian of R evaluated at equilibrium and $E(y) - Y$ is a simple dissipative linear relaxation operator. Note that, at variance with standard linearization techniques which operate on the short time scale, the operator is linearized on the asymptotically large time scale.

This decomposition permits to apply exponential techniques which solve exactly the linear part and are explicit in the non-linear part. The use of such techniques, as we will see, is essential in order to achieve some unconditional stability properties of the numerical schemes [29]. Such properties are usually characterized as unconditional asymptotic stability and asymptotic preservation.

Here we derive sufficient conditions for asymptotic stability and asymptotic preservation. This permits to introduce asymptotically stable and asymptotic preserving methods up to order 4 which are exact for BGK-type kinetic equations.

The rest of the paper is organized as follows. First in Section 2 we present the Boltzmann equation and its fluid-dynamic limit. Operator splitting methods and various notions of stability are illustrated in Section 3. Next, in Section 4, we introduce the explicit exponential Runge-Kutta methods and derive conditions for asymptotic stability and asymptotic preservation. Examples of methods up to order 4 are also constructed. In Section 5 we describe numerical experiments for homogeneous problems and one application to the full Boltzmann equation. In a final section we discuss conclusions and perspectives of the methods proposed in this article.

2. The Boltzmann equation. We consider the Boltzmann equation of rarefied gas dynamics [6]

$$\partial_t f + v \cdot \nabla_x f = Q(f, f) \tag{2.1}$$

with initial data

$$f|_{t=0} = f_0. \tag{2.2}$$

Here $f(x, v, t)$ is a non negative function describing the time evolution of the distribution of particles with velocity $v \in \mathbb{R}^3$ and position $x \in \Omega \subset \mathbb{R}^{d_x}$ at time $t > 0$. In the sequel for notation simplicity

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we will omit the dependence of f from the independent variables x, v, t unless strictly necessary. The operator $Q(f, f)$ which describes particles interactions has the form

$$Q(f, f) = \int_{\mathbb{R}^3 \times S^2} B(|v - v_*|, n)[f(v')f(v'_*) - f(v)f(v_*)]\, dv_*\, dn \tag{2.3}$$

where

$$v' = v + \frac{1}{2}(v - v_*) + \frac{1}{2}|v - v_*|n, \quad v'_* = v + \frac{1}{2}(v - v_*) - \frac{1}{2}|v - v_*|n, \tag{2.4}$$

and $B(|v - v_*|, n)$ is a nonnegative collision kernel characterizing the microscopic details of the collision given by

$$B(|v - v_*|, n) = \sigma\left(\frac{(v - v_*)}{|v - v_*|} \cdot n\right)|v - v_*|^\gamma,$$

with $\gamma \in [0, 3)$. The case $\gamma = 1$ is referred to as hard spheres case, whereas the simplified situation $\gamma = 0$, is referred to as Maxwell case. Note that in most applications the angle dependence is ignored and σ is assumed constant [3].

The operator $Q(f, f)$ is such that the the local conservation properties are satisfied

$$\int_{\mathbb{R}^3} mQ(f, f)dv =: \langle mQ(f, f)\rangle = 0 \tag{2.5}$$

where $m(v) = (1, v, \frac{|v|^2}{2})$ are the collision invariants. In addition it satisfies the entropy inequality

$$\frac{d}{dt}\int_{\mathbb{R}^3} f \log f\, dv = \int_{\mathbb{R}^3} Q(f, f)\log f dv \leq 0. \tag{2.6}$$

Integrating (2.1) against its invariants in the velocity space leads to the following set of non closed conservations laws

$$\partial_t\langle mf\rangle + \nabla_x\langle vmf\rangle = 0. \tag{2.7}$$

Equilibrium functions for the operator $Q(f, f)$ (i.e. solutions of $Q(f, f) = 0$) are local Maxwellian of the form

$$M_f(\rho, u, T) = \frac{\rho}{(2\pi T)^{3/2}} \exp\left(\frac{-|u - v|^2}{2T}\right), \tag{2.8}$$

where ρ, u, T are the density, mean velocity and temperature of the gas in the x-position and at time t defined as

$$(\rho, \rho u, E) = \langle mf\rangle, \qquad T = \frac{1}{3\rho}(E - \rho|u|^2). \tag{2.9}$$

We will denote by

$$U = (\rho, u, E), \qquad M[U] = M_f, \tag{2.10}$$

clearly we have

$$U = \langle mM[U]\rangle. \tag{2.11}$$

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Now, when the mean free path between particles is very small the operator $Q(f, f)$ is large and we can rescale the space and time variables in (2.1) as

$$x' = \varepsilon x, \quad t' = \varepsilon t \tag{2.12}$$

to obtain

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f, f) \tag{2.13}$$

where ε is a small parameter proportional to the mean free path and the primes have been omitted to keep notation simple.

Passing to the limit for $\varepsilon \to 0$ leads to $f \to M[U]$ and thus we obtain the closed hyperbolic system of compressible Euler equations for the macroscopic variables U

$$\partial_t U + \nabla_x F(U) = 0 \tag{2.14}$$

with

$$F(U) = \langle vmM[U] \rangle = (\rho u, \varrho u \otimes u + pI, Eu + pu), \quad p = \rho T,$$

where I is the identity matrix.

For small but non zero values of the Knudsen number, the evolution equation for the moments can be derived by the so-called Chapman-Enskog expansion [6]. This approach gives the compressible Navier-Stokes equations as a second order approximation with respect to ε to the solution to the Boltzmann equation.

2.1. Operator splitting and stability definitions. Here we restrict ourselves to operator splitting based schemes. It is well-known, in fact, that most numerical methods for the Boltzmann equation are based on a splitting in time between free particle transport and collisions [3, 15]. Possible extension of the present theory to non-splitting schemes is actually under study and it will be considered elsewhere.

Even if it is difficult to give a rigorous definition of asymptotic preserving scheme since the concept has been used for a long time in the physics and mathematics literature and may refer to different discretization parameters, here following [25, 26, 33] we formalize this notion for the time discretization of equation (2.1).

DEFINITION 2.1. *A consistent time discretization method for (2.1) of stepsize Δt is* asymptotic preserving (AP) *if, independently of the initial data (2.2) and of the stepsize Δt, in the limit $\varepsilon \to 0$ becomes a consistent time discretization method for the reduced system (2.14).*

Note that this definition does not imply that the scheme preserves the order of accuracy in t in the stiff limit $\varepsilon \to 0$. In the latter case the scheme is said *asymptotically accurate.*

As discussed above, the starting point in the solution of the kinetic equations is given by an operator splitting of (2.1) in a time interval $[0, \Delta t]$ between relaxation

$$\partial_t f = \frac{1}{\varepsilon} Q(f, f), \tag{2.15}$$

and free transport

$$\partial_t f + v \cdot \nabla_x f = 0. \tag{2.16}$$

This situation is typical of Monte Carlo methods and of several other numerical codes used in realistic simulations. Even if this splitting, usually referred to as Lie-Trotter splitting, is limited

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to first order it permits to treat separately the hyperbolic free transport from the stiff relaxation step which often is of paramount importance in applications.

Higher order splitting formulas can be derived in different ways (see [19]). Let us denote by $\mathcal{T}_{\Delta t}(f)$ and $\mathcal{C}_{\Delta t}(f)$ the above transport and collision steps in a time interval $[0, \Delta t]$ starting from the initial data f_0, then the well-known second order Strang splitting [34] can be written as

$$\mathcal{C}_{\Delta t/2}(\mathcal{T}_{\Delta t}(\mathcal{C}_{\Delta t/2}(f_0))). \tag{2.17}$$

Unfortunately for splitting methods of order higher then two it can be shown that it is impossible to avoid negative time steps both in the transport as well as in the collision [19]. Higher order formulas which avoid negative time stepping can be obtained as suitable combination of splitting steps [11]. Note, however, that the appearance of negative coefficients or negative time steps in high order formulas may lead to several drawbacks in practical applications like the lack of positivity of the solution which makes very difficult their use in Monte Carlo schemes.

As mentioned before the study of the accuracy properties of the different splitting methods in stiff regimes, although important, is beyond the scopes of the present manuscript and we refer to [24, 35] for an error analysis of some splitting methods in stiff conditions.

Now we can reformulate the asymptotic preserving property and prove that

PROPOSITION 2.2. *A sufficient condition for a consistent time discretization method of stepsize* Δt *applied to the operator splitting approximation of (2.1), given by (2.15)-(2.16), to be AP is that the time discretization of step (2.15), independently of the initial data (2.2) and of the stepsize* Δt, *in the limit* $\varepsilon \to 0$ *projects the solution* f *over the local Maxwellian equilibrium* $M[U_0]$, $U_0 = \langle m f_0 \rangle$.

The proof of the above proposition is an immediate consequence of the fact that as $\varepsilon \to 0$ step (2.15) degenerates into the projection $\mathcal{C}_{\Delta t}(f_0) = M[U_0]$ which coupled with the transport step (2.16) originates a so-called kinetic approximation [8] to the Euler equation (2.14) given by $\mathcal{T}_{\Delta t}(M[U_0])$. We omit further details.

In other words, Proposition 2.2 states that if the relaxation step (2.15) is AP then the whole splitting (2.15)-(2.16) is AP. Analogous results hold true for higher order splitting methods.

In the sequel we will focus on the solution to the space homogeneous Boltzmann equation (2.15). In fact, most computational challenges related to the behavior of the full equation for small values of ε depend on the time discretization of the homogeneous step.

Of course AP is an important property in term of stability of the numerical scheme in stiff regimes. For implicit Runge-Kutta methods applied to (2.15) it has been shown in [33] that AP is equivalent to the notion of L-stability [18].

For general unconditionally stable schemes a weaker requirement is the notion of *asymptotic stability* (AS) [29]. Let us denote by $f(t)$ and $g(t)$ two solutions of (2.15) corresponding respectively to the initial data f_0 and g_0 such that $U_0 = \langle m f_0 \rangle = \langle m g_0 \rangle$. It can be proved that, for a suitable distance $d(\cdot, \cdot)$, system (2.15) is contractive $d(f(t), g(t)) \le d(f_0, g_0)$, and asymptotically stable since $f(t) \to M[U_0]$ and $g(t) \to M[U_0]$ as $t \to \infty$ (see [6] for example) and so $f(t) - g(t) \to 0$ as $t \to \infty$. We refer to [37] for more details and examples of contractive metrics for problem (2.15) in the case of Maxwell molecules. A particular metric is presented in Section 3.3.

Let us denote by f^n and g^n, $n \ge 1$ the numerical solution at $t = n\Delta t$ obtained with a given time discretization method applied to (2.15) with initial data f_0 and g_0 respectively. Now we can introduce the following definition.

DEFINITION 2.3. *A time discretization method for (2.15) is called* unconditionally contractive *with respect to the distance* $d(\cdot, \cdot)$ *if* $d(f^1, g^1) \le d(f_0, g_0)$ *holds for all* f_0, g_0 *such that* $\langle m f_0 \rangle = \langle m g_0 \rangle$ *and for all stepsizes* Δt. *Furthermore, it is called* unconditionally asymptotically stable *if* $f^n - g^n \to 0$ *as* $n \to \infty$ *independently of the step size* Δt.

Note that unlike contractivity, asymptotic stability is not related to a specific metric. Contractivity of Runge-Kutta methods has been studied in [27] and it is well-known that such methods

have limited order of accuracy. Clearly for implicit Runge-Kutta methods, asymptotic stability is closely related to A-stability.

For the sake of completeness we finally introduce the notion of entropic stability, namely schemes that preserve the entropy inequality (2.6).

DEFINITION 2.4. *A time discretization method for (2.15) is called* unconditionally entropic *if* $H(f^{n+1}) \leq H(f^n)$, *where* $H(f) = \int_{\mathbb{R}^3} f \log f \, dv$, *independently of the step size* Δt.

The above monotonicity property for Runge-Kutta schemes is essentially equivalent to the so-called strong stability preserving (SSP) property which is often used when dealing with the time discretization of partial differential equations [17]. Let us recall that Implicit Euler is the sole unconditional SSP method. All SSP Runge-Kutta methods of order greater than one suffer from some time-step restriction [20]. Thus, except for first order implicit Euler, the entropy inequality is not satisfied by high order unconditionally stable Runge-Kutta schemes applied to (2.15) unless a suitable time step restriction is considered.

3. Exponential methods. Since we aim at developing unconditionally stable schemes, the most natural choice would be to use implicit solvers applied to (2.15). Unfortunately the use of fully implicit schemes for (2.15) is unpracticable due to the prohibitive computational cost required by the solution of the very large non-linear algebraic system originated by the five fold integral appearing in $Q(f, f)$ which has to be computed in each spatial cell at each time step in the inhomogeneous cases. We will see in this section a possible way to overcome these difficulties.

First we rewrite the homogeneous equation (2.15) in the form

$$\partial_t f = \frac{1}{\varepsilon}(P(f, f) - \mu f), \tag{3.1}$$

where $P(f, f) = Q(f, f) + \mu f$ and $\mu > 0$ is a constant such that $P(f, f) \geq 0$. Typically μ is an estimate of the largest spectrum of the loss part of the collision operator. Let us emphasize that most of the subsequent theory can be generalized to the case where $P(f, f)$ is not strictly positive and μ is an arbitrary nonnegative constant. However such an assumption makes the presentation easier and we will discuss possible generalizations later.

By construction we have the following

$$\frac{1}{\mu}\langle mP(f, f)\rangle = \langle mf \rangle = U. \tag{3.2}$$

Thus $P(f, f)/\mu$ is a density function and we can consider the following decomposition

$$P(f, f)/\mu = M[U] + g. \tag{3.3}$$

The function g represents the non equilibrium part of the distribution function and from the definition above it follows that g is in general non positive. Moreover since $P(f, f)/\mu$ and $M[U]$ have the same moments we have

$$\langle mg \rangle = 0. \tag{3.4}$$

The homogeneous equation can be written in the form

$$\partial_t f = \frac{\mu}{\varepsilon}g + \frac{\mu}{\varepsilon}(M[U] - f) = \frac{\mu}{\varepsilon}\left(\frac{P(f, f)}{\mu} - M[U]\right) + \frac{\mu}{\varepsilon}(M[U] - f). \tag{3.5}$$

The above system is equivalent to the penalization method for the collision operator recently introduced in [14]. Note that even if $M[U]$ is nonlinear in f, thanks to the conservation properties (2.5), it remains constant during the relaxation process. The main feature of such formulation is

that on the right hand side we have a stiff dissipative linear part $\mu(M[U] - f)/\varepsilon$ which characterizes the asymptotic behavior of f and a stiff non dissipative non linear part $(P(f, f)/\mu - M[U])/\varepsilon$ which describes the deviations of $P(f, f)/\mu$ from $M[U]$, or equivalently the deviations of the Boltzmann operator from a BGK-like relaxation term.

We remark that the decision whether problem (2.15), or equivalently (3.1) and (3.5), should be regarded as stiff or nonstiff does not depend only on the ratio μ/ε but depends also on the chosen initial conditions. If the initial data is close to local equilibrium $f = M[U] + O(\varepsilon)$, then the problem is clearly nonstiff. In fact, if $f = M[U] + \varepsilon f_1$ we have

$$Q(f, f) = \varepsilon[Q(f_1, f) + Q(f, f_1) + \varepsilon Q(f_1, f_1)],$$

and so $Q(f, f) = O(\varepsilon)$ and there is no need of using a specific stiff solver. For this reason, and the fact that the transport step (2.16) drives the solution far from local equilibrium, in the sequel we concentrate our analysis to non equilibrium initial data.

In such case the problem is stiff as a whole and a fully implicit method should be used in the numerical integration to avoid stability constraints of the type $\Delta t = O(\varepsilon)$. On the other hand the linear part itself suffices to characterize the correct large time behavior of f. Therefore, instead of fully implicit methods, one should hopefully use methods which are implicit in the linear part and explicit in the non-linear part. One class of such methods is given by the IMEX Runge-Kutta schemes [1, 33]. Note, however, that standard IMEX schemes may lose their stability properties since here also the explicit part is stiff. An alternative approach is based on the so-called exponential integrators where the exact solution of the linear part is used in the construction of the numerical methods [21, 22].

3.1. Explicit exponential Runge-Kutta schemes. In order to derive the methods it is useful to rewrite (3.5) as

$$\frac{\partial (f - M) e^{\mu t/\varepsilon}}{\partial t} = \frac{1}{\varepsilon}(P(f, f) - \mu M) e^{\mu t/\varepsilon}. \tag{3.6}$$

The above form is readily obtained if one multiplies (3.5) by the integrating factor $\exp(\mu t/\varepsilon)$ and takes into account the fact that M does not depend of time. A class of explicit exponential Runge-Kutta schemes is then obtained by direct application of an explicit Runge-Kutta method to (3.6). More in general we can consider the family of methods characterized by

$$F^{(i)} = e^{-c_i \mu \Delta t/\varepsilon} f^n + \frac{\mu \Delta t}{\varepsilon} \sum_{j=1}^{i-1} A_{ij}(\mu \Delta t/\varepsilon) \left(\frac{P(F^{(j)}, F^{(j)})}{\mu} - M^n \right)$$

$$+ \left(1 - e^{-c_i \mu \Delta t/\varepsilon} \right) M^n, \qquad i = 1, \dots, \nu \tag{3.7}$$

$$f^{n+1} = e^{-\mu \Delta t/\varepsilon} f^n + \frac{\mu \Delta t}{\varepsilon} \sum_{i=1}^{\nu} W_i(\mu \Delta t/\varepsilon) \left(\frac{P(F^{(i)}, F^{(i)})}{\mu} - M^n \right)$$

$$+ \left(1 - e^{-\mu \Delta t/\varepsilon} \right) M^n, \tag{3.8}$$

where Δt is the time step, $f^n = f(t^n)$, $M^n = M(t^n)$, $c_i \geq 0$, and the coefficients A_{ij} and the weights W_i are such that

$$A_{ij}(0) = a_{ij}, \quad W_i(0) = w_i, \quad i, j = 1, \dots, \nu$$

with coefficients a_{ij} and weights w_i given by a standard explicit Runge-Kutta method called the underlying method. Various schemes come from the different choices of the underlying method.

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The most popular approaches are the integrating factor (IF) and the exponential time differencing (ETD) methods [21, 22, 29]. Since M^n does not depend on time during the collision process in the sequel we will omit the index n.

For the so-called Integrating Factor methods, which correspond to a direct application of the underlying method to (3.6), we have

$$A_{ij}(\lambda) = a_{ij}e^{-(c_i-c_j)\lambda}, \quad i,j = 1,\ldots,\nu, \quad i > j$$

$$W_i(\lambda) = w_i e^{-(1-c_i)\lambda}, \quad i = 1,\ldots,\nu,$$

(3.9)

with $\lambda = \mu\Delta t/\varepsilon$.

The first order IF scheme reads

$$f^{n+1} = e^{-\frac{\mu\Delta t}{\varepsilon}}f^n + \frac{\mu\Delta t}{\varepsilon}e^{-\frac{\mu\Delta t}{\varepsilon}}\left(\frac{P(f^n,f^n)}{\mu} - M\right) + \left(1 - e^{-\frac{\mu\Delta t}{\varepsilon}}\right)M,$$

(3.10)

which is based on explicit Euler. For such methods the order of accuracy is the same as the order of the underlying method.

The Exponential Time Differencing methods are strictly connected with the integral representation of (3.6). In the general case the coefficients for ETD methods have the form

$$A_{ij}(\lambda) = \int_0^1 e^{(1-s)c_i\lambda}p_{ij}(s)\,ds, \quad i,j = 1,\ldots,\nu, \quad i > j$$

$$W_i(\lambda) = \int_0^1 e^{(1-s)\lambda}p_i(s)\,ds, \quad i = 1,\ldots,\nu,$$

where p_i and p_{ij} are suitable polynomials.

The standard first order ETD method based on explicit Euler in our case gives [21]

$$f^{n+1} = e^{-\frac{\mu\Delta t}{\varepsilon}}f^n + \frac{\mu\Delta t}{\varepsilon}\varphi\left(\frac{\mu\Delta t}{\varepsilon}\right)\frac{P(f^n,f^n)}{\mu},$$

(3.11)

where $\varphi(z) = (1 - e^{-z})/z$.

3.2. Time Relaxed methods. A class of exponential methods for kinetic equations, the so-called time relaxed (TR) methods, has been introduced in [16] as a combination of an exponential expansion (or Wild sum) together with a suitable Maxwellian truncation. In this paragraph we show that these schemes included already decomposition (3.5) and can be derived directly from a suitable Taylor expansion of (3.6).

To show this, let us first introduce the change of variables

$$\tau = 1 - \exp(-\mu t/\varepsilon),$$

which, using the bilinearity of $P(f,f)$, gives the equation

$$\frac{\partial}{\partial\tau}\left[(f - M)\frac{1}{1-\tau}\right] = (P(f,f) - \mu M)\frac{1}{\mu(1-\tau)^2}.$$

(3.12)

The application of an explicit Runge-Kutta scheme to (3.12) with time step $\Delta\tau = 1 - \exp(-\mu\Delta t/\varepsilon)$ leads to a class of ETD methods. For example the first order scheme based on explicit Euler in the original variables yields

$$f^{n+1} = e^{-\frac{\mu\Delta t}{\varepsilon}}f^n + \frac{\mu\Delta t}{\varepsilon}\varphi_1\left(\frac{\mu\Delta t}{\varepsilon}\right)\left(\frac{P(f^n,f^n)}{\mu} - M\right) + (1 - e^{-\frac{\mu\Delta t}{\varepsilon}})M,$$

(3.13)

where $\varphi_k(z) = e^{-z}(1 - e^{-z})^k/z$, $k = 1, 2, \ldots$.

Note that such scheme coincides with the first order exponential time relaxed method derived in [16] and differs from the standard ETD method based on explicit Euler. Higher order ETD methods can be derived as well simply applying higher order explicit Runge-Kutta methods to (3.12). Although interesting, here we do not explore further this class of schemes.

Now let us consider a different approach by taking the Taylor expansion of $(f - M)/(1 - \tau)$ around $\tau = 0$ in (3.12). This leads to

$$(f - M)/(1 - \tau) = (f_0 - M) + \tau \left[\frac{P(f_0, f_0)}{\mu} - M \right]$$
$$+ \frac{\tau^2}{2} \left[\frac{P(P(f_0, f_0), f_0) + P(f_0, P(f_0, f_0))}{\mu^2} - 2M \right] + O(\tau^3)$$

where we have used the bilinearity of the operator $P(f, f)$.

If we compute all the terms in the expansion and use recursively the bilinearity of $P(f, f)$ we can state the following

PROPOSITION 3.1. *The solution to problem (3.5) or equivalently (3.6) or (3.12) can be represented as*

$$f(v, t) = (1 - \tau)f_0(v) + (1 - \tau)\sum_{k=1}^{\infty} \tau^k(f_k^n(v) - M(v)) + \tau M(v), \qquad (3.14)$$

where the functions f_k are given by the recurrence formula

$$f_{k+1}(v) = \frac{1}{k+1}\sum_{h=0}^{k} \frac{1}{\mu} P(f_h, f_{k-h})(v), \quad k = 0, 1, \ldots. \qquad (3.15)$$

By truncating expansion (3.14) at the order m, and reverting to the old variables, we recover exactly the time relaxed schemes presented in [16]

$$f^{n+1} = e^{-\mu\Delta t/\varepsilon} f^n + e^{-\mu\Delta t/\varepsilon} \sum_{k=1}^{m} (1 - e^{-\mu\Delta t/\varepsilon})^k (f_k^n - M) + (1 - e^{-\mu\Delta t/\varepsilon})M, \qquad (3.16)$$

which, using the fact that

$$1 - e^{-\mu\Delta t/\varepsilon} \sum_{k=0}^{m} (1 - e^{-\mu\Delta t/\varepsilon})^k = (1 - e^{-\mu\Delta t/\varepsilon})^{m+1},$$

can be rewritten in the usual form emphasizing their convexity properties

$$f^{n+1} = e^{-\mu\Delta t/\varepsilon} \sum_{k=0}^{m} (1 - e^{-\mu\Delta t/\varepsilon})^k f_k^n + (1 - e^{-\mu\Delta t/\varepsilon})^{m+1} M. \qquad (3.17)$$

A remarkable feature of these methods is that the functions $f_k(v)$ are density functions with the same moments of the initial data $\langle m f_k \rangle = \langle f_0 \rangle$. Such property, together with unconditional nonnegativity and convexity of the weights, is enough to guarantee asymptotic preservation of the schemes as well as nonnegativity, contractivity and entropic stability (see [16] for details).

Clearly TR schemes do not belong to the general family of methods defined by (3.7)-(3.8), since they are based on the assumption that $P(f, f)$ is a bilinear operator.

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3.3. Main properties. In this section we will establish the main properties for a general exponential scheme in the form (3.7)-(3.8). For some of the properties, like contractivity and entropic stability, we give proofs in the case of Maxwellian molecules.

Since solutions to (2.15) are nonnegative and preserves the mass in our analysis we will restrict, without loss of generality, to probability density functions.

Let us denote by $\mathcal{P}_2(\mathbb{R}^3)$, the class of all probability density functions f on $\mathbb{R}^3$, such that

$$\int_{\mathbb{R}^3} |v|^2 dF(v) < \infty. \tag{3.18}$$

We introduce a metric on $\mathcal{P}_2(\mathbb{R}^3)$ by

$$d_2(f, g) = \sup_{\xi \in \mathbb{R}^3} \frac{|\hat{f}(\xi) - \hat{g}(\xi)|}{|\xi|^2} \tag{3.19}$$

where $\hat{f}$ is the Fourier transform of f

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-i\xi \cdot v} dF(v). \tag{3.20}$$

The metric $d_2(\cdot, \cdot)$ is nonexpanding with time along trajectories of the Boltzmann equation; that is, if f and g are two such solutions $d_2(f(t), g(t)) \leq d_2(f_0, g_0)$. The above property is a consequence of the fact that for Maxwell's molecules we have

$$Q(f, f) = \int_{\mathbb{R}^3 \times S^2} \sigma \left(\frac{v - v_*}{|v - v_*|} \cdot n \right) [f(v')f(v'_*) - f(v)f(v_*)] \, dv_* \, dn \tag{3.21}$$

where for any fixed unit vector $\bar{e}$

$$\int_{S^2} \sigma(\bar{e} \cdot n) \, dn = S,$$

with $S > 0$ a constant. Taking $\mu = S$ and

$$P(f, f) = \int_{\mathbb{R}^3 \times S^2} \sigma \left(\frac{v - v_*}{|v - v_*|} \cdot n \right) f(v')f(v'_*) \, dv_* \, dn,$$

we have $Q(f, f) = P(f, f) + \mu f$ and [37]

$$d_2(P(f, f), P(g, g)) \leq \mu d_2(f, g). \tag{3.22}$$

We refer to [37] for more information on the above metric and other contractive metrics for the Boltzmann equation in the case of Maxwellian molecules.

Now let us denote by f^n and g^n the corresponding solutions obtained with an explicit exponential Runge-Kutta method. We have

$$F^{(i)} - G^{(i)} = e^{-c_i \mu \Delta t/\varepsilon}(f^n - g^n) + \frac{\Delta t}{\varepsilon} \sum_{j=1}^{i-1} A_{ij}(\mu \Delta t/\varepsilon)(P(F^{(j)}, F^{(j)}) - P(G^{(j)}, G^{(j)}))$$

$$f^{n+1} - g^{n+1} = e^{-\mu \Delta t/\varepsilon}(f^n - g^n) + \frac{\Delta t}{\varepsilon} \sum_{i=1}^{\nu} W_i(\mu \Delta t/\varepsilon)(P(F^{(i)}, F^{(i)}) - P(G^{(i)}, G^{(i)})),$$

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and then

$$d_2(F^{(i)}, G^{(i)}) \le e^{-c_i \mu \Delta t / \varepsilon} d_2(f^n, g^n) + \frac{\mu \Delta t}{\varepsilon} \sum_{j=1}^{i-1} |A_{ij}(\mu \Delta t / \varepsilon)| d_2(F^{(j)}, G^{(j)})$$

$$d_2(f^{n+1}, g^{n+1}) \le e^{-\mu \Delta t / \varepsilon} d_2(f^n, g^n) + \frac{\mu \Delta t}{\varepsilon} \sum_{i=1}^{\nu} |W_i(\mu \Delta t / \varepsilon)| d_2(F^{(i)}, G^{(i)}).$$

Let us denote by $\lambda = \mu \Delta t / \varepsilon$, $\bar{A}(\lambda)$ the $\nu \times \nu$ matrix of elements $|A_{ij}(\lambda)|$, $\bar{w}(\lambda)$ the $\nu \times 1$ vector of elements $|W_i(\lambda)|$, $\bar{d}$ the $\nu \times 1$ vector of elements $d_2(F^{(i)}, G^{(i)})$ and $\bar{e}$ the $\nu \times 1$ unit vector we can write

$$\left(I - \lambda \bar{A}(\lambda) \right) \bar{d} \le \bar{E}(\lambda) d_2(f^n, g^n) \bar{e}$$
$$d_2(f^{n+1}, g^{n+1}) \le e^{-\lambda} d_2(f^n, g^n) + \lambda \bar{w}(\lambda)^T \bar{d},$$

where $\bar{E}(\lambda) = \text{diag}(e^{-c_1 \lambda}, \ldots, e^{-c_\nu \lambda})$.

Then we obtain

$$d_2(f^{n+1}, g^{n+1}) \le R(\lambda) d_2(f^n, g^n), \tag{3.23}$$

where

$$R(\lambda) = e^{-\lambda} + \lambda \bar{w}(\lambda)^T (I - \lambda \bar{A}(\lambda))^{-1} \bar{E}(\lambda) \bar{e} \tag{3.24}$$

$$= e^{-\lambda} + \sum_{k=0}^{\nu-1} \lambda^{k+1} \bar{w}(\lambda)^T \bar{A}(\lambda)^k \bar{E}(\lambda) \bar{e}. \tag{3.25}$$

To derive the last expression we expanded $(I - \lambda \bar{A}(\lambda))^{-1}$ by the geometric series and use the fact that $\bar{A}(\lambda)$ is strictly lower triangular and so it is a nilpotent matrix of degree ν.

We can state [29]

THEOREM 3.2. *If an explicit exponential Runge-Kutta method in the form (3.7)-(3.8) satisfies*

$$R(\lambda) \le 1, \quad \forall \lambda \ge 0, \tag{3.26}$$

with $R(\lambda)$ given by (3.25) then it is unconditionally contractive and unconditionally stable with respect to the metric $d_2(\cdot, \cdot)$.

Note that for an IF method we have

$$|A_{ij}(\lambda)| \le |a_{ij}| e^{-(c_i - c_j)\lambda}, \quad |W_i(\lambda)| \le |w_i| e^{-(1 - c_i)\lambda},$$

thus we require

$$0 = c_1 \le c_2 \ldots \le c_\nu \le 1, \tag{3.27}$$

in order for the above quantities to be bounded independently of λ.

We have

$$R(\lambda) = e^{-\lambda} \left(1 + \sum_{k=0}^{\nu-1} \lambda^{k+1} \bar{w}^T \bar{A}^k \bar{e} \right). \tag{3.28}$$

Thus condition (3.26) is satisfied when

$$1 + \sum_{k=0}^{\nu-1} \lambda^{k+1} \bar{w}^T \bar{A}^k \bar{e} \le e^{\lambda} = 1 + \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{(k+1)!}. \tag{3.29}$$

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Now, if we consider an underlying Runge-Kutta method with a $\nu \times \nu$ non negative coefficient matrix A and a $\nu \times 1$ non negative vector of weights w then (3.29) holds if

$$w^T A^k \bar{e} \leq \frac{1}{(k+1)!}, \quad k = 0, 1, \ldots, \nu - 1.$$

The above condition is clearly satisfied if the underlying Runge-Kutta method is a ν-stage explicit Runge-Kutta method of order ν.

Thus we have proved

PROPOSITION 3.3. *An explicit IF method is unconditionally contractive and asymptotically stable with respect to the metric $d_2(\cdot, \cdot)$ if the underlying Runge-Kutta method is a ν-stage explicit Runge-Kutta method of order ν with nonnegative coefficients and weights satisfying (3.27).*

As pointed out in [27, 29] examples of such methods are well-known up to $\nu = 4$ and the classical RK method of order four is the sole method with four stages. Moreover, it is also known that there does not exist explicit methods of order greater than four satisfying (3.27) with positive weights. For methods which are not of IF type there are no accuracy barrier, for example all time relaxed method satisfy immediately condition (3.26). Other examples are reported in [22, 29].

We have

THEOREM 3.4. *If an explicit exponential Runge-Kutta method in the form (3.7)-(3.8) satisfies*

$$\lim_{\lambda \to \infty} R(\lambda) = 0, \tag{3.30}$$

with $R(\lambda)$ given by (3.25) then it is asymptotic preserving.

In fact taking $g_0 = M$ we have

$$d_2(f^{n+1}, M) \leq R(\lambda) d_2(f^n, M),$$

and so $d_2(f^{n+1}, M)$ goes to 0 as $\lambda \to \infty$.

For an IF method $R(\lambda)$ is given by (3.28) and condition (3.30) is always satisfied. Thus

PROPOSITION 3.5. *An explicit IF method is asymptotic preserving if the underlying Runge-Kutta method satisfies (3.27).*

For practical applications it may be convenient to require that as $\lambda \to \infty$ the numerical solution f^{n+1} and each level $F^{(i)}$ of the IF method are projected towards the local Maxwellian without using explicitly the structure of $P(f, f)$. It is straightforward to verify that this stronger AP property is satisfied if we replace condition (3.27) by

$$0 = c_1 < c_2 < \ldots < c_\nu < 1. \tag{3.31}$$

We remark that the usual concept of stiff order [29] for exponential methods is in contrast with the latter strong AP property. For example for a stiff order one method the condition [22]

$$\sum_{i=1}^{\nu} W_i(\lambda) = \frac{1 - e^{-\lambda}}{\lambda}$$

implies that the Maxwellian term in (3.8) vanishes. So classical stiff order two ETD methods do not satisfy the AP property (see [29] for example).

We conclude this section with a results concerning an important convexity property of the schemes.

THEOREM 3.6. *If an explicit exponential Runge-Kutta method in the form (3.7)-(3.8) satisfies*

$$\sum_{j=1}^{i-1} A_{ij}(\lambda) \leq \frac{1 - e^{-c_i \lambda}}{\lambda}, \quad \forall \lambda \geq 0, \quad i = 1, \ldots, \nu \tag{3.32}$$

$$\sum_{i=1}^{\nu} W_i(\lambda) \leq \frac{1 - e^{-\lambda}}{\lambda}, \quad \forall \lambda \geq 0, \tag{3.33}$$

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with $A_{ij}(\lambda) \geq 0$ and $W_i(\lambda) \geq 0$ then it is unconditionally positive and entropic.

In fact, the operator $P(f, f)$ is nonnegative and convexity of the schemes is guaranteed if (3.32) and (3.33) hold. Moreover since $H(M) \leq H(f^n)$ and (see [38])

$$H\left(\frac{P(f, f)}{\mu}\right) \leq H(f), \tag{3.34}$$

by convexity we have also

$$H(F^{(i)}) \leq e^{-c_i \lambda} H(f^n) + \lambda \sum_{j=1}^{i-1} A_{ij}(\lambda) H(F^{(j)})$$

$$+ \left(1 - e^{-c_i \lambda} - \lambda \sum_{j=1}^{i-1} A_{ij}(\lambda)\right) H(M)$$

$$\leq H(f^n), \qquad i = 1, \dots, \nu$$

$$H(f^{n+1}) \leq e^{-\lambda} H(f^n) + \lambda \sum_{i=1}^{\nu} W_i(\lambda) H(F^{(i)})$$

$$+ \left(1 - e^{-\lambda} - \lambda \sum_{i=1}^{\nu} W_i(\lambda)\right) H(M)$$

$$\leq H(f^n).$$

For convexity of an IF method we require

$$\sum_{j=1}^{i-1} a_{ij} e^{c_j \lambda} \leq \frac{e^{c_i \lambda} - 1}{\lambda}, \quad \forall \, \lambda \geq 0, \quad i = 1, \dots, \nu$$

$$\sum_{i=1}^{\nu} w_i e^{c_i \lambda} \leq \frac{e^{\lambda} - 1}{\lambda}, \quad \forall \, \lambda \geq 0.$$

By Taylor expansion we obtain conditions

$$\sum_{j=1}^{i-1} a_{ij} c_j^k \leq \frac{c_i^k}{k+1}, \quad k = 0, 1, 2, \dots, \quad i = 1, \dots, \nu \tag{3.35}$$

$$\sum_{i=1}^{\nu} w_i c_i^k \leq \frac{1}{k+1}, \quad k = 0, 1, 2, \dots, \tag{3.36}$$

This allows to state the following.

PROPOSITION 3.7. *An explicit IF method is unconditionally positive and entropic if the underlying Runge-Kutta method has nonnegative coefficients and weights satisfying (3.35)-(3.36).*

Note that the above conditions on the choice of the underlying method are quite restrictive and that we are not using the bilinearity of $P(f, f)$ which would lead to weaker constraints on a_{ij} and w_i. For example, if we consider the family of second order Runge-Kutta methods with two levels characterized by $w_1 = 1 - w$, $w_2 = w$ and $a_{21} = (2w)^{-1}$ with $w \in [0, 1]$, Proposition 3.7 is satisfied when

$$w^{k-1} \geq \frac{k+1}{2^k}, \quad k = 0, 1, \dots$$

which implies $w \in [\frac{3}{4}, 1]$. Examples of methods that satisfy convexity are the second order Midpoint or Runge method ($w = 1$) and the third order Heun method but not the classical fourth order Runge-Kutta method [18].

REMARK 1. *Convexity is an essential property if one wants to rely on Monte Carlo techniques for the computation of the collision operator. In fact, the resulting scheme is a convex combination of probability densities and then can be evaluated using the same methods described in [31].*

3.4. Generalizations and implementation. An essential aspect in the reformulation of the problem given by (3.28) is the choice of the value of the constant μ used in estimating the spectrum of the collision operator. Of course such constant can be chosen at each time step in order to improve our estimate. In the sequel we show different choices in the case of variable hard spheres.

The choice of an upper bound for the loss part of the collision term leads to take $\mu = \mu_p$ where

$$\mu_p = \sup_v \int_{\mathbb{R}^3} |v - v_*|^\gamma f(v_*) \, dv_*. \tag{3.37}$$

Positivity is guaranteed since this choice implies clearly $P(f, f) \geq 0$. From a practical viewpoint computation of (3.37) can be done at $O(N \log N)$ for a deterministic method based on N parameters for representing $f(v)$ on a regular mesh. This can be done using the FFT algorithm thanks to the convolution structure of the loss term in (3.37). Thus, in general, the computation of μ_p will not affect the computational cost of the scheme.

For Monte Carlo methods based on $v_1, \ldots, v_N$ particles one should estimate

$$\mu_p \approx \max_{v_i} \frac{1}{N} \sum_j |v_i - v_j|^\gamma.$$

To avoid the $O(N^2)$ cost it is a common choice to consider the following upper bound

$$\tilde{\mu}_p = 2^\gamma \max_i |v_i - u|^\gamma \geq \max_{v_i} \frac{1}{N} \sum_j |v_i - v_j|^\gamma, \quad u = \frac{1}{N} \sum_i v_i. \tag{3.38}$$

However, such positivity constraint on $P(f, f)$ typically leads to overestimates of the true spectrum of the collision operator, especially in Monte Carlo simulations. A better estimate of μ would be given by the average collision frequency

$$\mu_a = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v - v_*|^\gamma f(v) f(v_*) \, dv_* \, dv. \tag{3.39}$$

This can be computed again at $O(N \log N)$ cost in a deterministic setting whereas in a Monte Carlo simulation we have

$$\mu_a \approx \frac{1}{N^2} \sum_{i,j} |v_i - v_j|^\gamma,$$

which, to avoid the quadratic cost, can be overestimated by

$$\tilde{\mu}_a = \frac{2^\gamma}{N} \sum_i |v_i - u|^\gamma. \tag{3.40}$$

Finally, as suggested in [14], μ can be chosen as an estimate of the spectral radius of the linearized operator Q around the Maxwellian M. In fact

$$Q(f, f) \approx Q(M, M) + \nabla Q(M, M)(M - f) = \nabla Q(M)(M - f),$$

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where $\nabla Q(M, M)$ is the Frechet derivative of Q evaluated at M. For example one can take

$$\mu_s = \sup_v \left| \frac{Q(f, f)}{f - M} \right|. \tag{3.41}$$

Note however that the above estimate can be computed easily only in a deterministic framework.

The choices $\mu = \mu_a$ or $\mu = \mu_s$, although more accurate, pose the question of stability of the resulting scheme since they do not guarantee $P(f, f) \geq 0$. Note that, if we denote by $P_p(f, f) = Q(f, f) + \mu_p f$ and by $\lambda_p = \mu_p \Delta t / \varepsilon$, for an arbitrary $\mu \geq 0$, we can rewrite (3.7)-(3.8) as

$$
F^{(i)} = e^{-c_i \lambda} f^n - (\lambda_p - \lambda) \sum_{j=1}^{i-1} A_{ij}(\lambda) F^{(j)} + \lambda_p \sum_{j=1}^{i-1} A_{ij}(\lambda) \frac{P_p(F^{(j)}, F^{(j)})}{\mu_p}
$$

$$
+ \left(1 - e^{-c_i \lambda} - \lambda \sum_{j=1}^{i-1} A_{ij}(\lambda) \right) M, \qquad i = 1, \ldots, \nu \tag{3.42}
$$

$$
f^{n+1} = e^{-\lambda} f^n - (\lambda_p - \lambda) \sum_{i=1}^{\nu} W_i(\lambda) F^{(j)} + \lambda_p \sum_{i=1}^{\nu} W_i(\lambda) \frac{P_p(F^{(i)}, F^{(i)})}{\mu_p}
$$

$$
+ \left(1 - e^{-\lambda} - \lambda \sum_{i=1}^{\nu} W_i(\lambda) \right) M. \tag{3.43}
$$

For stability now we must perform the same analysis of Section 3.3 on system (3.42)-(3.43). For brevity here we omit the resulting conditions which typically cannot be satisfied without introducing a stability restriction on the time step.

To illustrate this let us consider the case $\nu = 1$ for IF methods. We have

$$
f^{n+1} = (1 - \lambda_p + \lambda) e^{-\lambda} f^n + \lambda_p e^{-\lambda} \frac{P_p(f^n, f^n)}{\mu_p} + \left(1 - e^{-\lambda} - \lambda e^{-\lambda} \right) M. \tag{3.44}
$$

Convexity is guaranteed as soon as Δt satisfies

$$
\Delta t \leq \frac{\varepsilon}{\mu_p - \mu},
$$

or equivalently if we take $\mu \geq \mu_p - \varepsilon / \Delta t$ which represents the lower bound for μ that makes the scheme unconditionally positive. On the other hand contractivity and asymptotic stability remain valid under the weaker restriction

$$
\frac{\mu}{2} + \frac{\varepsilon}{\Delta t}(1 + e^{\mu \Delta t / \varepsilon}) \geq \mu_p.
$$

Similar considerations hold for higher order IF schemes. In such cases the AP property is guaranteed provided that the underlying method satisfy Proposition 3.5.

REMARK 2. *Along this paragraph we have assumed μ constant during the time stepping. In practice it is clear that the computation of μ_p from (3.37) on the initial data does not guarantee positivity of all terms in the vector $\bar{f}_1$. In principle one can take $\mu = \mu(t)$ and rewrite the exponential methods for a time dependent μ from*

$$
\frac{\partial (f - M) e^{\frac{1}{\varepsilon} \int_0^t \mu(s)\, ds}}{\partial t} = \frac{1}{\varepsilon}(P(f, f) - \mu(t)M) e^{\frac{1}{\varepsilon} \int_0^t \mu(s)\, ds}, \tag{3.45}
$$

and then recompute μ_p at each stage of the Runge-Kutta level. This procedure however may be quite expensive and in practice violation of positivity is rarely observed. In such circumstances one can set initially a numerical tolerance on μ_p or simply repeat the computation with a larger value of μ_p.

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4. Numerical results. In this section we perform some numerical tests for the exponential Runge Kutta schemes applied to the case of the full Boltzmann equation. We consider the first order IF scheme (3.10) and the second and third order IF schemes obtained using second order Midpoint and third order Heun methods respectively. All methods satisfy Proposition 3.7 and thus can be implemented using Monte Carlo strategies for the evaluation of the five fold collision integral [31]. Since we are interested in measuring the time discretization error of the different schemes we need to cancel the other sources of error in the computations. This is realized using a very large number of particles and statistical averages.

We consider two different test cases: first the evolution of the fourth order moment in a space homogeneous case and then the heat flux behavior in a space inhomogeneous shock problem.

4.1. Homogeneous relaxation. As initial data we consider an equilibrium Maxwellian distribution with temperature $T = 6$, density $\varrho = 1$ and mean velocity $u = -0.5$. To this distribution we add a bump on the right tail along the x-axis. The bump is realized adding a Maxwellian with mass $\varrho_b = 0.5\ \varrho$, mean velocity $u_b = 4\ \sqrt{T}$ and temperature $T_b = 0.5\ T$ to the initial Maxwellian distribution. We consider the case of Maxwell interactions and hard spheres. The simulations are run till the equilibrium is approximately reached, which means $t = 0.4$ in the case of hard spheres and $t = 0.8$ for Maxwell molecules. The reference solution is computed by the Bird method [3] which converges toward the exact solution when the number of particles goes to infinity.

In Figure 4.1 we show the L_2 error for the fourth order moment of the distribution function f for Maxwellian molecules and hard spheres. In each of the plots the error is depicted for different choices of the time step: respectively $\mu\Delta t/\varepsilon = 0.5, 1, 2$ and 4. In the case of Maxwell molecules $\mu = 1$ while for hard spheres $\mu \gg 1$ is an upper bound for the collision cross section computed using the simple choice (3.38). As a consequence for the same values of $\mu\Delta t/\varepsilon$ the L_2 norm of the error is larger for hard spheres with respect to Maxwell molecules. Here we do not use any strategy to reduce the effect of possible overestimation of μ as described in section 3.4. We leave this possibility to further research. The expected convergence rate is observed for all schemes.

4.2. A shock wave computation. We consider a Sod shock tube test with initial values

$$\mathbf{u}_L = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}, \text{ if } 0 \leq x < 0.5 \quad \mathbf{u}_R = \begin{pmatrix} 0.125 \\ 0 \\ 4 \end{pmatrix}, \text{ if } 0.5 \leq x \leq 1.$$

The solution is computed using 150 grid points in space, the final time is $t = 0.05$. The transport step is solved exactly by particle transport as it is usual in Monte Carlo methods. As before we consider a very large number of particles and averaged the solution over several runs. Again the reference solution is obtained letting the time step go to zero and the number of particles to infinity using Bird's DSMC method.

We report the heat flux profile for the first and the second order Runge Kutta integration factor scheme in figures 4.2. A first order splitting is employed for the first order IF while a second order Strang splitting is used in combination with the second order IF method. From top to bottom of the figures the Knudsen number values are $\varepsilon = 10^{-3}, 5\ 10^{-4}$ and 10^{-4}. The time step is fixed equal to 10^{-3}.

Both schemes show a good agreement with the reference solution for $\varepsilon = 10^{-3}$. Then, when the Knudsen number is halved we start to see some discrepancies between the profiles of the first order method and the reference solution. On the contrary the second order method is still in good agreement with the reference solution. When the Knudsen number becomes 1/10 of the time step both the schemes present larger errors but they are still able to catch correctly the departure from equilibrium. This is especially true for the second order scheme which is able to reproduce the correct profile with sufficient accuracy.
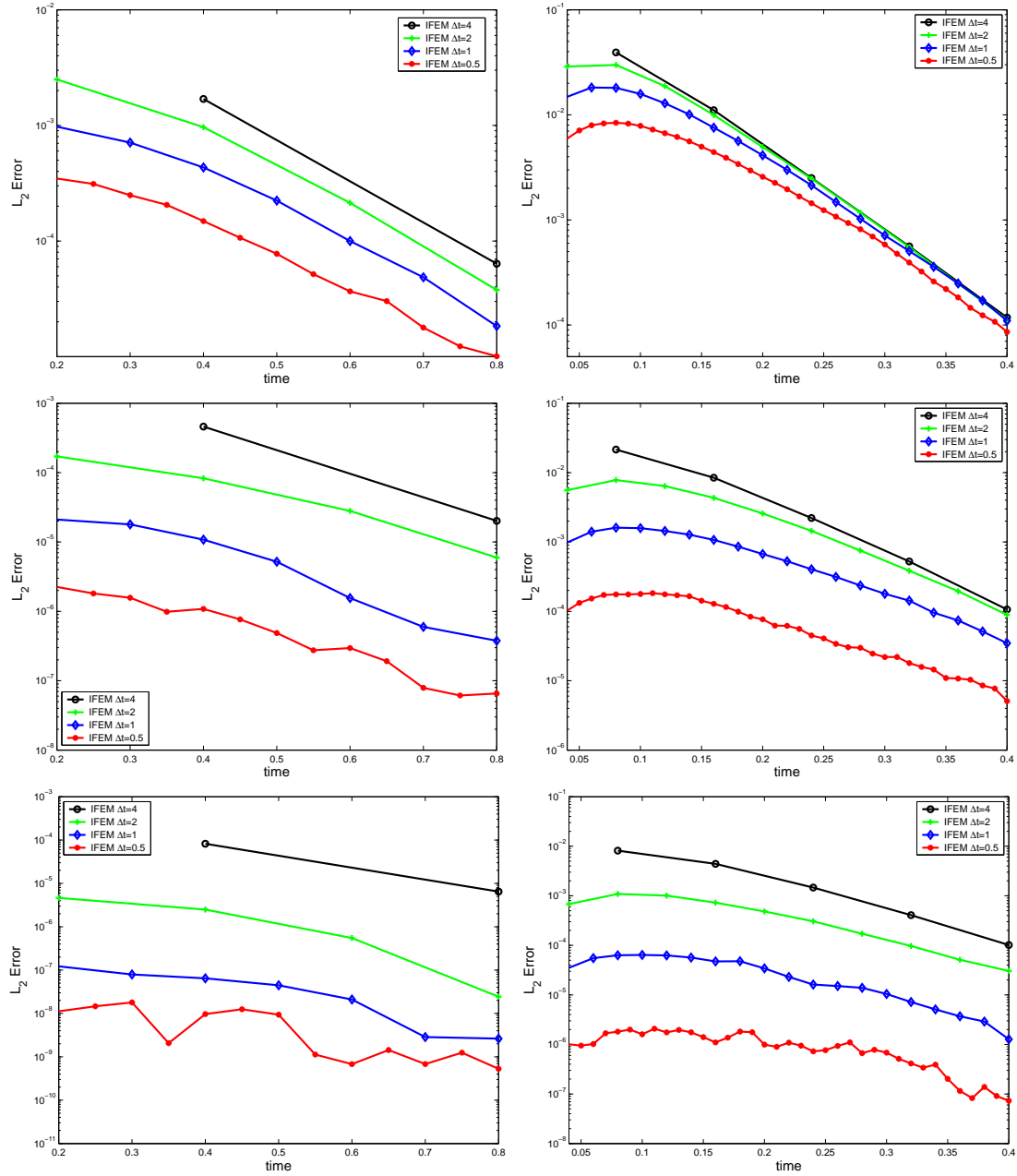
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Fig. 4.1. L_2 Error for the Fourth Moment Relaxation for the homogeneous relaxation problem with Maxwellian particles (left) and hard spheres (right).

5. Conclusions and further developments. We have presented a general class of exponential schemes for the numerical solution of nonlinear kinetic equations in stiff regimes. The schemes generalize the class of schemes previously developed in [16] and share the fundamental property of asymptotic preservation. Even if the schemes have been developed in the case of the Boltzmann equation for dilute gases, extension of the schemes to other collisional kinetic equations
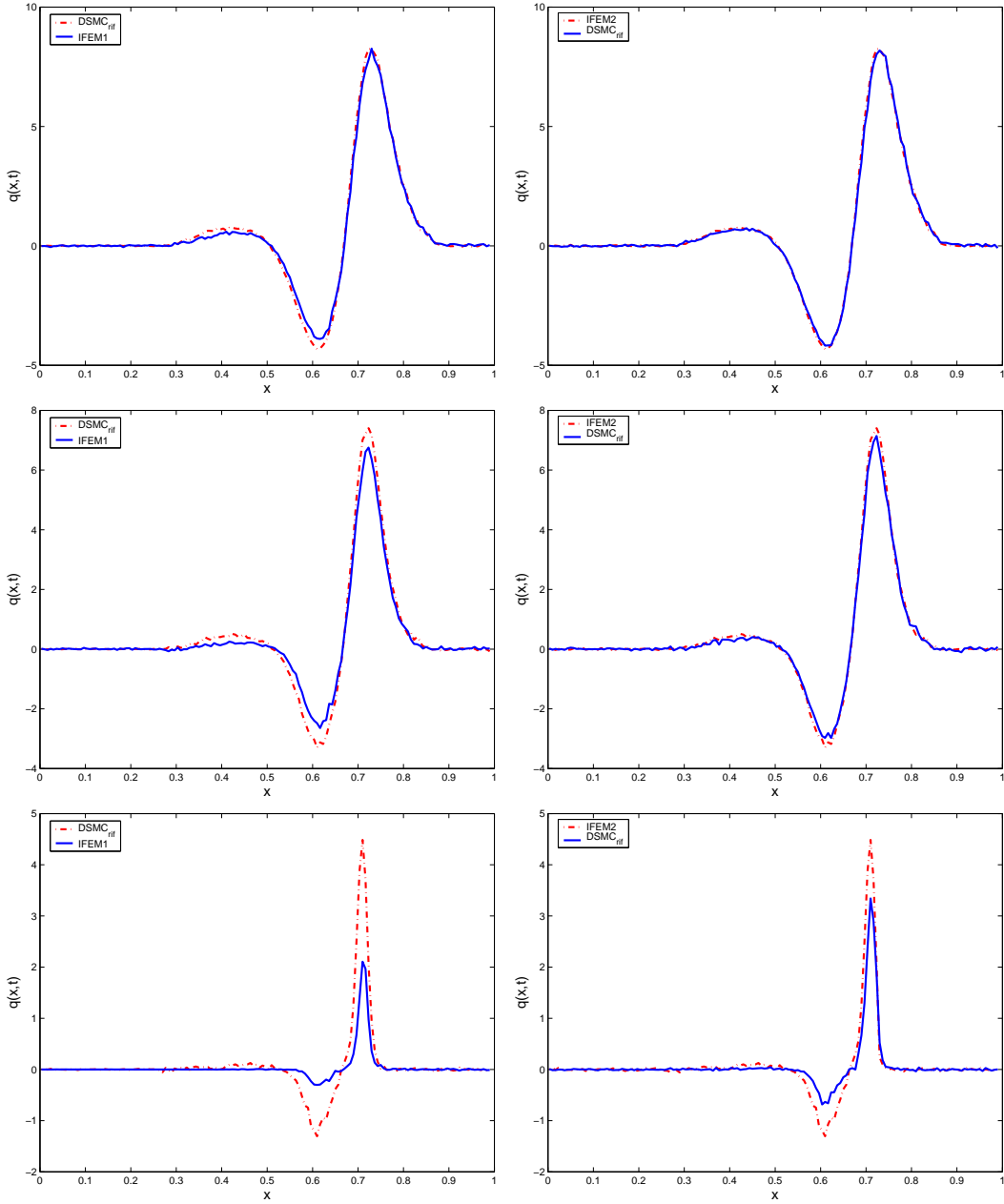
FIG. 4.2. *Heat flux profile for first order (left) and second order (right) IF schemes. Top Kundsen number* $\varepsilon = 10^{-3}$*, middle* $\varepsilon = 5\ 10^{-4}$ *and bottom* $\varepsilon = 10^{-4}$*.* $\Delta t = 10^{-3}$*.*

which possess a smooth equilibrium state are straightforward. Let us also mention that decomposition (3.3) represents only one of the possible choices in order to linearize the collision operator close to equilibrium and then using it as a penalization factor in the construction of the numerical methods. For example a more accurate penalization can be obtained using the so called *ES-BGK* equilibrium function [23], instead of the standard Maxwellian, which is well known to provide a better approximation of the collision dynamics. Let us finally mention that in principle the schemes

can be extended to the non-splitting method case for (2.1) using the integral representation

$$f(x, v, t) = f(x, v, 0)e^{-\mu t/\varepsilon} + \frac{\mu}{\varepsilon} \int_0^t e^{-\mu(t-s)/\varepsilon} G(y(s)) \, ds + \frac{\mu}{\varepsilon} \int_0^t e^{-\mu(t-s)/\varepsilon} M(x, v, s) \, ds,$$

where

$$G(f, s) = -\frac{\varepsilon}{\mu} v \nabla_x f + \frac{P(f, f)}{\mu} - M.$$

Here we do not explore further this direction and we leave it to future researches.

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